Solutions Final Exam M & I, 23-6-11

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Problem 1 [25 pt]. For $j = 1, 2, \ldots$ let $X_j := \mathbb{R}$ and let $\mathcal{A}_j := \mathcal{B}(\mathbb{R})$. Then the set $X := \Pi_{j=1}^{\infty} X_j$ consists of all infinite sequences $(x_j)_{j=1}^{\infty}$ of real numbers. Let $\mathcal{C} \subset 2^X$ be the collection of all sets $C \subset X$ of the following form: there exists a finite set of indices $K \subset \mathbb{N}$ (K may vary with the set C and may even be empty) and associated sets $B_k \in \mathcal{A}_k$, $k \in K$, such that $C = \{(x_j)_{j=1}^{\infty} : \forall_{k \in K} x_k \in B_k\}$. Define $\mathcal{A} := \sigma(\mathcal{C})$ to be the σ-algebra on X.
a. Prove: for every $\alpha \in \mathbb{R}$ the set $A := \{(x_j)_{j=1}^{\infty} \in X : \limsup_j x_j \leq \alpha\}$ belongs to $\mathcal{A}$.
b. Prove: the set $D := \{(x_j)_{j=1}^{\infty} \in X : \lim_j x_j$ exists and is finite$\}$ belongs to $\mathcal{A}$. *Hint:* a sequence converges in $\mathbb{R}$ if and only if it is a Cauchy sequence.

SOLUTION. Below it is handy to write $\mathbf{x} := (x_j)_{j=1}^{\infty}$, similar to what you usually do for n-tuples.

a. **Method 1: use of Corollary 8.9.** Define for each $p \in \mathbb{N}$ the function $u_p : \mathbf{x} \mapsto x_p$ from X into $\mathbb{R}$. Each u_p is obviously $\mathcal{A}$-measurable (because $u_p^{-1}(B) \in \mathcal{C} \subset \mathcal{A}$ for every $B \in \mathcal{B}(\mathbb{R})$ – take $K := \{p\}$ and $B_p := B$ in the above definition). So by Corollary 8.9, the function $u := \limsup_p u_p$ is also $\mathcal{A}$-measurable. Because of $u(\mathbf{x}) := \limsup_p u_p(\mathbf{x}) = \limsup_p x_p$, this implies $A = u^{-1}((-\infty, \alpha]) \in \mathcal{A}$.

Method 2: direct proof. The following proof "from scratch" repeats a lot of what goes into Corollary 8.9. By

$$\mathbf{x} \in A \Leftrightarrow \forall_{p \in \mathbb{N}} \limsup_j x_j < \alpha + \frac{1}{p} \Leftrightarrow \forall_p \inf_{m \in \mathbb{N}} \sup_{j \geq m} x_j < \alpha + \frac{1}{p} \Leftrightarrow \forall_p \exists_m \sup_{j \geq m} x_j < \alpha + \frac{1}{p}$$

you get $A = \cap_p \cup_m B_{p,m}$, with $B_{p,m} := \{\mathbf{x} \in X : \sup_{j \geq m} x_j < \alpha + \frac{1}{p}\}$. Next, for every $p, m \in \mathbb{N}$

$$\mathbf{x} \in B_{p,m} \Leftrightarrow \exists_{k \in \mathbb{N}} \sup_{j \geq m} x_j \leq \alpha + \frac{1}{p} - \frac{1}{k} \Leftrightarrow \exists_{k \in \mathbb{N}} \forall_{j \geq m} x_j \leq \alpha + \frac{1}{p} - \frac{1}{k}$$

shows that $B_{p,m} = \cup_k \cap_{j \geq m} C_{p,k,j}$, with $C_{p,k,j} := \{\mathbf{x} \in X : x_j \leq \alpha + \frac{1}{p} - \frac{1}{k}\}$. Together with the preceding this proves $A = \cap_p \cup_m \cup_k \cap_{j \geq m} C_{p,k,j}$ and each $C_{p,k,j}$ obviously belongs to $\mathcal{C} \subset \mathcal{A}$. From this identity you obtain $A \in \mathcal{A}$, because $\mathcal{A}$ is closed for countable intersections and unions.

