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A Unifying Pair of Cournot–Nash Equilibrium Existence Results

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Pluralitas non est ponenda sine necessitate.
William of Ockham

For games with a measure space of players a tandem pair, consisting of a mixed and a pure Cournot–Nash equilibrium existence result, is presented. Their generality causes them to be completely mutually equivalent. This provides a unifying pair of Cournot–Nash existence results that goes considerably beyond the central result of E. J. Balder (1995, *Int. J. Game Theory* **24**, 79–94, Theorem 2.1). The versatility of this pair is demonstrated by the following new applications: (i) unification and generalization of the two equilibrium distribution existence results by K. P. Rath (1996, *J. Math. Econ.* **26**, 305–324) for anonymous games, (ii) generalization of the equilibrium existence result of T. Kim and N. C. Yannelis (1997, *J. Econ. Theory* **77**, 330–353) for Bayesian differential information games, (iii) inclusion of the Bayesian Nash equilibrium existence results of P. R. Milgrom and R. J. Weber (1985, *Math. Oper. Res.* **10**, 619–632) and E. J. Balder (1988, *Math. Operations Res.* **13**, 265–276) for games with private information in the sense of J. C. Harsanyi (1967, *Manage. Sci.* **14**, 159–182). *Journal of Economic Literature* Classification Number: C72.

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1. INTRODUCTION

The present paper unifies existence results for Cournot–Nash equilibria (CNE for short) in the following three rather different areas: (1) games with a measure space of players, as used by Schmeidler and others, (2) anonymous games *à la* Mas-Colell and (3) games with private information in the sense of Harsanyi and others. This will be done by means of a unifying tandem pair of CNE existence results, consisting of Theorem 2.1.1 and 2.2.1.

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First, Theorem 2.1.1, the mixed half of the just mentioned pair, is used to generalize and unify two separate CNE distribution existence results for anonymous games obtained by Rath in [42]; in turn, these results generalize Mas-Colell's original existence result in [37]. This continues the approach to CNE equilibrium distributions introduced in [9, 11, 13], which views CNE distributions as special mixed CNE's (i.e., equilibria which correspond to a special externality that aggregates over players). Secondly, a generalization of the main existence result for Bayesian pure CNE's of Kim and Yannelis [36], who use a model with priors regarding interim beliefs, is given by means of Theorem 2.2.1. The latter result is the pure half of the pair mentioned above. A complete reduction of the Kim–Yannelis model to the pseudogame form used in this paper is achieved via the formulation of a suitable "global" σ-algebra that keeps track of differential information. Thirdly, the same Theorem 2.2.1 is shown to generalize the existence results in [6, 20]. Those results, in turn, are know to extend the well-known Bayesian CNE existence result of Milgrom–Weber [39] for games with private information *à la* Harsanyi [31] to the more natural situation where players' type spaces are non-topological.

The unifying tandem pair of CNE existence results in pseudogames with a measure space of players, which implies these three different applications, is obtained by extending the approach of [11]. This approach is based on a direct application to the best reply correspondence of topological fixed point results in the space of mixed action profiles itself. In the present paper this approach will be considerably strengthened. One half of this pair, Theorem 2.1.1, is a mixed CNE existence result, just as Theorem 2.1 in [11], which it improves in several respects. It is teamed with Theorem 2.2.1, an extension of a recent pure CNE existence result, [15, Theorem 2.1] that is based on a new, so-called *feeble* topology. This topology owes its importance to the compactness conditions for player's individual action spaces that we impose throughout. In such a setting the feeble topology is a very flexible instrument: Examples 2.2.1 and 2.2.2 show that the feeble topology *simultaneously* is capable of subsuming the two usual situations in the literature or games with a measure space of players, which work either with the weak topology $\sigma(L^1, L^\infty)$ or with its weak star counterpart $\sigma(L^\infty, L^1)$. As is shown here, under those same compactness conditions for player's individual action spaces the feeble topology also subsumes tile narrow topology that lies at the base of Theorem 2.1.1, the twin mixed CNE existence result mentioned before. This causes Theorems 2.1.1 and 2.2.1 to be *equivalent* (see Proposition 2.3.1), which testifies to their high level of generality.

Other applications, already discussed in [9, 11, 13, 15], follow *a fortiori* from our results and will not be repeated here. In forthcoming work the methods of this paper will be devoted to a very general treatment of upper semicontinuity of the CNE correspondence [22].

2. MAIN EQUILIBRIUM EXISTENCE RESULTS

This section presents a tandem pair of mixed/pure Cournot–Nash equilibrium (CNE) existence results for continuum pseudogames, that is to say, pseudogames with an abstract measure space of players. We start with Theorem 2.1.1, the mixed CNE existence result, which is formulated for a pseudogame Γ in Section 2.1. Theorem 2.2.1, the pure CNE existence result, is given in Section 2.2 for a pseudogame Γ'. Common elements of these pseudogames Γ and Γ' are as follows. Both have a measure space $(T, \mathscr{T}, \mu)$ of *players* (or, if so desired, *player's types*) with $\mu(T) < +\infty$; thus, the models are in the spirit of Aumann and Schmeidler [3, 43].

Assumption 2.1. $(T, \mathscr{T}, \mu)$ is a separable complete measure space.

Recall that $(T, \mathscr{T}, \mu)$ is said to be separable if the (prequotient) space $\mathscr{L}^1(T, \mathscr{T}, \mu)$ is separable for the usual $\mathscr{L}^1$-seminorm. This separability assumption plays an important part in the proofs below (essentially, by allowing sequential arguments, which is sometimes very critical in measure theory). However, by exploiting an astute trick of Castaing and Valadier [24, p. 78], the separability assumption can be removed from all the existence results below—as opposed to their proofs—at the cost of only a slight strengthening of the measurability conditions (see Remark 2.1.1(iii) for some details—this trick has been worked out in [12] and in [15, Remark 4.2]). The present paper only discusses this trick in connection with (step 2 of) the proof of Theorem 3.4.1. The completeness assumption for $(T, \mathscr{T}, \mu)$ can be removed from the existence results as well. This goes by well-known reasoning involving measurable modifications, based on the fact that the central existence results allow for an exceptional null set (see again [15, Remark 4.2]). Both Γ and Γ' have for each player t a set S_t of (*individually*) *feasible actions*. All sets S_t, $t \in T$, are supposed to lie in an *action universe* S.

2.1. *Mixed Cournot–Nash Equilibrium Existence Result*

This section centers around Theorem 2.1.1, a mixed CNE existence result for the pseudogame $\Gamma := (S_t, A_t, U_t)_{t \in T}$. The following assumptions must hold:

Assumption 2.1.1. S is a completely regular Suslin space.

Recall that S is said to be *completely regular* if for every s in S and every open neighborhood O of s there exists a continuous function $c: S \to \mathbb{R}$ with

$c(s) = 1$ and $c = 0$ on $S \setminus O$. Any metric space is completely regular, and so is any locally convex topological vector space. Recall that a topological space is said to be *Suslin* if it is a topological Hausdorff space that is the surjective image of a Polish space under a continuous mapping; cf. [27, III], [44, II]. For instance, any Polish (i.e., a separable metric and complete) space S or any Borel subset S of a Polish space meets the above assumption, and it continues to do so when equipped with a completely regular topology that is coarser than the original one. E.g., a. separable Banach space meets Assumption 2.1.1, both for the norm-topology, for which it is a Polish space, and for the usual weak topology. Other examples include spaces that are countable unions of Polish spaces, such as the dual of a separable Banach space, when equipped with the weak star topology.

Assumption 2.1.2. (i) For every $t \in T$ the set $S_t \subset S$ is nonempty and compact.

(ii) $\text{gph}\, \Sigma \in \mathscr{T} \times \mathscr{B}(S)$.

Here $\Sigma: T \to 2^S$ is defined by $\Sigma(t) := S_t$ and its *graph* is given by $\text{gph}\, \Sigma := \{(t, s) \in T \times S : s \in S_t\}$. As usual, the symbol $\mathscr{B}(S)$ refers to the *Borel* σ-algebra on S and $\mathscr{T} \times \mathscr{B}(S)$ denotes the product σ-algebra. The trace of the latter σ-algebra on $\text{gph}\, \Sigma$ is denoted by $(\mathscr{T} \times \mathscr{B}(S)) \cap \text{gph}\, \Sigma$. By Assumption 2.1.1, S has metric ρ that is not finer than its original topology (apply [24, III.32] or [27, III.66]—see [15, Section 3] for an explicit description). Hence, Assumption 2.1.2(i) ensures that on the compact sets S_t, $t \in T$, these two topologies coincide. Therefore, from now on we can freely use the metric ρ in some topological arguments that pertain only to the sets of individually feasible actions.

The *mixed action universe* of Γ is the set $M_1^+(S)$, consisting of all probability measures on $(S, \mathscr{B}(S))$. This set is equipped with the classical narrow topology; cf. [23], [27, III]. Recall that this is the coarsest topology on $M_1^+(S)$ for which the functionals $\nu \mapsto \int_S c\, d\nu$ on $M_1^+(S)$ are continuous for every c in the set $\mathscr{C}_b(S)$ of all bounded continuous functions on S. The canonical *mixed action profiles* of Γ are the functions $\delta: T \to M_1^+(S)$, measurable with respect to $\mathscr{T}$ and $\mathscr{B}(M_1^+(S))$. Such δ's can be seen as descriptions/prescriptions of how all the players could or should act (in a mixed way) in the game. The set of all such mixed action profiles is denoted by $\mathscr{R}$. A mixed action profile $\delta \in \mathscr{R}$ is said to be *feasible* if $\delta(t)(S_t) = 1$ for a.e. (meaning μ-almost every) t in T; notice that there is an exceptional null set involved in this definition. The set of all such feasible profiles is denoted by $\mathscr{R}_\Sigma$. Observe that Assumptions 2.1.1, 2.1.2 entail that $\mathscr{R}_\Sigma$ is nonempty. Indeed, the von Neumann–Aumann measurable selection theorem [24, III.22] can be applied here. This gives the existence of a function $f: T \to S$, measurable with respect to $\mathscr{T}$ and $\mathscr{B}(S)$, such that

$f(t) \in S_t$ for every $t \in T$; hence setting $\delta(t) := \varepsilon_f(t) :=$ Dirac point measure at $f(t)$ defines a feasible mixed action profile. Attention is called to the fact that, seen from a mathematical viewpoint, the mixed action profiles in $\mathscr{R}$ are precisely transition probabilities with respect to $(T, \mathscr{T})$ and $(S, \mathscr{B}(S))$ in the sense of [40, III] (see also [1, 2.6]). More precisely, for a function $\delta: T \to M_1^+(S)$ the following are equivalent: (1) $\delta \in \mathscr{R}$, (2) for every $B \in \mathscr{B}(S)$ the function $t \mapsto \delta(t)(B)$ is $\mathscr{T}$-measurable. This is easy to prove by the fact that, although the metric ρ on S generates a topology that is coarser than the original one, the corresponding Borel σ-algebras coincide (apply Corollary 2 on p. 101 of [44]). Using the fact that S is separable metric for ρ, the equivalence proof then follows by [27, III.60] and [40, Proposition III.2.1]. This equivalence makes the results in [40, III.2] about integration and measurability available for the mixed action profiles. In connection with the topology to be introduced shortly, the elements of $\mathscr{R}$ are also often referred to as *Young measures*. Recall from [5, 6, 7] that the *narrow* topology on $\mathscr{R}$ (and on its subset $\mathscr{R}_\Sigma$) is defined as the coarsest topology on $\mathscr{R}$ for which all functionals

$$I_g: \delta \mapsto \int_T \left[\int_S g(t, s)\, \delta(t)(ds) \right] \mu(dt), \qquad g \in \mathscr{G}_C(T; S)$$

are continuous. Note that these integrals are well-defined by [40, III]. Here $\mathscr{G}_C(T; S)$ stands for the collection of all *Carathéodory integrands* on $T \times S$. Recall that this is the set of all $\mathscr{T} \times \mathscr{B}(S)$-measurable functions $g: T \times S \to \mathbb{R}$ for which $g(t, \cdot)$ is continuous on S for every $t \in T$ and for which there is an integrable function $\phi_g \in \mathscr{L}^1_{\mathbb{R}}(T, \mathscr{T}, \mu)$ with $\sup_{s \in S} |g(t, s)| \leqslant \phi_g(t)$ for all $t \in T$. Equivalently (apply [6, Theorem 2.2]), the narrow topology on $\mathscr{R}$ is the coarsest topology for which all functionals

$$I_g: \delta \mapsto \int_T \left[\int_S g(t, s)\, \delta(t)(ds) \right] \mu(dt), \qquad g \in \mathscr{G}^{bb}(T; S)$$

are lower semicontinuous. Here $\mathscr{G}^{bb}(T; S)$ is the collection of all *normal integrands* on $T \times S$ that are *integrably bounded below*; these are the $\mathscr{T} \times \mathscr{B}(S)$-measurable functions $g: T \times S \to \mathbb{R}$ such that $g(t, \cdot)$ is lower semicontinuous on S for every $t \in T$ and for which there is an integrable function $\phi_g \in \mathscr{L}^1_{\mathbb{R}}(T, \mathscr{T}, \mu)$ with $\inf_{s \in S} g(t, s) \geqslant \phi_g(t)$ for all $t \in T$. Evidently, the narrow topology on $M_1^+(S)$, to which reference was already made, can be seen as a special case of the above narrow topology on $\mathscr{R}$ (e.g., consider what happens to the constant mixed action profiles or, more particularly, what happens when T is a singleton). To distinguish it from the latter, it will from now on consistently be called the *classical* narrow topology. In connection with Section 2.3, the following addition fact is useful: The restriction of the narrow topology to $\mathscr{R}_\Sigma \subset \mathscr{R}$ is precisely the coarsest topology for which all functionals

$$I_g\colon \delta \mapsto \int_T \left[\int_{S_t} g(t, s)\, \delta(t)(ds) \right] \mu(dt), \qquad g \in \mathscr{G}_{C,\Sigma}(T)$$

are continuous on $\mathscr{R}_\Sigma$. Here $\mathscr{G}_{C,\Sigma}(T)$ is the set of all $(\mathscr{T} \times \mathscr{B}(S)) \cap \operatorname{gph} \Sigma$-measurable functions $g\colon \operatorname{gph} \Sigma \to \mathbb{R}$ for which $g(t, \cdot)$ is continuous on S_t for every $t \in T$ and for which there exists an integrable function $\phi_g \in \mathscr{L}^1_{\mathbb{R}}(T, \mathscr{T}, \mu)$ with $\sup_{s \in S_t} |g(t, s)| \leqslant \phi_g(t)$ for all $t \in T$. This is a direct consequence of the above equivalence: one has $g_1, g_2 \in \mathscr{G}^{bb}(T; S)$ by setting $g_i := (-1)^i g$ on $\operatorname{gph} \Sigma$ and $g_i := +\infty$ on $(T \times S) \setminus \operatorname{gph} \Sigma$, with $I_{g_1}(\delta) = -I_{g_2}(\delta)$ for all $\delta \in \mathscr{R}_\Sigma$.

