Journal of Economic Theory **93**, 110–117 (2000)
doi:10.1006/jeth.2000.2646, available online at http://www.idealibrary.com on IDEAL®

NOTES, COMMENTS, AND LETTERS TO THE EDITOR

Incompatibility of Usual Conditions for Equilibrium Existence in Continuum Economies without Ordered Preferences[1]

Erik J. Balder

Mathematical Institute, University of Utrecht, Utrecht, The Netherlands

Received October 25, 1996; final version received January 3, 2000

In recent years attempts were made to extend the equilibrium existence results of Gale and Mas-Colell (1975, *J. Math. Econ.* **2**, 9–15) and Shafer and Sonnenschein (1975, *J. Math. Econ.* **2**, 345–348) to continuum economies. Here it is shown that the usual conditions used for these attempts force the "preferred to" correspondence to be *empty-valued* almost everywhere on the nonatomic part of the measure space of agents. Several published continuum extensions of the existence results of Shafer and Sonnenschein are thus rendered pointless. *Journal of Economic Literature* Classification Number: C72. © 2000 Academic Press

Key Words: economies without ordered preferences; Nash equilibrium; continuum games; measurable selections.

1. INTRODUCTION

In several recent papers attempts were made to extend the seminal equilibrium existence results of Gale and Mas-Colell [9] and Shafer and Sonnenschein [16], formulated for abstract economies/games without ordered preferences and a finite number of agents/players, to similar so-called *continuum economies/games* with a measure space $(T, \mathscr{T}, \mu)$ of agents (a satisfactory extension to a countable set of agents was given in [19, Theorem 6.1]). This note submits that such an extension to continuum economies is highly problematic because of measurability and continuity complications inherent to the central primitive concept, the "*preferred to*" *correspondence* (PTC). We do so by considering in Section 2 the measurability and continuity properties of the PTC for a continuum economy

[1] This work was started during a visit to the Department of Economics, University of Bergen, Norway. Financial support from the universities of Bergen and Utrecht is gratefully acknowledged.

0022-0531/00 $35.00

in the usual, non-abstract situation where each agent has a utility/payoff function (i.e., the preferences are ordered). The idea is that the properties discovered in this way for the ordered PTC should also function as a yardstick for the fully abstract model considered in the literature cited above. The motivation for this comes from the very close ties between the ordered and unordered preference situations when the set of agents is finite [16].

We show that under standard measurability and continuity conditions for the utilities/payoffs the resulting "natural" properties of the ordered PTC are quite *non-standard*. By themselves, those properties would not seem to be adequate at all for any equilibrium existence proof by means of current measure theory and functional analysis. Yet, at the same time, existence results are well-known to hold in this context, but only by means of proofs that thoroughly exploit the payoff function structure and its standard-type conditions.

In Section 3 we contrast the "natural" properties for the PTC, as gleaned from Section 2, with the usual measurability and continuity conditions for the abstract PTC, as used in the cited literature. Those conditions are not only much stronger than the "natural" ones, but in fact much too strong. This is demonstrated by the Incompatibility Theorem 3.1, which shows that they cause the PTC to have empty values almost everywhere on the nonatomic part of T.

It thus follows that several extensions of existence results of Shafer–Sonnenschein-type to continuum economies/games in the literature [10–14, 17, 18] are quite pointless. But also similar extensions in the literature, such as [3], that use somewhat different conditions and thus do not formally fall under the theorem, seem to be affected by the fundamental problems indicated here.

2. THE ORDERED "PREFERRED TO" CORRESPONDENCE

The primary purpose of the present section is to learn what measurability and continuity properties of the PTC could be considered natural in a model with a measure space of players (we adopt from now on game-theoretical terminology). We do this by studying the ordered PTC. That is to say the PTC is considered in the concrete, standard situation when the players have ordinary payoffs (ordered preferences case) that satisfy the usual measurability and continuity conditions [5]. In (2) and (3) we shall see that in this situation the standard measurability and continuity conditions for the payoffs translate into highly nonstandard measurability and continuity properties of the ordered PTC.