b. **Method 1: use the hint.** Let $D := \{\mathbf{x} \in X : \lim_j x_j$ exists and is finite$\}$. The hint suggests you to use that $\mathbf{x} = (x_j)_{j=1}^{\infty}$ belongs to D if and only if it is Cauchy, which holds evidently if and only if

$$\forall_{p \in \mathbb{N}} \exists_{m \in \mathbb{N}} \forall_{k,l \geq m} |x_k - x_l| < \frac{1}{p}.$$

Therefore, $D = \cap_p \cup_m \cap_{k,l \geq m} E_{p,k,l}$, where $E_{p,k,l} := \{\mathbf{x} \in X : |x_k - x_l| < \frac{1}{p}\}$. By

$$|x_k - x_l| < \frac{1}{p} \Leftrightarrow \exists_{q,q' \in \mathbb{Q}}\ x_l - \frac{1}{p} < q < x_k \text{ and } x_k < q' < x_l + \frac{1}{p}$$

it follows that $E_{p,k,l} = \cup_{q,q'\in\mathbb{Q}} F_{p,k,l,q} \cap F'_{p,k,l,q'}$, with

$$F_{p,k,l,q} := \{\mathbf{x} \in X : x_k \in (q, +\infty), x_l \in (-\infty, q + \frac{1}{p})\} \in \mathcal{C} \subset \mathcal{A},$$

$$F'_{p,k,l,q} := \{\mathbf{x} \in X : x_k \in (-\infty, q'), x_l \in (q' - \frac{1}{p}, +\infty)\} \in \mathcal{C} \subset \mathcal{A}.$$

By repeated use of the fact that $\mathcal{A}$ is closed for countable intersections and unions it thus follows that D belongs to $\mathcal{A}$.

Method 2: use part a. Let $D := \{\mathbf{x} \in X : \lim_j x_j$ exists and is finite$\}$. Then

$$D = \{\mathbf{x} \in X : li(\mathbf{x}) = ls(\mathbf{x})\} \cap [\cup_k \{\mathbf{x} \in X : ls(\mathbf{x}) \le k\}] \cap [\cup_k \{\mathbf{x} \in X : li(\mathbf{x}) \ge -k\}],$$

by defining $ls(\mathbf{x}) := \limsup_j x_j$ and $li(\mathbf{x}) := \liminf_j x_j$ for $x = (x_j)_j \in X$. By part a the function $ls : X \to [-\infty, +\infty]$ is measurable and so is the other function $li : X \to [-\infty, +\infty]$. The latter is seen by first observing that $\mathbf{x} := (x_j)_j \mapsto (-x_j)_j =: -\mathbf{x}$ is measurable from X into X (you can use Lemma 7.2 for this) and then using the well-known identity $li(\mathbf{x}) = -ls(-\mathbf{x})$. So $D \in \mathcal{A}$ follows by Corollary 8.12 and Lemma 8.1.

Problem 2 [25 pt] Consider $X := \mathbb{R}$, equipped with the Borel σ-algebra and the Lebesgue measure λ. Let $(f_n)_n$ be a sequence of functions $f_n : X \to \mathbb{R}_+$ with $\int_X f_n \, d\lambda = 1$ and $\{x \in X : f_n(x) \neq 0\} \subset \{x \in X : |x - r_n| < 2^{-n}\} =: S_n$ for each n. Here $\mathbb{Q} := \{r_n : n \in \mathbb{N}\}$ denotes an arbitrary but fixed enumeration of the rational numbers. Define $\mu(A) := \sum_{n=1}^{\infty} \int_A f_n \, d\lambda$. Prove successively:

i. μ is a measure on $(X, \mathcal{A})$,

ii. $\sum_{n=1}^{\infty} f_n(x) < +\infty$ holds λ-a.e. [*Hint:* Use the following result **and give its proof as well**: if $\sum_n \lambda(B_n) < +\infty$ holds for a sequence $(B_n)_n$ in $\mathcal{A}$, then $\lambda(\cap_m \cup_{n\ge m} B_n) = 0$.]

iii. μ is σ-finite on $(X, \mathcal{A})$.

iv. $\mu(A) = \infty$ holds for every open subset A of X. [*Hint:* An open subset of $\mathbb{R}$ contains two concentric open intervals.]

