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A GENERAL APPROACH TO LOWER SEMICONTINUITY AND LOWER CLOSURE IN OPTIMAL CONTROL THEORY*

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Abstract. A self-contained approach to lower semicontinuity and lower closure evolves from an extension of relaxed control theory, which is based on a central relative weak compactness criterion (called tightness) and relaxation in all but one variable. Two lower closure results for outer integral functionals with variable abstract time domain are developed. The first of these has a convexity condition for the integrand and generalizes all similar results in the literature. The second lower closure result is of a new kind; among other things, it implies a quite general version of Fatou's lemma in several dimensions.

Key words. relaxed control theory, tightness, normal integrands, outer integral functionals, lower semicontinuity, lower closure, Fatou's lemma in several dimensions

1. Introduction. This paper presents a rather self-contained approach to lower closure—and lower semicontinuity—in optimal control theory. (An excellent description of the role of lower closure in the existence theory for optimal control has been given in [27a].) Quite similar approaches lead to existence results in other areas of the decision sciences, notably in economics (competitive equilibria, optimal growth theory) and statistics (statistical decision theory); cf. e.g. [3d, h, i, l]. Essentially, this approach is an extension of relaxed control theory [33], [23], [32], [6], [3]. Thus, in our approach the subjects of relaxed control theory and lower closure are brought together.

We shall obtain here essentially two quite general lower closure results. The first step in either proof is the same; it depends on relaxation in all but one variable (cf. [22]) and an associated relative weak compactness criterion, called *tightness*, for sets of measurable functions (considered as parametrized measures); cf. [3]. It turns out that tightness can hold naturally for the trajectories, time domains (variable), "derivative functions", "singular component functions" and control functions of a control problem; cf. Examples 2.1–2.5. To facilitate the presentation of our results, central results of relaxed control theory have been concentrated in Theorem I, which can be regarded as an extension of the classical theorem by Yu. V. Prokhorov in topological measure theory [7]; cf. [3i].

After this common step the proofs are quickly finished by two quite different continuations. Theorem 3.1 follows by an application of Jensen's inequality, and Theorem 3.7 by applying Lyapunov's theorem. These results are then expanded by the consideration of variable time domains (as introduced in [3g]) and nonmeasurable integrands (by means of Lemma II, first formulated in [3k]). This leads to Theorems 3.3 and 3.8. The former result is a generalization of a well-known lower semicontinuity result (e.g. [16]); cf. Corollary 3.5. In § 4 it is shown to be equivalent to a lower closure result for abstract finite-dimensional orientor fields (Theorem 4.3). As such it generalizes all similar lower closure results in the literature (e.g. [10d, (6.i)], [30a, Thm. 3.1], [11b, Thm. 4.1], [11c, Thm. 3.1], [11d, Thm. 3.1], [3e, Thm. 5]). For a more concrete orientor field it leads to Theorem 4.6, a lower closure result of Lagrange type; this, too, subsumes all similar results in the literature, such as [10d, (6.ii)], [11b, Thm. 4.2], [3e, Thms. 7, 8, 10, Prop. 9]. The other main lower closure result, Theorem 3.8, is more general than Corollary 3.9, the multidimensional Fatou lemma of [31], which in

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turn subsumes all previous versions of this lemma ([29], [11c], [1b]) as well as certain existence results for allocation problems ([2], [6, Prop. III.2.1], [1a], [3a]).

In our presentation of the major results on the subject of lower closure we have tried to keep the principal results as uncluttered as possible. In subsequent remarks we then present alternative formulations, "extension modules", etc. We hope that this enables the reader to see the main lines of thought more clearly.

Let us now introduce some conventions and definitions concerning (outer) integration. Let $(T, \mathcal{T}, \mu)$ be a finite measure space. The set of all $\mathcal{T}$-measurable functions from T into $[-\infty, +\infty]$ will be denoted by $\mathcal{M}(T; [-\infty, +\infty])$. For any nonnegative $\phi \in \mathcal{M}(T; [-\infty, +\infty])$ the integral $\int_T \phi \, d\mu$—possibly equal to $+\infty$—is defined in the classical sense [26]. For any $\phi \in \mathcal{M}(T; [-\infty, +\infty])$ we define

$$\int_T \phi \, d\mu \equiv \int_T \phi^+ \, d\mu - \int_T \phi^- \, d\mu,$$

where $\phi^+ \equiv \max(\phi, 0)$, $\phi^- \equiv \max(-\phi, 0)$, with the convention $(+\infty) - (+\infty) = +\infty$. For any $\psi: T \to [-\infty, +\infty]$, possibly not $\mathcal{T}$-measurable, the outer integral of ψ over T with respect to μ is defined by

$$\bar{\int_T} \psi \, d\mu \equiv \inf \left\{ \int_T \phi \, d\mu \colon \phi \in \mathcal{M}(T; [-\infty, +\infty]),\ \phi \geqq \psi \text{ a.e. in } T \right\};$$

it is easy to see that this infimum is attained for some $\phi \in \mathcal{M}(T; [-\infty, +\infty])$, $\phi \geqq \psi$ a.e. in T. Also, it is obvious that outer and ordinary integration coincide on $\mathcal{M}(T; [-\infty, +\infty])$.

2. Tightness. Let $(T, \mathcal{T}, \mu)$ be a finite measure space and S a standard Borel space (alias metrizable Lusin space [12]). Let $\mathcal{B}(S)$ stand for the Borel σ-algebra on S, and $M_1^+(S)$ for the set of all probability measures on $(S, \mathcal{B}(S))$; equipped with the usual weak (alias narrow) topology, $M_1^+(S)$ is also standard Borel [12, III.60].

The set of all $\mathcal{B}(S)$-measurable functions from T into S will be denoted by $\mathcal{M}(T; S)$. Instead of $\mathcal{M}(T; M_1^+(S))$ we shall write $\mathcal{R}(T; S)$; the elements of this set are frequently referred to as "parametrized measures", "relaxed controls", etc. [23], [32], [6].

An *integrand* on $T \times S$ is a function from $T \times S$ into $(-\infty, +\infty]$. An integrand g on $T \times S$ is said to be *lower semicontinuous* if $g(t, \cdot): t \mapsto g(t, s)$ is lower semicontinuous on S for every $t \in T$ and it is said to be *normal* if it is lower semicontinuous and $\mathcal{T} \times \mathcal{B}(S)$-measurable. Let $\mathcal{G}(T; S)$ denote the set of all normal integrands on $T \times S$; $\mathcal{G}^+(T; S)$ will then stand for the set of all nonnegative normal integrands on $T \times S$. The subset $\mathcal{H}(T; S)$ of $\mathcal{G}^+(T; S)$ is defined to consist of all $h \in \mathcal{G}^+(T; S)$ such that for every $t \in T$, $\gamma \in \mathbb{R}$

$$\{s \in S \colon h(t, s) \leqq \gamma\} \text{ is compact.}$$

For $s_0 \in \mathcal{M}(T; S)$, $g \in \mathcal{G}(T; S)$ we shall frequently denote the function $t \mapsto g(t, s_0(t))$ by $g(\cdot, s_0)$. A sequence $\{s_k\}_1^\infty \subset \mathcal{M}(T; S)$ is defined to be *tight* if there exists $h \in \mathcal{H}(T; S)$ such that

$$\sup_k \int_T h(t, s_k(t)) \, \mu(dt) < +\infty.$$

When formulated in terms of $\mathcal{R}(T; S)$, this concept is a generalization of tightness in topological measure theory [7], [3i]; cf. Appendix A and Remark 2.6 below.

The following examples illustrate the various forms in which tightness can manifest itself in the existence theory for optimal control. Let X and V be standard Borel spaces and r, $\bar{r}$ given dimensions.

Example 2.1. Let $\{x_k\}_0^\infty \subset \mathcal{M}(T; X)$ be such that

$$x_k(t) \to x_0(t) \text{ a.e. in } T. \tag{2.1}$$

Then $\{x_k\}_1^\infty$ is tight, as we can see as follows. Let N stand for the exceptional null set in (2.1). For $t \in T \backslash N$ we define

$$h(t, x) \equiv \begin{cases} 0 & \text{if } x \in \{x_k(t)\}_0^\infty, \\ +\infty & \text{else.} \end{cases}$$

For $t \in N$ we define

$$h(t, x) \equiv \begin{cases} 0 & \text{if } x = x_0(t), \\ +\infty & \text{else.} \end{cases}$$

Then $h \in \mathcal{H}(T; X)$, as is easy to see. Also,

$$\sup_k \int_T h(\cdot, x_k)\, d\mu = 0.$$

Example 2.2. Let $\{\xi_k\}_0^\infty \subset \mathcal{L}_1(T; \mathbb{R}^r)$ be such that

$$\{\xi_k\}_1^\infty \text{ converges weakly in } \sigma(\mathcal{L}_1^r, \mathcal{L}_\infty^r) \text{ to } \xi_0. \tag{2.2}$$

Then $\{\xi_k\}_1^\infty$ is tight, as can be seen, for instance, by applying de la Vallée-Poussin's theorem [12, II.22, 25]. By this result there exists a lower semicontinuous function $h': \mathbb{R}_+ \to \mathbb{R}_+$ such that $h'(\gamma)/\gamma \to +\infty$ as $\gamma \to +\infty$ and

$$\sup_k \int_T h'(|\xi_k(t)|)\mu(dt) < +\infty,$$

where $|\cdot|$ indicates the usual Euclidean norm. Now take $h(t, \xi) \equiv h'(|\xi|)$.

Example 2.3. Let $\{\eta_k\}_1^\infty \subset \mathcal{L}_1(T; \mathbb{R}^{\bar{r}})$ be such that

$$\sup_k \int_T |\eta_k|\, d\mu < +\infty. \tag{2.3}$$

Then $\{\eta_k\}_1^\infty$ is tight, as is evident from setting $h(t, \eta) \equiv |\eta|$.

Example 2.4. Let $\{v_k\}_1^\infty \subset \mathcal{M}(T; V)$. Then we shall have

$$\{v_k\}_1^\infty \text{ is tight,} \tag{2.4}$$

when, for instance,

$$\{v_k(t)\}_1^\infty \text{ is relatively compact a.e. in } T.$$

This is seen by introducing

$$h(t, v) \equiv \begin{cases} 0 & \text{if } v \in \text{cl}\,\{v_k(t) \colon k \in \mathbb{N}\}, \\ +\infty & \text{else.} \end{cases}$$

The above examples will play roles further on. So as to illustrate the connection of tightness with topological measure theory, we consider one more example.

Example 2.5. Let Z be a Polish space and let $\{z_k\}_0^\infty \subset \mathcal{M}(T; Z)$ be such that

$$\{\mu_k\}_1^\infty \text{ converges weakly to } \mu_0,$$

where μ_k stands for the image of the measure μ under z_k. Then $\{z_k\}_1^\infty$ is tight, as seen by applying Prokhorov's theorem [7, Thm. 6.2]. By this result there exists for every $p\in\mathbb{N}$ a compact subset K_p of Z such that

$$\sup_k \mu_k(Z\backslash K_p)\leqq 3^{-p}.$$

Setting

$$h(t,z)\equiv\begin{cases}2^p & \text{if } z\in K_p\Big\backslash\Big(\bigcup_{j\leqq p-1} K_j\Big), \quad p\in\mathbb{N},\\ +\infty & \text{if } z\in Z\Big\backslash\Big(\bigcup_{p=1}^{\infty} K_p\Big),\end{cases}$$

we see that h belongs to $\mathcal{H}(T;Z)$ and

$$\sup_k \int_T h(\cdot, z_k)\, d\mu \leqq 2\mu(T)+4.$$

Remark 2.6. It is easy to see from the previous example that a subset Ω of $M_1^+(Z)$ is tight in the sense of topological measure theory [7, p. 37] if and only if there exists a function $h: Z\to[0,+\infty]$ such that $\{z\in Z: h(z)\leqq\gamma\}$ is compact for every $\gamma\in\mathbb{R}$ and

$$\sup_{\nu\in\Omega}\int_Z h\, d\nu<+\infty.$$

This shows that our definition of tightness is a generalization of the classical one.

Good reasons for considering the tightness property will be produced now (and in later proofs). In Appendix A it is shown that tightness implies relative sequential compactness in some suitable topology on $\mathcal{R}(T;S)$; this actually generalizes one half of Prokhorov's theorem [7, Thms. 6.1, 6.2]. (Generalization of the other half is almost trivial; cf. Example 2.5.) The gist of this can be formulated as follows.

THEOREM I. *Suppose that $\{s_k\}_1^\infty\subset\mathcal{M}(T;S)$ is tight. Then there exist a subsequence $\{\ell\}$ of $\{k\}$ and a parametrized measure $\delta_*\in\mathcal{R}(T;S)$ such that for every $g\in\mathcal{G}(T;S)$*

$$\lim_\ell \int_T g(\cdot, s_\ell)\, d\mu \geqq \int_T g(\cdot,\delta_*)\, d\mu = \int_T\left[\int_S g(t,s)\delta_*(t)(ds)\right]\mu(dt), \tag{2.5}$$

provided that

$$\{g^-(\cdot, s_\ell)\} \text{ is uniformly integrable.} \tag{2.6}$$

Moreover, for a.e. $t\in T$ the measure $\delta_*(t)$ is carried by the set

$$\bigcap_{p=1}^{\infty} \operatorname{cl}\{s_\ell(t): \ell\geqq p\}$$

of all limit points of $\{s_\ell(t)\}$.

Remark 2.7. The following obvious addition can be made in Theorem I: the generalized limit δ_* of the subsequence $\{s_\ell\}$ satisfies

$$\int_T h(\cdot,\delta_*)\, d\mu\leqq \sup_k\int_T h(\cdot, s_k)\, d\mu<+\infty, \tag{2.7}$$

where $h\in\mathcal{H}(T;S)$ is as in the definition of tightness for $\{s_k\}_1^\infty$.

An important property of tightness is the following. Let S' be another standard Borel space. Then marginal tightness implies joint tightness, as is expressed more formally below.

PROPOSITION 2.8. *Suppose that* $\{s_k\}_1^\infty \subset \mathcal{M}(T; S)$ *and* $\{s'_k\}_1^\infty \subset \mathcal{M}(T; S')$ *are tight. then* $\{(s_k, s'_k)\}_1^\infty$ *is tight in* $\mathcal{M}(T; S\times S')$.

Proof. It is enough to remark that for $h \in \mathcal{H}(T; S)$, $h' \in \mathcal{H}(T; S')$ an integrand $\tilde{h} \in \mathcal{H}(T; S\times S')$ is defined by

$$\tilde{h}(t, (s, s')) \equiv h(t, s) + h'(t, s'). \qquad \text{QED}$$

Let us see what more can be said about the "generalized limits" of a tight sequence by considering again the above examples.