Let $(\mathscr{T}, \mathscr{T}, \mu)$ be a finite measure space of players and let $\mathbb{R}^d$ be the space of actions (the choice of more complicated action spaces would only aggravate the complications to be exhibited). Also, let $\mathscr{F}$ be a space of measurable functions from T into $\mathbb{R}^d$; these functions can be thought of as the canonical action profiles of the game. Now suppose each player t has an ordinary payoff function $U_t: \mathbb{R}^d \times \mathscr{F} \to \mathbb{R}$. Then the canonical *ordered* "preferred to" correspondence (see [16] or [5, p. 714]) is the multifunction $P_U: T \times \mathscr{F} \to 2^{\mathbb{R}^d}$ given by

$$P_U(t, f) := \{x \in \mathbb{R}^d : U_t(x, f) > U_t(f(t), f)\}. \tag{1}$$

On the right we see the striking appearance of the joint evaluation mapping ev: $(t, f) \mapsto f(t) : T \times \mathscr{F} \to \mathbb{R}^d$. This brings us to the first complication of the ordered PTC, which concerns measurability[2]

$$\text{the joint evaluation } ev \text{ has nonstandard measurability properties.} \tag{2}$$

Naturally, such nonstandard measurability then also affects the ordered PTC P_U of (1) itself.[3] More precisely, even when T is merely the unit interval *cum* Lebesgue measure, it is well-known (cf. [7, 8]) that ev is only measurable with respect to a product σ-algebra that is *not* the product $\mathscr{T} \times \mathscr{B}(\mathscr{F})$ of $\mathscr{T}$ and the Borel σ-algebra on $\mathscr{F}$.

Regarding the above appearance of $\mathscr{B}(\mathscr{F})$, it is necessary to recall that in the context of the papers under consideration the desired equilibrium profile is invariably obtained as some fixed point in $\mathscr{F}$ by means of a Kakutani-type fixed point theorem. For this purpose the space $\mathscr{F}$ has to be endowed with some topology of choice. In the present, prequotient setting of (1) the standard choice corresponding to the literature would be to take $\mathscr{F}$ as a subset of $\mathscr{L}^1_{\mathbb{R}^d}(T, \mathscr{T}, \mu)$ and to equip it with the weak topology $\sigma(\mathscr{L}^1_{\mathbb{R}^d}, \mathscr{L}^\infty_{\mathbb{R}^d})$. That brings us to two additional complications that affect the PTC in the papers under consideration.[4] A major topological complication is that for fixed $t \in T$

$$\text{the pointwise evaluation } ev_t \text{ is not weakly continuous} \tag{3}$$

(actually, observe that ev_t is not even continuous with respect to the usual $\mathscr{L}^1$-seminorm). Here $ev_t: \mathscr{F} \to \mathbb{R}^d$ is given by $ev_t(f) := f(t)$. Thus, (1) shows that there is at least no natural way in which continuity of $f \mapsto U_t(x, f)$, known to be indispensable for application of the usual fixed point theorems,[5] transfers into continuity properties of the ordered PTC.

A less serious complication concerns modelling. It has been discussed before in the literature, although in our opinion it was not resolved entirely

[2] Note that this complication does not hold when T is finite or countable.

[3] This also means that the proof of Corollary 3.1 in [3] is completely wrong.

[4] Again, these do not hold when T is finite or countable.