SOLUTION. *i.* **Method 1.** A direct consequence of Lemma 10.8, which guarantees that each $\nu_n : A \mapsto \int_A f_n$ is a measure, and Problem 4.6(*ii*), which then implies that $\sum_n \nu_n$ is a measure.

Method 2. First, $\mu(\emptyset) = \sum_n 0 = 0$ is obvious and for any mutually disjoint collection $\{A_j\}$ in $\mathcal{A}$

$$\mu(\cup_j A_j) = \sum_n \int_X \sum_j 1_{A_j} f_n \overset{\text{Cor 9.9}}{=} \sum_n \sum_j \int_{A_j} f_n \overset{\text{Tonelli}}{=} \sum_j \sum_n \int_{A_j} f_n = \sum_j \mu(A_j),$$

where "Tonelli" refers to the swith of the summation indices, applied to two counting measures (validity of that switch also follows by Problem 4.6(*ii*), p 24).

ii. First you must prove the hint (= actual homework problem 6.9, p. 47) about the Borel-Cantelli theorem. Let $C_m := \cup_{n\ge m} B_n$; then $C_m \downarrow \cap_m \cup_{n\ge m} B_n =: N$. Now $0 \le \mu(N) \le \mu(C_m) \le \sum_{n\ge m} \mu(B_n)$ and for $m \to \infty$ the right hand side converges to zero by the hypothesis $\sum_n \lambda(B_n) < +\infty$. Conclusion: $\mu(N) = 0$. Now apply this hint to the sets $B_n := \{f_n \neq 0\}$. Note that $\lambda(B_n) < 2 * 2^{-n}$ is given, so $\sum_n \lambda(B_n)$ is certainly finite. By the hint, it then follows that $\lambda(N) = 0$ for $N := \cap_m \cup_{n\ge m} \{f_n \neq 0\}$. Now for any $x \notin N$ you have, by definition of N, that there is $m = m(x) \in \mathbb{N}$ such that $x \notin B_n$ (i.e., such that $f_n(x) = 0$) for all $n \ge m(x)$; hence, it follows that $\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{m(x)} f_n(x) < +\infty$.

iii. Define $E_m := \{x \in X : \sum_{n=1}^{m} f_n(x) < +\infty\}$ for every $m \in \mathbb{N}$; then by part b the set $E_0 := X \backslash (\cup_m E_m)$, being contained in N, has $\mu(E_0) = 0$. Now $\cup_{m=0}^{\infty} E_m = X$, where

$\mu(E_0) = 0 < +\infty$ and $\mu(E_m) = \sum_{n=1}^{\infty} \int_{E_m} f_n \leq \sum_{n=1}^{m} \int_X f_n = m < +\infty$ for every $m \geq 1$. This proves μ to be σ-finite.

iv. Correction: The open set A should – obviously – be supposed nonempty in addition. Then A has some point x_0 and there is $\delta > 0$ such that the interval $I_1 := (x_0 - \delta, x_0 + \delta)$, whence also $I_2 := (x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2})$, is contained in A. Choose n from now on so large that $2^{-n} < \delta/2$. Now the nonempty open interval I_2 contains *infinitely many* r_n's (i.e., they satisfy $|r_n - x_0| < \delta/2$) and for each of the corresponding indices n the set $\{f_n \neq 0\} \subset S_n$ is entirely contained in A (use $|x - x_0| \leq |x_0 - r_n| + |x - r_n| < \frac{\delta}{2} + 2^{-n} < \delta$). The definition of μ now gives $\mu(A) = +\infty$, because $\int_A f_n = 1$ for infinitely many n's.