Example 2.1 (continued). Every generalized limit δ_* of $\{x_\ell\}$ can be identified with x_0, in that

(2.8) $\delta_*(t)$ is the Dirac (or point) measure at $x_0(t)$ a.e. in T.

This follows from Theorem I by observing that for a.e. $t \in T$ the only limit point of $\{x_\ell(t)\}$ is $x_0(t)$.

Example 2.2 (continued). Every generalized limit δ_* of $\{\xi_\ell\}$ is such that

(2.9) $$\text{bar } \delta_*(t) \equiv \int_{\mathbb{R}^r} \xi\delta_*(t)(d\xi) \text{ exists and equals } \xi_0(t) \text{ a.e. in } T.$$

To see this connection, note first that for a.e. $t \in T$

$$\int_{\mathbb{R}^r} |\xi|\delta_*(t)(d\xi) < +\infty$$

by (2.7) and the properties of h'. These same properties imply that for every $\tilde{\xi} \in \mathbb{R}^r$, $B \in \mathcal{T}$ the sequences $\{g^-(\cdot, \xi_\ell)\}$ and $\{g^+(\cdot, \xi_\ell)\}$ are uniformly integrable, where

$$g(t, \xi) \equiv \begin{cases} \langle \tilde{\xi}, \xi\rangle & \text{if } t \in B, \\ 0 & \text{else.} \end{cases}$$

Here $\langle\cdot,\cdot\rangle$ stands for the usual inner product. Hence, we may invoke Theorem I for both g and $-g$. By (2.2) it follows that

$$\int_B \langle \tilde{\xi}, \xi_0(t)\rangle \mu(dt) = \int_B \langle \tilde{\xi}, \text{bar } \delta_*(t)\rangle \mu(dt).$$

Since $\tilde{\xi}$ and B were arbitrary, the result follows.

Example 2.3 (continued). Every generalized limit δ_* of $\{\eta_\ell\}$ is such that

(2.10) $\eta_*(t) \equiv \text{bar } \delta_*(t)$ exists a.e. in T,

(2.11) $\eta_*(t) \in \bigcap_{p=1}^{\infty} \text{cl co}\{\eta_\ell(t) : \ell \geqq p\}$ a.e. in T,

(2.12) $\eta_* \in \mathcal{L}_1(T; \mathbb{R}^r)$.

To prove this, let us first note that by (2.7)

$$\int_T \left[\int_{\mathbb{R}^r} |\eta|\delta_*(t)(d\eta)\right]\mu(dt) < +\infty;$$

therefore (2.10) and (2.12) hold. By Theorem I we also have that the probability measure $\delta_*(t)$ is carried by the set $\bigcap_{p=1}^\infty \text{cl}\{\eta_\ell(t) : \ell \geqq p\}$ a.e. in T; hence, the barycenter of $\delta_*(t)$ belongs to the closed convex hull of that same set. This proves (2.11).

Example 2.3 (variant). In addition to the usual suppositions in Example 2.3, suppose that T is the unit interval, $\mathcal{T}$ the Lebesgue σ-algebra and μ the Lebesgue measure on T. Suppose now also that there exists $F: T \to \mathbb{R}_+^{\bar{r}}$, componentwise nondecreasing and right-continuous on T, such that for every $t \in T$

$$\lim_k \int_0^t \eta_k^+(\tau)\, d\tau = F(t), \tag{2.13}$$

where $(\eta_k^+)^j \equiv \max(\eta_k^j, 0)$ defines η_k^+ in terms of its component functions, $j = 1, \cdots, \bar{r}$. Then we have in addition to (2.10)–(2.12) that

$$\eta_*^+(t) \leqq \frac{dF^{ac}}{dt}(t) \text{ a.e. in } T, \tag{2.14}$$

where F^{ac} stands for the absolutely continuous part (componentwise) of F with respect to the Lebesgue measure μ (Lebesgue decomposition). This is demonstrated by an application of Theorem I to

$$g(t, \eta) \equiv \begin{cases} \max(\eta^j, 0) & \text{if } \alpha \leqq t \leqq \beta, \\ 0 & \text{else,} \end{cases}$$

for arbitrary $\alpha, \beta \in T$, $j = 1, \cdots, \bar{r}$. In view of (2.13) it follows then that

$$\nu_F^j((\alpha, \beta]) \equiv F^j(\beta) - F^j(\alpha) \geqq \int_\alpha^\beta (\eta_*^+)^j(t)\, dt.$$

Since the collection of finite disjoint unions of intervals $(\alpha, \beta]$ forms an algebra generating the Borel σ-algebra on T, it follows by Carathéodory's extension theorem that for every Borel set B in T

$$\nu_F(B) \geqq \int_B \eta_*^+(t)\, dt.$$

Augmenting B by a negligible set can only increase the left side of this inequality. Therefore the inequality also holds for $B \in \mathcal{T}$. Now (2.14) follows from a well-known property of Lebesgue decomposition [26, IV.1.3].

Example 2.4 (continued). There exists a subsequence of $\{v_k\}_1^\infty$ of which every generalized limit δ_* is such that

$$\delta_*(t) \text{ is a Dirac measure a.e. in } T^{\mathrm{pa}},$$

where T^{pa} denotes the purely atomic part of T. (For the sake of clarity we remark that this statement is made under the mere assumption (2.4) of tightness.) We prove this by fixing a collection of atoms A_p of which T^{pa} is the union; of course, this collection can be taken so as to be at most countable. Let $h \in \mathcal{H}(T; V)$ be as in the definition of tightness; it follows from Lemma A.1 (Appendix A) that for every atom A_p there is $h_p: V \to [0, +\infty]$ such that $h(t, \cdot) = h_p$ a.e. on A_p. Also, since a standard Borel space is isomorphic to a Borel set in $\mathbb{R}$ [12, III.20], every function v_k is equal to a constant a.e. on A_p; this constant will be denoted by $v_{k,p}$. It is now easy to see from the definition of tightness that for every atom A_p the sequence $\{v_{k,p}\}$ is relatively compact. Hence, by a diagonal extraction argument we can find a subsequence $\{k'\}$ of $\{k\}$ such that for every p the sequence $\{v_{k',p}\}$ converges. We conclude that on T^{pa}, with $\{k\}$ replaced by $\{k'\}$, the situation of Example 2.1 prevails. The proof is now easily finished.

Example 2.5 (continued). Every generalized limit δ_* of $\{z_k\}$ is such that the marginal on Z of the product measure of μ and δ_* equals μ_0. This is seen as follows. Let $c \in \mathscr{C}_b(Z)$ be arbitrary, where $\mathscr{C}_b(Z)$ stands for the set of all bounded continuous functions on Z. We can apply Theorem I to c and $-c$. This gives

$$\int_Z c(z)\mu_0(dz) = \int_T \left[\int_Z c(z)\delta_*(t)(dz)\right]\mu(dt),$$

and since c is arbitrary the result has been proven.

3. Lower closure for outer integral functionals. The first in a series of lower closure results for integral functionals will now be formulated.

THEOREM 3.1. *Suppose that* $\{x_k\}_0^\infty$, $\{\xi_k\}_0^\infty$, $\{\eta_k\}_1^\infty$ *satisfy* (2.1)–(2.3). *Then there exist a subsequence* $\{\ell\}$ *of* $\{k\}$ *and* $\eta_* \in \mathscr{L}_1(T; \mathbb{R}^{\bar{r}})$ *such that*

$$\underline{\lim}_{\ell} \int_T g(\cdot, x_\ell, \xi_\ell, \eta_\ell)\, d\mu \geqq \int_T g(\cdot, x_0, \xi_0, \eta_*)\, d\mu \tag{3.1}$$

for every normal integrand g *on* $T\times(X\times\mathbb{R}^r\times\mathbb{R}^{\bar{r}})$ *satisfying*

$$g(t, x_0(t), \cdot, \cdot) \text{ is convex on } \mathbb{R}^r\times\mathbb{R}^{\bar{r}} \text{ a.e. in } T, \tag{3.2}$$

$$\{g^-(\cdot, x_\ell, \xi_\ell, \eta_\ell)\} \text{ is uniformly integrable.} \tag{3.3}$$

Moreover, η_* *is such that*

$$\eta_*(t) \in \bigcap_{p=1}^{\infty} \operatorname{cl}\operatorname{co}\{\eta_\ell(t) : \ell \geqq p\} \text{ a.e. in } T. \tag{3.4}$$

Proof. By what was proven for Examples 2.1–2.3 the sequence $\{(x_k, \xi_k, \eta_k)\}_1^\infty \subset \mathscr{M}(T; X\times\mathbb{R}^r\times\mathbb{R}^{\bar{r}})$ is tight (Proposition 2.8). It follows from Theorem I that there exist a subsequence $\{\ell\}$ of $\{k\}$ and $\delta_* \in \mathscr{R}(T; X\times\mathbb{R}^r\times\mathbb{R}^{\bar{r}})$ such that for every normal integrand g satisfying (3.3)

$$\alpha \geqq \int_T g(\cdot, \delta_*)\, d\mu, \tag{3.5}$$

where α denotes the left side of (3.1). Moreover, we know that for a.e. $t\in T$ the measure $\delta_*(t)$ is carried by the set of limit points of $\{(x_\ell(t), \xi_\ell(t), \eta_\ell(t))\}$, i.e., by the Cartesian product of $\{x_0(t)\}$ and the set of limit points of $\{(\xi_\ell(t), \eta_\ell(t))\}$, in view of (2.1). Denote by $\delta^*(t)$ the marginal probability measure of $\delta_*(t)$ on $\mathbb{R}^r\times\mathbb{R}^{\bar{r}}$ and the submarginals on $\mathbb{R}^r$ and $\mathbb{R}^{\bar{r}}$ by $\delta_1^*(t)$ and $\delta_2^*(t)$ respectively. For g as in (3.5) it now follows that

$$\alpha \geqq \int_T g(\cdot, x_0, \delta^*)\, d\mu. \tag{3.6}$$

A fortiori, we now have for every $g' \in \mathscr{G}(T; \mathbb{R}^r)$ for which $\{g'^-(\cdot, \xi_\ell)\}$ is uniformly integrable, that marginally

$$\underline{\lim}_{\ell} \int_T g'(\cdot, \xi_\ell)\, d\mu \geqq \int_T g'(\cdot, \delta_1^*)\, d\mu.$$

A similar situation is found for δ_2^*. Thus, marginally we find precisely the situations investigated in Examples 2.2–2.3. this means

$$\xi_0(t) = \operatorname{bar} \delta_1^*(t) \text{ a.e. in } T,$$

$$\eta_*(t) \equiv \operatorname{bar} \delta_2^*(t) \text{ exists a.e. in } T,$$

with η_* satisfying (2.11)–(2.12). Hence, by definition of barycenter

$$\operatorname{bar} \delta^*(t) = (\xi_0(t), \eta_*(t)) \text{ a.e. in } T. \tag{3.7}$$

We finish the proof by applying Jensen's inequality. Suppose for a moment that g is bounded from below by a constant. Then for a.e. $t \in T$ the function $g(t, x_0(t), \cdot, \cdot)$ is proper convex, unless it is identically equal to $+\infty$. The latter possiblity is trivial to deal with, so let us only look at the former. By applying a well-known result about proper convex functions and their affine minorants [9, I.4] and using (3.7) it follows that

$$g(t, x_0(t), \delta^*(t)) \geqq g(t, x_0(t), \xi_0(t), \eta_*(t)) \text{ a.e. in } T.$$

In view of (3.6) the desired inequality (3.1) then follows. Thus far, we worked under the extra assumption that g is bounded from below. For general g it follows easily from (3.3)—cf. [16]—that for every $\varepsilon > 0$ there exists $\gamma > 0$ such that

$$\int_T g(\cdot, x_{\mathscr{k}}, \xi_{\mathscr{k}}, \eta_{\mathscr{k}})\, d\mu \geqq \int_T \max(-\gamma, g(\cdot, x_{\mathscr{k}}, \xi_{\mathscr{k}}, \eta_{\mathscr{k}}))\, d\mu - \varepsilon$$

for all $\mathscr{k}$. By the inequality (3.1) for the normal integrand $\max(-\gamma, g)$ established above, by the inequality $\max(-\gamma, g) \geqq g$ and the arbitrary choice of ε, the inequality (3.1) must now also hold for g. QED

This is essentially our main lower closure result "with convexity". Its relation to other results in the literature will be discussed after Theorem 4.3. Next we shall derive more general and useful versions of this result; these apply to general integrands, whether measurable or not. The following lemma is instrumental [3k]; its proof can be found in Appendix A.

LEMMA II. *Suppose that $\mathscr{g}$ is a lower semicontinuous integrand on $T \times (X \times \mathbb{R}^r \times \mathbb{R}^{\bar{r}})$ with*

(3.8) $$\mathscr{g}(t, x_0(t), \cdot, \cdot) \text{ is convex on } \mathbb{R}^r \times \mathbb{R}^{\bar{r}} \text{ a.e. in } T.$$

Then there exists a normal integrand g on $T \times (X \times \mathbb{R}^r \times \mathbb{R}^{\bar{r}})$ such that

(3.9) $$g \geqq \mathscr{g},$$

(3.10) $$g(t, x_0(t), \cdot, \cdot) \text{ is convex on } \mathbb{R}^r \times \mathbb{R}^{\bar{r}} \text{ a.e. in } T$$

and for every $x \in \mathcal{M}(T; X)$, $\xi \in \mathcal{M}(T; \mathbb{R}^r)$, $\eta \in \mathcal{M}(T; \mathbb{R}^{\bar{r}})$

(3.11) $$\bar{\int}_T \mathscr{g}(t, x(t), \xi(t), \eta(t))\mu(dt) = \int_T g(t, x(t), \xi(t), \eta(t))\mu(dt),$$

where outer integration and integration are subject to conventions introduced in § 1.

THEOREM 3.2. *Suppose that $\{x_k\}_0^\infty$, $\{\xi_k\}_0^\infty$, $\{\eta_k\}_1^\infty$ satisfy* (2.1)–(2.3). *Then there exist a subsequence $\{\mathscr{k}\}$ of $\{k\}$ and $\eta_* \in \mathscr{L}_1(T; \mathbb{R}^{\bar{r}})$ such that*

(3.12) $$\underline{\lim}_{\mathscr{k}} \bar{\int}_T l(\cdot, x_{\mathscr{k}}, \xi_{\mathscr{k}}, \eta_{\mathscr{k}})\, d\mu \geqq \bar{\int}_T l(\cdot, x_0, \xi_0, \eta_*)\, d\mu$$

for every integrand l on $T \times X \times \mathbb{R}^r \times \mathbb{R}^{\bar{r}}$ satisfying

(3.13) *$l(t, \cdot, \cdot, \cdot)$ is lower semicontinuous at every point in $\{x_0(t)\} \times \mathbb{R}^r \times \mathbb{R}^{\bar{r}}$ a.e. in T,*

(3.14) *$l(t, x_0(t), \cdot, \cdot)$ is convex on $\mathbb{R}^r \times \mathbb{R}^{\bar{r}}$ a.e. in T,*

(3.15) *there is a uniformly integrable sequence $\{\lambda_{\mathscr{k}}\} \subset \mathscr{L}_1(T; \mathbb{R})$ with $l(\cdot, x_{\mathscr{k}}, \xi_{\mathscr{k}}, \eta_{\mathscr{k}}) \geqq \lambda_{\mathscr{k}}$ for all $\mathscr{k}$.*

Moreover, η_* satisfies (3.4).