[5] Actually, one uses even continuity of $(x, f) \mapsto U_t(x, f)$.

adequately [10, p. 94]. Instead of defining the PTC on $T \times \mathscr{F}$, as done above, the papers under consideration define the PTC on $T \times F$, where $F := \pi(\mathscr{F})$ is the *quotient* of the space $\mathscr{F}$ with respect to the usual "equal almost everywhere" equivalence relation. This follows Schmeidler [15], who worked with payoff functions of the type $U_t(x, f) := V_t(x, \pi(f))$; in his model this is actually a quite natural form because of externality considerations (recall that in [15] T is the nonatomic unit interval *cum* Lebesgue measure). However, for the model having the PTC as its primitive concept quotients are considerably less natural: The very reference point of player t's "preferred to" set under the profile f ought to be his/her action $f(t)$, but this is no longer well-defined in terms of quotient profiles (simply observe that $\pi(f)(t)$ does not specify the action $f(t)$ for any player t in the nonatomic part of T). Nevertheless, the latter complication could be removed by going back to the original space $\mathscr{F}$ of true functions (prequotient setting). For equilibrium existence proofs this should go hand in hand with the use of a non-Hausdorff version of some classical results.[6]

3. INCOMPATIBILITY OF STANDARD CONDITIONS

We now state an incompatibility result. Let $(T, \mathscr{T}, \mu)$ be a finite measure space, that is complete and separable and consider $\mathscr{F} \subset \mathscr{L}^1_{\mathbb{R}^d}(T, \mathscr{T}, \mu)$ equipped with the weak topology, which is as in Section 2. We shall suppose that $\mathscr{F}$ is closed with respect to the usual seminorm $f \mapsto \|f\|_1 := \int_T |f| \, d\mu$ and *decomposable* in the following sense [4, VII]: for any measurable set $A \in \mathscr{T}$ and any pair f, f' in $\mathscr{F}$ the (quotient of the), concatenation f'', defined by $f''(t) := f(t)$ on A, and $f''(t) := f'(t)$ on $T \backslash A$, belongs to $\mathscr{F}$. In contrast to Section 2, we shall now consider an abstract "preferred to" correspondence (PTC), denoted by P: $T \times \pi(\mathscr{F}) \to 2^{\mathbb{R}^d}$. Observe that $F := \pi(\mathscr{F})$ is a closed subset of $L^1_{\mathbb{R}^d}(T, \mathscr{T}, \mu)$, the quotient of $\mathscr{L}^1_{\mathbb{R}^d}(T, \mathscr{T}, \mu)$. Since our assumptions cause $L^1_{\mathbb{R}^d}(T, \mathscr{T}, \mu)$ to be complete separable and metric for the L^1-norm, we have the important technical fact that F is a Polish space.[7] A natural and fundamental aspect of the abstract PTC is the following *nonreflexivity* property: for every $f \in \mathscr{F}$

$$f(t) \notin P(t, \pi(f)) \qquad \text{for a.e. } t \text{ in } T. \tag{4}$$

[6] See for instance [6] or [2] for a non-Hausdorff version of Ky Fan's inequality or Kakutani's theorem. But additional non-Hausdorff versions (such as one of the measurable projection theorem) should also be developed for this purpose.

[7] This explains why we conveniently adopted the quotient setting, even though it was criticized in Section 2! Bear in mind, however, that we only wish to produce a negative result that can be applied to the relevant literature.

Of course, this holds *a fortiori* if $f(t) \notin$ convex hull of $P(t, \pi(f))$ a.e., and in this form the condition can be found in the original PTC-model of Shafer and Sonnenschein [16] for a finite number of players.[8] As argued in (2), a highly unnatural condition is measurability of the graph of P in standard terms:

$$\{(t, \pi(f), x) \in T \times F \times \mathbb{R}^d : x \in P(t, \pi(f))\} \text{ is } \mathscr{T} \times \mathscr{B}(F \times \mathbb{R}^d)\text{-measurable.} \tag{5}$$

Also quite unnatural in the continuum game situation (cf. (3) ff.) is the following *open lower section* condition:

$$\{\pi(f) \in F : y \in P(t, \pi(f))\} \text{ is relatively } \|\cdot\|_1\text{-open in } F \subset L^1_{\mathbb{R}^d}(T, \mathscr{T}, \mu). \tag{6}$$

In the original PTC-model of Shafer and Sonnenschein measurability in the variable t is trivial and F is Euclidean. So (5) and (6) could hold quite naturally there, because in [16] the graph of $P(t, \cdot)$ is supposed to be open for every $t \in T$. In contrast, in the continuum game context conditions (6) and (5), which have been directly adopted from [16], strengthen the "natural" properties of P beyond endurance, as the following demonstrates:

THEOREM 3.1 (incompatibility of standard conditions). *If P satisfies* (4), (5) *and* (6), *then P must be trivial on the nonatomic part T^{na} of T. That is to say, then*

$$P(t, \pi(f)) = \varnothing \qquad \text{for a.e. } t \text{ in } T^{na}$$

for every $f \in \mathscr{F}$.

