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A Note on Strong Convergence for Pettis Integrable Functions*

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Abstract. A partial answer is given to an open question of Amrani, Castaing and Valadier [1] about conditions under which a weakly converging sequence of Pettis integrable functions actually converges strongly.

1. Introduction

In a recent paper [1] Amrani, Castaing and Valadier obtained a convergence result for a sequence of Pettis integrable functions that continues a series of results started by Olech [11], Tartar [12] and Visintin [13]. These results are of the following type. Given a weakly convergent sequence of integrable functions, one formulates an extreme point condition (pointwise) for the limit function. This eliminates persistent oscillations, and forces the convergence to the limit function to be strong as well (for instance, convergence in measure). By the use of Young measures, Visintin's results were improved in [4] and [14] and extended to Bochner integrable functions with values in a separable Banach space. In this note it is shown that Theorem 2.4, the main result of [1], can be approached in a manner that is very similar to the method introduced in [4]. This leads to a considerable improvement of that result, and in a direction that is suggested on p. 331 of [1]. However, we have not been able to solve the open question on p. 331 completely.

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2. Preliminaries

Let (Ω, Σ, μ) a complete probability space. Let E be a separable Banach space. By E' we denote the topological dual of E. Occasionally, the weak topology $\sigma(E, E')$ will be referred to by adding the symbol "w" in the mathematical formulas, and, likewise, the symbol "s" will indicate the strong (i.e., norm) topology on E. Let $\mathcal{B}(E)$ be the Borel σ-algebra on E; observe that $\mathcal{B}(E) := \mathcal{B}(E_s) = \mathcal{B}(E_w)$. Let $C_b(E_s)$ be the set of all bounded s-continuous functions on E and let $\mathcal{P}(E)$ be the set of all probability measures on $(E, \mathcal{B}(E))$. We denote by $\mathcal{R}(\Omega; E)$ the set of all Young measures from (Ω, Σ) into $(E, \mathcal{B}(E))$, i.e., the set of all functions $\delta : \Omega \to \mathcal{P}(E)$ such that $\omega \mapsto \delta(\omega)(B)$ is Σ-measurable for every $B \in \mathcal{B}(S)$. A sequence $\{\delta_n\}_n$ in $\mathcal{R}(\Omega; E)$ converges *narrowly* (with respect to the s-topology on E) to another Young measure $\delta \in \mathcal{R}(\Omega; E)$ if

$$\liminf_{n\to\infty} \int_\Omega \Big[\int_E g(\omega, x)\delta_n(\omega)(dx)\Big]\mu(d\omega) \geq \int_\Omega \Big[\int_E g(\omega, x)\delta(\omega)(dx)\Big]\mu(d\omega) \tag{1}$$

for every measurable function $g : \Omega \times E \to [0, +\infty]$ such that $g(\omega, \cdot)$ is s-lower semicontinuous on E for every $\omega \in \Omega$. We shall denote such convergence by $\delta_n \stackrel{s}{\Longrightarrow} \delta$. Several equivalent statements of $\delta_n \stackrel{s}{\Longrightarrow} \delta$ can be given; see [7, Theorem 4.7]. Observe that to any measurable function $f : \Omega \to E$ there corresponds a Young measure ε_f in $\mathcal{R}(\Omega; E)$, called the *relaxation* of f. This is given by $\varepsilon_f(\omega)(B) := 1_B(f(\omega))$; i.e., $\varepsilon_f(\omega)$ is the point measure (alias Dirac measure) at $f(\omega)$. The following well-known result (e.g., see [6, Proposition 4.16]) will be very important:

Proposition 2.1. *For a sequence $\{f_n\}_n$ of measurable functions $f_n : \Omega \to E$ and another measurable function $f : \Omega \to E$ the following are equivalent:*
(a) $\varepsilon_{f_n} \stackrel{s}{\Longrightarrow} \varepsilon_f$,
(b) $\{f_n\}_n$ *converges in measure to* f.

