New fundamentals of Young measure convergence*

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1 Introduction

This paper presents a new, penetrating approach to Young measure convergence in an abstract, measure theoretical setting. It was started in [12, 13, 14] and given its definitive shape in [18, 22]. This approach is based on *K-convergence*, a device by which narrow convergence on $\mathcal{P}(\mathbb{R}^d)$ can be systematically transferred to Young measure convergence. Here $\mathcal{P}(\mathbb{R}^d)$ stands for the set of all probability measures on $\mathbb{R}^d$ (in the sequel, a much more general topological space S is used instead of $\mathbb{R}^d$). Recall that in this context Young measures are measurable functions from an underlying finite measure space $(\Omega, \mathcal{A}, \mu)$ into $\mathcal{P}(\mathbb{R}^d)$. Recall also from [12], [13] (see also [24]) that K-convergence takes the following form when applied to Young measures (see Definition 3.1): A sequence (δ_k) of Young measures *K-converges* to a Young measure δ_0 [notation: $\delta_k \xrightarrow{K} \delta_0$] if for every subsequence (δ_{k_j}) of (δ_k) the following *pointwise* Cesaro-type convergence takes places

$$\frac{1}{N}\sum_{j=1}^{N} \delta_{k_j}(\omega) \Rightarrow \delta_0(\omega) \text{ as } N \to \infty$$

at μ-almost every point ω in Ω. Here "$\Rightarrow$" means classical narrow convergence on $\mathcal{P}(\mathbb{R}^d)$ (see Definition 2.1). As is shown much more completely in Proposition 3.6 and Theorem 4.8, the following fundamental relationship holds between Young measure convergence, denoted by "$\Longrightarrow$", and K-convergence as just defined [18, Corollary 3.16]:

Theorem 1.1 *Let (δ_n) be a sequence of Young measures. The following are equivalent:*
(a) $\delta_n \Longrightarrow \delta_0$.
(b) Every subsequence $(\delta_{n'})$ of (δ_n) contains a further subsequence $(\delta_{n''})$ such that $\delta_{n''} \xrightarrow{K} \delta_0$.

Both the nature of this equivalence result and the way in which we shall employ it are rather reminiscent of the well-known characterization of convergence in measure in terms of convergence almost everywhere. But while the latter result is simple, the former one is deep, as will become clear in the sequel. Nevertheless, thanks to this result several fundamental results on (sequential) Young measure

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convergence become simple to derive and can be stretched to what are arguably their most general versions in an abstract setting. These include: (1) the Prohorov-type criterion for relative sequential narrow compactness (Theorem 4.10), (2) the support theorem (Theorem 4.12, (3) the lower closure theorem (Theorem 4.13), (4) the denseness theorem for Dirac Young measures. The power of the apparatus thus developed is demonstrated by a selection of advanced applications in section 5, some of which are new as well (see also [18, 22] for references to applications in economics, such as [19, 21]). To the interested reader we also recommend [44, 48, 49, 56, 57, 59] for further background material and orientations towards various applications in applied analysis and optimal control.

2 Narrow convergence of probability measures

This section recapitulates some results on narrow convergence of probability measures on a metric space; cf. [2, 27, 28, 35, 46]. Let S be a completely regular Suslin space, whose topology is denoted by τ. On such a space there exists a metric ρ whose topology τ_ρ is not stronger than τ, with the property that the Borel σ-algebras $\mathcal{B}(S, \tau_\rho)$ and $\mathcal{B}(S, \tau)$ coincide. To see this, recall that in a completely regular space the points are separated by the collection $\mathcal{C}_b(S, \tau)$ of all bounded continuous functions on S. Since S is also Suslin, it follows by [32, III.32] that there exists a countable subset (c_i) of $\mathcal{C}_b(S, \tau)$, with $\sup_{x \in S} |c_i(x)| = 1$ for each i, that still separates the points of S. A metric as desired is then given by $\rho(x, y) := \sum_{i=1}^{\infty} 2^{-i} |c_i(x) - c_i(y)|$. This is because $\tau_\rho \subset \tau$ is obvious and by another well-known property of Suslin spaces, the Borel σ-algebras $\mathcal{B}(S, \rho)$ and $\mathcal{B}(S, \tau)$ coincide [51, Corollary 2, p. 101]. Of course, if S is a *metrizable* Suslin space to begin with, then for ρ one can simply take any metric on S that is compatible with τ.