Proof. Rather than restricting all considerations to T^{na}, we suppose without loss of generality that $(T, \mathscr{T}, \mu)$ itself is nonatomic. Suppose there were $\bar{f} \in \mathscr{F}$ and a nonnull set $A \in \mathscr{T}$ such that $Q(t) := P(t, \pi(\bar{f})) \neq \varnothing$ for all $t \in A$. Since the graph of $Q\colon A \to 2^{\mathbb{R}^d}$ is the section at $\pi(\bar{f})$ of the graph of $P\colon A \times F \to 2^{\mathbb{R}^d}$, it follows by (5) that the graph of Q is measurable. Hence, there exists a $\mathscr{T} \cap A$-measurable selection $g\colon A \to \mathbb{R}^d$ of Q, i.e., $g(t) \in P(t, \pi(\bar{f}))$ for all $t \in A$ (apply [4, III.30, III.22]). Since $\bar{f}$ and g are measurable functions, one has $\mu(B) > 0$ for $K > 0$ large enough, where

$$B := \{t \in A : |g(t)| \leqslant K, |\bar{f}(t)| \leqslant K\}.$$

[8] Since we only wish to produce a negative result, the present formulation of nonreflexivity suffices. For positive results one would expect a Shafer–Sonnenschein-like convex hull operation on the purely atomic part of T.

Define for $n \in \mathbb{N}$

$$C_n := \{t \in B : \exists_{\pi(f) \in F}\, g(t) \notin P(t, \pi(f)) \text{ and } \|\pi(f) - \pi(\bar{f})\|_1 < 1/n\}.$$

The measurability of such C_n is demonstrated at the conclusion of this proof. Denote $B_n := B \backslash C_n$; then of course $B \supset \bigcup_n B_n$, but we have in fact identity: $\bigcup_n B_n = B$. To see this, let $t \in B \subset A$ be arbitrary. Then $\pi(\bar{f})$ belongs to the set $S := \{\pi(f) \in F : g(t) \in P(t, \pi(f))\}$, which is relatively $\|\cdot\|_1$-open by (6). So for some n the $\|\cdot\|_1$-ball around $\pi(\bar{f})$ with radius $1/n$ is contained in S, which is to say that t cannot belong to C_n, i.e., $t \in B_n$. We had $\mu(B) > 0$, so it follows that $\mu(B_{\bar{n}}) > 0$ for $\bar{n}$ sufficiently large. By nonatomicity of μ there exists $B^* \subset B_{\bar{n}}$ such that $0 < \mu(B^*) < (2K\bar{n})^{-1}$. Using decomposability of $\mathscr{F}$, we define $f^* \in \mathscr{F}$ by setting

$$f^*(t) := \begin{cases} g(t) & \text{if } t \in B^*, \\ \bar{f}(t) & \text{if } t \in T \backslash B^*, \end{cases}$$

Now $\|\pi(f^*) - \pi(\bar{f})\|_1 = \int_T |f^* - \bar{f}|\, d\mu \leqslant 1/\bar{n}$; this gives $f^*(t) := g(t) \in P(t, \pi(f^*))$ for all t in the nonnull set $B^* \subset B_{\bar{n}}$. Clearly, this contradicts the nonreflexivity property (4).