Proof. Suppose first that l is bounded from below by a constant. The lower semicontinuous integrand $\bar{l}$ on $T\times(X\times\mathbb{R}^r\times\mathbb{R}^r)$ is defined by

$$\bar{l}(t,x,\xi,\eta)\equiv \varliminf_{x'\to x,\xi'\to\xi,\eta'\to\eta} l(t,x',\xi',\eta'). \tag{3.16}$$

By (3.13), (3.16) we have

$$l(t,x_0(t),\cdot,\cdot)=\bar{l}(t,x_0(t),\cdot,\cdot) \text{ a.e. in } T. \tag{3.17}$$

Clearly, (3.8) now holds for $\mathscr{g}=\bar{l}$ in view of (3.14). Applying Lemma II to $\mathscr{g}=\bar{l}$ gives that there exists a normal integrand g on $T\times(X\times\mathbb{R}^r\times\mathbb{R}^r)$ such that (3.9)–(3.11) hold. We can now apply Theorem 3.1 to g, since (3.2) holds by (3.10) and (3.3) by (3.9), (3.16) and the extra supposition. Thus, (3.1) holds for g. By using successively the inequality $l\geqq\mathscr{g}$, (3.11), (3.1) and (3.17), the inequality (3.12) follows. Secondly, consider the general case. By elementary properties of outer integration it follows easily from (3.15) that for every $\varepsilon>0$ there exists $\gamma>0$ such that for all ℓ

$$\tilde{\int_T} l(\cdot,x_\ell,\xi_\ell,\eta_\ell)\,d\mu\geqq\tilde{\int_T}\max(-\gamma,l(\cdot,x_\ell,\xi_\ell,\eta_\ell))\,d\mu-\varepsilon.$$

(Let $\phi_\ell\in\mathcal{M}(T;(-\infty,+\infty])$ correspond to $l(\cdot,x_\ell,\xi_\ell,\eta_\ell)$ as in the attainment property for outer integrals mentioned in § 1, $\phi_\ell\geqq l(\cdot,x_\ell,\xi_\ell,\eta_\ell)$ a.e. in T, and consider the identity

$$\tilde{\int_T} l(\cdot,x_\ell,\xi_\ell,\eta_\ell)\,d\mu=\int_{\{\phi_\ell\geqq-\gamma\}}\phi_\ell\,d\mu+\int_{\{\phi_\ell<-\gamma\}}\phi_\ell\,d\mu.$$

From this the above inequality follows quickly.) Just as in the proof of Theorem 3.1, the inequality (3.12) now follows by the previous step. QED

It is easy to convert Theorem 3.2 into a more useful result by "recombination of variables" [3e, g]. In particular, this will lead to closure results for models with variable time domain.

Let $\{T_k\}_0^\infty\subset\mathscr{T}$ be such that

$$1_{T_k}(t)\to 1_{T_0}(t) \text{ a.e. in } T, \tag{3.18}$$

where 1_{T_k} stands for the characteristic function of the set T_k. Note that this is equivalent to saying that the set $T_0\equiv\lim_k T_k$ exists modulo a null set; cf. [26, I.4]. Further, $\{x_k\}_0^\infty\subset\mathcal{M}(T;X)$, $\{\xi_k\}_0^\infty\subset\mathscr{L}_1(T;\mathbb{R}^r)$ and $\{\eta_k\}_1^\infty\subset\mathscr{L}_1(T;\mathbb{R}^{\bar{r}})$ are now also allowed to satisfy

$$x_k(t)\to x_0(t) \text{ a.e. in } T_0, \tag{3.19}$$

$$\{1_{T_k}\xi_k\}_1^\infty \text{ converges weakly in } \sigma(\mathscr{L}_1^r,\mathscr{L}_\infty^r) \text{ to } 1_{T_0}\xi_0, \tag{3.20}$$

$$\sup_k\int_{T_k}|\eta_k|\,d\mu<+\infty. \tag{3.21}$$

Note that (3.20) is already implied by (2.2) and (3.18); this follows by a simple imitation of the proof of Theorem 3.3 below. Further, let $\{d_k\}_1^\infty\subset\mathcal{M}(T;\mathbb{R}^r)$, $\{\bar{d}_k\}_1^\infty\subset\mathcal{M}(T;\mathbb{R}^{\bar{r}})$ and $\{\bar{e}_k\}_1^\infty\subset\mathscr{L}_1(T;\mathbb{R}^r)$ be such that

$$d_k(t)\to 0 \text{ a.e. in } T_0, \tag{3.22}$$

$$\bar{d}_k(t)\to 0 \text{ a.e. in } T_0, \tag{3.23}$$

$$\{1_{T_k}\bar{e}_k\}_1^\infty \text{ converges weakly in } \sigma(\mathscr{L}_1^{\bar{r}},\mathscr{L}_\infty^{\bar{r}}) \text{ to } 0. \tag{3.24}$$

THEOREM 3.3. *Suppose that* $\{T_k\}_0^\infty, \{x_k\}_0^\infty, \{\xi_k\}_0^\infty, \{\eta_k\}_1^\infty$ *satisfy* (3.18)–(3.21) *and that* $\{d_k\}_1^\infty, \{\bar{d}_k\}_1^\infty, \{\bar{e}_k\}_1^\infty$ *satisfy* (3.22)–(3.24). *Then there exist a subsequence* $\{\mathscr{k}\}$ *of* $\{k\}$ *and* $\eta_* \in \mathscr{L}_1(T; \mathbb{R}^{\bar{r}})$ *such that*

$$(3.25)\qquad \varliminf_{\mathscr{k}} \int_{T_{\mathscr{k}}}^{\sim} l(\cdot, x_{\mathscr{k}}, \xi_{\mathscr{k}}+d_{\mathscr{k}}, \eta_{\mathscr{k}}+\bar{d}_{\mathscr{k}}+\bar{e}_{\mathscr{k}})\, d\mu \geqq \int_{T_0}^{\sim} l(\cdot, x_0, \xi_0, \eta_*)\, d\mu,$$

for every integrand l *on* $T\times X\times\mathbb{R}^r\times\mathbb{R}^{\bar{r}}$ *satisfying*

(3.26) $l(t,\cdot,\cdot,\cdot)$ *is lower semicontinuous at every point in* $\{x_0(t)\}\times\mathbb{R}^r\times\mathbb{R}^{\bar{r}}$ *a.e. in* T_0,

(3.27) $l(t, x_0(t), \cdot, \cdot)$ *is convex on* $\mathbb{R}^r\times\mathbb{R}^{\bar{r}}$ *a.e. in* T_0,

(3.28) *there is a uniformly integrable sequence* $\{\lambda_{\mathscr{k}}\}\subset\mathscr{L}_1(T;\mathbb{R})$ *with* $1_{T_{\mathscr{k}}} l(\cdot, x_{\mathscr{k}}, \xi_{\mathscr{k}}+d_{\mathscr{k}}, \eta_{\mathscr{k}}+\bar{d}_{\mathscr{k}}+\bar{e}_{\mathscr{k}})\geqq\lambda_{\mathscr{k}}$ *for all* $\mathscr{k}$.

Moreover, η_* *satisfies*

$$(3.29)\qquad \eta_*(t)\in\bigcap_{p=1}^\infty \operatorname{cl}\operatorname{co}\{\eta_{\mathscr{k}}(t)\colon \mathscr{k}\geqq p\}\ \text{a.e. in } T_0.$$

Proof. Let $\bar{x}\in X$ be arbitrary but fixed. Let $\bar{x}_k\in\mathcal{M}(T;X)$ be such that it coincides with x_k on T_k and has the constant value $\bar{x}$ on $T\setminus T_k$; then $\bar{x}_k(t)\to\bar{x}_0(t)$ a.e. in T by (3.18)–(3.19). Let us now define $\tilde{x}_k\in\mathcal{M}(T; X\times\{0,1\}\times\mathbb{R}^r\times\mathbb{R}^{\bar{r}})$ by $\tilde{x}_k\equiv(\bar{x}_k, 1_{T_k}, 1_{T_k}d_k, 1_{T_k}\bar{d}_k)$ for $k\in\mathbb{N}$ and $\tilde{x}_0\equiv(\bar{x}_0, 1_{T_0}, 0, 0)$ for $k=0$. Also, we define $\{\tilde{\xi}_k\}_0^\infty\subset\mathscr{L}_1(T;\mathbb{R}^{r+\bar{r}})$ by $\tilde{\xi}_k\equiv(1_{T_k}\xi_k, 1_{T_k}\bar{e}_k)$ and $\tilde{\xi}_0\equiv(1_{T_0}\xi_0, 0)$, and finally $\{\tilde{\eta}_k\}_1^\infty$ by $\tilde{\eta}_k\equiv 1_{T_k}\eta_k$. By (3.18)–(3.24) and the above we have

$$\tilde{x}_k(t)\to\tilde{x}_0(t)\ \text{a.e. in } T,$$

$$\{\tilde{\xi}_k\}_1^\infty \text{ converges weakly in } \sigma(\mathscr{L}_1^{r+\bar{r}}, \mathscr{L}_\infty^{r+\bar{r}}) \text{ to } \tilde{\xi}_0,$$

$$\sup_k \int_T |\tilde{\eta}_k|\, d\mu < +\infty.$$

Given l with (3.26)–(3.28), we define the integrand $\tilde{l}$ by

$$\tilde{l}(t, \tilde{x}, \tilde{\xi}, \eta)\equiv\gamma l(t, x, \xi+d, \eta+\bar{d}+\bar{e})$$

for $\tilde{x}\equiv(x,\gamma,d,\bar{d})\in X\times\{0,1\}\times\mathbb{R}^r\times\mathbb{R}^{\bar{r}}$, $\tilde{\xi}\equiv(\xi,\bar{e})\in\mathbb{R}^r\times\mathbb{R}^{\bar{r}}$. By (3.26)

$$\tilde{l}(t,\cdot,\cdot,\cdot) \text{ is lower semicontinuous at every point in } \{\tilde{x}_0(t)\}\times\mathbb{R}^{r+\bar{r}}\times\mathbb{R}^{\bar{r}} \text{ a.e. in } T.$$

Also, in the same notation, we have

$$\tilde{l}(t, \tilde{x}_0(t), \tilde{\xi}, \eta)=\begin{cases} l(t, x_0(t), \xi, \eta+\bar{e}) & \text{if } t\in T_0,\\ 0 & \text{else.}\end{cases}$$

This shows that by (3.27)

$$\tilde{l}(t, \tilde{x}_0(t), \cdot, \cdot) \text{ is convex on } \mathbb{R}^{r+\bar{r}}\times\mathbb{R}^{\bar{r}} \text{ a.e. in } T.$$

Finally, it follows from (3.28) by definition of $\tilde{l}$ that for all $\mathscr{k}$

$$\tilde{l}(\cdot, \tilde{x}_{\mathscr{k}}, \tilde{\xi}_{\mathscr{k}}, \tilde{\eta}_{\mathscr{k}})\geqq\lambda_{\mathscr{k}}.$$

Hence the situation found in the statement of this theorem has been reduced completely to that of Theorem 3.2. It remains to invoke this result. □

Remark 3.4. Suppose that instead of the suppositions (2.1), (3.18)–(3.19) regarding convergence a.e. we make the following weaker suppositions about convergence in measure:

(2.1′) $\mu(\{t \in T: \operatorname{dist}(x_k(t), x_0(t)) > \varepsilon\}) \to 0$ for every $\varepsilon > 0$,

(3.18′) $\mu((T_k \backslash T_0) \cup (T_0 \backslash T_k)) \to 0$ for every $\varepsilon > 0$,

(3.19′) $\mu(\{t \in T_0: \operatorname{dist}(x_k(t), x_0(t)) > \varepsilon\}) \to 0$ for every $\varepsilon > 0$,

to be used instead of (2.1), (3.18) and (3.19) respectively. Then our previous results will remain valid, since we will now have (2.1), (3.18) and (3.19) respectively for suitable subsequences of $\{x_k\}_0^\infty$ and $\{T_k\}_0^\infty$. The same can be said if instead of (2.2), (3.20) we suppose

(2.2′) $\{\xi_k\}_1^\infty$ is uniformly integrable,

(3.20′) $\{1_{T_k}\xi_k\}_1^\infty$ is uniformly integrable,

provided that we denote any weak limit point in either case by ξ_0; here we use the Dunford–Pettis theorem [12, II.25] (or, alternatively, Theorem I and Example 2.2).

COROLLARY 3.5. *Suppose that $\{T_k\}_0^\infty$, $\{x_k\}_0^\infty$, $\{\xi_k\}_0^\infty$ satisfy (3.18)–(3.20). Then*

$$\varliminf_k \int_T^{\sim} l(\cdot, x_k, \xi_k)\, d\mu \geqq \int_{T_0}^{\sim} l(\cdot, x_0, \xi_0)\, d\mu$$

for every integrand l on $T \times X \times \mathbb{R}^r$ satisfying

$l(t, \cdot, \cdot)$ is lower semicontinuous at every point in $\{x_0(t)\} \times X \times \mathbb{R}^r$ a.e. in T_0,

$l(t, x_0(t), \cdot)$ is convex on $\mathbb{R}^r$ a.e. T_0,

there is a uniformly integrable sequence $\{\lambda_k\}_1^\infty \subset \mathcal{L}_1(T; \mathbb{R})$ with $1_{T_k} l(\cdot, x_k, \xi_k) \geqq \lambda_k$ for all $k \in \mathbb{N}$.

This generalizes a classical lower semicontinuity result for integral functionals to outer integral functionals with variable time domain [13, VIII.2.2], [17, 9.1.4], [10d, (1.iii)], [27c, d], [16], [3c, f].

Necessary conditions for lower semicontinuity in similar setups have been obtained in e.g. [16], [27d]; they indicate that the conditions of Corollary 3.5—and Theorem 3.3 by implication—are quite sharp.