Problem 3 [25 pt]. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $(u_j)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{L}^\infty(X)$ and let u be in $\mathcal{L}^\infty(X)$ as well. As usual, the essential supremum seminorm on $\mathcal{L}^\infty(X)$ is denoted by $\|\cdot\|_\infty$. Prove the following equivalence $\lim_j \|u_j - u\|_\infty = 0 \Leftrightarrow \exists_{A\in\mathcal{A},\mu(A)=0} \lim_j \sup_{x\in X\setminus A} |u_j(x) - u(x)| = 0$. *Hint:* $\lim_j \|u_j - u\|_\infty = 0$ means that for every $m \in \mathbb{N}$ $\|u_j - u\|_\infty \leq m^{-1}$ for large enough j.

Solution. $\Rightarrow$: By the hint the hypothesis can be stated as follows: for every $m \in \mathbb{N}$ there exists J_m such that $\|u_j - u\|_\infty := \inf\{C : \mu(|u_j - u| > C) = 0\} < 1/m$ for all $j \geq J_m$. Hence, for every $m \in \mathbb{N}$ there exists $C_m < 1/m$ such that $\mu(|u_j - u| > C_m) = 0$ for all $j \geq J_m$. Form $A := \cup_m \cup_{j\geq J_m} \{|u_j - u| > C_m\}$, a countable union of null sets; then $\mu(A) = 0$. Given any $\epsilon > 0$, let m be so large that $1/m < \epsilon$. Then one has for $j \geq J_m$ that $X\setminus A \subset \{|u_j - u| \leq C_m\}$, which implies $\sup_{x\notin A} |u_j(x) - u(x)| \leq C_m < 1/m < \epsilon$.

$\Leftarrow$: Let the null set A be as in the statement and let $\epsilon > 0$ be arbitrary. The following is given: there exists J such that $\sup_{x\notin A} |u(x) - u_j(x)| \leq \epsilon$ for every $j \geq J$. Then for every $j \geq J$ it follows from $x \notin A \Rightarrow |u(x) - u_j(x)| \leq \epsilon$ that $\{|u_j - u| > \epsilon\} \subset A$, whence $\mu(\{|u_j - u| > \epsilon\}) = 0$. This proves $\|u_j - u\|_\infty \leq \epsilon$ for all $j \geq J$.

Problem 4 [25 pt] Let μ be a finite measure on $(X, \mathcal{A}) := (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Prove that for every $A \in \mathcal{A}$

$$\mu(A) = \inf\{\mu(G) : G \supset A \text{ and } G \text{ is open }\} =: \iota(A)$$

and

$$\mu(A) = \sup\{\mu(F) : F \subset A \text{ and } F \text{ is closed}\} =: \sigma(A)$$

Hint: Used the good sets principle and recall that every closed subset of X is a countable intersection of open sets **(prove this as well).**

Solution. *Step 1.* Naturally, you still remember homework Problem 10.12, so you begin by mimicking its part (*i*). By monotonicity of μ, the above definitions give $\iota(A) \geq \mu(A) \geq \sigma(A)$ for every $A \in \mathcal{A}$. Notice also the following property: for any $A \in \mathcal{A}$

$$\iota(X\setminus A) = \inf_{G \text{ open}, G\supset X\setminus A} \mu(G) = \inf_{F \text{ closed}, F\subset A} \mu(X\setminus F) = \inf_{F \text{ closed}, F\subset A} \mu(X) - \mu(F),$$

which implies

$$\iota(X\setminus A) = \mu(X) - \sup_{F \text{ closed}, F\subset A} \mu(F) = \mu(X) - \sigma(A). \tag{1}$$

Step 2. Next, you follow the hint: let $\mathcal{C}$ be the class of all "good sets", i.e., the class of all $A \in \mathcal{A}$ for which $\mu(A) = \iota(A) = \sigma(A)$ (also this is still similar to the spirit of Problem 10.12). Then the inequalities above imply that $\mathcal{C}$ is actually the set of all $A \in \mathcal{A}$ for which $\iota(A) \leq \sigma(A)$. The strategy suggested by the hint is to try to prove that $\mathcal{C}$ is a σ-algebra (see step 4) and to prove also that $\mathcal{C}$ contains any closed set $F \subset X$ (see step 3). After that, the proof is obviously finished, for then you have achieved $\mathcal{A} \supset \mathcal{C} = \sigma(\mathcal{C}) \supset \mathcal{B}(\mathbb{R}^d) = \mathcal{A}$, i.e., $\mathcal{C} = \mathcal{A}$.