The proof is finished by proving $\mathscr{T}$-measurability of the sets C_n; let n be arbitrary. Observe that C_n is the projection onto B of $D' \cap D''$. Here D' is the set of all pairs $(t, \pi(f)) \in B \times F$ such that $g(t) \notin P(t, \pi(f))$ and D'' is the Cartesian product of B and the $\|\cdot\|_1$-ball around $\pi(\bar{f})$ with radius $1/n$. Clearly, D'' is $\mathscr{T} \times \mathscr{B}(F)$-measurable. To see that D' is also $\mathscr{T} \times \mathscr{B}(F)$-measurable requires an elementary argument. Let $\mathscr{H}$ be the collection of all sets $H \subset T \times F \times \mathbb{R}^d$ for which $\{(t, \pi(f)) : (t, \pi(f), g(t)) \in H\}$ belongs to $\mathscr{T} \times \mathscr{B}(F)$. Then $\mathscr{H}$ is a σ-algebra over $T \times F \times \mathbb{R}^d$ containing the measurable rectangles. Therefore, $\mathscr{H}$ must coincide with $(\mathscr{T} \cap W) \times \mathscr{B}(F \times \mathbb{R}^d)$, the σ-algebra generated by all such rectangles. Hence, (5) implies that D' belongs to $\mathscr{T} \times \mathscr{B}(F)$. So we conclude that C_n is the projection onto B of the $\mathscr{T} \times \mathscr{B}(F)$-measurable set $D' \cap D''$. Previously, we already observed that F is Polish; *a fortiori* this makes F a Suslin space. We can now invoke the classical measurable projection theorem [4, III.23] to conclude that C_n is $\mathscr{T}$-measurable. Q.E.D

With very minor adaptations, Theorem 3.1 continues to hold if the action space $\mathbb{R}^d$ is replaced by a separable Banach space.

Remarkably, very much of the above incompatibility result, including its proof, can already be found in an article by M. A. Khan and N. S Papageorgiou. In [11] they obtained a very curious existence result, apparently in response to a question by B. Grodal. From Remark 1 in [11] it is quite clear that at the time its authors did not realize the full implication of their results: In Remark 1 they only admit that, in contrast to the usual literature, strictly preferred actions *may* be infeasible. However, if one applies their claim 4 of [11, p. 507] to the special choice $A(t, x) \equiv X(t)$ (here $X(t)$ is the set of player t's feasible actions), then it is evident that all strictly preferred actions in [11] *must* be infeasible!

If we take $\mathscr{F}$ to be the set of all integrable a.e. selections of a multifunction with measurable graph and nonempty, convex and weakly compact values in some separable Banach space, then the corresponding quotient $F := \pi(\mathscr{F})$ is what is used in the referenced literature (we already observed above that instead of $\mathbb{R}^d$ we could have used such a separable Banach space without any difficulties). Observe that such a set $\mathscr{F}$ is decomposable. Also, it is weakly compact by Diestel's theorem [18]. Hence, it is then also weakly closed. So $\mathscr{F}$ is also closed for $\|\cdot\|_1$, which was our starting assumption. This means that the conditions which we impose upon $\mathscr{F}$ in are amply fulfilled in [10–14, 17, 18]. Also, the usual measurability condition (5) for P holds in all those references except for [11].

Let us now relate the other conditions of Theorem 3.1 to the literature. References [13, 17] and [18, Section 8] work with the convex version of (4) (similar to [16]); as already remarked before, (4) then follows *a fortiori.* Those same references also meet (6), as a direct consequence of their condition that for each t the graph of $P(t, \cdot)$ be open (similar to [16]). In [11] (4) is imposed as such, and (6) holds as a consequence of the assumption that the complementary multifunction $P^c(t, \cdot)$ be upper semicontinuous. In that reference the PTC is required to have nonempty values, but the values $P(t, f)$ are not required to be contained in the feasible sets $X(t)$. In [14] and [12, Theorem 2] the convex version of (4) is used again (see above), and (6) is assumed to hold for the weak topology (of course, this implies the validity of (6) itself). Moreover, in [12] the PTC is also required to have nonempty values. Theorem 3.1 therefore implies that the basic assumptions of the model in [12] are inconsistent (i.e., self-contradictory) on the nonatomic part of the set of players/agents.