COROLLARY 3.6. *Suppose that $(T, \mathcal{T}, \mu)$ is as in the variant of Example 2.3, that $\{\alpha_k\}_0^\infty$, $\{\beta_k\}_0^\infty \subset T$ are such that*

$$\alpha_k \to \alpha_0, \qquad \beta_k \to \beta_0$$

and that for $T_k \equiv [\alpha_k, \beta_k]$ the sequences $\{x_k\}_0^\infty$, $\{\xi_k\}_0^\infty$, $\{\eta_k\}_1^\infty$ satisfy (3.18)–(3.21) and (2.13). Then

$$\varliminf_k \int_{\alpha_k}^{\beta_k} g(\cdot, x_k, \xi_k, \eta_k)\, d\mu \geqq \int_{\alpha_0}^{\beta_0} g\left(t, x_0(t), \xi_0(t), \frac{dF^{ac}}{dt}(t)\right) dt$$

for every normal integrand g on $T \times (X \times \mathbb{R}^r \times \mathbb{R}^{\bar{r}})$ satisfying

$g(t, x_0(t), \cdot, \cdot)$ is convex on $\mathbb{R}^r \times \mathbb{R}^{\bar{r}}$ a.e. in T_0,

$g(t, x_0(t), \xi_0(t), \cdot)$ is nonincreasing on $\mathbb{R}^{\bar{r}}$ a.e. in T_0,

$\{g^-(\cdot, x_k, \xi_k, \eta_k)\}_1^\infty$ is uniformly integrable.

It would seem that this corollary is a new result; it can also be derived from [3c, Thm. 5], as was already argued in [3f, Thm. 3.2] for a slightly less general version.

Let us return to the proof of Theorem 3.1, which has been essential for the developments thus far. We can see that in deriving the inequality (3.6) the convexity property (3.2) was not used. Therefore (3.6) suggests a radically different way to obtain a lower closure result for integral functionals; namely, we can try to trade the convexity condition (3.2) for "extreme point considerations and Lyapunov's theorem". Thus, we enter the domain of "existence without convexity", explored for the first time in optimal control theory by L. W. Neustadt [25], from a completely new angle. The following lemma, proven in Appendix A by extreme point considerations and Lyapunov's theorem, will bring us the desired results.

LEMMA III. *Suppose that* $(T, \mathcal{T}, \mu)$ *is nonatomic and that* $\delta^* \in \mathcal{R}(T; V)$ *is tight with respect to* $h \in \mathcal{H}(T; V)$, *i.e.*,

$$\int_T h(\cdot, \delta^*)\, d\mu < +\infty.$$

Suppose that the normal integrands $g_1, \cdots, g_m$ *on* $T \times V$ *satisfy the following growth condition with respect to* h: *for every* $\varepsilon > 0$ *there is* $\phi_\varepsilon \in \mathcal{L}_1(T; \mathbb{R})$ *with*

$$g_j^-(t, v) \leqq \varepsilon h(t, v) + \phi_\varepsilon(t), \qquad j = 1, \cdots, m. \tag{3.30}$$

Then there exists $v^* \in \mathcal{M}(T; V)$ *such that*

$$\int_T g_j(\cdot, v^*)\, d\mu \leqq \int_T g_j(\cdot, \delta^*)\, d\mu, \qquad j = 1, \cdots, m.$$

THEOREM 3.7. *Suppose that* $\{x_k\}_0^\infty$, $\{v_k\}_1^\infty$ *satisfy* (2.1), (2.4). *Then there exists a subsequence* $\{\ell\}$ *of* $\{k\}$ *such that to every finite collection* $g_1, \cdots, g_m$ *of normal integrands on* $T \times (X \times V)$ *there corresponds* $v_* \in \mathcal{M}(T; V)$ *with*

$$\varliminf_{\ell} \int_T g_j(\cdot, x_\ell, v_\ell)\, d\mu \geqq \int_T g_j(\cdot, x_0, v_*)\, d\mu, \qquad j = 1, \cdots, m, \tag{3.31}$$

provided that

$$\{g_j^-(\cdot, x_\ell, v_\ell)\} \text{ is uniformly integrable}, \qquad j = 1, \cdots, m. \tag{3.32}$$

Moreover, v_* *satisfies*

$$v_*(t) \in \bigcap_{p=1}^{\infty} \operatorname{cl}\{v_\ell(t) : \ell \geqq p\} \text{ a.e. in } T. \tag{3.33}$$

Proof. We start by considering a preliminary subsequence $\{k'\}$ of $\{k\}$ which is such that $\{v_{k'}(t)\}$ converges a.e. on T^{pa} (Example 2.4). It is left to the reader to see that the proof of Theorem 3.1 can now be imitated up to formula (3.6). Here $\mathbb{R}^r \times \mathbb{R}^r$ is replaced by V and $\{\xi_k, \eta_k\}$ by $v_{k'}$; also we use tightness of $\{x_{k'}\}$, $\{v_{k'}\}$ and condition (3.32). Hence, there exist a subsequence $\{\ell\}$ of $\{k'\}$ and $v^* \in \mathcal{M}(T^{\text{pa}}; V)$, $\delta^* \in \mathcal{R}(T^{\text{na}}; V)$ such that for all j

$$\varliminf_{\ell} \int_T g_j(\cdot, x_\ell, v_\ell)\, d\mu \geqq \int_{T^{\text{pa}}} g_j(\cdot, x_0, v^*)\, d\mu + \int_{T^{\text{na}}} g_j(\cdot, x_0, \delta^*)\, d\mu, \tag{3.34}$$

where $T^{na} \equiv T \backslash T^{pa}$ denotes the nonatomic part of T and where the pointwise convergence on T^{pa} has been taken into account (cf. Example 2.1). Moreover, we have that

(3.35) $$v^*(t) \in V_1(t) \equiv \bigcap_{p=1}^{\infty} \text{cl}\,\{v_\ell(t) \colon \ell \geqq p\} \text{ a.e. in } T^{pa},$$

(3.36) $$\delta^*(t) \text{ is carried by } V_1(t) \text{ a.e. in } T^{na},$$

$$\int_{T^{na}} h(\cdot, \delta^*)\, d\mu < +\infty,$$

where h is as in Example 2.4. Momentarily we shall make an extra assumption: we assume that for every $\varepsilon > 0$ there exists $\phi_\varepsilon \in \mathcal{L}_1(T; \mathbb{R})$ with

(3.37) $$g_j^-(t, x_0(t), v) \leqq \varepsilon h(t, v) T \phi_\varepsilon(t), \qquad j = 1, \cdots, m.$$

This allows us to invoke Lemma III: there exists $v^{**} \in \mathcal{M}(T^{na}; V)$ such that

(3.38) $$\int_{T^{na}} g_j(\cdot, x_0, v^{**})\, d\mu \leqq \int_{T^{na}} g_j(\cdot, x_0, \delta^*)\, d\mu, \qquad j = 1, \cdots, m,$$

(3.39) $$\int_{T^{na}} g_{m+1}(\cdot, v^{**})\, d\mu \leqq 0,$$

where $g_{m+1} \in \mathcal{G}^+(T; V)$ is defined by

$$g_{m+1}(t, v) \equiv \begin{cases} 0 & \text{if } v \in V_1(t), \\ +\infty & \text{else.} \end{cases}$$

Defining $v_* \equiv v^*$ on T^{pa}, $v_* \equiv v^{**}$ on T^{na}, we see that (3.34) and (3.38) imply (3.31). Also, (3.35), (3.36) and (3.39) imply (3.33). Let us now see how the extra assumption (3.37) can be revoked. We shall apply the result established under (3.37) to the normal integrands $\tilde{g}_1, \cdots, \tilde{g}_m$ on $T \times (X \times V \times \mathbb{R})$ defined by

(3.40) $$\tilde{g}_j(t, x, v, \lambda) \equiv \max\,(g_j(t, x, v), \lambda).$$

We define also

$$\lambda_k(t) \equiv -|g^-(t, x_k(t), v_k(t))|,$$

where $(g^-)^j \equiv (g^j)^-$. By de la Vallée-Poussin's theorem there exists $h' : \mathbb{R}_+ \to \mathbb{R}_+$ with with $h'(\gamma)/\gamma \to +\infty$ as $\gamma \to +\infty$ and

$$\sup_k \int_T h'(|\lambda_k(t)|)\mu(dt) < +\infty,$$

in view of (3.32); cf. Example 2.2. Hence, by (2.4) the sequence $\{(v_k, \lambda_k)\}_1^\infty$ satisfies

(3.41) $$\sup_k \int_T \tilde{h}(\cdot, v_k, \lambda_k)\, d\mu < +\infty,$$

where $\tilde{h}(t, v, \lambda) \equiv h(t, v) + h'(|\lambda|)$, and

(3.42) $$\tilde{g}_j(\cdot, x_k, v_k, \lambda_k) = g_j(\cdot, x_k, v_k) \quad \text{for all } k \text{ and } j.$$

We may now apply the result established above. Note that (3.41) replaces (2.4) and that (3.37) obviously holds for $\tilde{g}_j^-$ with respect to $\tilde{h}$: in view of (3.40) and the properties of h', we have for every $\varepsilon > 0$ that there exists $\gamma_\varepsilon > 0$ with

$$\tilde{g}_j^-(t, x, v, \lambda) \leqq \max\,(-\lambda, 0) \leqq \varepsilon h'(|\lambda|) + \gamma_\varepsilon \leqq \varepsilon \tilde{h}(t, v, \lambda) + \gamma_\varepsilon.$$

Thus, we find that there exists $(v_*, \lambda_*) \in \mathcal{M}(T; V \times \mathbb{R})$ with

$$\underline{\lim}_{\mathscr{k}} \int_T \tilde{g}_j(\cdot, x_{\mathscr{k}}, v_{\mathscr{k}}, \lambda_{\mathscr{k}})\, d\mu \geqq \int_T \tilde{g}_j(\cdot, x_0, v_*, \lambda_*)\, d\mu, \qquad j = 1, \cdots, m.$$

In view of (3.40), (3.42), this amounts to (3.31). Moreover, we have

$$(v_*(t), \lambda_*(t)) \text{ is a limit point of } \{(v_{\mathscr{k}}(t), \lambda_{\mathscr{k}}(t))\} \text{ a.e. in } T,$$

and this implies (3.33). QED

Theorem 3.7 is a quite new result. Just as Theorem 3.1 was upgraded by using Lemma II and recombination of variables, so can this be done with Theorem 3.7.

Let $\{T_k\}_0^\infty$ be as in (3.18) ff. We shall now also consider a sequence $\{v_k\}_1^\infty$ of measurable functions, $v_k \in \mathcal{M}(T_k; V)$, such that there exists $h \in \mathcal{H}(T; V)$ with

$$\sup_k \int_{T_k} h(\cdot, v_k)\, d\mu < +\infty. \tag{3.43}$$

THEOREM 3.8. *Suppose that* $\{T_k\}_0^\infty$, $\{x_k\}_0^\infty$, $\{v_k\}_1^\infty$ *satisfy* (3.18), (3.19), (3.43). *Then there exists a subsequence* $\{\mathscr{k}\}$ *of* $\{k\}$ *such that to every finite collection* $l_1, \cdots, l_m$ *of integrands on* $T \times (X \times V)$ *there corresponds* $v_* \in \mathcal{M}(T_0; V)$ *with*

$$\underline{\lim}_{\mathscr{k}} \tilde{\int}_{T_{\mathscr{k}}} l_j(\cdot, x_{\mathscr{k}}, v_{\mathscr{k}})\, d\mu \geqq \tilde{\int}_{T_0} l_j(\cdot, x_0, v_*)\, d\mu, \qquad j = 1, \cdots, m,$$

provided that

there is a uniformly integrable sequence $\{\lambda_{\mathscr{k}}\} \subset \mathcal{L}_1(T; \mathbb{R})$ *with* $\bar{l}_j(\cdot, x_{\mathscr{k}}, v_{\mathscr{k}}) \geqq \lambda_{\mathscr{k}}$ *for all* $\mathscr{k}$, $j = 1, \cdots, m$,

where $\bar{l}_j$ *is defined by*

$$\bar{l}_j(t, x, v) \equiv \underline{\lim}_{x' \to x, v' \to v} l_j(t, x', v').$$

Moreover, v_* *satisfies*

$$v_*(t) \in \bigcap_{p=1}^{\infty} \operatorname{cl}\{v_k(t) : \mathscr{k} \geqq p\} \text{ a.e. in } T_0.$$

Proof. The proof is quite similar to the argument by which Theorem 3.1 was transformed—via Theorem 3.2—into Theorem 3.3. It will be left to the reader, except for the following point. Define $\bar{h} : T \to [0, +\infty]$ by $\bar{h}(t) \equiv \inf\{h(t, v) : v \in V\}$. Then $\bar{h}$ is measurable with respect to the completion $\bar{\mathcal{T}}$ of $\mathcal{T}$ [9, III.39]. By Fatou's lemma it follows from (3.18) that

$$\int_{T_0} \bar{h}\, d\mu \leqq \underline{\lim}_k \int_{T_k} h(\cdot, v_k)\, d\mu. \tag{3.44}$$

By inf-compactness of $h(t, \cdot)$ there exists for every $t \in T$ an element $v_t \in V$ with $h(t, v_t) = \bar{h}(t)$. Since the set of all $(t, v) \in T \times V$ for which $h(t, v) = \bar{h}(t)$ is $\bar{\mathcal{T}} \times \mathcal{B}(V)$-measurable, it follows from Aumann's theorem [14] that there exists $\bar{v} : T \to V$, $\bar{\mathcal{T}}$-measurable, such that $h(t, \bar{v}(t)) = \bar{h}(t)$ a.e. in T. Since V is isomorphic to a Borel subset of $\mathbb{R}$ [12], it follows that there exists a $\mathcal{T}$-measurable modification $\bar{\bar{v}} \in \mathcal{M}(T; V)$ of $\bar{v}$. In view of (3.44) we conclude that

$$\int_{T_0} h(\cdot, \bar{\bar{v}})\, d\mu \leqq \sup_k \int_{T_k} h(\cdot, v_k)\, d\mu.$$

Hence we obtain a tight sequence $\{\bar{v}_k\}_1^\infty \subset \mathcal{M}(T_0; V)$ by defining $\bar{v}_k \equiv v_k$ on $T_0 \cap T_k$ and $\bar{v}_k \equiv \bar{v}$ on $T_0 \backslash T_k$. As explained above, the rest of the proof is quite simple. QED

COROLLARY 3.9 (Fatou's lemma in several dimensions). *Suppose that* $\{\phi_k\}_1^\infty \subset \mathcal{L}_1(T; \mathbb{R}^m)$ *is such that*

$$\lim_k \int_T \phi_k \, d\mu \text{ exists (in } \mathbb{R}^m), \tag{3.45}$$

$$\{\phi_k^-\}_1^\infty \text{ is uniformly integrable.} \tag{3.46}$$

Then there exists $\phi_* \in \mathcal{L}_1(T; \mathbb{R}^m)$ *such that*

$$\int_T \phi_* \, d\mu \leqq \lim_k \int_T \phi_k \, d\mu,$$

$$\phi_*(t) \text{ is a limit point of } \{\phi_k(t)\}_1^\infty \text{ a.e. in } T.$$

Proof. We define the normal integrands $g_1, \cdots, g_{3m}$ on $T \times V$ by $g_j(t, v) \equiv \max(v^j, 0)$, $g_{m+j}(t, v) \equiv \max(-v^j, 0)$ and $g_{2m+j}(t, v) \equiv \min(v^j, 0)$, $j = 1, \cdots, m$. Also, we set $v_k \equiv \phi_k$. Note that (3.45)–(3.46) imply

$$\sup_k \int_T |\phi_k| \, d\mu < +\infty.$$

Hence (2.4) holds. Since (3.46) implies that (3.32) is fulfilled, we have by Theorem 3.7 the desired result, as is seen at once. QED

The above corollary was given in [31], where it was shown to be equivalent to slightly weaker form of Theorem 3.7. It generalizes the multidimensional Fatou lemmas of [29], [11c] and [1b], as well as a number of existence results for allocation problems arising in economics ([2], [6, Prop. III.2.1], [1a], [3a]). Corollary 3.9 can also be used directly to obtain existence results "without convexity conditions" for the optimal control of certain linear dynamical systems. In this way an existence result has been derived in [3m] for the optimal control of a linear integral equation having singular components; this generalizes the existence results of [25], [19], [4], [10b] and essentially also that of [30b].