As a social feature of Γ, each player must choose his or her actions as follows in accordance with the other players: given the profile $\delta \in \mathscr{R}_\Sigma$, player t's *socially feasible* actions constitute a given subset $A_t(\delta) \subset S_t$. In a truly noncooperative situation one simply eliminates such social interaction by choosing

$$A_t(\delta) := S_t \qquad \text{for all} \quad t \in T \qquad \text{and} \qquad \delta \in \mathscr{R}_\Sigma. \tag{2.1}$$

Assumption 2.1.3. (i) For every $(t, \delta) \in T \times \mathscr{R}_\Sigma$ the set $A_t(\delta) \subset S_t$ is nonempty and closed.

(ii) For every $t \in T$ the multifunction $A_t\colon \mathscr{R}_\Sigma \to 2^{S_t}$ is (narrowly) upper semicontinuous.

(iii) For every $\delta \in \mathscr{R}_\Sigma$ the graph of the multifunction $t \mapsto A_t(\delta)$ belongs to $\mathscr{T} \times \mathscr{B}(S)$.

To measure the consequences of player t's actions in the face of his or her opponents, one introduces the *payoff function* $U_t\colon S_t \times \mathscr{R}_\Sigma \to [-\infty, +\infty]$. Given the mixed action profile $\delta \in \mathscr{R}_\Sigma$, player t receives $U_t(s, \delta)$ for taking action $s \in S_t$ (see also the comments following Theorem 2.1.1).

Assumption 2.1.4. (i) For every $t \in T$ the function $U_t\colon S_t \times \mathscr{R}_\Sigma \to [-\infty, +\infty]$ is upper semicontinuous.

(ii) For every $\delta \in \mathscr{R}_\Sigma$ the function $(t, s) \mapsto U_t(s, \delta)$ is $(\mathscr{T} \times \mathscr{B}(S)) \cap \operatorname{gph} \Sigma$-measurable.

The final assumption requires a certain interplay between social feasibility and payoff:

Assumption 2.1.5. For every $t \in T$ the function $\delta \mapsto \sup_{s \in A_t(\delta)} U_t(s, \delta)$ is (narrowly) lower semi-continuous on $\mathscr{R}_\Sigma$.

Remark 2.1.1. (i) In the strictly noncooperative situation of (2.1) Assumption 2.1.3 holds automatically by Assumption 2.1.2. In that same situation Assumption 2.1.5 certainly holds if $U_t(s, \cdot)$ is narrowly lower semicontinuous on $\mathscr{R}_\Sigma$ for every $(t, s) \in \operatorname{gph} \Sigma$ (then, together with Assumption 2.1.4(i), this implies that $U_t(s, \cdot)$ is narrowly continuous).

(ii) In the terminology of the highly tautological [45, Proposition 1], Assumption 2.1.5 states that $U_t(s, \delta)$ is *feasible path transfer lower semicontinuous in s with respect to* A_t for every $t \in T$.

(iii) The measurability Assumptions 2.1.3(iii) and 2.1.4(ii) serve exclusively to make that the multifunction $t \mapsto \operatorname{argmax}_{s \in A_t(\delta)} U_t(s, \delta)$ has a $\mathscr{T} \times \mathscr{B}(S)$-measurable graph for $\delta \in \mathscr{R}_\Sigma$. If, moreover, $(t, \delta) \mapsto \operatorname{argmax}_{s \in A_t(\delta)} U_t(s, \delta)$ has a $\mathscr{T} \times \mathscr{B}(S) \times \mathscr{B}(\mathscr{R}_\Sigma)$-measurable graph, then separability in Assumption 2.1 in no longer needed, for it can be re-created by following [24, p. 78], which has also been followed in step 2 of the proof of Theorem 3.4.1. Observe that such a measurability condition for the graph certainly holds if joint measurability in (t, s, δ), instead of (t, s), is required in Assumptions 2.1.2(iii) and 2.1.4(ii)

The main result of this section, a result about existence of a mixed Cournot–Nash equilibrium profile in Γ, can now be stated. Observe below that under such an equilibrium profile μ-almost every player t randomizes over actions that maximize his/her own payoff in a socially feasible way. The proof of this result will be given in Section 4.1.

THEOREM 2.1.1 (Mixed equilibrium existence result). *Under Assumption* 2.1 *and Assumptions* 2.1.1 *to* 2.1.5 *there exists a* mixed Cournot–Nash equilibrium *for the above pseudogame* Γ. *That is, there exists a mixed action profile* $\delta_* \in \mathscr{R}_\Sigma$ *such that*

$$\delta_*(t)(\operatorname*{argmax}_{s \in A_t(\delta_*)} U_t(s, \delta_*)) = 1 \qquad \text{for } \mu\text{-a.e. } t \text{ in } T.$$

Taking into account the comments following Assumption 2.1, this improves Theorem 2.1, the main result of [11], in the following respects: (1) Assumption 2.1.5 improves upon the continuity requirement in [11, Assumption 2.6]; cf. Remark 2.1.1(i). (2) In [11] only the purely noncooperative situation with (2.1) is considered. (3) Theorem 2.1.1 deals directly with $U_t(s, \delta)$. In contrast, in [11] a $U_t(s, \delta)$ of the form $U_t(s, e_t(\delta))$ is used, with the technical complication that all mappings e_t, $t \in T$, on $\mathscr{R}_\Sigma$ should map into a common space that is itself Suslin and metric.

2.2. *Pure Cournot–Nash Equilibrium Existence Result*

In this section a pure counterpart to the above existence result Theorem 2.1.1 is presented. This result (partially) allows for purification by nonatomicity. The counterpart to Γ is now a pseudogame $\Gamma' := (S_t, A'_t, U'_t)_{t \in T}$ in pure actions. Let us suppose that T is partitioned into two different groups of players, i.e., suppose there are $\bar{T}, \hat{T} \in \mathscr{T}$ such that $T = \bar{T} \cup \hat{T}$ and $\bar{T} \cap \hat{T} = \varnothing$.

Assumption 2.2.1. $\hat{T}$ is contained in the nonatomic part of the measure space $(T, \mathscr{T}, \mu)$.

Purification by nonatomicity is to take place on the part $\hat{T}$.

Assumption 2.2.2. S is a Suslin locally convex topological vector space.

This assumption entails that S is completely regular as well, which makes it a specialization of Assumption 2.1.1. As before, we define $\Sigma: T \to 2^S$ by $\Sigma(t) := S_t$ and denote its graph by gph Σ.

Assumption 2.2.3. (i) For every $t \in \bar{T}$ the set $S_t \subset S$ is nonempty, convex and compact.

(ii) For every $t \in \hat{T}$ the set $S_t \subset S$ is nonempty and compact.

(iii) gph $\Sigma \in \mathscr{T} \times \mathscr{B}(S)$.

A *pure action profile* of Γ' is a function $f: T \to S$ that is measurable with respect to $\mathscr{T}$ and $\mathscr{B}(S)$ or, equivalently [24, III.36], that is *scalarly measurable*, i.e., for which all scalar functions $t \mapsto \langle f(t), s^* \rangle$, $s^* \in S^*$, are $\mathscr{T}$-measurable. Here S^* stands for the topological dual of S. Such equivalence of ordinary and scalar measurability is due to Assumption 2.2.2. Let $\mathscr{S}$ denote the set of all such action profiles. A pure action profile $f \in \mathscr{S}$ is *feasible* if $f(t) \in S_t$ for μ-a.e. t in T. The set of all feasible action profiles is denoted by $\mathscr{S}_\Sigma$. Also, let $\bar{\mathscr{S}}_\Sigma$ be the set of all restrictions to $\bar{T}$ of functions in $\mathscr{S}_\Sigma$; it is only this set that needs to be topologized. Recall from [15] that the *feeble topology* on $\bar{\mathscr{S}}_\Sigma$ is defined as the coarsest topology for which all functionals

$$J_g: f \mapsto \int_{\bar{T}} g(t, f(t))\, \mu(dt), \qquad g \in \bar{\mathscr{G}}_{LC, \Sigma}$$

are continuous. Here $\bar{\mathscr{G}}_{LC, \Sigma}$ is the collection of all $(\mathscr{T} \cap \bar{T}) \times \mathscr{B}(S)$-measurable functions $g: \bar{T} \times S \to \mathbb{R}$ for which $g(t, \cdot)$ is linear and continuous on S for every $t \in \bar{T}$ and for which there is an integrable function $\phi_g \in \mathscr{L}^1_{\mathbb{R}}(\bar{T}, \mathscr{T} \cap \bar{T}, \mu)$ with $\sup_{s \in S_t} |g(t, s)| \leqslant \phi_g(t)$ for all $t \in \bar{T}$. Note that this causes the above functional to be well-defined. In the special case $\bar{T} = T$ we shall write $\mathscr{G}_{LC, \Sigma}$ instead of $\bar{\mathscr{G}}_{LC, \Sigma}$. The following two examples show that, quite remarkably, the feeble topology can simultaneously subsume the two customary topologies that have been used in the literature on games with a measure space of players.

EXAMPLE 2.2.1. Let S be a separable Banach space, equipped with either the norm topology or the weak topology $\sigma(S, S^*)$ (in both cases the dual space is S^*). In addition to what is required in Assumption 2.1.2, let $\Sigma: T \to 2^S$ be *integrably bounded*; that is to say, there exists $\phi_\Sigma \in \mathscr{L}^1_{\mathbb{R}}(T, \mathscr{T}, \mu)$ such that $\sup_{s \in S_t} \|s\| \leqslant \phi_\Sigma(t)$ for every $t \in T$. Here $\|\cdot\|$ stands for the norm on S. In this situation S is a locally convex topological vector space that is Suslin (in fact, for the norm topology S is even a Polish space, so it is Suslin for the weaker topology $\sigma(S, S^*)$). Clearly, because of the integrable boundedness condition, $\mathscr{S}_\Sigma$ is precisely the prequotient space $\mathscr{L}^1_\Sigma$, i.e., the space consisting of all Bochner-integrable μ-a.e.-selectors of the multifunction Σ. Also, on $\mathscr{S}_\Sigma = \mathscr{L}^1_\Sigma$ the feeble topology coincides in this situation with the usual (prequotient) weak $\mathscr{L}^1$-topology $\sigma(\mathscr{L}^1_S, \mathscr{L}^\infty_{S^*}[S])$. Recall here that $\mathscr{L}^\infty_{S^*}[S] := \mathscr{L}^\infty_{S^*}(T, \mathscr{T}, \mu)$ is the space of all functions $b: T \to S^*$ that are S-scalarly measurable and bounded (i.e., $\sup_{t \in T} \|b(t)\|^* < +\infty$, with $\|\cdot\|^*$ denoting the dual norm, as usual). The space $\mathscr{L}^\infty_{S^*}[S]$ can be identified with the dual of $\mathscr{L}^1_S := \mathscr{L}^1_S(T, \mathscr{T}, \mu)$, when the latter is equipped with the usual $\mathscr{L}^1$-seminorm [33, IV]. The equality of these two topologies can be seen as follows. First, observe that on $\mathscr{S}_\Sigma$ the feeble topology is at least as fine as the topology $\sigma(\mathscr{L}^1_S, \mathscr{L}^\infty_{S^*}[S])$, because to every $b \in \mathscr{L}^\infty_{S^*}[S]$ there corresponds a canonical $g_b \in \mathscr{G}_{LC, \Sigma}$ which is given by $g_b(t, s) := \langle s, b(t) \rangle$ (by [24, III.14] g_b is $\mathscr{T} \times \mathscr{B}(S)$-measurable; note also the inequality $\sup_{s \in S_t} |g_b(t, s)| \leqslant \phi_\Sigma(t) \sup_T \|b(\cdot)\|^*$). Further, by [15, Proposition 3.2], which is a corollary of Theorems 4.1.1 and 4.2.2 used below, $\mathscr{S}_\Sigma$ is feebly compact. Unlike the feeble topology itself, the quotient of the feeble topology for the usual equivalence relation "equality μ-almost everywhere" is Hausdorff (denote this equivalence relation on the set of all measurable functions from T into S by π). So on the quotient-feebly compact set $\pi(\mathscr{S}_\Sigma)$ the quotient-feeble topology coincides with the usual quotient topology $\sigma(L^1_S, L^\infty_{S^*}[S])$. Since the defining functionals J_g, $g \in \mathscr{G}_{LC, \Sigma}$, for the feeble topology are constant on every π-equivalence class, it follows that the coincidence of these topologies can be carried back to the original prequotient setting.

