Solution of Quiz 1 ECRMMAT, 7-10-2011

Problem Consider the optimal consumption problem to maximize $\int_0^T e^{-\rho t}\sqrt{y(t)}dt$ over all control functions $y(\cdot) : [0,T] \to \mathbb{R}_+$ such that $x(0) = 1$ and $x(T) \geq x_{min}$. Here $\rho > 0$ is a given discount factor, $x_{min} \in \mathbb{R}$ is a given real number and the dynamical system is $\dot{x} = -y$.

a. [10 pts] Discuss briefly how the above problem could be interpreted as an optimal consumption problem. Discuss in this connection in particular the economic meaning of the values $x(t)$ and the dynamical system.

b. [60 pts] Derive, under Dowling's "heroic interiority assumption", all possible candidate(s) $y^*(\cdot)$ for optimality, based on Dowling's conditions 1., 2., 3., as adapted on pp. 498-499.

c. [20 pts] Check that the Suffiency Theorem for optimal control theory, as treated in class, is applicable here. Use this to identify all possible optimal control functions $y^*(\cdot)$ (if any) which satisfy the "heroic interiority assumption".

d. [10 pts] Give an economic interpretation of your outcomes; explain your outcomes in particular for the special case where $x_{min} = 2$ and $\rho = 0$.

Solution. a. We interpret $y(t)$ as the consumption *rate*, i.e., in a small time interval of length Δt around time t the amount consumed of a certain good is $y(t)\Delta t$. This causes $x(t + \Delta t) - x(t)$ to be approximately equal to $-y(t)\Delta t$, where $x(t)$ represents the total supply at time t of the good. This explains the dynamical system. The initial supply of the good is 1 and the terminal supply should at least be x_{min}. Moreover, during that same time interval the discounted utility value of $y(t)$ is set at $e^{-\rho t}\sqrt{y(t)}$, so the time interval itself contributes $e^{-\rho t}\sqrt{y(t)}\Delta t$, where ρ is a discount factor. Summing over all time intervals and letting $\Delta t \to 0$ then shows that $\int_0^T e^{-\rho t}\sqrt{y(t)}dt$ can be interpreted as the total discounted utility enjoyed by consuming at the rate $y(\cdot)$ over the time interval $[0,T]$.

Observation: From this economic interpretation it is already obvious that the case $x_{min} > 1$ is impossible, because negative consumption is not allowed by the model: $\sqrt{y(t)}$ only makes sense for $y(t) \geq 0$. The mathematics below will completely confirm this economic intuition.

b. These necessary conditions for y^* to be an optimal control are as follows:

1. $\forall_t H_\theta(x^*(t), y^*(t), \lambda(t), t) = 0$,
2. $\forall_t \dot{\lambda}(t) = -H_\xi(x^*(t), y^*(t), \lambda(t), t)$ and $\dot{x}^*(t) = g(x^*(t), y^*(t), t)$,
3. $x^*(0) = x_0$ and either (i) $x^*(T) > x_{min}$ and $\lambda(T) = 0$ or (ii) $x^*(T) = x_{min}$ and $\lambda(T) \geq 0$,

Here $f(\xi, \theta, t) = e^{-\rho t}\sqrt{\theta}$ and $g(\xi, \theta, t) = -\theta$, so the Hamiltonian is

$$H(\xi, \theta, \lambda(t), t) := e^{-\rho t}\sqrt{\theta} - \lambda(t)\theta. \tag{1}$$

This gives $H_\xi = 0$ and $H_\theta = \frac{1}{2}e^{-\rho t}\theta^{-1/2} - \lambda(t)$. Hence, 2. gives $\dot{\lambda} = 0$, so $\lambda = c_1$, a constant. Then 1. implies $y^*(t) = \frac{1}{4c_1^2}e^{-2\rho t}$ (provided that $c_1 \neq 0$ of course).[1] By the dynamical system this implies $x^*(t) = \frac{1}{8c_1^2}e^{-2\rho t} + c_2$, where c_2 is an integration constant. For $t = 0$ this yields $1 = x^*(0) = \frac{1}{8c_1^2} + c_2$, i.e., $c_2 = 1 - \frac{1}{8c_1^2}$. So

$$x^*(t) = \frac{1}{8c_1^2}(e^{-2\rho t} - 1) + 1,$$

provided that $c_1 \neq 0$. For $t = T$ we distinguish the two cases (i) and (ii) as in 3. above:

Case (i). We have $c_1 = \lambda(1) = 0$, so we must go back to *"provided that* $c_1 \neq 0$*"* above. This provision was made in connection with 1., which now runs as follows: because of $\lambda(t) = c_1 = 0$, $H_\theta = \frac{1}{2}e^{-\rho t}\theta^{-1/2} = 0$ must be solved. However, this equation has no solution, so case (i) cannot happen at all.