4. Lower closure for orientor fields. It turns out that each of the main lower closure results of the previous section can be expressed in an alternative form, involving multifunctions. Here we shall only work out such a procedure for the lower closure result "with convexity", i.e. Theorem 3.3. It will lead us to the so-called lower closure results for orientor fields.

For a multifunction $Q: T \times X \rightrightarrows \mathbb{R}^r \times \mathbb{R}^r$ we define dom Q to be the set of those $(t, x) \in T \times X$ for which the set $Q(t, x)$ is nonempty. We shall say that Q has *property* (K) at a point $(t, x) \in T \times X$ if

$$Q(t, x) = \bigcap_{\gamma > 0} \operatorname{cl} \bigcup \{Q(t, x') : x' \in X, \operatorname{dist}(x', x) < \gamma\}, \tag{4.1}$$

where "dist" refers to any compatible metric on X. Note that x' runs effectively in the section at t of dom Q.

LEMMA 4.1. *Suppose* $Q: T \times X \rightrightarrows \mathbb{R}^r \times \mathbb{R}^r$ *and* $(t, x^0) \in T \times X$ *are given. Let* $l_1^Q, \cdots, l_r^Q$ *be integrands on* $T \times X \times \mathbb{R}^r \times \mathbb{R}^r$*, defined by*

$$l_j^Q(t, x, \xi, \eta) = \begin{cases} \eta^j & \text{if } (\xi, \eta) \in Q(t, x), \\ +\infty & \text{else.} \end{cases} \tag{4.2}$$

(a) *The following are equivalent*:

(4.3) *Q has property (K) at (t, x^0)*,

(4.4) *$l_j^Q(t, \cdot, \cdot, \cdot)$ is lower semicontinuous at every point of $\{x^0\}\times\mathbb{R}^r\times\mathbb{R}^{\bar{r}}$, $j=1, \cdots, \bar{r}$.*

(b) *The following are also equivalent*:

$$Q(t, x^0) \text{ is convex},$$

$$l_j^Q(t, x^0, \cdot, \cdot) \text{ is convex on } \mathbb{R}^r\times\mathbb{R}^{\bar{r}}, j=1, \cdots, \bar{r}.$$

Proof. Suppose first that (4.3) holds. Let $\{(x^k, \xi^k, \eta^k)\}_1^\infty$ be arbitrary and such that $x^k \to x^0$, $\xi^k \to \xi^0$, $\eta^k \to \eta^0$ for certain ξ^0, η^0. Without loss of generality we may assume that $\zeta \equiv \underline{\lim}_k l_j^k$ is finite, where $l_j^k \equiv l_j^Q(t, x^k, \xi^k \eta^k)$. Further, we can assume that $\zeta = \lim_k l_j^k$, instead of restricting ourselves to a suitable subsequence. It now follows that eventually l_j^k is finite, so without loss of generality we can suppose that $(\xi^k, \eta^k) \in Q(t, x^k)$ and $(\eta^k)^j = l_j^k$ for all $k \in \mathbb{N}$. Also, it follows that $\zeta = (\eta^0)^j$. For every $\gamma > 0$ we have now evidently

$$(\xi^0, \eta^0) \in \text{cl} \bigcup \{Q(t, x): x \in X, \text{dist}(x, x^0) < \gamma\},$$

so it follows from (4.3) that $(\xi^0, \eta^0) \in Q(t, x^0)$. By (4.2) we find $l_j^Q(t, x^0, \xi^0, \eta^0) = (\eta^0)^j = \zeta$. This shows that (4.4) holds.

Conversely, suppose that (4.4) holds. One inclusion in (4.1) is always trivial (take $x' = x$). To prove the other inclusion, let (ξ^0, η^0) belong to the right side in (4.1) (with $x = x^0$). This is easily seen to be equivalent to the following: for every $k \in \mathbb{N}$ there exist $x^k \in X$, $(\xi^k, \eta^k) \in Q(t, x^k)$ such that dist (x^k, x^0), $|\xi^k - \xi^0|$ and $|\eta^k - \eta^0|$ are all smaller than k^{-1}. Hence $x^k \to x^0$, $\xi^k \to \xi^0$ and $\eta^k \to \eta^0$. By (4.4) we have for any j

$$(\eta^0)^j = \underline{\lim}_k \, l_j^Q(t, x^k, \xi^k, \eta^k) \geqq l_j^Q(t, x^0, \xi^0, \eta^0).$$

Since the left side is finite, (4.2) gives that $(\xi^0, \eta^0) \in Q(t, x^0)$.

(b) The demonstration of this part is trivial. QED

Remark 4.2. From the final step in the proof of Lemma 4.1(a) it appears clearly that (4.3)–(4.4) are also equivalent to

(4.4′) $l_j^Q(t, \cdot, \cdot, \cdot)$ is lower semicontinuous at every point of $\{x^0\}\times\mathbb{R}^r\times\mathbb{R}^{\bar{r}}$ for some j, $1 \leqq j \leqq \bar{r}$.

A similar remark holds with regard to part (b) of Lemma 4.1.

We shall now state our main lower closure result for orientor fields and show that it is equivalent to Theorem 3.3.

THEOREM 4.3. *Suppose that $\{T_k\}_0^\infty$, $\{x_k\}_0^\infty$, $\{\xi_k\}_0^\infty$, $\{\eta_k\}_1^\infty$ satisfy* (3.18)–(3.21) *and that $\{d_k\}_1^\infty$, $\{\bar{d}_k\}_1^\infty$, $\{\bar{e}_k\}_1^\infty$ satisfy* (3.22)–(3.24). *Then there exist a subsequence $\{\ell\}$ of $\{k\}$ and $\eta_* \in \mathcal{L}_1(T, \mathbb{R}^{\bar{r}})$ such that*

(4.5) $$(\xi_0(t), \eta_*(t)) \in Q(t, x_0(t)) \text{ a.e. in } T_0,$$

(4.6) $$\underline{\lim}_{\ell} \int_{T_\ell} \eta_\ell^j \, d\mu \geqq \int_{T_0} \eta_*^j \, d\mu,$$

for every multifunction $Q: T \times X \rightrightarrows \mathbb{R}^r \times \mathbb{R}^{\bar{r}}$ and every j, $1 \leqq j \leqq \bar{r}$, such that

(4.7) *Q has property (K) at $(t, x_0(t))$ a.e. in T_0,* (4.8) $Q(t, x_0(t))$ *is a convex subset of* $\mathbb{R}^r \times \mathbb{R}^r$ *a.e. in* T_0,

(4.9) $(\xi_\ell(t) + d_\ell(t), \eta_\ell(t) + \bar{d}_\ell(t) + \bar{e}_\ell(t)) \in Q(t, x_\ell(t))$ *a.e. in* T_ℓ,

(4.10) $\{(1_{T_\ell}\eta_\ell^j)^-\}$ *is uniformly integrable.*

Moreover, η_* *satisfies* (3.29).

Proof. Suppose Q and j satisfy (4.7)–(4.10). Define $\tilde{x}_k \equiv (x_k, \bar{d}_k)$ for $k \in \mathbb{N}$ and $\tilde{x}_0 \equiv (x_0, 0)$. Then by (3.19), (3.22)

$$\tilde{x}_k(t) \to \tilde{x}_0(t) \text{ a.e. in } T_0.$$

We shall apply Theorem 3.3 to l_j defined by

$$l_j(t, \tilde{x}, \xi, \eta) \equiv l_j^Q(t, x, \xi, \eta + \bar{d}) - \bar{d}^j$$

for $\tilde{x} \equiv (x, \bar{d})$. Hence, in view of (4.2), (4.9),

$$1_{T_\ell} l_j(\cdot, \tilde{x}_\ell, \xi_\ell, \eta_\ell + \bar{e}_\ell) = 1_{T_\ell}(\eta_\ell^j + \bar{e}_\ell^j).$$

Further, by Lemma 4.1 it follows from (4.7)–(4.8) that

$l_j(t, \cdot, \cdot, \cdot)$ is lower semicontinuous at every point in $\{\tilde{x}_0(t)\} \times \mathbb{R}^r \times \mathbb{R}^r$,

$l_j(t, \tilde{x}_0(t), \cdot, \cdot)$ is convex on $\mathbb{R}^r \times \mathbb{R}^r$.

Hence we conclude that the conditions of Theorem 3.3 are fulfilled. We find therefore

(4.11) $$\varliminf_\ell \int_{T_\ell} (\eta_\ell^j + \bar{e}_\ell^j)\, d\mu \geqq \tilde{\int}_{T_0} l_j(\cdot, \tilde{x}_0, \xi_0, \eta_*)\, d\mu.$$

In view of (3.21), (3.24) and elementary properties of the outer integral it follows that

$$l_j(t, \tilde{x}_0(t), \xi_0(t), \eta_*(t)) < +\infty \text{ a.e. in } T_0.$$

This gives (4.5) by definition of l_j. Finally, (4.11) implies now (4.6). QED

Remark 4.4. Evidently, it follows from (4.5) that

(4.12) $$(t, x_0(t)) \in \operatorname{dom} Q \text{ a.e. in } T_0.$$

In the literature one usually considers only the restriction Q' of Q to dom Q. Let $A(t)$ denote the set of all $x' \in X$ with $(t, x') \in \operatorname{dom} Q$. The multifunction $Q_D : \operatorname{dom} Q \rightrightarrows \mathbb{R}^r \times \mathbb{R}^r$ is said to have *property* (K) *with respect to* $A(t)$ at $(t, x) \in \operatorname{dom} Q$ if

$$Q_D(t, x) = \bigcap_{\gamma > 0} \operatorname{cl} \bigcup \{Q_D(t, x') : x' \in A(t), \operatorname{dist}(x', x) < \gamma\}.$$

To connect the formulation of results in the literature with that employed here, it is enough to observe that Q has property (K) at a point $(t, x) \in T \times X) \setminus \operatorname{dom} Q$ if (but not only if!) $A(t)$ is closed, whereas Q has property (K) at $(t, x) \in \operatorname{dom} Q$ if and only if Q_D has property (K) with respect to $A(t)$ at (t, x). This explains also why (4.12) is a final conclusion in our somewhat more general approach, while it is a necessary preliminary step for the usual approach in the literature.

Before discussing Theorem 4.3, we show that Theorems 3.2, 3.3 are in fact equivalent to it.

PROPOSITION 4.5. *The lower closure results obtained in Theorems* 3.2, 3.3 *and* 4.3 *are equivalent.*

Proof. Theorem 3.3 was shown to follow from Theorem 3.2 by "recombination of variables". Theorem 4.3 was derived from Theorem 3.3. Hence it is enough to show that Theorem 4.3 implies Theorem 3.2.

Suppose that l satisfies (3.13)–(3.15). If the left side in (3.12) equals $+\infty$, there is nothing left to prove. Thus, we shall assume that this is not the case. We shall apply Theorem 4.3 to Q: $T\times X \rightrightarrows \mathbb{R}^r\times\mathbb{R}^{\bar r+1}$, defined by

$$Q(t,x)\equiv\{(\xi,\eta,\gamma)\in\mathbb{R}^r\times\mathbb{R}^{\bar r}\times\mathbb{R}: \gamma\geqq l(t,x,\xi,\eta)\}; \tag{4.13}$$

in particular, we shall look at the coordinate $j=\bar r+1$. In the present case (4.2) gives

$$l^Q_{\bar r+1}(t,x,\xi,\eta,\gamma)=\begin{cases}\gamma & \text{if } \gamma\geqq l(t,x,\xi,\eta),\\ +\infty & \text{else.}\end{cases}$$

Hence, (3.13)–(3.14) imply (4.7)–(4.8), as follows by Lemma 4.1 and Remark 4.2. By the elementary properties of outer integrals, established in § 1, there exists for every $k\in\mathbb{N}$ a function $\gamma_k\in\mathcal{M}(T;(-\infty,+\infty])$ with

$$\gamma_k(t)\geqq l(t,x_k(t),\xi_k(t),\eta_k(t)) \text{ a.e. in } T, \tag{4.14}$$

$$\bar{\int_T} l(\cdot,x_k,\xi_k,\eta_k)\,d\mu=\int_T\gamma_k\,d\mu. \tag{4.15}$$

Since we work under the assumption that the left side in (3.12) is not equal to $+\infty$, we can suppose without loss of generality that

$$\sup_k\int_T|\gamma_k|\,d\mu<+\infty,$$

in view of (3.15). A fortiori, we have $\gamma_k(t)<+\infty$ a.e. in T for all $k\in\mathbb{N}$. Thus, (4.13)–(4.14) entail that for all $k\in\mathbb{N}$

$$(\xi_k(t),\eta_k(t),\gamma_k(t))\in Q(t,x_k(t)) \text{ a.e. in } T.$$

Hence, condition (4.9)—with $T_k=T$, $d_k=0$, $\bar d_k=0$, $\bar e_k=0$—is also satisfied. Further, (3.15) and (4.14) imply that (4.10) holds. Application of Theorem 4.3 gives the existence of a subsequence $\{\ell\}$ of $\{k\}$ and of $(\eta_*,\gamma_*)\in\mathcal{L}_1(T;\mathbb{R}^{\bar r+1})$ such that

$$\gamma_*(t)\geqq l(t,x_0(t),\xi_0(t),\eta_*(t)) \text{ a.e. in } T,$$

$$\underline{\lim}_{\ell}\int_T\gamma_\ell\,d\mu\geqq\int_T\gamma_*\,d\mu.$$

In view of (4.15) this establishes the inequality (3.12). Note that the subsequence $\{\ell\}$ and the function η_* would seem to depend upon the choice of the integrand l. Although this certainly applies to γ_*, it is easy to see from Example 2.3 and formula (3.7) that $\{\ell\}$ and η_* can indeed be chosen independently from l. QED

A result which is very closely related to Theorem 4.3 is due to Cesari and Suryanarayana [11c, Thm. 3.1] (rather similar results already figure in [10a]). In several respects this result is generalized by our present result. In [11c] the orientor field Q has to have the following property:

$$\eta'\geqq\eta \text{ and } (\xi,\eta)\in Q(t,x) \text{ imply } (\xi,\eta')\in Q(t,x).$$

Also, it is assumed there that for all k $T_k=T$, $d_k=0$, $\bar d_k=0$, $\bar e_k=0$. Other restrictions are that $(T,\mathcal{T},\mu)$ must be nonatomic, complete and that X must be finite-dimensional. There are other, less significant differences; with respect to each of these Theorem 4.3 is the more general result. (Apart from this, it should be pointed out that the argument in [11c] is incomplete: the proof of [11c, 2.2] is not given, even though this concerns a quite nontrivial extension of Fatou's lemma in several dimensions.)