By the above example, the referenced compactness result of [15, Proposition 3.2] forms an extension of Diestel's theorem [46, Theorem 3.1]. The following situation is considered on some occasions (e.g., cf. [34, p. 101]):

EXAMPLE 2.2.2. Let S be the dual of a separable Banach space R and let S be equipped with the weak star topology $\sigma(S, R)$. Then S is locally convex. Also, it is the countable union of the balls $S_n := \{s \in S : \|s\|^* \leqslant n\}$, $n \in \mathbb{N}$, each of which is weak star compact (by the Alaoglu–Bourbaki theorem) and metrizable (by separability of R). Hence, S is a Suslin space.

Following [34, p. 101], consider the situation where Assumption 2.2.3 holds and where all sets S_t, $t \in T$, are contained in a single closed ball K around the origin (so K is compact and metrizable for the reasons just explained). Then $\mathscr{S}_\Sigma$ is obviously the prequotient space $\mathscr{L}^\infty_\Sigma[R](T, \mathscr{T}, \mu)$ that consists of all bounded and R-scalarly measurable μ-a.e.-selectors of Σ. On $\mathscr{S}_\Sigma$ the feeble topology coincides in this situation with the weak star topology $\sigma(\mathscr{L}^\infty_S[R], \mathscr{L}^1_R)$. To see this, notice that on $\mathscr{S}_\Sigma$ the feeble topology is at least as fine as the weak star topology, simply because to every $\ell \in \mathscr{L}^1_R$ there corresponds a canonical $g_\ell \in \mathscr{G}_{LC, \Sigma}$, given by $g_\ell(t, s) := \langle \ell(t), s \rangle$ (note that on every closed ball S_n, which is metrizable and compact, the function g_ℓ is $\mathscr{T} \times \mathscr{B}(S)$-measurable by [24, III.14] and observe also the inequality $\sup_{s \in S_t} |g_\ell(t, s)| \leqslant \alpha_K \|\ell(t)\|_R$, where α_K denotes the diameter of the set K). Again, the compactness result [15, Proposition 3.2] and a quotient argument can be used to show that these two topologies on $\mathscr{S}_\Sigma$ actually coincide.

Yet another possibility is presented by a modification of the previous example:

EXAMPLE 2.2.3. As in Example 2.2.2, let S be the dual of a separable Banach space R and let S be equipped with the weak star topology $\sigma(S, R)$. Then S is a Suslin space, as shown in the previous example. Consider now the situation where Assumption 2.2.3 holds and where there exists $\phi_\Sigma \in \mathscr{L}^1_{\mathbb{R}}(T, \mathscr{T}, \mu)$ such that $\sup_{s \in S_t} \|s\|^* \leqslant \phi_\Sigma(t)$ for every $t \in T$. Then $\mathscr{S}_\Sigma$ is the prequotient space $\mathscr{L}^1_\Sigma[R](T, \mathscr{T}, \mu)$ that consists of all R-scalarly integrable (also called *Gelfand-integrable*) μ-a.e.-selectors of the multifunction Σ. To see this, observe in one direction that $|\langle r, f(t) \rangle| \leqslant \|r\|_R \, \phi_\Sigma(t)$ for every $r \in R$, and observe in the opposite direction that scalar and ordinary measurability are equivalent by [24, III.36], as pointed out earlier (note that R is the topological dual of $(S, \sigma(S, R))$). Similar to Example 2.2.1, note that the feeble topology on $\mathscr{S}_\Sigma$ is at least as fine as the topology $\sigma(\mathscr{L}^1_S[R], \mathscr{L}^\infty_R[S])$. This is because for every $b \in \mathscr{L}^\infty_R[S](T, \mathscr{T}, \mu)$ the function $g_b\colon (t, s) \mapsto \langle b(t), s \rangle$ defines an element in $\mathscr{G}_{LC, \Sigma}$ (product measurability of g_b is proven as in Example 2.2.2 and the inequality $|g_b(t, s)| \leqslant \psi_\Sigma(t) \sup_T \|b(\cdot)\|_R$ is similar to the one obtained in Example 2.2.1). After this, the actual identity of these two topologies is proven in the same way as in Example 2.2.1.

Let us now define as the *externality* of each player $t \in T$ the mapping $d := (\bar{d}, \hat{d})\colon \mathscr{S}_\Sigma \to \bar{\mathscr{S}}_\Sigma \times \mathbb{R}^m$, which is defined by

$$\bar{d}(f) := f|_{\bar{T}}, \qquad \hat{d}(f) := \left(\int_{\hat{T}} g_i(t, f(t)) \, \mu(dt) \right)_{i=1}^m.$$

Here $f|_{\bar{T}} \in \bar{\mathscr{S}}_\Sigma$ stands for the restriction to $\bar{T}$ of $f \in \mathscr{S}_\Sigma$. Also, $g_1, ..., g_m$: $\text{gph}\, \Sigma \cap (\hat{T} \times S) \to \mathbb{R}$ are given functions that satisfy the following condition.

Assumption 2.2.4. $g_1, ..., g_m \in \mathscr{G}_{C,\Sigma}(\hat{T})$.

Thus, the externality d is such that on $\hat{T}$ the restriction $f|_{\hat{T}}$ of $f \in \mathscr{S}_\Sigma$, which completely describes the action $f(t)$ by each player t in $\hat{T}$, is replaced by the aggregate $\hat{d}(f)$ over all of $\hat{T}$. Observe that in the special situation with $\hat{T} := \emptyset$ and $\bar{T} := T$ we have $d(f) = f$ and $\bar{\mathscr{S}}_\Sigma = \mathscr{S}_\Sigma$. Each player $t \in T$ must choose his/her actions in accordance with the other players as follows: given the pure action profile $f \in \mathscr{S}_\Sigma$, player t's *socially feasible* actions constitute a given subset $A'_t(d(f)) \subset S_t$. Observe that the externality intervenes here. Of course, for a truly noncooperative situation one can always choose $A'_t \equiv S_t$, quite similar to (2.1). Further, every player $t \in T$ has a *payoff function* U'_t: $S_t \times \bar{\mathscr{S}}_\Sigma \times \mathbb{R}^m \to [-\infty, +\infty]$.

Assumption 2.2.5. (i) For every $(t, \bar{f}, y) \in T \times \bar{\mathscr{S}}_\Sigma \times \mathbb{R}^m$ the set $A'_t(\bar{f}, y) \subset S_t$ is nonempty and closed.

(ii) For every $t \in T$ the multifunction A'_t: $\bar{\mathscr{S}}_\Sigma \times \mathbb{R}^m \to 2^{S_t}$ is upper semicontinuous.

(iii) For every $(\bar{f}, y) \in \bar{\mathscr{S}}_\Sigma \times \mathbb{R}^m$ the graph of the multifunction $t \mapsto A'_t(\bar{f}, y)$ belongs to $\mathscr{T} \times \mathscr{B}(S)$.

Assumption 2.2.6. (i) For every $t \in T$ the function U'_t: $S_t \times \bar{\mathscr{S}}_\Sigma \times \mathbb{R}^m \to [-\infty, +\infty]$ is upper semicontinuous.

(ii) For every $(\bar{f}, y) \in \bar{\mathscr{S}}_\Sigma \times \mathbb{R}^m$ the function $(t, s) \mapsto U'_t(s, \bar{f}, y)$ is $(\mathscr{T} \times \mathscr{B}(S)) \cap \text{gph}\, \Sigma$-measurable.

Similar to what went on in the mixed equilibrium model, the final assumption requires certain relationships between A' and U' to hold; this time, a convexity condition is added to what was required in tile corresponding Assumption 2.1.5, but only for players in $\bar{T}$:

Assumption 2.2.7. (i) For every $t \in T$ the function $(\bar{f}, y) \mapsto \sup_{s \in A'_t(\bar{f}, y)} U'_t(s, \bar{f}, y)$ is lower semi-continuous on $\bar{\mathscr{S}}_\Sigma \times \mathbb{R}^m$.

(ii) For every $(t, \bar{f}, y) \in \bar{T} \times \bar{\mathscr{S}}_\Sigma \times \mathbb{R}^m$ the set $\text{argmax}_{s \in A'_t(\bar{f}, y)} U'_t(s, \bar{f}, y)$ is convex.

Here a counterpart to Remark 2.1.1 applies:

Remark 2.2.1. (i) If $A'_t \equiv S_t$ for all $t \in T$ (noncooperative situation), Assumption 2.2.5 holds automatically and Assumption 2.2.7(i) holds if $U'_t(s, \cdot, \cdot)$ is continuous on $\bar{\mathscr{S}}_\Sigma \times \mathbb{R}^m$ for every $s \in S_t$.

(ii) In the terminology of [45, Proposition 1], Assumption 2.2.7(i) states that $U'_t(s, \bar{f}, y)$ is *feasible path transfer lower semicontinuous in s with respect to* A'_t for every $t \in T$.

(iii) Assumption 2.2.7(ii) holds if for every $t \in \bar{T}$ and $(\bar{f}, y) \in \bar{\mathscr{S}}_\Sigma \times \mathbb{R}^m$ the set $A'_t(\bar{f}, y)$ is convex and the function $U'_t(\cdot, \bar{f}, y)$ is quasiconcave on $A'_t(\bar{f}, y)$.

(iv) Assumptions 2.2.5(iii) and 2.2.6(ii) purely serve to guarantee that the graph of the multifunction $t \mapsto \operatorname{argmax}_{s \in A'_t(\bar{f}, y)} U'_t(s, \bar{f}, y)$ is $(\mathscr{T} \times \mathscr{B}(S)) \cap \operatorname{gph} \Sigma$-measurable for every $(\bar{f}, y) \in \bar{\mathscr{S}}_\Sigma \times \mathbb{R}^m$. In connection with the removal of Assumption 2.1 a similar comment holds as in Remark 2.1.1(iii). Under stronger (semi)continuity conditions for the payoffs and social feasibility multifunctions, such a removal has been implemented in [15].

(v) As will become clear in the proof, the vector structure of the action universe S, postulated in Assumption 2.2.2, is really only needed for players $t \in \bar{T}$ so as to obtain barycenters (i.e., expectations) of their mixed actions. Therefore, the following extension is possible: one could introduce two separate action universes, viz. $\bar{S}$ (for players $t \in \bar{T}$) and $\hat{S}$ (for players $t \in \hat{T}$). In such a setup only $\bar{S}$ would have to be as in Assumption 2.2.2, and $\hat{S}$ could be of the same type as in Assumption 2.1.1. In particular, this means that for the special case $\bar{T} := \varnothing, \hat{T} := T$ in Theorem 2.2.1 we can replace Assumption 2.2.2 by Assumption 2.1.1.

(vi) An obvious extension, inspired by the way purification is used in the proof of Theorem 2.2.1, is as follows. Instead of the externality component $\hat{d}$, defined above, we could also have $\hat{d}$ equal to a countable sequence $(\hat{d}_j)$. This sequence would correspond to some countable measurable partitioning $(\hat{T}_j)$ of $\hat{T}$, and each $\hat{d}_j$ would have the same structure as $\hat{d}$ studied above, but relative to $\hat{T}_j$ instead of $\hat{T}$. Thus, for each j there would be m_j integrands $g^j_1, \ldots, g^j_{m_j}$ in $\mathscr{G}_{\Sigma, C}(\hat{T}_j)$, with $\hat{d}_j(f) := (\int_{\hat{T}_j} g^j_i(t, f(t)) \mu(dt))_{i=1}^{m_j}$, and the new externality $\hat{d}(f)$ would now be $(\hat{d}_j(f))_j$.

THEOREM 2.2.1 (Pure equilibrium existence result). *Under Assumption* 2.1 *and Assumptions* 2.2.1 *to* 2.2.7 *there exists a* pure Cournot–Nash equilibrium *for the above pseudogame Γ'. That is, there exists a pure action profile $f_* \in \mathscr{S}_\Sigma$ such that*

$$f_*(t) \in \operatorname*{argmax}_{s \in A'_t(d(f_*))} U'_t(s, d(f_*)) \qquad \textit{for } \mu\textit{-a.e. } t \textit{ in } T.$$

Observe that in the extreme case $\hat{T} = T$, with $(T, \mathscr{T}, \mu)$ nonatomic, this result is entirely about purification by nonatomicity. In this capacity, for instance, it was shown in [11] to generalize the main result of

[41, Theorem 2]; that result has a finite-dimensional action universe S, uses $g_i(t, s) := i$th coordinate of s and works with integrable boundedness assumptions for Σ, as in Example 2.2.1. See, however, [35, Theorem 1] for a rather special equilibrium result by purification that is apparently not covered by Theorem 2.2.1. The above result, which will be proven in Section 4.2, is [15, Theorem 2.1], but with two additional improvements: (1) The current formulation of Assumption 2.2.7 means that certain continuity conditions that appear in [15, Assumptions 2.4, 2.5] can be replaced by mere upper semicontinuity conditions. (2) Assumption 2.2.5(iii) is less demanding than the corresponding part of [15, Assumption 2.9]. As explained in [15], Theorem 2.2.1 subsumes the extensions of Schmeidler's original result, obtained in [34, Theorems 7.1, 7.8, 7.11, 7.13] and [32, Theorem 4.7.3].

2.3. *Equivalence of Theorems* 2.1.1 *and* 2.2.1

After the proof of Theorem 2.1.1 in Section 4.2 has been completed, it will be shown that Theorem 2.1.1 implies Theorem 2.2.1. This is a standard kind of implication. However, to find the converse implication would seem to be extremely rare (if not totally new), since the feasible mixed action spaces, i.e., the sets $M_1^+(S_t)$, $t \in T$, are usually stationed at a much higher level of generality than the action spaces S_t themselves. Yet this converse implication holds, as is shown by the following result:

PROPOSITION 2.3.1 (Equivalence). *Each of Theorems* 2.1.1 *and* 2.2.1 *implies the other result.*

The proof of the nonstandard implication in this result is given in Section 4.3.