Case (ii). In this case $x^*(T) = x_{min}$ holds. By the above this gives

$$x_{min} = x^*(T) = \frac{1}{8c_1^2}e^{-2\rho T} + 1 - \frac{1}{8c_1^2},$$

from which it follows that $\frac{1}{8c_1^2}$ is equal to $(1 - x_{min})/(1 - e^{-2\rho T})$. Because $\frac{1}{8c_1^2}$ is strictly positive, this equality can only occur if $1 - x_{min} > 0$, i.e., if x_{min} is strictly less than the initial supply $x_0 = 1$. So we must distinguish between $x_{min} < 1$ (case (iia)) and $x_{min} \geq 1$ (case (iib).

Case (iia)*:* $x_{min} < 1$. The above gives

$$x^*(t) = \frac{1 - x_{min}}{1 - e^{-2\rho T}}(e^{-2\rho t} - 1) + 1,$$

so

$$y^*(t) = -\dot{x}^*(t) = \frac{2\rho(1 - x_{min})}{1 - e^{-2\rho T}}e^{-2\rho t}$$

is the candidate-optimal control function.

Case (iib)*:* $x_{min} \geq 1$. In this case the above reasoning does not provide a candidate for optimality. Actually, if $x_{min} > 1$ we see immediately that no control function $y(\cdot)$ at all can have an associated trajectory $x(\cdot)$ with $x(0) = 1 < x_{min} \leq x(T)$, because $\dot{x} = -y \leq 0$ implies that $x(\cdot)$ cannot increase strictly. The case $x_{min} = 1$ is a little more subtle: again by $\dot{x} = -y \leq 0$, the function $y \equiv 0$ constantly equal to zero is the *only* control function for which the associated trajectory $x(\cdot)$ can meet both $x(0) = 1$ and $x(T) \geq 1$. Therefore, $y \equiv 0$ is trivially also the optimal control. However, it is not detected by 1., 2., 3. because $y \equiv 0$ violates the "heroic interiority assumption": by $\Omega = \mathbb{R}_+$ such interiority requires $y(t) > 0$ for all t.[2]

[1] Use of the current value Hamiltonian $H_c(\xi, \theta, \mu(t), t) := \sqrt{\theta} - \mu(t)\theta$ leads to the same expression via $\dot{\mu} = \rho\mu$ (whence $\mu(t) = C_1 e^{\rho t}$) and $(H_c)_\theta = \frac{1}{2}\theta^{-1/2} - \mu(t) = 0$, causing $\frac{1}{2}y^*(t)^{-1/2} = C_1 e^{\rho t}$.

[2] Observe: $y(t) > 0$ for all t is also needed for another reason. Namely, the underlying technical conditions of the model would be violated by allowing $y(t) = 0$, because $f(t, \xi, \theta) = \exp^{-\rho t}\sqrt{\theta}$ is not partially differentiable for $\theta = 0$.

c. The Hamiltonian of (1) has as its Hessian in the variables ξ, θ a 2×2-matrix whose only nonzero element is in the bottom right corner: it is $-\frac{1}{4}\theta^{-3/2}$, which is strictly negative. Hence, it follows immediately (from the definition) that this Hessian is negative semi-definite. For this reason $H(\xi, \theta, \lambda(t), t)$ is concave in (ξ, θ). Therefore, the Sufficiency Theorem guarantees that the candidate-optimal control function $y^*(\cdot)$, found in case (iia), gives indeed a global maximum.

d. We now consider what happens if $x_{min} = 2$ and $\rho = 0$. The special case $\rho = 0$ violates the original assumption $\rho > 0$. So we must trace back the previous arguments: for $\rho = 0$ the above derivation gives $\lambda \equiv c_1$ and $y^* \equiv \frac{1}{4c_1^2}$, provided $c_1 \neq 0$ (thus, case (i) in 3. goes just as above: it cannot occur). This causes the supply to decrease *linearly* rather than exponentially: $x^*(t) = -\frac{1}{4c_1^2}t + c_2$ and $c_2 = 1$ follows from $x^*(0) = 1$. As only case (ii) can occur, we get $x_{min} = x^*(T) = -\frac{1}{4c_1^2}T + 1 < 1$. So $x_{min} = 2 > 1$ gives a meaningless problem, as was to be expected from the previous considerations in part b (and see also the observation made at the end of part a).