As a consequence of the above, we shall write from now on

$$\mathcal{B}(S) := \mathcal{B}(S, \rho) = \mathcal{B}(S, \tau),\ \mathcal{P}(S) := \mathcal{P}(S, \rho) = \mathcal{P}(S, \tau)$$

for respectively the Borel σ-algebra and the set of all probability measures on $(S, \mathcal{B}(S))$.

Definition 2.1 (narrow convergence in $\mathcal{P}(S)$) A sequence (ν_n) in $\mathcal{P}(S)$ converges τ_ρ*-narrowly* to $\nu_0 \in \mathcal{P}(S)$ (notation: $\nu_n \overset{\rho}{\Rightarrow} \nu_0$) if $\lim_n \int_S c\, d\nu_n = \int_S c\, d\nu_0$ for every c in $\mathcal{C}_b(S, \tau_\rho)$.

Here $\mathcal{C}_b(S, \tau_\rho)$ stands for the set of all bounded τ_ρ-continuous functions from S into $\mathbb{R}$. Although τ_ρ-narrow convergence is more fundamental for our purposes, we shall often be able to use the stronger form of narrow convergence that arises when $\mathcal{C}_b(S, \tau_\rho)$ in the above definition is replaced by the larger set $\mathcal{C}_b(S, \tau)$. This will be denoted by " $\overset{\tau}{\Rightarrow}$ ". Definition 2.1 obviously extends to a definition of the τ- and τ_ρ-narrow topologies on $\mathcal{P}(S)$; we indicate these by $\mathcal{T}_\tau$ and $\mathcal{T}_\rho$). By [51, Appendix, Theorem 7] $\mathcal{P}(S)$ is a Suslin space for $\mathcal{T}_\tau$ (it is also Suslin and even metrizable for $\mathcal{T}_\rho$ – cf. [35, III.60]). Hence, completely analogous to what was observed above for S, the Borel σ-algebras coincide by [51, Corollary 2, p. 101]:

$$\mathcal{B}(\mathcal{P}(S)) := \mathcal{B}(\mathcal{P}(S), \mathcal{T}_\tau) = \mathcal{B}(\mathcal{P}(S), \mathcal{T}_\rho).$$

A vehicle by which we frequently manage to go from τ_ρ-convergence to the more general τ-convergence is τ-tightness:

Definition 2.2 (tightness in $\mathcal{P}(S)$) A sequence (ν_n) in $\mathcal{P}(S)$ is said to be τ*-tight* if there exists a sequentially τ-inf-compact function $h : S \to [0, +\infty]$ (i.e., all lower level sets $\{x \in S : h(x) \leq \beta\}$, $\beta \in \mathbb{R}$, are sequentially τ-compact) such that $\sup_n \int_S h\, d\nu_n < +\infty$.

Observe that *a fortiori* h must be τ_ρ-inf-compact on S (causing h to be Borel measurable); note here that τ_ρ is metrizable, so that the distinction between sequential and ordinary τ_ρ-inf-compactness vanishes.

Remark 2.3 *The above definition can be shown to be equivalent to the following one [5, Example 2.5]: (δ_n) is τ-tight if and only if for every $\epsilon > 0$ there exists a sequentially τ-compact set K_ϵ such that $\sup_n \nu_n(S \backslash K_\epsilon) \leq \epsilon$.*

Theorem 2.4 (portmanteau theorem for $\Rightarrow$) *(i) Let (ν_n) and ν_0 be in $\mathcal{P}(S)$. The following are equivalent:*

(a) $\nu_n \overset{\rho}{\Rightarrow} \nu_0$.

(b) $\lim_n \int_S c\, d\nu_n = \int_S c\, d\nu_0$ for every $c \in \mathcal{C}_u(S, \rho)$.

(c) $\liminf_n \int_S q\, d\nu_n \geq \int_S q\, d\nu_0$ for every τ_ρ-lower semicontinuous function $q : S \to (-\infty, +\infty]$ which is bounded from below.

(ii) Moreover, if (ν_n) is τ-tight, then the above are also equivalent to

(d) $\nu_n \overset{\tau}{\Rightarrow} \nu_0$.