Further, Theorem 4.3 generalizes [3e, Thm. 5], which has $\bar{r}=1$. Therefore we can refer the reader to a number of comparisons with other results in the literature made in [3e]. (Note that the measurability conditions for the orientor field there are superfluous in the light of Theorem 4.3.)

The main point made in [3e] is that when $\bar{r}=1$ a large number of lower closure results follows from the lower semicontinuity result in Corollary 3.5. Conversely, it is well known that such lower semicontinuity results follow from lower closure results. Needless to say, the equivalence result of Proposition 4.5 advances such insights. Interestingly enough, by making use of R. V. Chacon's "biting lemma" [8] one can also obtain Theorem 4.3 from Corollary 3.5 (this has been observed independently by the referee and the author). From the above it will be clear that the necessary conditions for lower semicontinuity of e.g. [16], [27d] can also be converted into necessary conditions for lower closure in certain problems.

Let us now look at more concrete orientor fields. Similar fields figure in many existence problems of optimal control theory.

Let $q: T\times X\times V\to\mathbb{R}^r$ and $\bar{q}: T\times X\times V\to(-\infty,+\infty]^{\bar{r}}$ be $\mathcal{T}\times\mathcal{B}(X\times V)$-measurable functions. We shall consider the multifunction $\tilde{Q}: T\times X \rightrightarrows \mathbb{R}^r\times\mathbb{R}^{\bar{r}}$ defined by

$$\tilde{Q}(t,x) \equiv \{(q(t,x,v),\eta)\in\mathbb{R}^r\times\mathbb{R}^{\bar{r}}: v\in V, \eta\geqq\bar{q}(t,x,v)\}.$$

In what follows we shall consider the sequence $\{v_k\}_1^\infty$, $v_k\in\mathcal{M}(T_k;V)$, of the previous section. Let us agree to set $q(\cdot,x_k,v_k)=0$, $\bar{q}(\cdot,x_k,v_k)=0$ on $T\setminus T_k$.

THEOREM 4.6. *Suppose that* $\{T_k\}_0^\infty$, $\{x_k\}_0^\infty$ *satisfy* (3.18)–(3.19), *that*

(4.16) $\{q(\cdot,x_k,v_k)\}_1^\infty$ *converges weakly in* $\sigma(\mathcal{L}_1^r,\mathcal{L}_\infty^r)$ *to* $\xi_0\in\mathcal{L}_1(T;\mathbb{R}^r)$,

(4.17) $\{\bar{q}^-(\cdot,x_k,v_k)\}_1^\infty$ *is uniformly integrable,*

(4.18) $\lim_k \int_{T_k}\bar{q}(\cdot,x_k,v_k)\,d\mu$ *exists (in* $\mathbb{R}^{\bar{r}}$*),*

and that

(4.19) $\tilde{Q}$ *has property (K) at* $(t,x_0(t))$ *a.e. in* T_0,

(4.20) $\tilde{Q}(t,x_0(t))$ *is a convex subset of* $\mathbb{R}^r\times\mathbb{R}^{\bar{r}}$ *a.e. in* T_0.

Then there exists $v_*\in\mathcal{M}(T_0;V)$ *such that*

(4.21) $\xi_0=q(\cdot.\,x_0,v_*)$ *a.e. in* T_0,

(4.22) $\lim_k\int_{T_k}\bar{q}(\cdot,x_k,v_k)\,d\mu\geqq\int_{T_0}\bar{q}(\cdot,x_0,v_*)\,d\mu.$

Moreover, condition (4.19) *can be lifted altogether either: if* (3.43) *holds and a.e. in* T_0

(4.23) $q(t,\cdot,\cdot)$ *is continuous at every point of* $\{x_0(t)\}\times V$,

(4.24) $\bar{q}(t,\cdot,\cdot)$ *is lower semicontinuous at every point of* $\{x_0(t)\}\times V$,

or under the following set of conditions

(4.25) $q(t,x_k(t),v_k(t))-q(t,x_0(t),v_k(t))\to 0$ *a.e. in* T_0,

(4.26) $\bar{q}(t,x_k(t),v_k(t))-\bar{q}(t,x_0(t),v_k(t))\to 0$ *a.e. in* T_0.

Proof. It follows from (4.17)–(4.18) that

$$\sup_k \int_{T_k} |\bar{q}(\cdot, x_k, v_k)|\, d\mu < +\infty.$$

Also, by definition of $\tilde{Q}$

$$(q(t, x_k(t), v_k(t)), \bar{q}(t, x_k(t), v_k(t))) \in \tilde{Q}(t, x_k(t)) \text{ a.e. in } T_k.$$

Applying Theorem 4.3 (with $d_k = 0$, $\bar{d}_k = 0$, $\bar{e}_k = 0$) we find that there exists $\eta_* \in \mathscr{L}_1(T; \mathbb{R}^{\bar{r}})$ such that for a.e. $t \in T_0$ there is $v \in V$ with

$$\xi_0(t) = q(t, x_0(t), v), \qquad \eta_*(t) \geqq \bar{q}(t, x_0(t), v), \tag{4.27}$$

$$\lim_k \int_{T_k} \bar{q}(\cdot, x_k, v_k)\, d\mu \geqq \int_{T_0} \eta_*\, d\mu. \tag{4.28}$$

The set of all $(t, v) \in T_0 \times V$ for which (4.27) holds is $\mathscr{T} \times \mathscr{B}(V)$-measurable. Hence, by Aumann's measurable selection theorem [14] there exists $v_* \in \mathscr{M}(T_0; V)$ such that

$$\xi_0 = q(\cdot, x_0, v_*), \qquad \eta_* \geqq \bar{q}(\cdot, x_0, v_*) \quad \text{a.e. in } T_0.$$

Together with (4.28) this shows that (4.21)–(4.22) hold.

Next, we show that in the specified special cases condition (4.19) can be omitted. In the first case we define $\bar{\bar{q}} \equiv (\bar{q}, h)$, where h is as in (3.43). We then consider the multifunction $\tilde{Q}' : T \times X \rightrightarrows \mathbb{R}^r \times \mathbb{R}^{\bar{r}+1}$ defined as follows

$$\tilde{Q}'(t, x) \equiv \begin{cases} \tilde{Q}(t, x_0(t)) \times \mathbb{R} & \text{if } x = x_0(t), \\ \{(q(t, x, v), \eta) \in \mathbb{R}^r \times \mathbb{R}^{\bar{r}+1} : v \in V, \eta \geqq \bar{\bar{q}}(t, x, v)\} & \text{else.} \end{cases}$$

From the inf-compactness property of $h(t, \cdot)$ and (4.23)–(4.24) it follows by elementary reasoning that $\tilde{Q}'$ has property (K) at $(t, x_0(t))$, irrespective of condition (4.19). We now have

$$(q(t, x_k(t), v_k(t)), \bar{\bar{q}}(t, x_k(t), v_k(t))) \in \tilde{Q}'(t, x_k(t)) \quad \text{a.e. in } T_k,$$

and the remaining conditions of the previously established part of the theorem are easily seen to hold (with $\tilde{Q}'$, $\bar{\bar{q}}$ instead of $\tilde{Q}$, $\bar{q}$; note that in view of (3.43), condition (4.18) is fulfilled without loss of generality). Hence, there exists $v_* \in \mathscr{M}(T_0; V)$ such that (4.21) holds and

$$\lim_k \int_{T_k} \bar{\bar{q}}(\cdot, x_k, v_k)\, d\mu \geqq \int_{T_0} \bar{\bar{q}}(\cdot, x_0, v_*)\, d\mu.$$

By definition of $\bar{\bar{q}}$ this entails (4.22).

For the second case we define

$$d_k \equiv q(\cdot, x_0, v_k) - q(\cdot, x_k, v_k),$$

$$\bar{d}_k \equiv \bar{q}(\cdot, x_0, v_k) - \bar{q}(\cdot, x_k, v_k),$$

$$\tilde{Q}''(t, x) \equiv \tilde{Q}(t, x_0(t)).$$

Then (3.22)–(3.23) hold by (4.25)–(4.26). Evidently

$\tilde{Q}''$ has property (K) at every $(t, x) \in T \times X$,

$$(q(t, x_k(t), v_k(t)) + d_k(t), \bar{q}(t, x_k(t), v_k(t)) + \bar{d}_k(t)) \in \tilde{Q}''(t, x_k(t)) \text{ a.e. in } T_k.$$

It is now easy to verify that we may invoke Theorem 4.3 and Aumann's theorem as before to arrive at (4.21)–(4.22). QED

Remark 4.7. Define dom $\bar{q}^j$ to be the set of all $(t, x, v) \in T \times X \times V$ with $\bar{q}^j(t, x, v) < +\infty$ (dom $\tilde{Q}$ is precisely the projection of $\bigcap_{j=1}^{\bar{r}} \text{dom}\, \bar{q}^j$ on $T \times X$; cf. Remark 4.4). It is easy to verify that (4.23) can be replaced by the weaker condition

(4.23′) $q(t, \cdot, \cdot)$ is continuous at every point of $\{x_0(t)\} \times V$ relative to the section at t of $\bigcap_{j=1}^{\bar{r}} \text{dom}\, \bar{q}^j$ a.e. in T_0.

Remark 4.8. In the literature one usually restricts the considerations from the beginning to a $\mathcal{T} \times \mathcal{B}(X \times V)$-measurable subset D of $T \times X \times V$. One introduces functions $q_D: D \to \mathbb{R}^r$, $\bar{q}_D: D \to \mathbb{R}^{\bar{r}}$ and the multifunction $\tilde{Q}_D: D_0 \rightrightarrows \mathbb{R}^r \times \mathbb{R}^{\bar{r}}$ given by

$$\tilde{Q}_D(t, x) \equiv \{(q_D(t, x, v), \eta) \in \mathbb{R}^r \times \mathbb{R}^{\bar{r}} : (t, x, v) \in D, \eta \geqq \bar{q}_D(t, x, v)\},$$

where D_0 stands for the projection of D on $T \times X$. The present setup is regained by introducing the integrand $\bar{q}^{\bar{r}+1}$ on $T \times X \times V$, given by

$$\bar{q}^{\bar{r}+1}(t, x, v) \equiv \begin{cases} 0 & \text{if } (t, x, v) \in D, \\ +\infty & \text{else,} \end{cases}$$

by letting $q: T \times X \times V \to \mathbb{R}^r$ be the extension of q_D, obtained by setting $q = 0$ on $(T \times X \times V) \backslash D$, and by letting $\bar{q}^j$ be the extension of $\bar{q}_D^j$ with $\bar{q}^j = +\infty$ on $(T \times X \times V) \backslash D$. As for (4.19), Remark 4.4 holds. Concerning the use of (4.23′)–(4.24), we note that these are satisfied if a.e. in T_0

$q_D(t, \cdot, \cdot)$ is continuous at every point of $(\{x_0(t)\} \times V) \cap D_t$,

$\bar{q}_D^j(t, \cdot, \cdot)$ is lower semicontinuous at every point of $(\{x_0(t)\} \times V) \cap D_t$, $j = 1, \cdots, \bar{r}$,

$\bar{q}^{\bar{r}+1}(t, \cdot, \cdot)$ is lower semicontinuous at every point of $\{x_0(t)\} \times V$;

here D_t stands for the section at t of D. As for (4.25)–(4.26), note that they are equivalent to having a.e. in T_0

$$q_D(t, x_k(t), v_k(t)) - q_D(t, x_0(t), v_k(t)) \to 0,$$

$$\bar{q}_D(t, x_k(t), v_k(t)) - q_D(t, x_0(t), v_k(t)) \to 0,$$

$$(t, x_k(t), v_k(t)) \in D, (t, x_0(t), v_k(t)) \in D \text{ for large enough } k.$$

Remark 4.9. Conditions (3.43), (4.23)–(4.24) suggest that this special case of Theorem 4.6 can also be proven directly, in the style of § 3. Indeed, this is true. We shall leave it to the reader to work out the details for the integrands g_j on $T \times X \times \mathbb{R}^r \times V$, defined by

$$g_j(t, x, \xi, v) = \begin{cases} \bar{q}^j(t, x, v) & \text{if } q(t, x, v) = \xi, \\ +\infty & \text{else.} \end{cases}$$

Remark 4.10. An obvious extension of (3.43), (4.23)–(4.24) is obtained by letting h also depend on the x-variable: Suppose instead of (3.43) that there exists a nonnegative $\mathcal{T} \times \mathcal{B}(X \times V)$-measurable integrand h on $T \times X \times V$ such that a.e. in T_0

(3.43′) $h(t, \cdot, \cdot)$ is lower semicontinuous at every point of $\{x_0(t)\} \times V$,

(3.43″) $\sup_k h(t, x^k, v^k) < +\infty$ implies that $\{v^k\}_1^\infty$ has a limit point for every $\{x^k\}_1^\infty \subset X$, $\{v^k\}_1^\infty \subset V$ with $x^k \to x_0(t)$.

Suppose further that

$$\sup_k \int_{T_k} h(\cdot, x_k, v_k)\, d\mu < +\infty. \tag{3.43'''}$$

Then the conclusions of Theorem 4.6 regarding (3.43), (4.23)–(4.24) remain valid. This is seen by noting that the multifunction $\tilde{Q}'$ in the proof of Theorem 4.6 then still has property (K) at $(t, x_0(t))$ a.e. in T_0.