ACKNOWLEDGMENT

The author is grateful to an associate editor and two anonymous referees for suggestions and questions that helped him to clarify the issues addressed in this note.

REFERENCES

1. E. J. Balder, A unifying approach to existence of Nash equilibria, *Intern. J. Game Theory* **24** (1995), 79–94.
2. E. J. Balder, On the existence of Cournot–Nash equilibria in continuum games, *J. Math. Econ.* **32** (1999), 207–223.

3. E. J. Balder and N. C. Yannelis, Equilibria in random and Bayesion games with a continuum of players, *in* "Equilibrium Theory in Infinite Dimensional Spaces" (M. A. Khan and N. C. Yannelis, Eds.), pp. 333–350, Springer-Verlag, Berlin, 1991.
4. C. Castaing and M. Valadier, "Convex Analysis and Measurable Multifunctions," Lecture Notes in Math, Vol. 580, Springer-Verlag, Berlin, 1977.
5. G. Debreu, Existence of competitive equilibrium, *in* "Handbook of Mathematical Economics II" (K. J. Arrow and M. D. Intriligator, Eds.), pp. 697–743, North-Holland, Amsterdam, 1982.
6. X. P. Ding and K.-K. Tan, Generalized variational inequalities and generalized quasi-variational inequalities, *J. Math. Anal. Appl.* **148** (1990), 497–508.
7. R. M. Dudley, A counterexample on measurable processes, *in* "Proc. Sixth Berkeley Symp. Math. Statist. Prob.," Univ. of California Press, Berkeley, 1972.
8. R. M. Dudley, Correction to A counterexample on measurable processes, *Ann. Prob.* **1** (1973), 191–192.
9. D. Gale and A. Mas-Colell, An equilibrium existence result for a general model without ordered preferences, *J. Math. Econ.* **2** (1975), 9–15 [Erratum, *J. Math. Econ.* **6** (1979), 297–298].
10. M. A. Khan, On extensions of the Cournot–Nash theorem, *in* "Advances in Equilibrium Theory" (C. D. Aliprantis, O. Burkinshaw, and N. J. Rothman, Eds.), Lecture Notes in Economics and Math. Systems, Vol. 244, pp. 79–106, Springer-Verlag, Berlin, 1985.
11. M. A. Khan and N. S. Papageorgiou, On Cournot–Nash equilibria in generalized qualitative games with an atomless measure space of agents, *Proc. Amer. Math. Soc.* **100** (1987), 505–510.
12. M. A. Khan and N. S. Papageorgiou, On Cournot–Nash equilibria in generalized qualitative games with a continuum of players, *Nonlinear Anal.* **11** (1987), 741–755.
13. M. A. Khan and R. Vohra, Equilibrium in abstract economies without ordered preferences and with a measure spaces of agents, *J. Math. Econ.* **13** (1984), 133–142.
14. T. Kim, K. Prikry, and N. C. Yannelis, Equilibria in abstract economies with a measure space of agents and with an infinite dimensional strategy space, *J. Approx. Theory* **56** (1989), 256–266.
15. D. Schmeidler, Equilibrium points of non-atomic games, *J. Statist. Phys.* **7** (1973), 295–300.
16. W. Shafer and H. Sonnenschein, Equilibrium in abstract economies without ordered preferences, *J. Math. Econ.* **2** (1975), 345–348.
17. N. C. Yannelis, Equilibria in noncooperative models of competition, *J. Econ. Theory* **41** (1987), 96–111.
18. N. C. Yannelis, Integration of Banach-valued correspondences, *in* "Equilibrium Theory in Infinite Dimensional Spaces" (M. A. Khan and N. C. Yannelis, Eds.), pp. 2–35, Springer-Verlag, Berlin, 1991.
19. N. C. Yannelis and N. D. Prabhakar, Existence of maximal elements and equilibria in linear topological spaces, *J. Math. Econ.* **12** (1983), 233–245.