(e) $\liminf_n \int_S q\, d\nu_n \geq \int_S q\, d\nu_0$ for every sequentially τ-lower semicontinuous function $q : S \to (-\infty, +\infty]$ which is bounded from below.

Here $\mathcal{C}_u(S, \rho)$ stands for the set of all uniformly ρ-continuous and bounded functions from S into $\mathbb{R}$. The name "portmanteau theorem" comes from [28].

Proof. Part (i), which is stated in a metrizable context, is classical; cf. [2, 4.5.1], [27, Proposition 7.21] and [28, Theorem 2.1]. Next, we prove part (ii): $(d) \Rightarrow (a)$ holds *a fortiori*. $(e) \Rightarrow (d)$ is evident since for any $c \in \mathcal{C}_b(S, \tau)$ both c and $-c$ meet the conditions imposed on q in part (e). $(d) \Rightarrow (e)$: Let h be as in Definition 2.2 For any q as stated in (e) and for any $\epsilon > 0$ the function $q_\epsilon := q + \epsilon h$ is sequentially τ-inf-compact, whence τ_ρ-inf-compact. Hence, q_ϵ is τ_ρ-lower semicontinuous on S and bounded from below.[1] So (c) and an easy argument with $\epsilon \to 0$ give (e). QED

It turns out that tightness is a criterion for relative compactness in the narrow topology. Just as in Definition 2.2 we only state the sequential version.

Theorem 2.5 (Prohorov's theorem for $\Rightarrow$) *Let (ν_n) in $\mathcal{P}(S)$ be τ-tight. Then there exist a subsequence $(\nu_{n'})$ of (ν_n) and $\nu_* \in \mathcal{P}(S)$ such that $\nu_{n'} \overset{\tau}{\Rightarrow} \nu_*$.*

Proof. By $\tau \supset \tau_\rho$ we can apply Prohorov's classical theorem [28, Theorem 6.1]. Hence, there exist a subsequence $(\nu_{n'})$ of (ν_n) and $\nu_* \in \mathcal{P}(S)$ such that $\nu_{n'} \overset{\rho}{\Rightarrow} \nu_*$. Hereupon, we can invoke Theorem 2.4. QED

Let $\hat{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ be the usual Alexandrov-compactification of the natural numbers. This is a metrizable space, so let $\hat{\rho}$ be a fixed metric on $\hat{\mathbb{N}}$ and let $\tilde{S} := S \times \hat{\mathbb{N}}$. We can equip $\tilde{S}$ with the product metric $\tilde{\rho}$ or with the product topology $\tilde{\tau} := \tau \times \tau_{\hat{\rho}}$. For $n \in \hat{\mathbb{N}}$, let $\epsilon_n \in \mathcal{P}(\hat{\mathbb{N}})$ be Dirac measure concentrated at the point n. The proof of the next result is rather obvious by Theorem 2.4(b) and a triangle inequality argument.

Corollary 2.6 *Let (ν_n) and ν_0 be in $\mathcal{P}(S)$. If $\frac{1}{N}\sum_{n=1}^N \nu_n \overset{\rho}{\Rightarrow} \nu_0$ in $\mathcal{P}(S)$, then*

$$\frac{1}{N}\sum_{n=1}^N (\nu_n \times \epsilon_n) \overset{\tilde{\rho}}{\Rightarrow} \nu_0 \times \epsilon_\infty \text{ in } \mathcal{P}(\tilde{S}).$$

In particular, if $\nu_n \overset{\rho}{\Rightarrow} \nu_0$ in $\mathcal{P}(S)$, then

$$\nu_n \times \epsilon_n \overset{\tilde{\rho}}{\Rightarrow} \nu_0 \times \epsilon_\infty \text{ in } \mathcal{P}(\tilde{S}).$$

Recall that the *support* τ-supp ν of $\nu \in \mathcal{P}(S)$ defined to be the complement of the union of all open ν-null sets; hence, $\nu(\tau\text{-supp } \nu) = 1$ (note that every τ-open subset of S has the countable subcover property by [35, III.67]).

[1] This shows $\tilde{q} : S \to \mathbb{R}$ to be $\mathcal{B}(S)$-measurable, with $\tilde{q} := q$ on $\{h < +\infty\}$ and $\tilde{q} := +\infty$ on $\{h = +\infty\}$. Hence, the integrals in (e) are well-defined.