Remark 4.11. Note that the graph of the multifunction $\tilde{Q}$ is the projection on $T \times X \times \mathbb{R}^r \times \mathbb{R}^{\bar{r}}$ of a $\mathcal{T} \times \mathcal{B}(X \times \mathbb{R}^r \times \mathbb{R}^{\bar{r}} \times V)$-measurable set. In itself this does not bestow any useful sort of measurability on $\tilde{Q}$. Hence, our consideration of nonmeasurable integrands in § 3 is justified. Note also that conditions like (4.25)–(4.26) warrant the use of the perturbations d_k, $\bar{d}_k$ in §§ 3–4.

By taking into account our Remarks 4.4, 4.8 it is easy to see that Theorem 4.6 generalizes [3e, Thms. 7, 10] (where $\bar{r}=1$ among other things). This also means that a large number of lower closure results in the literature (e.g. [5], [10c, d], [11a, b, d]) follow from our result.

In conclusion, we wish to remark that a large number of existence results can be obtained as follows (applications include the optimal control of ordinary differential equations, functional-differential equations, nonlinear integral equations and elliptic boundary value problems): One applies Corollary 3.5 to the dynamical system and Theorem 4.6 to the orientor field $\tilde{Q}$, where $\bar{q}^1$ may stand for the usual cost functional (for instance). Details can be found in [3g] and in forthcoming work by the author.

Appendix A. In this appendix we shall gather some facts about relaxed control theory that were established in [3]. In particular, we shall prove Theorem I and Lemmas II, III.

Since S is a standard Borel space, we may identify it with a Borel subset of a compact metric space $\hat{S}$, the metric of which will be denoted by ρ [12, III]. Hence $\mathcal{M}(T; S)$ and $\mathcal{R}(T; S)$ are subsets of $\mathcal{M}(T; \hat{S})$ and $\mathcal{R}(T; \hat{S})$ respectively.

Define $\mathcal{C}_e(\hat{S}) \subset \mathcal{C}(\hat{S})$ as follows:

$$\mathcal{C}_e(\hat{S}) \equiv \{-n\rho(\cdot, s) + \eta \colon n \in \mathbb{N},\, s \in \hat{S},\, \eta \in \mathbb{R}\}.$$

LEMMA A.1. *For every $g \in \mathcal{G}^+(T; S)$ there exist a null set $N \subset T$ and sequences $\{T_p\}_1^\infty \subset \mathcal{T}$, $\{c_p\}_1^\infty \subset \mathcal{C}_e(\hat{S})$ such that*

$$g(t, s) = \sup_p 1_{T_p}(t) c_p(s) \text{ on } (T \backslash N) \times S. \tag{a.1}$$

Proof. Let $\{s^j\}_1^\infty$ be a countable dense sequence in S and let $\{\gamma^k\}_1^\infty$ be an enumeration of the rationals. For $j, k, m \in \mathbb{N}$ we define $c_{jkm} \in \mathcal{C}_e(\hat{S})$ by $c_{jkm} \equiv \gamma^k - m\rho(s^j, \cdot)$ and $B_{jkm} \equiv \{t \in T \colon c_{jkm}(s) \leqq g(t, s) \text{ on } S\}$. Then B_{jkm} is the projection of the set of all $(t, s) \in T \times S$ such that $c_{jkm}(s) > g(t, s)$ onto T. Hence, B_{jkm} belongs to the completion of $\mathcal{T}$ with respect to μ [9, III.23]; this implies that there exists $T_{jkm} \in \mathcal{T}$, $T_{jkm} \subset B_{jkm}$, such that $B_{jkm} \backslash T_{jkm}$ is contained in a null set N_{jkm}. Using nonnegativity and lower semicontinuity of $g(t, \cdot)$ it is entirely elementary to prove that $\sup_{j,k,m} 1_{B_{jkm}}(t) c_{jkm}(s) = g(t, s)$ on $T \times S$. By taking N to be the union of all N_{jkm} the result now follows. QED

A $\mathcal{T} \times \mathcal{B}(S)$-measurable integrand g on $T \times S$ is said to be a *Carathéodory integrand* if $g(t, \cdot)$ is continuous on S for every $t \in T$ and there exists $\phi \in \mathcal{L}_1(T; \mathbb{R})$ such that

$$|g(t, s)| \leqq \phi(t) \text{ on } T \times S.$$

The set of all Carathéodory integrands on $T \times S$ will be denoted by $\mathcal{G}_{C}(T; S)$.

We shall equip $\mathcal{R}(T;\hat{S})$ with the coarsest topology $\hat{\mathcal{O}}$ for which all functions $\delta \mapsto \int g(\cdot,\delta)\,d\mu$ are continuous on $\mathcal{R}(T;\hat{S})$, $g \in \mathcal{G}_C(T;\hat{S})$. Its relative topology on $\mathcal{R}(T;S)$ will be denoted $\mathcal{O}$.

LEMMA A.2. *The topology $\mathcal{O}$ is the coarsest topology for which all functions $\delta \mapsto \int_T g(\cdot,\delta)\,d\mu$ are lower semicontinuous on $\mathcal{R}(T;S)$, $g \in \mathcal{G}^+(T;S)$.*

Proof. Call the topology that figures in the statement above $\mathcal{O}'$. Given $g \in \mathcal{G}_C(T;\hat{S})$, its restriction to $T \times S$ is easily seen to belong to $\mathcal{G}_C(T;S)$. It follows easily that $\delta \mapsto \int_T g(\cdot,\delta)\,d\mu$ is $\mathcal{O}'$-continuous on $\mathcal{R}(T;S)$. Conversely, given $g \in \mathcal{G}^+(T;S)$, let $\{T_p\}_1^\infty$ and $\{c_p\}_1^\infty$ be as in Lemma A.1. Define $\kappa_p:\mathbb{R}\to[0,p]$ by $\kappa_p(\gamma) = \max(\min(\gamma,p),0)$ and set

$$g_p(t,s) \equiv \kappa_p(\sup_{j \le p} 1_{T_j}(t)c_j(s)).$$

Then it follows by the monotone convergence theorem and (a.1) that for every $\delta \in \mathcal{R}(T;S)$

$$\int_T g(\cdot,\delta)\,d\mu = \sup_p \int_T g_p(\cdot,\delta)\,d\mu.$$

Hence, $\delta \mapsto \int_T g(\cdot,\delta)\,d\mu$ is $\mathcal{O}$-lower semicontinuous on $\mathcal{R}(T;S)$ for every $g \in \mathcal{G}^+(T;S)$. This finishes the proof. QED

Let $M^+(\hat{S})$ denote the set of all bounded nonnegative measures on $\hat{S}$; set $M(\hat{S}) = M^+(\hat{S}) - M^+(\hat{S})$. It is well known that the usual L_∞-space $L_\infty(T,\mathcal{T},\mu;M(\hat{S}))$ of essentially bounded $\mathcal{T}$-measurable functions from T into $M(\hat{S})$ has as its dual the usual L_1-space $L_1(T,\mathcal{T},\mu;\mathcal{C}(\hat{S}))$ of integrable functions from T into $\mathcal{C}(\hat{S})$ [18, VII.7]; cf. [24, p. 301] for a short proof. (For a good understanding we note that $L_\infty(T,\mathcal{T},\mu;M(\hat{S}))$ consists of (equivalence classes of) functions $\sigma: T \to M(\hat{S})$ that are Borel measurable with respect to the usual weak topology on $M(\hat{S})$ and have ess $\sup_t |\sigma(t)|_v < +\infty$, where $|\cdot|_v$ stands for the total variation norm.) Let Σ be the set of all $\sigma \in L_\infty(T,\mathcal{T},\mu;M(\hat{S}))$ for which $\sigma(t) \in M_1^+(\hat{S})$ a.e. in T. It will be equipped with the relative $\sigma(L_\infty, L_1)$-topology.

LEMMA A.3. *Σ is compact and sequentially compact for the topology $\sigma(L_\infty, L_1)$.*

Proof. Compactness of Σ is well known; it follows from the above by a simple application of the Alaoglu–Bourbaki theorem [9, V.2], [32, IV]. Further, it is well known that Σ is metrizable if the σ-algebra $\mathcal{T}$ is countably generated, since in that case $L_1(T,\mathcal{T},\mu;\mathcal{C}(\hat{S}))$ is separable [15, 12.F]; cf. [32, IV]. Sequential compactness is proven next (cf. [20]). Let $\{\sigma_k\}_1^\infty \subset \Sigma$ be arbitrary and let $\mathcal{T}_0$ stand for the σ-algebra generated by this sequence; it is countably generated, since $M_1^+(\hat{S})$ is metrizable and separable for the weak topology. Hence, it follows from the above that there exist a subsequence $\{\ell\}$ of $\{k\}$ and a $\mathcal{T}_0$-measurable $\sigma_* \in \Sigma$ such that $\{\sigma_\ell\}$ converges to σ_* in the topology $\sigma(L_\infty(T,\mathcal{T}_0,\mu;M(\hat{S})), L_1(T,\mathcal{T}_0,\mu;\mathcal{C}(\hat{S})))$. Since each element of $L_1(T,\mathcal{T},\mu;\mathcal{C}(\hat{S}))$ has a conditional expectation in $L_1(T,\mathcal{T}_0,\mu;\mathcal{C}(\hat{S}))$ [9, VIII.32], it follows directly that $\{\sigma_\ell\}$ also converges to σ_* in $\sigma(L_\infty, L_1)$. QED

LEMMA A.4. *$\mathcal{R}(T;\hat{S})$ is compact and sequentially compact for the topology $\hat{\mathcal{O}}$.*

Proof. Denote by χ the usual quotient mapping from the set of all $\delta \in \mathcal{M}(T;M(\hat{S}))$ such that $\sup_{t\in T}|\delta(t)|_v < +\infty$ into $L_\infty(T,\mathcal{T},\mu;M(\hat{S}))$. It is easy to see that $\hat{\mathcal{O}}$ is the coarsest topology for which χ is continuous with respect to $\sigma(L_\infty, L_1)$. Since χ is a surjection from $\mathcal{R}(T;\hat{S})$ into Σ, the desired result now follows directly from Lemma A.3. QED

Proof of Theorem I. By supposition there exists $h \in \mathcal{H}(T;S)$ such that $\sup_k \int_T h(\cdot,s_k)\,d\mu < +\infty$. For every k a parametrized measure $\delta_k \in \mathcal{R}(T;\hat{S})$ is defined by taking $\delta_k(t)$ to be the Dirac measure (point mass) at $s_k(t)$. By Lemma A.4 there exist a subsequence $\{\ell\}$ of $\{k\}$ and $\delta_* \in \mathcal{R}(T; \hat{S})$ such that $\{\delta_\ell\}$ converges to δ_* in the topology $\hat{\mathcal{O}}$. We shall show that in fact $\delta_* \in \mathcal{R}(T; S)$, which would mean that $\{\delta_\ell\}$ converges to δ_* in the relative topology $\mathcal{O}$. To see this, we define $\hat{h}: T \times \hat{S} \to [0, +\infty]$ by

$$\hat{h}(t, s) \equiv \begin{cases} h(t, s) & \text{on } T \times S, \\ +\infty & \text{on } T \times (\hat{S} \backslash S). \end{cases}$$

Since S is Borel in $\hat{S}$, $\hat{h}$ is $\mathcal{T} \times \mathcal{B}(\hat{S})$-measurable. Also, the topological homeomorphism that makes S into a subset of $\hat{S}$ turns compact subsets of S into compact subsets of $\hat{S}$. Hence, for every $t \in T$, $\gamma \in \mathbb{R}$ the set $\{s \in \hat{S}: \hat{h}(t, s) \leqq \gamma\}$ is compact. We conclude that $\hat{h} \in \mathcal{H}(T; \hat{S})$. By compactness of $\hat{S}$ the latter set is precisely $\mathcal{G}^+(T; \hat{S})$, so it follows from the above that

$$\int_T \hat{h}(\cdot, \delta_*)\, d\mu \leqq \underline{\lim}_{\ell} \int_T \hat{h}(\cdot, \delta_\ell)\, d\mu = \underline{\lim}_{\ell} \int_T h(\cdot, \delta_\ell)\, d\mu < +\infty.$$

By definition of $\hat{h}$ this implies that for a.e. $t \in T$ the probability measure $\delta_*(t)$ is carried by S. Of course, we can modify $\delta_*(t)$ on the exceptional set, and this does not affect the values of integrals. Thus we may conclude that $\delta_* \in \mathcal{R}(T; S)$ without loss of generality. By definition of $\mathcal{O}$ we have now for every $g \in \mathcal{G}^+(T; S)$ that (2.5) holds. For $g \in \mathcal{G}(T; S)$ such that there exists $\phi \in \mathcal{L}_1(T; \mathbb{R})$ with

$$g(t, s) \geqq \phi(t) \quad \text{on } T \times S,$$

(2.5) is also valid, as is easily seen by working with $g - \phi \in \mathcal{G}^+(T; S)$. Further, for $g \in \mathcal{G}(T; S)$ with (2.6) there exists for every $\varepsilon > 0$ a constant $\gamma > 0$ such that

$$\int_T g(\cdot, s_\ell)\, d\mu \geqq \int_T \max(-\gamma, g(\cdot, s_\ell))\, d\mu - \varepsilon \quad \text{for all } \ell.$$

In view of the above, (2.5) is now easy to derive.

