Homework for WISB372/ECRMMAT, week 38

At this stage all examples in Chapter 20, except for the ones dealing with the skipped section 20.7 (i.e., problems 20.34, 20.35), are part of the quizz and exam material.

Comment. Dowling's treatment of sufficiency (section 20.5) is too timid, as explained in class. In this connection, you should always follow the sufficiency theorem as presented in class.

Exercise 1 a. Let A be a possibly *non-symmetric* $n \times n$ matrix for $n = 2$. Show that there is a symmetric matrix B such that

$$\forall_{\xi,\eta}\ q_A(\xi,\eta) := (\xi,\eta)A\begin{pmatrix}\xi\\ \eta\end{pmatrix} = (\xi,\eta)B\begin{pmatrix}\xi\\ \eta\end{pmatrix}$$

Mathematics students: Extend this result to $n \geq 2$.

Hint: Prove first that $(\xi,\eta)A\begin{pmatrix}\xi\\ \eta\end{pmatrix} = (\xi,\eta)A^T\begin{pmatrix}\xi\\ \eta\end{pmatrix}$, where A^T is the transpose of A.

b. Now use part a and the test discussed on 21-9 to investigate for which values $c \in \mathbb{R}$ (if any) the matrix $A := \begin{pmatrix}1 & -10\\ 6 & c\end{pmatrix}$ is positive definite.

c. Test correctness of your answer in part b by finding a *concrete* value for both ξ and η such that $q_A(\xi,\eta) \leq 0$ for $c = 2$.

Exercise 2 [only for mathematics students] a. Consider the two modifications "Mod 1" and "Mod 2" of the simplest problem in the calculus of variations, as discussed in class. Formulate and prove the corresponding versions of the Sufficiency Theorem for these two modifications. *Hint:* By introducing additional convexity conditions, the former necessary conditions for "Mod 1" and "Mod 2" should be shown to be also sufficient.

b. Prove the necessary optimality conditions for the modification "Mod 3", as stated in class on 21-9.

Exercise 3 Consider the following CoV problem: minimize $\int_0^2 (4\dot{x}^2(t)+x^2(t))dt$ oiver all $x \in \mathcal{C}^1$ such that $x(0) = 0$, $x(2) = 6$ and $\int_0^2 (x(t) + \dot{x}(t))dt = 12$.

a. Find candidate-optimal solutions for this problem with a functional constraint, by following the method of section 20.6. *Note for economics students:* This requires your command of the sections on second order differential equations in Dowling.

b. Show that the function $H(t, \xi, \eta)$ that is used to handle this problem is convex in (ξ, η). Use this fact to argue that the candidate-optimal solution found in part a is indeed optimal. *Warning:* You cannot blindly invoke the sufficiency theorem here.

Exercise 4 Consider the general OCT problem when one applies it to the case $g(\xi, \theta, t) = \theta$ (i.e., when $\dot{x} = y$ is the dynamical system), with $\Omega = \mathbb{R}$ (causing Dowling's "heroic assumption" to hold automatically). Then prove that Dowling's conditions 1, 2 and 3 on p. 494 amount to the necessary conditions for the simplest problem in the CoV.

Exercise 5 Let $\Omega := [0, 1]$. For the problem to minimize $\int_0^3 (x(t) - y(t))dt$ over all piecewise continuous functions $y : [0, 3] \to \Omega$ such that $\dot{x} = y$ and $x(0) = 0$, it was already shown that $\int_0^3 (x(t) - y(t))dt = \int_0^3 (2 - s)y(s)ds$ follows by changing the order of integration. Deduce from this that y^*, defined by $y^*(s) := 1$ on $[0, 2]$ and $y^*(s) := 0$ on $(2, 3]$ is an optimal control function for this problem. Thus, for this problem Dowling's "heroic assumption" is too heroic.

Exercise 6 Consider the problem of minimizing $\int_0^\pi \dot{x}^2(t)dt$ over all $x \in \mathcal{C}^1$ such that $x(0) = 0$, $x(\pi) = 0$ and $\int_0^\pi x(t) \sin t \, dt = 1$. Find the absolute minimizer $x^*(\cdot)$ for this problem.

Exercise 7 Consider the problem of minimizing $\int_0^1 \dot{x}^2(t)dt$ over all $x \in \mathcal{C}^1$ such that $x(0) = -4$, $x(1) = 4$ and $\int_0^1 tx(t) \, dt = 0$. Find the absolute minimizer $x^*(\cdot)$ for this problem. *Check your own answer:* $x^*(t) = 5t^3 + 3t - 4$.