Solutions Quiz 2, WISB372, 28-10-2011 (closed book)

Problem 1. [50 pts] Consider the following optimal control problem with variable end time. Minimize $\int_0^T \frac{1}{2}(u^2(t)+1)dt$ over all $T > 0$ and all control functions $u : [0,T] \to [-2,2]$ such that $x(0) = 5$ and $x(T) = 0$. Here $\dot{x} = u$ is the dynamical system. Determine all candidate-optimal control functions, using the minimum principle.

Solution. The Hamiltonian is $H(x,u,p(t)) = \frac{1}{2}(u^2+1) + p(t)u$. It has $H_x = 0$, so the adjoint equation gives $\dot{p} = 0$, whence $p(t) \equiv c_1$, a constant. Also, the minimum principle gives $u^*(t) \equiv -c_1$, with the implicit condition $|c_1| \leq 2$; ; however, for $c_1 > 2$ minimization would still give $u^*(t) = -2$ whereas $c_1 < -2$ would give $x^*(t) = 2$. Equivalently, we can say that $u^*(t) \equiv -c_1$ holds, with $|c_1| \leq 2$. The associated trajectory is therefore $x^*(t) = -c_1 t + 5$, in view of the initial condition. Because of the variable end time and the stationarity of the problem, we must also have $H(x^*(t), u^*(t), p(t)) \equiv 0$; this gives $\frac{1}{2}(c_1^2+1) - c_1^2 = 0$. This equation yields $c_1^2 = 1$, whence either $c_1 = 1$ or $c_1 = -1$ (note that the above condition $|c_1| \leq 2$ is satisfied in either case).

Case 1: $c_1 = 1$. This gives $u^* \equiv -1$ and $x^*(t) = 5 - t$. So $T^* = 5$ follows by the end time condition.

Case 2: $c_1 = -1$. This gives $u^* \equiv 1$ and $x^*(t) = 5 + t$. But now the end time condition leads to $T^* = -5 < 0$, which is not allowed.

Conclusion: there is essentially one candidate-optimal solution and it is $u^* \equiv -1$.

Problem 2. [50 pts] Consider the following optimal control problem: maximize $6x_1(5) - 3x_2(5)$ over all control functions $u : [0,5] \to [0,1]$ such that $x_1(0) = 1$ and $x_2(0) = 2$. Here the dynamical system is $\dot{x}_1 = x_1 + x_2 + u$ and $\dot{x}_2 = 2x_1 - u$.

a. Determine all candidate-optimal control functions, using the minimum principle. *Hint:* Note that the adjoint equation is actually a system of two differential equations, similar to your solution method for the water reservoir homework problem from Bertsekas.

b. Check if the sufficiency theorem (which continues to hold when the state space is multi-dimensional) can be applied here.

Solution. a. The Hamiltonian is $H(x_1, x_2, u, p_1(t), p_2(t)) = p_1(t)(x_1 + x_2 + u) + p_2(t)(2x_1 - u)$, so the partial derivatives $H_{x_1} = p_1 + 2p_2$ and $H_{x_2} = p_1$ follow. Therefore, the adjoint equations are $\dot{p}_1 = -p_1 - 2p_2$ and $\dot{p}_2 = -p_1$, which form a homogeneous system. By differentiating again (this method was demonstrated in class), this gives $\ddot{p}_1 + \dot{p}_1 - 2p_1 = 0$. The associated characteristic equation is $r^2 + r - 2 = 0$ and it has roots $r_1 = -2$ and $r_2 = 1$. Hence, the first solution of the adjoint equation system is $p_1(t) = A_1 e^{-2t} + A_2 e^t$ and then $p_2(t) = \frac{1}{2}(-p_1(t) - \dot{p}_1(t)) = \frac{1}{2} A_1 e^{-2t} - A_2 e^t$ follows with ease. By transversality we also have $p_1(5) = 6$ and $p_2(5) = -3$. Therefore, $\alpha_1 := A_1 e^{-10}$ and $\alpha_2 := A_2 e^5$ must satisfy $\alpha_1 + \alpha_2 = 6$ and $\frac{1}{2}\alpha_1 - \alpha_2 = -3$. This gives $\alpha_1 = 2$ and $\alpha_2 = 4$, so $p_1(t) = 2e^{10-2t} + 4e^{t-5}$ and $p_2(t) = e^{10-2t} - 4e^{t-5}$ follow. Next, by the maximum principle we know that $u^*(t)$ maximizes $(p_1(t) - p_2(t))u$ over $u \in U := [0,1]$. Now $p_1(t) - p_2(t) = e^{10-2t} + 8e^{t-5}$ is strictly positive for any t. Hence, $u^* \equiv 1$ is the desired candidate-optimal solution. As an extra, the associated trajectory – formally the problem does not ask for it, so it can be ignored – follows from solving the system $\dot{x}_1 = x_1 + x_2 + 1$ and $\dot{x}_2 = 2x_1 - 1$. This gives first $\ddot{x}_1 - \dot{x}_1 - 2x_1 = 0$, with characteristic equation $r^2 - r - 2 = 0$ (roots: -1 and 2). Then $x_1(t) = B_1 e^{-t} + B_2 e^{2t}$ and $x_2(t) = \dot{x}_1(t) - x_1(t) - 1 = -2B_1 e^{-t} + B_2 e^{2t} - 1$ follow. Finally, the two integration constants are determined by the two initial conditions: $1 = x_1(0) = B_1 + B_2$ and $2 = -2B_1 + B_2 - 1$ give $B_1 = -2/3$ and $B_2 = 5/3$. This leads to $x_1^*(t) = -\frac{2}{3}e^{-t} + \frac{5}{3}e^{2t}$ and $x_2^*(t) = \frac{4}{3}e^{-t} + \frac{5}{3}e^{2t} - 1$.

b. The function $H(x_1, x_2, u, p_1(t), p_2(t))$ is clearly linear in (x_1, x_2, u), so a fortiori it is concave in (x_1, x_2, u). Hence, the sufficiency theorem applies; this implies that $u^* \equiv 1$, as found in part a, is optimal.