Proof. By Theorem 4.7(c) in [7], (1) remains valid if instead of requiring g there to have values in $[0, +\infty]$, we require g to have values in $[-1, +\infty]$. So (a) $\Rightarrow$ (b) follows in an elementary way by using, for arbitrary $\epsilon > 0$, the function $g_\epsilon : \Omega \times E \to [-1, +\infty]$ given by

$$g_\epsilon(\omega, x) := \begin{cases} -1 & \text{if } \|x - f(\omega)\| \geq \epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, (b) $\Rightarrow$ (a) follows by obvious arguments involving the extraction of an a.e. convergent subsequence from $\{f_n\}_n$ and Fatou's lemma.

Following [2], where a completely equivalent definition was introduced (see Remark 3.4 in [7]), a sequence $\{\delta_n\}_n$ in $\mathcal{R}(\Omega; E)$ is called *s-tight* (or simply *tight*) if for every $\epsilon > 0$ there exists a measurable multifunction $\Gamma_\epsilon : \Omega \to k(E)$ such that

$$\sup_n \int_\Omega \delta_n(\omega)(S \setminus \Gamma_\epsilon(\omega))d\mu \leq \epsilon.$$

Here $k(E)$ denotes the set of all s-compact sets in E. Also, a sequence $\{f_n\}_n$ of measurable functions $f_n : \Omega \to E$ is said to be *s-tight* if and only if $\{\varepsilon_{f_n}\}_n$, the corresponding sequence of relaxations, is s-tight [2], i.e., if and only if for every $\epsilon > 0$ there exists a measurable multifunction $\Gamma_\epsilon : \Omega \to k(E)$ such that

$$\sup_{n\in\mathbb{N}} \mu(\{\omega \in \Omega : f_n(\omega) \notin \Gamma_\epsilon(\omega)\}) \leq \epsilon.$$

Specializing to the present context, the fundamental lower closure theorem for Young measures [7, Theorem 4.13], which is equivalent to Prohorov's theorem for Young measures that was given in [2, 5], takes the following form.

Theorem 2.2. *Let $\{f_n\}_n$ be a sequence of measurable functions $f_n : \Omega \to E$. If $\{\varepsilon_{f_n}\}_n$ in $\mathcal{R}(\Omega; E)$ is s-tight, then there exist a subsequence $\{f_{n_j}\}_{n_j}$ and an associated $\delta_* \in \mathcal{R}(\Omega; E)$ such that*

$$\liminf_{j\to\infty} \int_\Omega g(\omega, f_{n_j}(\omega))\mu(d\omega) \geq \int_\Omega \Big[\int_E g(\omega, x)\delta_*(\omega)(dx)\Big]\mu(d\omega)$$

for every measurable function $g : \Omega \times E \to (-\infty, +\infty]$ for which $g(\omega, \cdot)$ is lower semicontinuous on E_s and for which

$$\lim_{\alpha\to\infty} \sup_n \int_{\{g(\cdot,f_n(\cdot))\leq-\alpha\}} \max(0, -g(\cdot, f_n(\cdot))d\mu = 0$$

(i.e., $\{\max(0, -g(\cdot, f_n(\cdot)))\}_n$ is uniformly integrable). In particular, this means $\varepsilon_{f_{n_j}} \overset{s}{\Longrightarrow} \delta_*$. Moreover,

$$\delta_*(\omega)(s\text{-Ls}\{f_n(\omega)\}) = 1 \quad for\ a.e.\ \ \omega \text{ in } \Omega,$$

where

$$s\text{-Ls } \{f_n(\omega)\} := \bigcap_{p=1}^{\infty} s\text{-cl } \{f_n(\omega) : n \geq p\}$$

is the pointwise Kuratowski limes superior of the sequence $\{f_n(\omega)\}_n$.

Let Ω^{pa} denote the purely atomic part of (Ω, Σ, μ).