2.4. *A Consistency Question*

A consistency question regarding the modeling of the payoffs in the pseudogames Γ and Γ' seems to have received only scant attention in the literature. Let us discuss this question only in terms of Γ'; a quite similar discussion can also be given for Γ. The point is that, any given action profile f completely specifies player t's action $f(t)$, which could affect the freedom of choice for the variable s in the payoff function $U'_t(s, d(f))$. In response, let us observe first that for players $t \in \hat{T}$ (these are "nonatomic players" by Assumption 2.2.1) the consistency issue does not arise: The action profile $g|_{\hat{T}}$ only influences the payoff $U'_t(s, d(f))$ via the aggregate $\hat{d}(f) = \hat{d}(f|_{\hat{T}})$ and this clearly does not determine the action $f(t)$ to be made by a nonatomic player t. However, for players t in $\bar{T}$ the response has to be more subtle, since $\bar{d}(f) := f|_{\bar{T}}$. For such players the model used in this paper still reflects proper modeling practice if $U'_t(s, d(f))$ is in addition supposed to be of a composite form, say $U'_t(s, d(f)) := U''_t(s, \pi_t(\bar{d}(f)), \hat{d}(f))$, where the mapping π_t is such that $\pi_t(\bar{d}(f))$ does *not* determine the value $f(t)$, i.e., player t's own action under f. So, rather than directly depending on $\bar{d}(f)$, the payoff depends on some "abstract" $\pi_t(\bar{d}(f))$ of $\bar{d}(f)$. As a concrete example, let us note that in the original model of Schmeidler [43], who works with $\bar{T} = T$ and $\hat{T} = \varnothing$, all such mappings π_t can be taken identically equal to the canonical L^1-space quotient mapping π. That is to say, Schmeidler's "abstract" of $d(f) = f$ is simply the L^1-equivalence class $\pi(f)$ consisting of all functions that are a.e. equal to f. This choice reflects proper modeling, because knowledge of the equivalence class $\pi(f)$ does not specify anything about the action $f(t)$ taken by any particular player t under the profile f (recall that [43] works with $T = [0, 1]$ and Lebesgue measure, so that, each player is nonatomic). Much of the subsequent literature on continuum games has more or less adopted this model, although not always with the understanding that the measure space $(T, \mathscr{T}, \mu)$ is nonatomic. In contrast, in games or pseudo-games with at most countably many players the consistency question surfaces very keenly, because each player would be given positive μ-measure with $\mathscr{T} := 2^T$—e.g., cf. [11, Theorem 3.1.1] (a quite similar situation arises if one would introduce "atomic players" in the above continuum game model). The standard formulations of such games simply realize consistency by working with $\pi_t(f) := f^{-t} := (f(\tau))_{\tau \neq t}$, etc. One might well ask why such an effective device has not been used for games with a measure space of players. The answer is that in the continuum setting those same functions $\pi_t(f) := f^{-t}$ would suddenly present formidable technical complications, because of the fact that the joint evaluation map $(t, f) \mapsto f(t)$ need not be measurable in any standard way [28]. This fact has been overlooked in the strand of the continuum game literature that deals with models with unordered preferences *à la* Shafer–Sonnenschein, where, as a consequence, incompatibilities occur [19].

2.5. *Nonmeasurable Versions*

Let us very briefly consider two cases where measurability plays no role, either because T is at most countable and $\mathscr{T} = 2^T$ (call this case (i)), causing measurability of the profiles to be automatic, or because measurability of the profiles is no longer desired (call this case (ii)). In both cases the nonatomic part does not figure (so $\hat{T} = \varnothing$).

Case (i). Because the Suslin property is only instrumental for measurability with respect to $\mathscr{T}$, which is now automatic, the proofs of Theorems 2.1.1 and 2.2.1 show that one can remove the adjective "Suslin" from Assumptions 2.1.1 and 2.2.2, provided that one supposes them metrizable (recall the introduction of ρ in Section 2.1) or at least semimetrizable, as a closer inspection of the adapted proof shows. Also, there is no longer a need to keep all S_t contained in one and the same action universe S, although one could always re-create such inclusion by means of direct sums. Further, one can systematically replace "for μ-almost every t in T" by "for every $t \in T$", since one can now work with, say, $\mu(\{t_i\}) := 2^{-i}$. Observe that now $\mathscr{R}_\Sigma = \prod_{t \in T} M_1^+(S_t)$ and $\mathscr{S}_\Sigma = \prod_{t \in T} S_t$, and on these Cartesian products the narrow and feeble topologies simply coincide with the usual product topologies (here the factors $M_1^+(S_t)$, $t \in T$, are equipped with the classical narrow topology).

Case (ii). Unlike case (i), which is a special case of the general model considered here, new proofs have to be given of the counterparts of Theorems 2.1.1 and 2.2.1 that discard the measurability aspect. However, their *statements* take the form indicated in the previous case (i). Results of this kind are well-known and need not be repeated here; cf. [32, Theorem 4.7.2].

3. IMPLICATIONS OF THE MAIN EQUILIBRIUM EXISTENCE RESULTS

As mentioned before, the applications given in [11] and [15] are also applications of Theorems 2.1.1 and 2.2.1. Their details can be found in those papers. Here we shall continue two major lines from [11], viz. applications to existence of Cournot–Nash equilibrium distributions Section 3.1) and to existence of Cournot–Nash equilibria in more or less complicated games with a measure space of players (Section 3.2). In addition, in Section 3.4 new light is also shed on the connection of our model with games with incomplete information in the sense of Harsanyi.

3.1. *Generalization and Unification of Two Existence Results by Rath*

In Theorem 3.1.1 below two separate results by Rath on the existence of CNE distributions in anonymous games *à la* Mas-Colell [37], namely Theorems 1 and 3 in [42], will be unified and generalized (recall that, in turn, these results generalize Mas-Colell's original result in [37]). The contribution made by [42] is in the line of [26]; it consists of specifying conditions that allow the payoff functions to be discontinuous (and more so than similar results of this kind, given in [9]). However, just as other existence results involving CNE distributions [9, 13], these results can be seen as a specialization of Theorem 2.1.1, that is, of a model for existence of CNE that goes considerably beyond the CNE distribution setting.

Recall that in anonymous games in the sense of Mas-Colell a player's type is made up entirely of his/her payoff function. This payoff function only depends on the (mixed) action profile via some marginal probability distribution generated by that profile on the action universe. The latter causes the anonymity feature. Let us now specify the following anonymous game Δ. As before, let S denote an action universe. The following repeats Assumption 2.1.1:

Assumption 3.1.1. S is a completely regular Suslin space.

Let T be a set of functions $t: S_t \times M_1^+(S) \to \mathbb{R}$, where the factor $S_t \subset S$ determines the domain of definition $S_t \times M_1^+(S)$ of the function t. As before, the multifunction $\Sigma: t \mapsto S_t$ is used frequently and T is equipped with a σ-algebra $\mathscr{T}$ and a measure μ, which is now also supposed to be a *probability* measure (i.e., $\mu(T) = 1$). The following repeats Assumption 2.1.2:

Assumption 3.1.2. (i) For every $t \in T$ the set $S_t \subset S$ is nonempty and compact.

(ii) $\operatorname{gph} \Sigma \in \mathscr{T} \times \mathscr{B}(S)$.

Below the set of probability measures $M_1^+(S)$ is equipped with the classical narrow topology [23, 27, 44].

Assumption 3.1.3. (i) For every $t \in T$ the function $t: S_t \times M_1^+(S) \to \mathbb{R}$ is upper semicontinuous and such that

$$\nu \mapsto \sup_{s \in S_t} t(s, \nu) \text{ is lower semicontinuous on } M_1^+(S).$$

(ii) For every $\nu \in M_1^+(S)$ the function $(t, s) \mapsto t(s, \nu)$ is $(\mathscr{T} \times \mathscr{B}(S)) \cap \operatorname{gph} \Sigma$-measurable.

THEOREM 3.1.1. *Under Assumption* 2.1 *and the above Assumptions* 3.1.1 *to* 3.1.3 *there exists a* Cournot–Nash equilibrium distribution *for the anonymous game* Δ. *That is, there exists a probability measure* p_* *on* $T \times S$ *such that*

$$p_*(\cdot \times S) = \mu \quad \text{and} \quad p_*(\{(t, s) \in \operatorname{gph} \Sigma : s \in \operatorname*{argmax}_{s' \in S_t} t(s, p_*(T \times \cdot))\}) = 1.$$

Proof. Let us apply Theorem 2.1.1 by making the following substitutions: set $A_t(\delta) \equiv S_t$, as in (2.1), and $U_t(s, \delta) := t(s, \nu_\delta)$, where $\nu_\delta \in M_1^+(S)$ is defined by

$$\nu_\delta(B) := \int_T \delta(t)(B)\, \mu(dt), \qquad B \in \mathscr{B}(S)$$

(recall that $\mu(T) = 1$). Now the mapping $\delta \mapsto \nu_\delta$ is continuous from $\mathscr{R}$, equipped with the narrow topology, into $M_1^+(S)$, equipped with the classical narrow topology. This follows directly from the fact that $\int_S c\, d\nu_\delta = I_g(\delta)$

for any $c \in \mathscr{C}_b(S)$, where $g(t, s) := c(s)$ defines a Carathéodory integrand on $T \times S$. Hence, it is evident that Assumption 2.1.4(i) holds, and Assumption 2.1.4(ii) follows of course by Assumption 3.1.3(ii). Also, Assumption 2.1.5 holds by the above continuity result and Assumption 3.1.3(i). Finally, as observed in Remark 2.1.1(i), Assumption 2.1.3 holds here automatically. So the theorem can be applied, which gives the existence of $\delta_* \in \mathscr{R}_\Sigma$ with the properties as stated in Theorem 2.1.1. Now the canonical product measure that is generated by the "starting" probability measure μ and the transition probability δ_* [40, III.2] is immediately seen to form the desired equilibrium distribution p_*. Q.E.D.

The structural similarity between the above proof and the proof of Theorem 2.2.1 is worth noting. Apart from the separate purification on $\hat{T}$, in the latter proof it is continuity of the barycentric mapping $\delta \mapsto \text{bar}\,\delta$ that causes Theorem 2.1.1 to apply. Here this continuity is replaced by the continuity of the special externality mapping $\delta \mapsto \nu_\delta$ that characterizes the Mas-Colell model. Theorem 3.1.1 generalizes both Theorem 1 and 3 of Rath [42] (in turn, Rath's Theorem 1 generalizes the original result by Mas-Colell in [37]). Let us see why this is so. In [42] one has S compact metric and $S_t \equiv S$; this obviously meets the Assumptions 3.1.1 and 3.1.2. Let $\mathscr{P}$ be the set of *all* bounded upper semicontinuous functions $t: S \times M_1^+(S) \to \mathbb{R}$ such that $\nu \mapsto \sup_{s \in S} t(s, \nu)$ is lower semicontinuous on $M_1^+(S)$. In [42] the set $\mathscr{P}$ is endowed with a probability measure μ on $(\mathscr{P}, \mathscr{B}(\mathscr{P}))$; the Borel σ-algebra in [42] is taken with respect to either the usual supremum norm topology (Theorem 1) or the hypotopology (Theorem 3). In the first situation $\mathscr{P}$ is denoted as $\mathscr{P}^S$ in [42], and in the second situation as $\mathscr{P}^H$. Now observe that each of the above choices of topology causes the mapping $(t, s) \mapsto t(s, \nu)$ to be upper semicontinuous on $\mathscr{P} \times S$, whence $\mathscr{B}(\mathscr{P} \times S)$-measurable, for every fixed $\nu \in M_+^1(S)$.

First, let us bring Theorem 3.1.1 to bear on Theorem 1 of [42]. In that result μ is *tight*, so there exists a sequence of compacts $K_n \subset \mathscr{P}$ with $\mu(\mathscr{P} \backslash T) = 0$ for $T := \bigcup_{n=1}^\infty K_n$. Let us also set $\mathscr{T} := \mathscr{B}(T)$, then $\mathscr{T} = \mathscr{B}(\mathscr{P}) \cap T$ and Assumption 2.1 holds. Since $\mathscr{P} = \mathscr{P}^S$ is equipped with a metric (supremum norm), it follows that T is separable, whence second countable, so the restriction of the Borel σ-algebra $\mathscr{B}(\mathscr{P} \times S)$ to $T \times S$ is equal to the restriction of $\mathscr{B}(\mathscr{P}) \times \mathscr{B}(S)$ to that same set. In view of the preceding, this shows that the restriction of $(t, s) \mapsto t(s, \nu)$ to $T \times S$ is measurable with respect to $\mathscr{T} \times \mathscr{B}(S)$. Hence, also the remaining Assumption 3.1.3(ii) of Theorem 3.1.1 is met. Clearly, by $\mu(\mathscr{P} \backslash T) = 0$, one can add the remaining functions in $\mathscr{P} \backslash T$ to the statement resulting from Theorem 3.1.1 to regain Theorem 1 of [42]. Next, let us obtain Theorem 3 of [42]. In this case the choice $T := \mathscr{P}$ suffices, since $\mathscr{P} = \mathscr{P}^H$ is well-known to be separable and metrizable for the hypotopology (cf. Propositions 8, 9 in [42]). So this meets Assumption 2.1 and satisfaction of the remaining Assumption 3.1.3(ii) of Theorem 3.1.1 follows as above.

3.2. *Generalization of an Existence Result by Kim and Yannelis*

We generalize the main Bayesian–Nash equilibrium existence result of Kim and Yannelis in [36]. Consider the following Bayesian game $\tilde{\Gamma}$. Let S be an action universe. The following assumption replicates Assumption 2.2.2.

Assumption 3.2.1. S is a Suslin locally convex topological vector space.