Finally, we shall demonstrate that for a.e. $t \in T$ the measure $\delta_*(t)$ is carried by the limit points of $\{s_\ell(t)\}$. Let $\hat{\mathbb{N}} \equiv \mathbb{N} \cup \{\infty\}$ be the usual Alexandrov compactification of the natural numbers; this is a metrizable compact space. We define $\tilde{g}: T \times \hat{\mathbb{N}} \times S \to [0, +\infty]$ as follows:

$$\tilde{g}(t, p, s) \equiv \begin{cases} 0 & \text{if } s \in \operatorname{cl}\{s_\ell(t): \ell \geqq p\}, \\ +\infty & \text{else,} \end{cases}$$

for $p \in \mathbb{N}$. For $p = \infty$ we define

$$\tilde{g}(t, \infty, s) \equiv \begin{cases} 0 & \text{if } s \in \bigcap_{p=1}^{\infty} \operatorname{cl}\{s_\ell(t): \ell \geqq p\}, \\ +\infty & \text{else.} \end{cases}$$

It is easy to check that $\tilde{g}$ belongs to $\mathcal{G}^+(T; \hat{\mathbb{N}} \times S)$. Let ρ' be a compatible metric on $\hat{\mathbb{N}}$; we shall equip $\hat{\mathbb{N}} \times \hat{S}$ with the metric $\rho' + \rho$. Let $\{T_m\}_1^\infty \subset \mathcal{T}$ and $\{c_m\}_1^\infty \subset \mathcal{C}_e(\hat{\mathbb{N}} \times \hat{S})$ correspond to $\tilde{g}$ as asserted in Lemma A.1. Define $\tilde{g}_m \equiv \kappa_m(\sup_{j \leq m} 1_{T_j} c_j)$, where κ_m is as in the proof of Lemma A.2. Note that for every m there exists a Lipschitz constant γ_m such that

$$|\tilde{g}_m(t, p, s) - \tilde{g}_m(t, p', s')| \leqq \gamma_m(\rho'(p, p') + \rho(s, s')).$$

It follows that for every m

$$\lim_{\ell} \left(\int_T \tilde{g}_m(\cdot, \ell, s_\ell)\, d\mu - \int_T \tilde{g}_m(\cdot, \infty, s_\ell)\, d\mu \right) = 0.$$

Hence we find for every m

$$\underline{\lim}_{k} \int_T \tilde{g}_m(\cdot, k, s_k)\, d\mu = \underline{\lim}_{k} \int_T \tilde{g}_m(\cdot, \infty, s_k)\, d\mu \geqq \int_T \tilde{g}_m(\cdot, \infty, \delta_*)\, d\mu,$$

since $\{\delta_k\}$ converges in $\hat{\mathcal{O}}$ to δ_*. Also, we have by the monotone convergence theorem for every $\delta \in \mathcal{R}(T; \hat{\mathbb{N}} \times S)$

$$\int_T \tilde{g}(\cdot, \delta)\, d\mu = \sup_m \int_T \tilde{g}_m(\cdot, \delta)\, d\mu.$$

Combined with the above this gives

$$\int_T \tilde{g}(\cdot, \infty, \delta_*)\, d\mu \leqq \underline{\lim}_{k} \int_T \tilde{g}(\cdot, k, s_k)\, d\mu = 0,$$

where the latter equality follows by definition of $\tilde{g}$. The desired conclusion now also follows from the definition of $\tilde{g}$. QED

LEMMA A.5. *For every lower semicontinuous integrand l on $T \times S$ there exists a normal integrand $g \in \mathcal{G}(T; S)$, $g \geqq l$, such that for every $u \in \mathcal{M}(T; S)$, $\phi \in \mathcal{M}(T; [-\infty, +\infty])$*

$$l(t, u(t)) \leqq \phi(t) \text{ a.e. in } T \text{ implies that } g(t, u(t)) \leqq \phi(t) \text{ a.e. in } T.$$

Proof. Suppose first that $l \geqq 0$. Although l need not be $\mathcal{T} \times \mathcal{B}(S)$-measurable, the proof of Lemma A.1 shows that there exist a sequence $\{B_p\}_1^\infty$ of subsets of T and $\{c_p\}_1^\infty \subset \mathscr{C}_e(\hat{S})$ such that

$$l(t, s) = \sup_p 1_{B_p}(t) c_p(s) \text{ on } T \times S. \tag{a.2}$$

For every p there exists $T_p \in \mathcal{T}$, $T_p \supset B_p$, such that $\mu(T_p)$ equals the outer measure of B_p [26, I.4]. Define on $T \times S$

$$g(t, s) \equiv \sup_p 1_{T_p}(t) c_p(s);$$

then $g \geqq l$ and $g \in \mathcal{G}^+(T; S)$. Let $u \in \mathcal{M}(T; S)$, $\phi \in \mathcal{M}(T; [0, +\infty])$ be arbitrary with

$$l(t, u(t)) \leqq \phi(t) \text{ for all } t \in T.$$

(Evidently, it is enough to prove the desired implication in this case.) We now have by (a.2) that for every p the set B_p is contained in $A_p \equiv \{t \in T: c_p(u(t)) \leqq \phi(t)\}$. By $\mathcal{T}$-measurability of u, A_p is $\mathcal{T}$-measurable. By definition of T_p it follows that $T_p \backslash A_p$ is a null set; in other words, we must have

$$1_{T_p}(t) c_p(u(t)) \leqq \phi(t) \text{ a.e. in } T.$$

Thus, the desired implication holds if $l \geqq 0$. For general l we define $l' \equiv \exp(l)$. From the previous step the desired conclusion then follows easily by monotonicity and continuity of the transformation involved. QED

LEMMA A.6. *For every $\bar{\mathcal{T}} \times \mathcal{B}(S)$-measurable lower semicontinuous integrand g on $T \times S$ there exist a null set $N \subset T$ and $g' \in \mathcal{G}(T; S)$ such that*

$$g'(t, s) = g(t, s) \text{ on } (T \backslash N) \times S;$$

here $\bar{\mathcal{T}}$ stands for the completion of $\mathcal{T}$ with respect to μ.

Proof. Suppose first that $g \geqq 0$. From the proof given for Lemma A.1 it follows that there exist sequences $\{B_p\}_1^\infty \subset \bar{\mathcal{T}}$, $\{c_p\}_1^\infty \subset \mathscr{C}_e(\hat{S})$ with

$$g(t, s) = \sup_p 1_{B_p}(t) c_p(s) \text{ on } T \times S.$$

For every $p \in \mathbb{N}$ there exists $T_p \in \mathcal{T}$, $B_p \subset T_p$, such that $T_p \backslash B_p$ is a null set. Defining on $T \times S$

$$g'(t, s) \equiv \sup_p 1_{T_p}(t) c_p(s),$$

we see that $g' \geqq g \geqq 0$ and that g' has the required properties. For general g we apply the previous step to $\exp(g)$. QED

Proof of Lemma II. Let $\mathscr{g}$ be as given. By Lemma A.5 there exists $\tilde{g} \in \mathscr{G}(T; X \times \mathbb{R}^r \times \mathbb{R}^{\bar{r}})$, $\tilde{g} \geqq \mathscr{g}$, such that for every $x \in \mathscr{M}(T; X)$, $\xi \in \mathscr{M}(T; \mathbb{R}^r)$, $\eta \in \mathscr{M}(T; \mathbb{R}^{\bar{r}})$, $\phi \in \mathscr{M}(T; [-\infty, +\infty])$

$$\text{(a.3)} \qquad \mathscr{g}(\cdot, x, \xi, \eta) \leqq \phi \text{ a.e. in } T \text{ implies } \tilde{g}(\cdot, x, \xi, \eta) \leqq \phi \text{ a.e. in } T.$$

We define the following Fenchel conjugate functions:

$$\tilde{g}_0^*(t, \xi, \eta) = \sup \{\langle \xi, \xi' \rangle + \langle \eta, \eta' \rangle - \tilde{g}(t, x_0(t), \xi', \eta') : \xi' \in \mathbb{R}^r, \eta' \in \mathbb{R}^{\bar{r}}\},$$

$$\tilde{g}_0^{**}(t, \xi, \eta) = \sup \{\langle \xi, \xi' \rangle + \langle \eta, \eta' \rangle - \tilde{g}_0^*(t, \xi', \eta') : \xi' \in \mathbb{R}^r, \eta' \in \mathbb{R}^{\bar{r}}\}.$$

Since $(t, \xi, \eta) \mapsto \tilde{g}(t, x_0(t), \xi, \eta)$ is certainly $\bar{\mathcal{T}} \times \mathscr{B}(\mathbb{R}^r \times \mathbb{R}^{\bar{r}})$-measurable, it follows by [9, III.39] that $\tilde{g}_0^*$, $\tilde{g}_0^{**}$ are also $\bar{\mathcal{T}} \times \mathscr{B}(\mathbb{R}^r \times \mathbb{R}^{\bar{r}})$-measurable. It is well known that for every $t \in T$, $\tilde{g}_0^{**}(t, \cdot, \cdot)$ is the lower semicontinuous convex hull of $\tilde{g}(t, x_0(t), \cdot, \cdot)$. By (3.8) and $\tilde{g} \geqq \mathscr{g}$ this implies that for every $t \in T$

$$\text{(a.4)} \qquad \tilde{g}_0(t, x_0(t), \cdot, \cdot) \geqq \tilde{g}_0^{**}(t, \cdot, \cdot) \geqq \mathscr{g}(t, x_0(t), \cdot, \cdot).$$

Also, by Lemma A.6 the above imply that there exist a null set $N \subset T$ and $\tilde{\tilde{g}}_0 \in \mathscr{G}(T; \mathbb{R}^r \times \mathbb{R}^{\bar{r}})$ such that for every $t \in T \backslash N$

$$\text{(a.5)} \qquad \tilde{\tilde{g}}_0(t, \cdot, \cdot) = \tilde{g}_0^{**}(t, \cdot, \cdot).$$

For $t \in T \backslash N$ we now define

$$g(t, x, \xi, \eta) \equiv \begin{cases} \tilde{g}(t, x, \xi, \eta) & \text{if } x \neq x_0(t), \\ \tilde{\tilde{g}}_0(t, \xi, \eta) & \text{if } x = x_0(t), \end{cases}$$

and for $t \in N$ we set $g(t, x, \xi, \eta) \equiv +\infty$. Then $g \geqq \mathscr{g}$ and g is $\mathcal{T} \times \mathscr{B}(X \times \mathbb{R}^r \times \mathbb{R}^{\bar{r}})$-measurable. From the first inequality in (a.4), (a.5) and lower semicontinuity of $\tilde{g}(t, \cdot, \cdot, \cdot)$ it follows now by elementary reasoning that $g(t, \cdot, \cdot, \cdot)$ is lower semicontinuous. Hence $g \in \mathscr{G}(T; X \times \mathbb{R}^r \times \mathbb{R}^{\bar{r}})$ and (3.9)–(3.10) hold. To prove (3.11), let x, ξ, η be as in (a.3) and arbitrary. As remarked in § 1, there exists $\phi \in \mathscr{M}(T; [-\infty, +\infty])$, $\phi \geqq \mathscr{g}(\cdot, x, \xi, \eta)$ a.e. in T, such that

$$\int_T \phi \, d\mu = \bar{\int_T} \mathscr{g}(\cdot, x, \xi, \eta) \, d\mu.$$

Note that $\tilde{g}(t, \cdot, \cdot, \cdot) \geqq g(t, \cdot, \cdot, \cdot)$ for $t \in T \backslash N$, by definition of g. Hence, it follows from (a.3) that $\phi \geqq g(\cdot, x, \xi, \eta)$ a.e. in T and so

$$\bar{\int_T} \mathscr{g}(\cdot, x, \xi, \eta) \, d\mu \geqq \int_T g(\cdot, x, \xi, \eta) \, d\mu.$$

The converse inequality holds trivially. QED

Proof of Lemma III. We may work with $S \equiv V$, so that we can use the results obtained in this appendix. We shall have to work with Σ rather than $\mathscr{R}(T; \hat{S})$, since

Σ lies in the locally convex Hausdorff space $L_\infty(T, \mathcal{T}, \mu; M(\hat{S}))$. For this purpose we define, by abuse of notation, for any $\sigma \in \Sigma$, $g \in \mathcal{G}(T; \hat{S})$

$$\int_T g(\cdot, \sigma)\, d\mu \equiv \int_T g(\cdot, \delta)\, d\mu,$$

where δ stands for any $\delta \in \mathcal{R}(T; \hat{S})$ with $\chi(\delta) = \sigma$; cf. the proof of Lemma A.4. This definition makes sense and from what was said in proving Lemma A.4 it follows that lower semicontinuity of $\sigma \mapsto \int_T g(\cdot, \sigma)\, d\mu$ with respect to $\sigma(L_\infty, L_1)$ follows from lower semicontinuity of $\delta \mapsto \int_T g(\cdot, \delta)\, d\mu$ with respect to $\hat{\mathcal{O}}$ (and conversely). Let $h \in \mathcal{H}(T; S)$ be as given; we define $\hat{h} \in \mathcal{H}(T; \hat{S})$ to be its extension defined in the proof of Theorem I. Define $\Sigma(\hat{h})$ to be the (compact) set of all $\sigma \in \Sigma$ with $\int_T \hat{h}(\cdot, \sigma)\, d\mu \leqq \int_T h(\cdot, \delta^*)\, d\mu$. Note that by definition of $\hat{h}$ every $\sigma \in \Sigma(\hat{h})$ has $\sigma(t)$ carried by S a.e. in T. As follows from Theorem I (or at least its obvious nonsequential analogue), the functions $\sigma \mapsto \int_T g(\cdot, \sigma)\, d\mu$ are lower semicontinuous on $\Sigma(\hat{h})$; note that uniform integrability—as in (2.6)—is guaranteed by (3.30). Hence, the set of all $\sigma \in \Sigma(\hat{h})$ with

$$\int_T g_j(\cdot, \sigma)\, d\mu \leqq \int_T g_j(\cdot, \delta^*)\, d\mu, \qquad j = 1, \cdots, m,$$

is compact; therefore it has an extreme point σ_* by the Krein–Milman theorem. By [6, Prop. II.2]—a consequence of Carathéodory's theorem; cf. [21]—it follows that σ_* is a convex combination of at most $m+1$ extreme points of $\Sigma(\hat{h})$. By the same result every extreme point of $\Sigma(\hat{h})$ is a convex combination of at most two extreme points of Σ. We thus conclude that σ_* is a convex combination of at most $2m+2$ extreme points $\sigma_1, \cdots, \sigma_{2m+2}$ of Σ. By [14, Thms. 5.2, 9.3] there corresponds an $s_i \in \mathcal{M}(T; \hat{S})$ to each σ_i such that $\chi(\varepsilon_i) = \sigma_i$, where $\varepsilon_i(t)$ is the Dirac measure at $s_i(t)$; in fact, we have that $s_i \in \mathcal{M}(T; S)$, as follows easily by $\sigma_* \in \Sigma(\hat{h})$. We now find that for certain $\alpha_1, \cdots, \alpha_{2m+2} \geqq 0$, $\sum_{i=1}^{2m+2} \alpha_i = 1$,

$$\beta_j \equiv \sum_{i=1}^{2m+2} \alpha_i \int_T g_j(\cdot, s_i)\, d\mu \leqq \int_T g_j(\cdot, \delta^*)\, d\mu, \qquad j = 1, \cdots, m.$$

By a well-known extension of Lyapunov's theorem [9, IV.17] there exists, in view of the nonatomicity supposition, a function $s^* \in \mathcal{M}(T; S)$ with

$$\beta_j = \int_T g_j(\cdot, s^*)\, d\mu, \qquad j = 1, \cdots, m,$$

and the proof is thereby finished. QED

A different proof of Lemma II, based on [31, Prop. 14], has been given in [3k]. (In turn, the above result from [31] has been generalized in [3j].)

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