Proposition 2.3. *In Theorem 2.2 $\{f_{n_j}\}_{n_j}$ can always be chosen in such a way that for some measurable function $f_* : \Omega^{pa} \to E$*

$$\|f_{n_j}(\omega)) - f_*(\omega)\| \to 0 \ for\ a.e.\ \omega \ \ in\ \Omega^{pa}.$$

Proof. We argue as in [3] (see also the proof of Theorem 3.7 of [2]). The purely atomic part Ω^{pa} consists of at most countably many non-null sets that are atoms A_j. On each atom A_j all Σ-measurable functions from Ω into E are constant a.e. Moreover, by Castaing representation such constancy on atoms extends to measurable closed-valued multifunctions. Fix j and choose $\epsilon = \mu(A_j)/2$ in the definition of s-tightness for (ε_{f_n}); then there exists a $\Gamma_\epsilon : \Omega \to k(E)$ such that $\mu(f_n \notin \Gamma_\epsilon) \leq \epsilon$ for all $n \in \mathbb{N}$. Let $K_j \subset E$ be the a.e.-constant value of Γ_ϵ on

A_j; this is a s-compact set. Then it follows that for every n the a.e.-constant value on A_j of f_n belongs to K_j. Hence, by a standard diagonal extraction procedure we obtain a preliminary subsequence of $\{f_n\}_n$ that s-converges a.e. on $\Omega^{pa} = \cup_j A_j$ to some $f_* : \Omega^{pa} \to E$. After this, one applies Theorem 2.2. ∎

We finish these preparations with the following result. Here $\mathcal{L}wc(E)$ denotes the collection of all convex, closed sets in E that are weakly locally compact and do not contain any lines. Also, $\tau(E', E)$ stands for the Mackey topology on E'. Recall from [8, III.32] that there always exists a countable, $\tau(E', E)$-dense subset of E'.

Lemma 2.4. *Let $\nu \in \mathcal{P}(E)$ and $C \in \mathcal{L}wc(E)$ be such that $\nu(C) = 1$. If $a \in \partial_{ext} C$ and if for a $\tau(E', E)$-dense sequence $\{x_i'\}_i$ in E'*

$$\langle x_i', a\rangle = \int_C \langle x_i', x\rangle \nu(dx) \;\; for\ every\ \; i \in \mathbb{N},$$

then ν is the point measure at a.

Proof. Suppose that ν were not concentrated in $\{a\}$. Then there would certainly exist a closed, bounded and convex subset B of C such that $a \notin B$ and $\gamma := \nu(B) > 0$. Also, γ could not be equal to 1, or else one would get $a \in B$ by the Hahn-Banach theorem. So it would follow that γ lies in $(0, 1)$. Now define two probability measures on C by setting $\nu_1 := \nu(\cdot \cap B)/\gamma$ and $\nu_2 := \nu(\cdot \cap (C \setminus B))/(1-\gamma)$. Then $\nu = \gamma\nu_1 + (1-\gamma)\nu_2$. Since B is bounded and E is separable Banach, the barycenter a_1 of ν_1 exists in the form of the Bochner integral $a_1 := \int_B x\nu_1(dx)$. Clearly, $a_1 \in B \subset C$ by the Hahn-Banach theorem (since B is closed and convex) and a_1 satisfies

$$\langle x', a_1\rangle = \int_B \langle x', x\rangle \nu_1(dx) = \gamma^{-1} \int_B \langle x', x\rangle \nu(dx) \;\; \text{for every} \;\; x' \in E'.$$

It follows that

$$\int_{C\setminus B} \langle x_i', x\rangle \nu_2(dx) = (1-\gamma)^{-1} \Big[\int_C \langle x_i', x\rangle \nu(dx) - \int_B \langle x_i', x\rangle \nu(dx) \Big] =$$
$$= (1-\gamma)^{-1} \langle x_i', a - \gamma a_1\rangle \;\; \text{for every} \;\; i \in \mathbb{N}.$$

We denote by a_2 the element $(a - \gamma a_1)/(1-\gamma)$ in E. From the above we know

$$\langle x_i', a_2\rangle = \int_{C\setminus B} \langle x_i', x\rangle \nu_2(dx) \le \sup_{x\in C} \langle x_i', x\rangle \;\; \text{for every} \;\; i \in \mathbb{N}.$$

In view of the fact that C is closed, convex, weakly locally compact and contains no line, it follows from this that a_2 belongs to C, by [9] (see also [8, Lemma III.34]). Then $a = \gamma a_1 + (1-\gamma)a_2$, with $a_1, a_2 \in C$. This contradicts the hypothesis that a is an extreme point of C. ∎