Let $(\Omega, \mathscr{F}, P)$ be a probability space of *states of nature* and, as before, let $(T, \mathscr{T}, \mu)$ be a measure space of players. Every player $t \in T$ obtains information about the realized state of nature via his/her *informational σ-algebra* $\mathscr{F}_t$, which is a given sub-σ-algebra of $\mathscr{F}$ (differential information). As in [36], the following drastic assumption is unavoidable for technical reasons; these are mainly of a topological nature—cf. Proposition 3.3.1.

Assumption 3.2.2. The set Ω is at most countable.

Rather than relabeling the atoms of $\mathscr{F}$, we shall assume without loss of generality that $\mathscr{F}$ is the power set 2^{Ω}. Let $\tilde{\Sigma}: T \times \Omega \to 2^S$ be a given multifunction. For each player $t \in T$ his/her *feasible* actions constitute the subset $\tilde{\Sigma}(t, \omega)$, given that $\omega \in \Omega$ is the realized state of nature. Observe from part (ii) of the following assumption that this feasibility restriction is in accordance with player t's informational σ-algebra.

Assumption 3.2.3. (i) For every $(t, \omega) \in T \times \Omega$ the set $\tilde{\Sigma}(t, \omega) \subset S$ is nonempty, convex and compact.

(ii) $\text{gph}\, \tilde{\Sigma}(t, \cdot) \in \mathscr{F}_t \times \mathscr{B}(S)$ for every $t \in T$.

(iii) $\text{gph}\, \tilde{\Sigma} \in \mathscr{T} \times \mathscr{F} \times \mathscr{B}(S)$.

Let us write $\tilde{\Sigma}_\omega(t) := \tilde{\Sigma}(t, \omega)$ and denote the corresponding multifunctions by $\tilde{\Sigma}_\omega$, $\omega \in \Omega$. Then parts (i) and (iii) of the above assumption imply that these multifunctions have nonempty compact convex values and a $\mathscr{T} \times \mathscr{B}(S)$-measurable graph. Thus, for each $\omega \in \Omega$ the set $\mathscr{S}_{\tilde{\Sigma}_\omega}$ is defined in complete analogy to the set $\mathscr{S}_\Sigma$ of Section 2.2. Hence, from now on each such space can be considered to be equipped with the (i.e., its own) feeble topology. A *feasible pure action profile* is a function $\tilde{f}: T \times \Omega \to S$ that is measurable with respect to $\mathscr{T} \times \mathscr{F}$ and $\mathscr{B}(S)$, with $\tilde{f}(t, \cdot)$ being $\mathscr{F}_t$-measurable for every $t \in T$ and with $\tilde{f}(t, \omega) \in \tilde{\Sigma}(t, \omega)$ for μ-a.e. t in T and P-a.e. ω in Ω. Note that this means that for every player $t \in T$ the description $\tilde{f}(t, \cdot)$ of what player t could/should do under the various states of nature, takes into account the way in which player t processes information about that state (i.e., by way of $\mathscr{F}_t$-measurability). The set of all such feasible pure action profiles is denoted by $\mathcal{S}_{\tilde{\Sigma}}$. Note that for every $\omega \in \Omega$ and $\tilde{f} \in \mathcal{S}_{\tilde{\Sigma}}$ the ω-section $\tilde{f}(\cdot, \omega)$ of $\tilde{f}$ belongs to $\mathcal{S}_{\tilde{\Sigma}_\omega}$. The Bayesian nature of the model is reflected by the fact that each player $t \in T$ possesses a *Bayesian prior distribution*; this is a transition probability π_t which expresses player t's *interim beliefs* about the actually realized state of nature, that is to say, beliefs formulated after having gained him/herself information (i.e., partially, via $\mathcal{F}_t$) about it.

Assumption 3.2.4. (i) For every $t \in T$ π_t is a transition probability with respect to $(\Omega, \mathcal{F}_t)$ and $(\Omega, \mathcal{F})$, i.e., for every $A \in \mathcal{F}$ the function $\pi_t(\cdot)(A)$ is $\mathcal{F}_t$-measurable.

(ii) For every $A \in \mathcal{F}$ the function $(t, \omega) \mapsto \pi_t(\omega)(A)$ is $\mathcal{T} \times \mathcal{F}$-measurable.

For every $(t, \omega) \in T \times \Omega$ let $u_{t,\omega}\colon Z_t \times \mathcal{S}_{\tilde{\Sigma}_\omega} \to \mathbb{R}$ be a given *utility function*. Here $Z_t := \bigcup_{\omega \in \Omega} \tilde{\Sigma}(t, \omega)$. If player t in T were to know the realized state $\omega \in \Omega$ completely, he or she would assign utility value $u_{t,\omega}(s, \tilde{f}(\cdot, \omega))$ to his or her own action $s \in \tilde{\Sigma}(t, \omega)$ in the face of the action profile $\tilde{f} \in \mathcal{S}_{\tilde{\Sigma}}$. Shortly, we shall see how, using his or her prior distribution $\pi_t(\omega)$ as a Bayesian assessment of the realized state of nature, player t can convert this into an appraisal that is in line with his or her informational sub-σ-algebra $\mathcal{F}_t$.

Assumption 3.2.5. (i) For every $(t, \omega) \in T \times \Omega$ the function $u_{t,\omega}\colon Z_t \times \mathcal{S}_{\tilde{\Sigma}_\omega} \to \mathbb{R}$ is upper semicontinuous.

(ii) For every $(t, \omega) \in T \times \Omega$ and $s \in Z_t$ the function $u_{t,\omega}(s, \cdot)\colon \mathcal{S}_{\tilde{\Sigma}_\omega} \to \mathbb{R}$ is continuous.

(iii) For every $\omega \in \Omega$ and $f \in \mathcal{S}_{\tilde{\Sigma}_\omega}$ the function $(t, s) \mapsto u_{t,\omega}(s, f)$ is $(\mathcal{I} \times \mathcal{B}(S)) \cap \operatorname{gph} Z$-measurable.

(iv) For every $(t, \omega) \in T \times \Omega$ there exists $\phi_t \in \mathcal{L}^1_{\mathbb{R}}(\Omega, \mathcal{F}, \pi_t(\omega))$ such that for every $\omega' \in \Omega$

$$\sup_{s \in \tilde{\Sigma}(t, \omega'),\, f \in \mathcal{S}_{\tilde{\Sigma}_{\omega'}}} |u_{t,\omega'}(s, f)| \leqslant \phi_t(\omega').$$

In part (iii) above $\operatorname{gph} Z$ refers to the graph of the multifunction $t \mapsto Z_t := \bigcup_{\omega \in \Omega} \tilde{\Sigma}(t, \omega)$, i.e., the $\mathcal{T} \times \mathcal{B}(S)$ measurable countable union $\bigcup_{\omega \in \Omega} \operatorname{gph} \tilde{\Sigma}_\omega$.

Assumption 3.2.6. For every $(t, \omega) \in T \times \Omega$ and $f \in \mathcal{S}_{\tilde{\Sigma}_\omega}$ the function $u_{t,\omega}(\cdot, f)$ is concave.

Following [36], let us introduce the following Bayesian object to overcome the informational limitations inherent to the utility evaluation $(t, \omega, s, \tilde{f}) \mapsto u_{t,\omega}(s, \tilde{f}(\cdot, \omega))$, as mentioned above. Given the state $\omega \in \Omega$, player t's *conditional expected interim utility function* $U_{t,\omega}: \tilde{\Sigma}(t,\omega) \times \mathscr{S}_{\tilde{\Sigma}} \to \mathbb{R}$ is defined as follows:

$$U_{t,\omega}(s, \tilde{f}) := \int_{\Omega} u_{t,\omega'}(s, \tilde{f}(\cdot, \omega'))\, \pi_t(\omega)(d\omega'). \tag{3.1}$$

Existence of this integral (actually, by Assumption 3.2.2 it is an at most countable sum) is elementary, in view of Assumption 3.2.5(iv) and the fact that $\pi_t(\omega)$ is a probability on $\mathscr{F} = 2^{\Omega}$.

THEOREM 3.2.1. *Under Assumption* 2.1 *and Assumptions* 3.2.1 *to* 3.2.6 *there exists a Bayesian Nash equilibrium action profile for the game* $\tilde{\Gamma}$, *i.e., there exists* $\tilde{f}_* \in \mathscr{S}_{\tilde{\Sigma}}$ *such that*

$$\tilde{f}_*(t,\omega) \in \operatorname*{argmax}_{s \in \tilde{\Sigma}(t,\omega)} U_{t,\omega}(s, \tilde{f}_*) \qquad \textit{for} \quad \mu\textit{-a.e. } t \textit{ in } T \quad \textit{and} \quad P\textit{-a.e. } \omega \textit{ in } \Omega.$$

This result improves and generalizes the main result Theorem 5.2 of Kim and Yannelis [36] in several respects. They need S to be a separable Banach space (equipped with its weak topology), and their multifunction $\tilde{\Sigma}$ is, in addition to our conditions, also integrably bounded. This causes their counterpart of our $\mathscr{S}_{\Sigma}$ to be in an L^1-context (this is a complicated quotient context, borrowed from [21]). Hence, on their counterpart of $\mathscr{S}_{\Sigma}$ they can work with the weak topology $\sigma(L^1, L^\infty)$, which is generalized by the feeble topology used here (see Example 2.2.1). The remaining comparisons, which are all in favor of the above set of conditions, are left to the reader. We only point out that Assumption B.1(ii) of [36] (which would constitute an improvement over our Assumption 3.2.5) is a little too weak, since such an assumption of strong-weak continuity of $u_{t,\omega}$ does not by itself imply joint continuity for the weak topology, which is the topology they work with, even when $u_{t,\omega}(s, f)$ is concave in s (for instance, with their $T \times \Omega$ a singleton, consider the fact that the inner product mapping $(x, y) \mapsto \sum_{i=1}^{\infty} x_i y_i$ from the product of the unit ball in ℓ_2 with itself into the reals is strong-weak continuous, bilinear, but not jointly weakly continuous).

3.3. *Proof of Theorem* 3.2.1

Let us prepare for an application of Theorem 2.2.1, of course with $\bar{T} = T$ and with noncooperativity in force (i.e., $A_{t,\omega} \equiv \tilde{\Sigma}(t,\omega)$). To begin with, let $\tilde{\mathscr{T}} \subset 2^{T \times \Omega}$ be the collection of all $E \in \mathscr{T} \times \mathscr{F}$ such that for every $t \in T$ the t-section $E_t := \{\omega \in \Omega : (t, \omega) \in E\}$ belongs to $\mathscr{F}_t$. Observe that $\tilde{\mathscr{T}}$ defines a σ-algebra on $\tilde{T} := T \times \Omega$. It is called the *progressive* σ-algebra in stochastics. For further coherence, let us denote $\tilde{\mu} := \mu \times P$ for the product measure on $(\tilde{T}, \tilde{\mathscr{T}})$. The measure space $(\tilde{T}, \tilde{\mathscr{T}}, \tilde{\mu})$ will now take the place of $(T, \mathscr{T}, \mu)$ as used in Theorem 2.2.1. Hereupon, observe that Assumption 3.2.3(ii)–(iii) amounts precisely to having gph $\tilde{\Sigma} \in \tilde{\mathscr{T}} \times \mathscr{B}(S)$. So, together with Assumption 3.2.3(i), this means that Assumption 2.2.2 has been met. In view of our adoption of noncooperativity, Assumption 2.2.5 is met vacuously (Remark 2.2.1(i)). Denote $\tilde{\pi}(t, \omega)(A) := \pi_t(\omega)(A)$. Then, by definition of the progressive σ-algebra, Assumption 3.2.4 states precisely that $\tilde{\pi}$ is a transition probability with respect to $(\tilde{T}, \tilde{\mathscr{T}})$ and $(\Omega, \mathscr{F})$. Fix $f \in \mathscr{S}_{\tilde{\Sigma}}$. By Assumption 3.2.5(iii) and $\mathscr{F} = 2^{\Omega}$ (i.e., Assumption 3.2.2) the function $(t, s) \mapsto u_{t,\omega'}(s, \tilde{f}(\cdot, \omega'))$ of (3.1) is $(\mathscr{T} \times \mathscr{B}(S)) \cap$ gph Z-measurable for every ω' in the countable set Ω'. So the above progressive measurability property of $\tilde{\pi}$ implies that $(t, \omega, s) \mapsto U_{t,\omega}(s, \tilde{f})$ is $(\tilde{\mathscr{T}} \times \mathscr{B}(S)) \cap$gph $\tilde{\Sigma}$-measurable. Hence, Assumption 2.2.6(ii) has been met. Remark 4.3.1(i) below (see [15] for details) implies that $\mathscr{S}_{\tilde{\Sigma}}$ is semimetrizable for its feeble topology (note that $(\tilde{T}, \tilde{\mathscr{T}}, \tilde{\mu})$ is also separable by Assumption 3.2.2). This allows us to use only sequential arguments to verify continuity/semicontinuity in what follows. We also need the following feeble-to-feeble continuity property of the ω-section mapping, which again draws heavily on Assumption 3.2.2:

Proposition 3.3.1. *For every $\omega \in \Omega$ the mapping $\tilde{f} \mapsto \tilde{f}(\cdot, \omega)$ from $\mathscr{S}_{\tilde{\Sigma}}$, equipped with the feeble topology, into $\mathscr{S}_{\tilde{\Sigma}_\omega}$, also equipped with its own feeble topology, is continuous.*

Proof. Fix any $\omega \in \Omega$. Given an arbitrary $g \in \mathscr{G}_{LC, \tilde{\Sigma}_\omega}$, we define $\tilde{g}(t, \omega', s) := g(t, s)$ if $\omega' = \omega$ and $\tilde{g}(t, \omega', s) := 0$ if $\omega' \neq \omega$. Then $\tilde{g}$ is easily seen to belong to $\mathscr{G}_{LC, \tilde{\Sigma}}$. Since

$$\int_{\tilde{T}} \tilde{g}(t, \omega', \tilde{f}(t, \omega'))\, \tilde{\mu}(d(t, \omega')) = P(\{\omega\}) \int_T g(t, \tilde{f}(t, \omega))\, \mu(dt),$$

the result follows by definition of the respective feeble topologies. Q.E.D.