Step 3. You must show $\iota(F) \leq \sigma(F)$, for which it is enough to prove (1) $\sigma(F) \geq \mu(F)$ and (2) $\mu(F) \geq \iota(F)$. Here (1) follows immediately from the definition of $\sigma(F)$. You can prove (2) by using the hint, which says that $F = \cap_k G_k$ for some countable collection $\{G_k\}_k$ of open sets. Without loss of generality you may suppose monotonicity (or else consider the open sets $G'_m := \cap_{k \leq m} G_k \supset F$). Because the measure μ is finite, it follows that $\mu(G_k) \downarrow \mu(F)$. By $\mu(G_k) \geq \iota(F)$ this gives (2).

Step 4. You must show that $\mathcal{C}$ does the following: (*i*) it contains $\emptyset$, (*ii*) it is closed for taking complements and (*iii*) it is closed for taking countable unions. As for (*i*), this follows immediately from $\iota(\emptyset) = \sigma(\emptyset) = 0$. As for (*ii*), let $A \in \mathcal{C}$ be arbitrary. Then $\iota(A) \leq \sigma(A)$, so (1) implies $\iota(X \backslash A) = \mu(X) - \sigma(A) \leq \mu(X) - \iota(A)$. Because it also follows from (1) that $\sigma(X \backslash A) = \mu(X) - \iota(A)$, you get $\iota(X \backslash A) \leq \sigma(X \backslash A)$, which proves that $X \backslash A$ belongs to $\mathcal{C}$. As for (*iii*), you must prove that if $\{A_j\}_j \subset \mathcal{C}$ (i.e., $\iota(A_j) \leq \sigma(A_j)$ for every j) then $A := \cup_j A_j \in \mathcal{C}$, i.e., $\iota(A) \leq \sigma(A)$, and for this it is already enough to prove that $\iota(A) \leq \sigma(A) + \epsilon$ for an arbitrary, fixed $\epsilon > 0$. Now for every j the inequality $\iota(A_j) \leq \sigma(A_j)$ implies that there exist an open set G_j, $G_j \supset A_j$, and a closed set F_j, $F_j \subset A_j$, such that $\mu(G_j) < \mu(F_j) + \epsilon/2^j$. Then for the open set $G := \cup_j G_j \supset A$ and the set $F := \cup_j F_j$ you have $\mu(G \backslash F) \leq \sum_j \mu(G_j \backslash F_j)$ by $G \backslash F \subset \cup_j (G_j \backslash F_j) < \epsilon$. Although F need not be closed, each set $F'_m := \cup_{j=1}^m F_j$ is closed. Here $F'_m \uparrow F$ implies $G \backslash F'_m \downarrow G \backslash F$, so it follows from the above that $\mu(G \backslash F'_m) < \epsilon$ for large enough m. By $G \supset A \supset F'_m$ this implies the desired inequality $\iota(A) \leq \sigma(A) + \epsilon$. The proof is now finished, in view of what was observed in step 2.

Finally, the hint follows by taking for an arbitrary closed and nonempty set $F \subset X$ the sequence composed of the open sets G_k, $k \in \mathbb{N}$, where $G_k := \{x \in X : \inf_{z \in F} \|z - x\| < 1/k\}$ is open, being the union over $z \in F$ of all open balls $B_{z,1/k}$. Then $\cap_k G_k \supset F$ is trivial and $\cap_k G_k \subset F$ follows simply from the fact that F is closed (if $\tilde{x} \in \cap_k G_k$ then there exists a sequence $(z_k) \subset F$ such that $z_k \to \tilde{x}$).