3. From Weak to Strong Convergence for Pettis Integrable Functions

Recall that a scalarly μ-integrable function $f : \Omega \to E$ is *Pettis integrable* if for every $A \in \Sigma$ there exists $\nu_f(A) \in E$ such that

$$\langle x', \nu_f(A)\rangle = \int_A \langle x', f\rangle d\mu \ \text{ for all } \ x' \in E'.$$

We denote by $P_E^1(\mu)$ the set of all Pettis integrable functions $f : \Omega \to E$. A norm on $P_E^1(\mu)$ is defined by

$$\|f\|_{Pe} := \sup_{x' \in E'} \Big\{ \int_\Omega |\langle x', f\rangle| \, d\mu : \|x'\|' \leq 1 \Big\}.$$

In [1] a sequence $\{f_n\}_n$ in $P_E^1(\mu)$ is called *Pettis uniformly integrable* if for every $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that for every $A \in \Sigma$ with $\mu(A) \leq \delta_\epsilon$ the following inequality holds:

$$\sup_{n \in \mathbb{N}} \|1_A f_n\|_{Pe} \leq \epsilon.$$

We shall adhere to this name, even though a more proper name for this notion would have been *Pettis equi-absolute continuity* or *Pettis uniform absolute continuity.* Also, a sequence $\{f_n\}_n$ in $P_E^1(\mu)$ is said to converge *weakly* to $f \in P_E^1(\mu)$ if

$$\lim_{n\to\infty} \int_A \langle x', f_n\rangle d\mu = \int_A \langle x', f\rangle d\mu \ \text{ for every } \ A \in \Sigma \ \text{ and } \ x' \in E'.$$

Our main result is as follows:

Theorem 3.1. *Let $\{f_n\}_n$ be a sequence in $P_E^1(\mu)$ such that:*
(a) *$\{f_n\}_n$ is s-tight;*
(b) *$\{f_n\}_n$ is Pettis uniformly integrable;*
(c) *$\{f_n\}_n$ converges weakly to $f \in P_E^1(\mu)$ with $f(\omega) \in \partial_{ext}$ (co cl s-Ls $\{f_n(\omega)\}$) a.e.*

Suppose also that co cl s-Ls $\{f_n(\omega)\}$ *belongs to $\mathcal{L}wc(E)$ a.e. Then* $\lim_{n\to 0} \|f_n - f\|_{Pe} = 0$.

This extends Theorem 2.4, the main result of [1] in several ways. First, our condition (c) gives a partial answer to the open question formulated on p. 331 of [1]: we are able to relax the corresponding assumption in [1] that there is a multifunction $\Phi : \Omega \to \mathcal{L}wc(E)$ with $\{f_n(\omega)\}_n \subset \Phi(\omega)$ and $f(\omega) \in \partial_{ext}(\Phi(\omega))$ a.e. For under such a condition it is immediate that co cl s-Ls$\{f_n(\omega)\} \subset \Phi(\omega)$ a.e., and $f(\omega) \in$ co cl s-Ls $\{f_n(\omega)\} \subset \Phi(\omega)$ is a consequence of the same arguments as those that will be used in our proof. Second, notice that two other improvements have been made:
(i) A more demanding tightness assumption is made in [1], the multifunctions Γ_ϵ in the definition of tightness are not only to have values in the collection of all

convex sets in $k(E)$, but Γ_ϵ is also required to be Pettis-integrable itself, which involves uniform integrability.
(ii) While we are able to express our results in terms of the Kuratowski s-limes superior, the open question in [1] is only formulated in terms of the w-limes superior.

Our proof will make use of the following straightforward result in [1], which is a version of the Vitali-Lebesgue Theorem in $P^1_E(\mu)$.

Proposition 3.2. [1, Proposition 2.1] *Suppose that $\{f_n\}_n$ is a Pettis uniformly integrable sequence in $P^1_E(\mu)$ converging in measure to $f \in P^1_E(\mu)$. Then* $\lim_{n\to 0} \|f_n - f\|_{Pe} = 0$.