Using these two results, it is now easy to see by an application of Fatou's lemma that for every $(t, \omega) \in \tilde{T}$ the function $U_{t,\omega}$ is upper semicontinuous on $\tilde{\Sigma}(t, \omega) \times \mathscr{S}_{\tilde{\Sigma}}$. Here Assumption 3.2.5(iv) provides integrable boundedness from above for the sequence, and Assumption 3.2.5(i) should be combined with Proposition 3.3.1. Conversely, in view of Assumption 3.2.5(ii) a similar application of Fatou's lemma (or—what has the same effect—Lebesgue's dominated convergence theorem) gives that for every $(t, \omega) \in \tilde{T}$ and $s \in \tilde{\Sigma}(t, \omega)$ the function $U_{t,\omega}(s, \cdot)$ is continuous on $\mathscr{S}_{\tilde{\Sigma}}$. So Assumption 2.2.7(i) holds by Remark 2.2.1(i). Finally, the integration operation in (3.1) obviously preserves the concavity, as guaranteed by Assumption 3.2.6. So Assumption 2.2.7(ii) holds by Remark 2.2.1(iii). We conclude that all assumptions of Theorem 2.2.1 have been shown to hold, since the model of Kim and Yannelis has been shown to be a special version of the one used in Section 2.2. Application of Theorem 2.2.1 immediately implies that Theorem 3.2.1 holds.

3.4. *Generalization of an Existence Result by Balder and Rustichini*

Let us show how the extensions of Balder [6] and Balder–Rustichini [20] of the BNE existence result of Milgrom–Weber [39] (which is in mixed action profiles) follow from Theorem 2.2.1 (which is in pure action profiles!). This approach would seem to be somewhat in the spirit of [38]. By trivializing the observation spaces, one regains a rather classical mixed CNE existence result (for games with at most countably many players—of course, this also follows directly from Theorem 2.1.1). Consider the following Bayesian game $\hat{\Gamma}$ *à la* Harsanyi [31].

Assumption 3.4.1. The set T is at most countable.

Assumption 3.4.2. For every $t \in T$ the set S_t is a nonempty metrizable compact set.

For every $t \in T$ let $(\Omega_t, \mathscr{F}_t)$ be a measurable space forming player t's space of *private observations*. Let P be a probability measure on the countable product space $(\Omega, \mathscr{F}) := \prod_{t \in T} (\Omega_t, \mathscr{F}_t)$. The realizations in Ω are governed by P, but player $t \in T$ is only informed of his or her marginal outcome on Ω_t ("private information"). Clearly, this marginal outcome is governed by P_t, the marginal of P of the tth factor space; i.e., $P_t(B) := P(\prod_{\tau \in T, \tau \neq t} \Omega_\tau \times B)$. The following condition was also used in [39, 6, 20]:

Assumption 3.4.3. P is absolutely continuous with respect to the product measure $\prod_{t \in T} P_t$.

For each $t \in T$ let $\mathscr{R}_t$ be the space of all transition probabilities with respect to $(\Omega, \mathscr{F}_t)$ and $(S_t, \mathscr{B}(S_t))$; this space is equipped with the narrow topology, introduced in Section 2.1, and here the measure P_t is used on $(\Omega_t, \mathscr{F}_t)$. Clearly, in using $\delta_t \in \mathscr{R}_t$, player $t \in T$ keeps to his or her allowed private information restriction, and uses mixed actions in $M_1^+(S_t)$ (it would be possible to introduce ω_t-dependency of the feasible mixed actions in the usual way, but this will not be done to keep the presentation simple). Let $S := \prod_{t \in T} S_t$. We also use $S^{-t} := \prod_{\tau \in T, \tau \neq t} S_\tau$. Each player $t \in T$ has a payoff function $u_t \colon \Omega \times S \to \mathbb{R}$, of which the following is required.

Assumption 3.4.4. (i) For every $(t, \omega) \in T \times \Omega$ the function $u_t(\omega, \cdot)\colon S \to \mathbb{R}$ is continuous.

(ii) For every $t \in T$ the function $u_t \colon \Omega \times S \to \mathbb{R}$ is $\mathscr{F} \times \mathscr{B}(S)$-measurable.

(iii) For every $t \in T$ there exists $\phi_t \in \mathscr{L}^1_{\mathbb{R}}(\Omega, \mathscr{F}, P)$ such that for every $\omega \in \Omega$

$$\sup_{s \in S} |u_t(\omega, s)| \leqslant \phi_t(\omega).$$

These assumptions allow the introduction of the following *expected payoff* functions $V_t\colon \prod_{\tau \in T} \mathscr{R}_\tau \to \mathbb{R}$,

$$V_t((\delta_\tau)_{\tau \in T}) := \int_\Omega \int_S u_t(\omega, s) \left[\bigotimes_{\tau \in T} \delta_\tau \right](\omega)(ds)\, P(d\omega),$$

where $\bigotimes_{\tau \in T} \delta_\tau$ is the transition probability with respect to $(\Omega, \mathscr{F})$ and $(S, \mathscr{B}(S))$, defined by

$$\left[\bigotimes_{\tau \in T} \delta_\tau \right](\omega) := \prod_{\tau \in T} \delta_\tau(\omega_\tau)$$

for $\omega := (\omega_\tau)_{\tau \in T}$. Note that on the right one takes pointwise product measures.

THEOREM 3.4.1. *Under Assumptions* 3.4.1 *to* 3.4.4, *there exists a Bayesian Nash equilibrium action profile for the game* $\hat{\Gamma}$, *i.e., there exists* $\delta_* := (\delta_{*t})_{t \in T} \in \prod_{t \in T} \mathscr{R}_t$ *such that*[1]

$$\delta_{*t} \in \operatorname*{argmax}_{\delta_t \in \mathscr{R}_t} V_t(\delta_t, \delta_*^{-t}) \qquad \textit{for every} \quad t \in T.$$

Proof.

Step 1: *separable case.* Suppose in addition that for every $t \in T$ the σ-algebra $\mathscr{F}_t$ is countably generated. This implies that $(\Omega_t, \mathscr{F}_t, P_t)$ is separable; hence it follows from Proposition 4.1.1 that every $\mathscr{R}_t$ is semimetrizable for the topology of narrow convergence of transition probabilities. By Assumption 3.4.1, we can simplify the application of Theorem 2.2.1 as discussed in Section 2.5. In particular, this means that a common action universe is not *per se* a requirement for the feasible action sets. For $t \in T$ take $\hat{\Sigma}(t) := \hat{S}_t$, with $\hat{S}_t$ defined to be the quotient of $\mathscr{R}_t$ for the obvious equivalence relation "equality P_t-a.e."; then by the above $\hat{S}_t$ is metrizable. By Theorem 4.1.1 $\hat{S}_t$ is compact for every $t \in T$; also, it is trivially convex. So Assumption 2.2.3 has been shown to hold. Observe that $\mathscr{S}_{\hat{\Sigma}} = \prod_{t \in T} \hat{S}_t$ by Assumption 3.4.1; as already mentioned in Section 2.5, the feeble topology on $\mathscr{S}_{\hat{\Sigma}}$ now coincides with the product topology. We define $U_t\colon \hat{S}_t \times \mathscr{S}_{\hat{\Sigma}} \to \mathbb{R}$ by $U_t(\delta, \hat{f}) := V_t(\delta, \hat{f}^{-t})$. Here we adopt standard notation that is explained in Section 2.2 and the previous footnote. In addition, we

[1] As usual, $(\delta_t, \delta_*^{-t})$ stands for $(\eta_\tau)_{\tau \in T}$ defined by $\eta_t := \delta_t$ and $\eta_\tau := \delta_{*\tau}$ for $\tau \neq t$.

abuse the notation a little—in the accepted way—in connection with the quotient setting in which we actually work (for instance, a quotient counterpart of V_t should be defined on $\prod_{\tau \in T} \hat{S}_\tau$ in an evident manner). Let us equip $(\Omega, \mathscr{F})$ with the measure $\prod_{t \in T} P_t$, and let $\mathscr{R}_0$ be the set of all transition probabilities with respect to $(\Omega, \mathscr{F})$ and $(S, \mathscr{B}(S))$ (recall here $S := \prod_{t \in T} S_t$). By well-known facts about the tensor product of transition probabilities (see [6, Theorem 2.5] and [20, p. 389]), the mapping $(\delta_t)_{t \in T} \mapsto \bigotimes_{t \in T} \delta_t$, defined from $\mathscr{S}_{\hat{\Sigma}} = \prod_{t \in T} \mathscr{R}_t$ into $\mathscr{R}_0$, is continuous with respect to the narrow topology on the latter space (recall that earlier we equipped the spaces $\mathscr{R}_t$ already with their respective narrow topologies). Here Assumption 3.4.3 and the associated choice of $\prod_t P_t$ as the leading measure on Ω play an important, role. Fix any $t \in T$. Define $v_t(\omega, s) := u_t(\omega, s)\, r(\omega)$, where r is any fixed version of the Radon–Nikodym density of P with respect to $\prod_{\tau \in T} P_\tau$. Then Assumption 3.4.4 causes v_t to belong to the class $\mathscr{G}_C(\Omega; S)$ of Carathéodory integrands with respect to the measure $\prod_{\tau \in T} P_\tau$ on Ω. Hence, I_{v_t} is narrowly continuous on $\mathscr{R}_0$. So by continuity of the tensor product, observed above, and the obvious identity $V_t((\delta_\tau)_{\tau \in T}) = I_{v_t}(\bigotimes_{\tau \in T} \delta_\tau)$, the function U_t is continuous on $\hat{S}_t \times \mathscr{S}_{\hat{\Sigma}}$. Hence, Assumption 2.2.6 is met (note that measurability in the variable t is trivial here), and also Assumption 2.2.7(i) (invoke Remark 2.2.1(i)). Finally, Assumption 2.2.7(ii) holds by the obvious affinity of $U_t(\cdot, \hat{f})$ on $\mathscr{R}_t$ for every $(t, \hat{f}) \in T \times \mathscr{S}_{\hat{\Sigma}}$. By an application of Theorem 2.2.1 it now follows that there exists $\hat{f}_* := (\delta_{*t})_{t \in T} \in \mathscr{S}_{\hat{\Sigma}}$ such that $\hat{f}_*(t) \in \operatorname{argmax}_{\delta \in \hat{S}_t} U_t(\delta, \hat{f}_*)$ for every $t \in T$. This is precisely to say that $(\delta_{*t})_{t \in T}$ has the equilibrium property stated in the theorem.

Step 2: *general case.* We reduce this case to step 1 by imitating a clever argument stated on p. 78 of [24]. Let $\mathscr{C}$ be the collection of all sequences $(\mathscr{G}_\tau^0)_{\tau \in T}$, where $\mathscr{G}_\tau^0$ runs through the collection of all countably generated sub-σ-algebras of $\mathscr{F}_\tau$. The essential point to note is the following identity:

$$\mathscr{E} := \cup \left\{ \prod_{\tau \in T} \mathscr{G}_\tau^0 \times \mathscr{B}(S) \colon (\mathscr{G}_\tau^0)_{\tau \in T} \in \mathscr{C} \right\} = \mathscr{F} \times \mathscr{B}(S). \tag{3.2}$$

Here each $\prod_{\tau \in T} \mathscr{G}_\tau^0 \times \mathscr{B}(S)$ also indicates a product σ-algebra. To prove (3.2), observe that $\mathscr{E}$, as defined, is a σ-algebra. For instance, if (A_m) is a sequence in $\mathscr{E}$, then each A_m belongs to $\prod_{\tau \in T} \mathscr{G}_\tau^m \times \mathscr{B}(S)$ for some $(\mathscr{G}_\tau^m)_{\tau \in T} \in \mathscr{C}$. But then $\bigcup_m A_m$ belongs to $\prod_{\tau \in T} \mathscr{G}_\tau^0 \times \mathscr{B}(S)$, where $\mathscr{G}_\tau^0$ is the σ-algebra, generated by $(\mathscr{G}_\tau^m)_{m=1}^\infty$, etc. This fact immediately proves (3.2). Indeed, one inclusion in (3.2) is trivial, and the other one follows by the fact that for each product set $F := \prod_{\tau \in T} F_\tau$, with $F_\tau \in \mathscr{F}_\tau$ for all τ, one has $F \in \prod_{\tau \in T} \mathscr{G}_\tau^0$ with $\mathscr{G}_\tau^0 := \{\emptyset, \Omega_\tau, F_\tau, \Omega_\tau \setminus F_\tau\}$ (observe that such sets F form the generators of $\mathscr{F}$).

Given that the collection $(u_t)_{t \in T}$ is at most countable, (3.2) implies that there exists a sequence $(\mathscr{F}_\tau^0)$ in $\mathscr{C}$ such that for every $t \in T$ the function u_t is $\mathscr{F}^0 \times \mathscr{B}(S)$-measurable, with $\mathscr{F}^0$ defined as the product σ-algebra $\prod_{\tau \in T} \mathscr{F}_\tau^0$. To see this, observe that it is enough to prove this fact only for one of the u_t. Actually, it is enough to prove it just for a u_t that is of the characteristic function form $u_t = 1_G$, with G a $\mathscr{F} \times \mathscr{B}(S)$-measurable set, for, once this is proven, an obvious approximation of the original u_t by step functions finishes the argument. Since (3.2) implies $G \in \mathscr{E}$, the desired fact for $u_t = 1_G$ already follows and this finishes the proof. Q.E.D.

