Example for WISB372/ECRMMAT, week 39

Example in the course (continued). The following optimal extraction problem was considered: maximize $\int_0^T e^{-\rho t}(py(t) - \frac{y^2(t)}{2})dt$ over all control functions $y(\cdot) : [0,T] \to \mathbb{R}$ such that $x(0) = x_0 > 0$ and $x(T) \geq 0$. Here the DS is $\dot{x} = -y$ and the parameter $p > 0$ represents the market price of the extracted mineral.

Solving it by means of the current value Hamiltonian gave $x(t) = -pt + \frac{c_1}{\rho}e^{\rho t} + x_0$ and $\mu(t) = c_1 e^{\rho t}$, with the constant c_1 still to be determined. Now we distinguish the following cases:

Case 1: $x(T) > 0$. In this case $\mu(T) = c_1 e^{\rho T} = 0$, which implies $c_1 = 0$. So we get $x(t) = -pt + x_0$. In this case this can only happen if $0 < x(T) = -pT + x_0$, i.e. if $pT < x_0$.

Case 2: $x(T) = 0$. In this case the above formula for $x(t)$ gives $0 = x(T) = -pT + \frac{c_1}{\rho}e^{\rho t} + x_0$, which implies $c_1 = \rho(pT - x_0)e^{-\rho T}$. It follows from this that $\mu(t) = \rho(pT - x_0)$. Because $\mu(T) \geq 0$ must hold as well in this case, this implies $pT \geq x_0$.

Preliminary conclusion: 1. If $pT < x_0$, then the candidate-optimal trajectory is $x(t) = -pt + x_0$ and then the associated candidate-optimal control function is $y(t) = -\dot{x}(t) = p$, 2. If $pT \geq x_0$, then the candidate-optimal trajectory is $x(t) = -pt + (pT - x_0)e^{\rho(t-T)} + x_0$ and the associated candidate-optimal control is $y(t) = -\dot{x}(t) = p - (pT - x_0)\rho e^{\rho(t-T)}$. Upon further inspection, we see that above "candidate-optimal" can be replaced by "optimal". This follows by application of the Sufficiency Theorem, because $f(t, \xi, \theta) := e^{-\rho t}(p\theta - \frac{\theta^2}{2})$ is concave in θ and $g(t, \xi, \theta) := -\theta$ is linear in θ.