Proof of Theorem 3.1. We follow the ideas of [4]. Let $\alpha := \limsup_n \|f_n - f\|_{Pe}$. It is enough to show $\alpha = 0$. Elementarily, there exists a subsequence $\{f_m\}_m$ such that $\alpha = \lim_m \|f_m - f\|_{Pe}$. Consider the Young measure relaxations ε_{f_n} of the functions f_n. Then (a) is equivalent to saying that the sequence $\{\varepsilon_{f_n}\}_{n\in\mathbb{N}}$ is tight in $\mathcal{R}(\Omega; E)$; so of course $\{\varepsilon_{f_m}\}_m$ is tight as well. Therefore, by Theorem 2.2, there exist a subsequence $\{f_{m'}\}_{m'}$ of $\{f_m\}_m$ and a Young measure $\delta_* \in \mathcal{R}(\Omega; E)$ such that $\varepsilon_{f_{m'}} \stackrel{s}{\Longrightarrow} \delta_*$. We distinguish now between what happens on Ω^{pa}, the purely atomic part of (Ω, Σ, μ), and on $\Omega^{na} := \Omega \setminus \Omega^{pa}$, which is the nonatomic part of (Ω, Σ, μ). On Ω^{pa} we know from Proposition 2.3 that, for some measurable $f_* : \Omega^{pa} \to E$, we can suppose without loss of generality that $\{f_{m'}(\omega)\}_{m'}$ s-converges to $f_*(\omega)$ for a.e. ω in Ω^{pa}. By weak convergence of $\{f_{m'}\}_{m'}$ to the function f, it follows immediately that $f_* = f$ a.e. on Ω^{pa}. So actually $\{f_{m'}\}_{m'}$ converges to f a.e. on Ω^{pa}. Next, we consider what happens on Ω^{na}. Let $\{x'_i\}_i$ be any countable $\tau(E', E)$-dense subset of E'. Then for every $A \in \Sigma$, $A \subset \Omega^{na}$, we have that

$$\int_A \langle x'_i, f_{m'}\rangle d\mu \to \int_A \Big(\int_E \langle x'_i, x\rangle \delta_*(\omega)(dx)\Big)\mu(d\omega)$$

by applying Theorem 2.2 (use both $g(\omega, x) := 1_A(\omega)\langle x'_i, x\rangle$ and $g'(\omega, x) := -1_A(\omega)\langle x'_i, x\rangle$). Note also in this connection that $\{\langle x'_i, f_{m'}(\cdot)\rangle\}_{m'}$ is uniformly integrable, as a consequence of condition (b) and the nonatomicity of Ω^{na}. To be more precise, (b) by itself implies that $\{\langle x'_i, f_{m'}(\cdot)\rangle\}_{m'}$ is equi-absolutely continuous, and then Exercise II.5.5 of [10] guarantees its uniform integrability on the nonatomic space Ω^{na}. On the other hand, from the hypothesis we know that

$$\int_A \langle x'_i, f_{m'}\rangle d\mu \to \int_A \langle x'_i, f\rangle d\mu.$$

So for almost every $\omega \in \Omega^{na}$

$$\langle x'_i, f(\omega)\rangle = \int_E \langle x'_i, x\rangle \delta_*(\omega)(dx) \quad \text{for every} \quad i \in \mathbb{N}.$$

Moreover, Theorem 2.2 also gives $\delta_*(\omega)$ (cl s-Ls $\{f_n(\omega)\}$) $= 1$ for a.e. $\omega \in \Omega^{na}$. By Lemma 2.4, applied to $\nu := \delta_*(\omega)$, $C :=$ co cl s-Ls $\{f_n(\omega)\}$ and $a := f(\omega)$, we obtain that $\delta_*(\omega) = \varepsilon_{f(\omega)}$ for every non-exceptional ω in Ω^{na}. Recall that on Ω^{pa} we saw earlier that $\{f_m\}_m$ s-converges to f a.e. By Proposition 2.1 it follows that $\{f_{n'}\}_{n'}$ μ-converges to f. So Proposition 3.2 gives $\|f_{m'} - f\|_{Pe} \to 0$. Since $\alpha = \lim_{m'} \|f_{m'} - f\|_{Pe}$, it follows that $\alpha = 0$. ■

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