4. PROOFS OF THEOREMS 2.1.1 AND 2.2.1 AND THEIR EQUIVALENCE

In this section we first prove Theorem 2.1.1 in Section 4.1. Its proof is an application of Kakutani's fixed point theorem, which is topologically made possible by some of the most fundamental results of Young measure theory (these are recapitulated for the convenience of the reader). Recall that this theory centers around an extension of the classical narrow topology from probability measures to transition probabilities. Following this, Theorem 2.2.1 is proven in Section 4.2, essentially by reformulating the existence problem of Section 2.2 in terms of Theorem 2.1.1 and by adding some purification arguments. Finally, the equivalence of Theorems 2.1.1 and 2.2.1 is demonstrated in Proposition 2.3.1.

4.1. *Proof of Theorem* 2.1.1

In this section let us first recall the only three results about the narrow topology on $\mathscr{R}$ that we need in the proof of Theorem 2.1.1. These can be found in [17] or, alternatively, in [18].

PROPOSITION 4.1.1. *Under Assumptions* 2.1 *and* 2.1.1 *the narrow topology on $\mathscr{R}_\Sigma$ is semi-metrizable.*

Proof. In Section 2.1 a metric ρ was introduced. On the compact sets S_t, $t \in T$, its topology was seen to coincide with the original topology. So the narrow topology on $\mathscr{R}_\Sigma$ does not change when we equip S with the metric ρ. Now [17, Theorem 4.6] (this is Theorem 4.5 in [18]) states that $\mathscr{R}$ itself is semimetrizable if S is metrizable, It thus follows that $\mathscr{R}_\Sigma$ is semimetrizable. Q.E.D.

THEOREM 4.1.1. *Under Assumptions* 2.1.1 *and* 2.1.2 *the subset $\mathscr{R}_\Sigma$ is narrowly compact.*

Under the extra Assumption 2.1 this follows from [17, Theorem 4.10] (this is Theorem 4.8 in [18]), in combination with Proposition 4.1.1. Indeed, define $h: T \times S \to [0, +\infty]$ by setting $h(t, s) := 0$ if $(s, t) \in \operatorname{gph} \Sigma$ and $h(t, s) := +\infty$ if $(s, t) \in (S \times T) \setminus \operatorname{gph} \Sigma$. Then h meets the conditions of [17, Definition 2.2] thanks to Assumption 2.1.2. So by Theorem 10 of [17], every sequence in $\mathcal{R}_\Sigma$ contains a narrowly convergent subsequence. So $\mathcal{R}_\Sigma$ is relatively compact for the narrow topology. But $\mathcal{R}_\Sigma$ is also closed: it consists precisely of all $\delta \in \mathcal{R}$ for which $I_h(\delta) \leqslant 0$. Thus, $\mathcal{R}_\Sigma$ is compact. Alternatively and without the need of Assumption 2.1, Theorem 4.1.1 follows by [6, Theorem 2.3], using the comments about its extension to a completely regular Suslin spaces first given in [7]. The next result can be found in [17, Theorem 4.12] or [18, Theorem 4.15]. Here the remarks about ρ that were made in the proof of Proposition 4.1.1 again play a role.

THEOREM 4.1.2. *Under Assumptions* 2.1.1 *and* 2.1.2 *the following holds. If a sequence* (η_n) *converges narrowly to* $\bar{\eta}$ *in* $\mathcal{R}_\Sigma$*, then pointwise, for a.e. t in T, the support* $\operatorname{supp} \bar{\eta}(t)$ *of the probability measure* $\bar{\eta}(t)$ *is contained in the set* $\bigcap_{p=1}^{\infty} \operatorname{cl} \bigcup_{n \geqslant p} \operatorname{supp} \eta_n(t)$.

This expresses a kind of sequential upper semicontinuity property of the (pointwise) supports; the set figuring in the above statement is called the *Painlevé–Kuratowski limes superior* and it is denoted as $\operatorname{Ls}_n \operatorname{supp} \eta_n(t)$.

Proof of Theorem 2.1.1. Evidently, $\delta_* \in \mathcal{R}_\Sigma$ is a mixed CNE if and only if $\delta_* \in F(\delta_*)$, where $F(\delta)$ stands for the set of all $\eta \in \mathcal{R}_\Sigma$ such that $\eta(t)(M_\delta(t)) = 1$ for μ-a.e. t in T. Here $M_\delta(t) := \operatorname{argmax}_{s \in A_t(\delta)} U_t(s, \delta)$. Therefore, the proof revolves around an application of Kakutani's fixed point theorem to $F: \mathcal{R}_\Sigma \to 2^{\mathcal{R}_\Sigma}$. Steps 1–2 below guarantee that $\mathcal{R}_\Sigma$ has the right compactness and convexity properties for such an application, and steps 3–5 show that F has the right semicontinuity properties. Step 6 applies Kakutani's theorem.

Step 1: *compactness/convexity/nonemptiness of* $\mathcal{R}_\Sigma$. By Theorem 1.1.1, $\mathcal{R}_\Sigma$ is compact for the narrow topology. Also, $\mathcal{R}_\Sigma$ is trivially convex and it was already seen before that $\mathcal{R}_\Sigma$ is nonempty.

Step 2: *a vector space setting for* $\mathcal{R}_\Sigma$. The intended application of Kakutani's theorem requires a topological vector space setting. Obviously, the classical narrow topology can be extended from $M_1^+(S)$ to the space $M(S)$ of all signed bounded measures on $(S, \mathcal{B}(S))$. Then the vector space $\mathcal{M}$ spanned by $\mathcal{R}$ is the space of all functions from T into $M(S)$ that are measurable with respect to $\mathcal{T}$ and $\mathcal{B}(M(S))$. Equip $\mathcal{M}$ with the coarsest topology for which all functionals $I_g: \delta \mapsto \int_T [\int_S g(t, s)\, \delta(t)(ds)]\, \mu(dt)$, $g \in \mathcal{G}_C(T; S)$, are continuous (note that these functionals are well defined).

When restricted to $\mathscr{R}$, this topology is the narrow topology that was defined previously.

Step 3: *upper semicontinuity of* $M_{.}(t)$. By the Weierstrass theorem, $M_\delta(t)$ is a nonempty compact subset of S_t for every $t \in T$ and $\delta \in \mathscr{R}_\Sigma$ (use Assumptions 2.1.2(i), 2.1.4(i)). Moreover, $\delta \mapsto M_\delta(t)$ is upper semicontinuous for arbitrary $t \in T$. To see this, it is enough to prove that $M_{.}(t)$ has the closed graph property (by compactness of S_t): So let $(s_n, \delta_n) \to (\bar{s}, \bar{\delta})$ with $s_n \in M_{\delta_n}(t)$ for every n, i.e., $s_n \in A_t(\delta_n)$ and $U_t(s_n, \delta_n) = \sup_{s \in A_t(\delta_n)} U_t(s, \delta_n)$. By Assumptions 2.1.4(i) and 2.1.5 this identity leads to $U_t(\bar{s}, \bar{\delta}) \geqslant \sup_{s \in A_t(\bar{\delta})} U_t(s, \bar{\delta})$ in the limit. Also $\bar{s} \in A_t(\bar{\delta})$, because A_t has the closed graph property by Assumption 2.1.3(ii). So $\bar{s} \in M_{\bar{\delta}}(t)$, which proves the closed graph property of $M_{.}(t)$.

Step 4: *upper semicontinuity of* F. Similar to step 3, it is enough to prove the closed graph property for F, because the values of F are contained in the compact set $\mathscr{R}_\Sigma$ (step 1). Here it is essential to convince oneself first that this classical result continues to hold on the semimetric space $\mathscr{R}_\Sigma$. To prove the closed graph property of F, let $(\eta_n, \delta_n) \to (\bar{\eta}, \bar{\delta})$ with $\eta_n \in F(\delta_n)$ for every n, i.e., $\eta_n(t)(M_{\delta_n}(t)) = 1$ for μ-a.e. t in T. This also means that for a.e. t in T and every n the support $\operatorname{supp} \eta_n(t)$ of the probability measure $\eta_n(t)$ is contained in $M_{\delta_n}(t)$, for the latter set is closed by step 3. By Theorem 4.1.2, for a.e. t in T, this implies that $\operatorname{supp} \bar{\eta}(t)$ is contained in the Painlevé–Kuratowski limes superior $\mathrm{Ls}_n M_{\delta_n}(t)$. By step 3, the latter set is contained in $M_{\bar{\delta}}(t)$, which finishes the proof.

Step 5: *F has nonempty closed convex values.* Fix $\delta \in \mathscr{R}_\Sigma$. The closedness of $F(\delta)$ follows *a fortiori* from the proof of the closed graph property of F in step 4. Convexity of $F(\delta)$ is trivial. Next, we prove nonemptiness of $F(\delta)$ by the application of a measurable selection theorem. To begin with, Assumptions 2.1.3(i) and 2.1.4(i) imply that the set $M_\delta(t)$ is nonempty for every $t \in T$ (by the Weierstrass theorem). Secondly, we show that M_δ has a measurable graph. Note that $s \in M_\delta(t)$ if and only if $s \in A_t(\delta)$ and $U_t(s, \delta) = \gamma_\delta(t)$, where $\gamma_\delta(t) := \sup_{s \in A_t(\delta)} \arctan U_t(s, \delta)$. By [24, III.39] the function γ_δ is $\mathscr{T}$-measurable (here completeness of $(T, \mathscr{T}, \mu)$ is used), so $\operatorname{gph} M_\delta$ belongs to $\mathscr{T} \times \mathscr{B}(S)$ by Assumptions 2.1.3(iii) and 2.1.4(ii). It follows by the von Neumann–Aumann measurable selection theorem [24, III.22] that there exists a measurable $f: T \mapsto S$ with $f(t) \in M_\delta(t)$ for every t in T. This implies that the Dirac Young measure ε_f (defined earlier) belongs to $F(\delta)$, which is thus seen to be nonempty.

Step 6: *application of Kakutani's fixed point theorem.* It is well-known that Ky Fan's original arguments in [30] do not require the Hausdorff space hypothesis [29, pp. 500–501]. In [15, Theorem A.2] this was used to obtain a non-Hausdorff version of Kakutani's theorem (in all other respects it is standard). Above, we saw that all properties needed for this fixed point result hold. So there exists $\delta_* \in \mathscr{R}_\Sigma$ with $\delta_* \in F(\delta_*)$, as desired.

4.2. *Proof of Theorem* 2.2.1

The foremost results needed in the derivation of Theorem 2.2.1 from Theorem 2.1.1 are as follows. Recall that S^* stands for the topological dual of S; also, $\langle s, s^* \rangle := s^*(s)$ indicates the usual duality. The following result is [25, Proposition 26.3]

THEOREM 4.2.1. *If $K \subset S$ is nonempty compact and convex, then for every ν in $M_1^+(S)$ with* supp $\nu \subset K$ *there exists a unique $s_\nu \in K$ for which*

$$\langle s_\nu, s^* \rangle = \int_K \langle s, s^* \rangle \, \nu(ds) \qquad \text{for all} \quad s^* \in S^*.$$

This unique element s_ν is denoted by bar ν.

Recall that s_ν is called the *barycenter* of the probability measure ν.

COROLLARY 4.2.1. *To every feasible mixed action profile $\delta \in \mathscr{R}_\Sigma$ there corresponds a pure action profile $\bar{f} \in \bar{\mathscr{S}}_\Sigma$ that satisfies $\bar{f}(t) =$* bar $\delta(t)$ *for a.e. t in $\bar{T}$. This function $\bar{f}$ is essentially unique (i.e., but for null sets) and is denoted by* bar δ.

Proof. The definition of $\mathscr{R}_\Sigma$ allows for an exceptional null set N of t's with $\delta(t)(S_t) < 1$. For $t \in \bar{T} \setminus N$ the well-definedness of the point $\bar{f}(t) :=$ bar $\delta(t)$ in S_t follows from Assumption 2.2.3(i) by Theorem 4.2.1. For $t \in N \cap \bar{T}$ we can set $\bar{f}(t)$ equal to an arbitrary but fixed point of S. Then $\bar{f} : \bar{T} \to S$ and $\bar{f}(t) \in S_t$ for a.e. $t \in \bar{T}$. Measurability of $\bar{f}$ is seen as follows: For every $s^* \in S^*$ the above definition yields $\langle \bar{f}(t), s^* \rangle = \int_{S_t} \langle s, s^* \rangle \, \delta(t)(ds)$ for all t in $\bar{T} \setminus N$. By [40, Proposition III.2.1] (here Assumption 2.2.3(iii) is also used), one concludes that $\bar{f}$ is scalarly measurable. By what was observed in Section 2.2, it follows that $\bar{f}$ is also measurable in the ordinary sense, with respect to $\mathscr{T} \cap \bar{T}$ and $\mathscr{B}(S)$. We can now conclude that $\bar{f}$ belongs to $\bar{\mathscr{S}}_\Sigma$. Q.E.D.

THEOREM 4.2.2. *The mapping $\delta \mapsto$* bar δ *from $\mathscr{R}_\Sigma$ into $\bar{\mathscr{S}}_\Sigma$ is continuous with respect to the narrow and feeble topologies.*

Proof. Let $g \in \mathscr{G}_{LC, \Sigma}$ and $\delta \in \mathscr{R}_\Sigma$ be arbitrary. Recall from Section 2.2 that one has $g(t, \cdot) \in S^*$ for every $t \in \bar{T}$. So Theorem 4.2.1 and Corollary 4.2.1 give

$$g(t, \text{bar}\, \delta(t)) = \int_{S_t} g(t, s) \, \delta(t)(ds) = \int_S g(t, s) \, \delta(t)(ds) = \int_{S_t} g(t, s) \, \delta(t)(ds)$$

for a.e. t in $\bar{T}$. Integration over $\bar{T}$ therefore gives that $J_g(\text{bar}\,\delta) = I_{\tilde{g}}(\delta)$, where $\tilde{g}(t, s) := g(t, s)$ if $t \in \bar{T}$ and $s \in S_t$ and $\tilde{g}(t, s) = 0$ otherwise. Finally, note that $\tilde{g}$ belongs to the class $\mathcal{G}_{C,\Sigma}(T)$. Q.E.D.

THEOREM 4.2.3 (Lyapunov's theorem for Young measures). *If $\ell_1, \ldots, \ell_d$: $\hat{T} \times S \to \mathbb{R}$ are $(\mathcal{T} \times \mathcal{B}(S)) \cap \hat{T}$-measurable and if $\delta_0 \in \mathcal{R}$ is such that $\int_{\hat{T}} [\int_S |\ell_i(t, s)|\, \delta_0(t)(ds)]\, \mu(dt) < +\infty$ for all $1 \leqslant i \leqslant d$, then there exists a measurable function $f_0: \hat{T} \to S$ such that $J_{\ell_i}(f_0) = I_{\ell_i}(\delta_0)$ for all i and $f_0(t) \in \operatorname{supp} \delta_0(t)$ for a.e. t in $\hat{T}$.*

This result is well-known in less general forms. The present version is [17, Theorem 5.3].

Proof of Theorem 2.2.1. Theorem 2.2.1 will be derived from Theorem 2.1.1 by the introduction of a mixed version Γ of the pseudogame Γ', which meets all conditions of Theorem 2.1.1. Thereupon, the mixed CNE action profile is transformed, both by barycentric arguments (on $\hat{T}$) and purification (on $\bar{T}$) into a pure CNE action profile.

Step 1: *a continuous mixed externality.* Following [11], let us define a *mixed* externality mapping $e: \mathcal{R}_\Sigma \to \bar{\mathcal{S}}_\Sigma \times \mathbb{R}^m$ by setting $e := (\bar{e}, \hat{e})$ with

$$\bar{e}(\delta) := \text{bar}\,\delta, \qquad \hat{e}(\delta) := \left(\int_{\hat{T}} \int_S [g_i(t, s)\, \delta(t)(ds)]\, \mu(dt) \right)_{i=1}^{m}.$$

Observe that $\hat{e}(\delta) = (I_{\tilde{g}_i}(\delta))_{i=1}^m$, with $\tilde{g}_i \in \mathcal{G}_{C,\Sigma}$, where $\tilde{g}_i := g_i$ on $\operatorname{gph} \Sigma \cap (\hat{T} \times S)$ and $\tilde{g}_i := 0$ on $\operatorname{gph} \Sigma \cap (\bar{T} \times S)$. Thus, the function $\hat{e}$ is continuous by the facts about the narrow topology that were presented in Section 2.1. By Theorem 4.2.2 $\bar{e}$, the other component of e, is also continuous. Thus, e is continuous with respect to the narrow topology on $\mathcal{R}_\Sigma$ and the product of the feeble and Euclidean topologies on $\bar{\mathcal{S}}_\Sigma \times \mathbb{R}^m$.

Step 2: *definition of a mixed pseudogame* Γ. Define $A_t(\delta) := A'_t(e(\delta))$ and $U_t(s, \delta) := U'_t(s, e(\delta))$. Then all assumptions of Theorem 2.1.1 are easily seen to be met by the current Assumptions 2.2.2 to 2.2.7, in view of Step 1.

Step 3: *application of the mixed CNE existence result.* By Theorem 2.1.1 there exists $\delta_* \in \mathcal{R}_\Sigma$ such that $\delta_*(t)(\operatorname{argmax}_{s \in A'_t(e(\delta_*))} U'_t(s, e(\delta_*))) = 1$ for a.e. t in T.

Step 4: *purification on* $\hat{T}$. Theorem 4.2.3 can be applied in view of Assumption 2.2.1. So there exists a measurable function $\hat{f}_*: \hat{T} \to S$ such that

$$\hat{f}_*(t) \in \operatorname{supp} \delta_*(t) \subset \operatorname*{argmax}_{s \in A'_t(e(\delta_*))} U'_t(s, e(\delta_*)) \quad \text{a.e. on } \hat{T} \quad \text{and} \quad \hat{d}(\hat{f}_*) = \hat{e}(\delta_*).$$

Step 5: *construction of the pure CNE action profile.* Set $f_*(t) := \text{bar}\ \delta_*(t)$ on $\bar{T}$ and $f_*(t) := \hat{f}_*(t)$ on $\hat{T}$; then $e(\delta_*) = d(f_*)$. This gives

$$f_*(t) = \hat{f}_*(t) \in \underset{s \in A'_t(d(f_*))}{\operatorname{argmax}}\ U'_t(s, d(f_*)) \qquad \text{a.e. on } \hat{T}.$$

On $\bar{T}$ we can apply Theorem 4.2.1 to conclude that

$$f_*(t) = \text{bar}\ \delta_*(t) \in \underset{s \in A'_t(d(f_*))}{\operatorname{argmax}}\ U'_t(s, d(f_*)) \qquad \text{a.e. on } \bar{T}.$$

By the Hahn–Banach theorem this follows from the fact that the sets $\operatorname{argmax}_{s \in A'_t(d(f_*))} U'_t(s, d(f_*))$ are convex and compact for every $t \in \bar{T}$, in view of Assumptions 2.2.3(i), 2.2.6(i) and 2.2.7(ii). This finishes the proof. Q.E.D.

4.3. *Proof of Proposition* 2.3.1

Clearly, to prove Proposition 2.3.1 it remains to derive Theorem 2.1.1 from Theorem 2.2.1, since the converse implication was already proven in deriving the former result. Given the nature of the former result, it will be enough to make the special choice $\bar{T} = T$.

Step 1: *definition of S and* Σ'. Denote $S' := M(S)$, where $M(S)$ is the same space of signed measures as in Section 4.1. In view of Assumption 2.1.1, S' is a Suslin space for the classical narrow topology by [27, III.60] and [44, Theorem 3, p. 96]. Also, it is evident that S' is locally convex by definition of the classical narrow topology. So Assumption 2.2.2 is met. Denote also $S'_t := \{\nu \in M(S) : \nu \in M_1^+(S) \text{ and } \nu(S_t) = 1\}$. In view of Assumption 2.1.2(i), S'_t is (classically) narrowly compact for every $t \in T$ [27, III.60], and it is trivially convex. By [24, Theorem IV.12] and Assumption 2.1.2(ii), the graph of $\Sigma' : t \mapsto S'_t$ is measurable. So Assumption 2.2.3 holds.

Step 2: $\mathscr{S}_{\Sigma'}$ *is* $\mathscr{R}_\Sigma$. By the above definition of Σ', it follows that $\mathscr{S}_{\Sigma'}$ is precisely the set $\mathscr{R}_\Sigma$ (recall from Section 4.2 that scalar and ordinary measurability are the same for functions in $\mathscr{S}_{\Sigma'}$).

Step 3: *feeble topology on* $\mathscr{S}_{\Sigma'}$ *is narrow topology on* $\mathscr{R}_\Sigma$. Observe first that to every $g \in \mathscr{G}_C(T; S)$ there evidently corresponds $g' \in \bar{\mathscr{G}}_{LC, \Sigma'}$ via the formula $g'(t, \nu) := \int_S g(t, s)\, \nu(ds)$ (recall again that here $\bar{T} = T$). So all I_g, $g \in \mathscr{G}_C(T; S)$, are feebly continuous on $\mathscr{R}_\Sigma = \mathscr{S}_{\Sigma'}$. Conversely, let $g' \in \bar{\mathscr{G}}_{LC, \Sigma'}$ be arbitrary. By [25, Proposition 22.4] the topological dual $(S')^*$ of S is the set of all functionals $\nu \mapsto \int_S c\, d\nu$, $c \in \mathscr{C}_b(S)$, on $S' := M(S)$. Thus, by definition of $\bar{\mathscr{G}}_{LC,\Sigma'}$, for every $t \in T$ there exists $c_t \in \mathscr{C}_b(S)$ such that $g'(t, \nu) = \int_S c_t\, d\nu$ for all $\nu \in M(S)$. Observe that this gives $g'(t, \varepsilon_s) = c_t(s) =: g(t, s)$ for all $t \in T$ and $s \in S$. By evident measurability of $(t, s) \mapsto (t, \varepsilon_s)$, this implies that g is $\mathscr{T} \times \mathscr{B}(S)$-measurable. As in the previous case, the resulting formula is $g'(t, \nu) = \int_S g(t, s)\, \nu(ds)$. Finally, $g' \in \bar{\mathscr{G}}_{LC,\Sigma'}$ also implies that there exists $\phi_{g'} \in \mathscr{L}^1_{\mathbb{R}}(T, \mathscr{T}, \mu)$ such that $\phi_{g'}(t) \geqslant \sup_{\nu \in S'_t} |g'(t, \nu)| = \sup_{s \in S_t} |g'(t, \varepsilon_s)| = \sup_{s \in S_t} |g(t, s)|$ for every $t \in \bar{T}$. Hence, if one sets $\tilde{g}(t, s) := g(t, s)$ if $t \in \bar{T}$ and $s \in S_t$ and $\tilde{g}(t, s) := 0$ otherwise, then $\tilde{g}$ belongs to the class $\mathscr{G}_{C,\Sigma}$, defined in Section 2.1, and $J_{g'}(\delta) = I_{\tilde{g}}(\delta)$ for every $\delta \in \mathscr{S}_{\Sigma'} = \mathscr{R}_\Sigma$. The conclusion is that the two topologies on $\mathscr{S}_{\Sigma'} = \mathscr{R}_\Sigma$ are the same.

Step 4: *definition and properties of* A'. Recall again that here $\bar{T} = T$, so that $\bar{\mathscr{S}}_\Sigma = \mathscr{R}_\Sigma = \mathscr{S}_{\Sigma'}$. For $\delta \in \mathscr{S}_{\Sigma'}$ one sets $A'_t(\delta) := \{\nu \in M_1^+(S) : \nu(A_t(\delta)) = 1\}$. Then Assumption 2.2.5(i) holds by [27, III.58, III.60]. Also, in view of Assumption 2.1.3(ii), the corresponding Assumption 2.2.5(ii) holds by a well-known upper semicontinuity property *à la* Kuratowski for convergence in the classical narrow topology of the supports of a sequence in $M_1^+(S)$ [10, Corollary A.2]. Note that this is the "classical" analogue of a similar property used for mixed action profiles in the proof of Theorem 2.1.1 in Section 4.1.

Step 5: *definition and properties of* U'. Define $U'_t(\nu, \delta) := \int_{S_t} \arctan U_t(s, \delta)\, \nu(ds)$ for $\delta \in \mathscr{S}_{\Sigma'}$ (the arctangent transformation serves to keep the integrand bounded, whence integrable). By applying [23, Theorem 3.2] in a particular way introduced by the present author (see [17] for a systematic use of such arguments—the fact that $\mathscr{R}_\Sigma$ was shown to be semimetrizable in Proposition 4.1.1 is essential for this to work) it is easy to show that U'_t is upper semicontinuous on $S'_t \times \mathscr{R}_\Sigma$. Also, it is standard to prove that $(t, \nu) \mapsto U'_t(\nu, \delta)$ is product measurable for every $\delta \in \mathscr{S}_{\Sigma'}$. So the conclusion is that Assumption 2.2.6 also holds.

Step 6: *verification of Assumption* 2.2.7. By the above definitions,

$$\sup_{\nu \in A'_t(\delta)} U'_t(\nu, \delta) = \sup_{s \in A_t(\delta)} \arctan U_t(s, \delta) = \arctan \sup_{s \in A_t(\delta)} U_t(s, \delta)$$

and the latter expression is clearly lower semicontinuous in δ by Assumption 2.1.5(i) and monotonicity/continuity of the arctangent function. Also, the above shows that the set $\operatorname{argmax}_{\nu \in A'_t(\delta)} U'_t(\nu, \delta)$ is identical to the set $\{\nu \in M_1^+(S) : \nu(\operatorname{argmax}_{s \in A_t(\delta)} U_t(s, \delta)) = 1\}$, which is trivially convex.

The proof is now virtually finished: Steps 1–6 show that Theorem 2.2.1 applies. So, writing δ_* instead of f_*, there exists $\delta_* \in \mathscr{S}_{\Sigma'} = \mathscr{R}_\Sigma$ such that $\delta_*(t)(\operatorname{argmax}_{s \in A_t(\delta_*)} U_t(s, \delta_*)) = 1$ for a.e. t in $\bar{T} = T$ (here the last part of Step 6 is used again).

Remark 4.3.1. In view of the analogies displayed here, it is not hard to prove that Proposition 4.1.1 and Theorems 4.1.1, 4.1.2 have certain counterparts for action profiles in $\mathscr{S}_\Sigma$. Indeed, under Assumptions 2.1, 2.2.2 and 2.2.3 the following hold: (i) The feeble topology on $\mathscr{S}_\Sigma$ is semimetrizable, (ii) $\mathscr{S}_\Sigma$ is feebly compact (this follows already from Theorem 4.1.1 and 4.2.2), (iii) If a sequence (f_n) converges feebly to $\bar{f}$ in $\mathscr{S}_\Sigma$, then pointwise, for a.e. t in T, $\bar{f}(t) \in \mathrm{cl\,co} \bigcap_{p=1}^{\infty} \mathrm{cl} \bigcup_{n \geqslant p} f_n(t) \subset \bigcap_{p=1}^{\infty} \mathrm{cl\,co} \bigcup_{n \geqslant p} f_n(t)$. By using (i)–(iii) in a way similar to the proof of Theorem 2.1.1, an independent proof of Theorem 2.2.1 can also be given.

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