Theorem 2.7 (support theorem for $\Rightarrow$) *(i) Let* $\frac{1}{N}\sum_{n=1}^{N}\nu_n \overset{\rho}{\Rightarrow} \nu_0$ *for* (ν_n) *and* ν_0 *in* $\mathcal{P}(S)$ *(in particular, this holds when* $\nu_n \overset{\rho}{\Rightarrow} \nu_0$*). Then*

$$\tau_\rho\text{-supp } \nu_0 \subset \tau_\rho\text{-Ls}_n \tau_\rho\text{-supp } \nu_n.$$

(ii) Moreover, if (ν_n) τ*-tight, then*

$$\nu_0(\tau\text{-seq-cl } \tau\text{-Ls}_n\tau\text{-supp } \nu_n) = 1 \text{ and } \tau\text{-supp } \nu_0 \subset \tau\text{-cl } \tau\text{-Ls}_n\tau\text{-supp } \nu_n,$$

Here "τ-seq-cl" stands for *sequential closure* with respect to the topology τ and "τ-Ls$_n$" refers to the usual Kuratowski sequential τ-limes superior of a sequence of subsets. This set is τ-closed if $\tau = \tau_\rho$ (metrizable case).

Proof. (i) By Corollary 2.6 it follows that $\pi_N := \frac{1}{N}\sum_{n=1}^{N}(\nu_n \times \epsilon_n) \overset{\tilde{\rho}}{\Rightarrow} \nu_0 \times \epsilon_\infty$ in $\mathcal{P}(\tilde{S})$, where $\tilde{S} := S \times \hat{\mathbb{N}}$. Setting $S_n := \tau_\rho$-supp ν_n and $S_\infty := \tau_\rho$-Ls$_n\tau_\rho$-supp ν_n, we define $\tilde{q}_0 : \tilde{S} \to \{0, +\infty\}$ as follows: If $x \in S_k$ then $\tilde{q}_0(x,k) := 0$. If $x \notin S_k$ for all k, $1 \leq k \leq \infty$, then $\tilde{q}_0(x,k) := +\infty$. We claim that $\tilde{q}_0$ is $\tau_{\tilde{\rho}}$-lower semicontinuous in every point (x,k) of $S \times \hat{\mathbb{N}}$. For let $\tilde{\rho}((x^j,k^j),(x,k)) \to 0$. We must show that $\alpha := \liminf_n \tilde{q}_0(x^j,k^j) \geq q'_0(x,k)$. If $k < \infty$, then eventually $k^j \equiv k$, so $\alpha \geq \tilde{q}_0(x,k)$ follows since S_k is τ_ρ-closed. If $k = \infty$, we can have two cases: if eventually $k^j \equiv \infty$, then $\alpha \geq \tilde{q}_0(x,\infty)$ follows by τ_ρ-closedness of S_∞. On the other hand, if $k^j < \infty$ infinitely often, then the same inequality follows directly from the definition of S_∞. This shows that $\tilde{q}_0$ is indeed $\tau_{\tilde{\rho}}$-lower semicontinuous. Now $\int_{\tilde{S}} \tilde{q}_0 d(\nu_n \times \epsilon_n) = \int_S \tilde{q}_0(x,n)\nu_n(dx) = 0$ for every n. Hence, $\int_{\tilde{S}} \tilde{q}_0 d\pi_N = 0$ for every N. Thus, Theorem 2.4 gives $\int_S \tilde{q}_0(x,\infty)\nu_0(dx) = 0$, and the desired τ_ρ-support property for ν_0 follows.

(ii) Since $\hat{\mathbb{N}}$ is compact, $\tilde{\tau}$-tightness of $(\nu_n \times \epsilon_n)$ in $\mathcal{P}(\tilde{S})$ is evident. Hence, Theorem 2.4 gives $\frac{1}{N}\sum_{n=1}^{N}(\nu_n \times \epsilon_n) \overset{\tilde{\tau}}{\Rightarrow} \nu_0 \times \epsilon_\infty$ in $\mathcal{P}(\tilde{S})$. We now essentially proceed as in the proof of (i), but a little more carefully: the additional sequential closure operation in the definition of S_∞) is needed because τ-Ls$_n\tau$-supp ν_n need not be sequentially τ-closed on its own accord. QED

Theorem 2.8 *Let* $\nu_n \overset{\rho}{\Rightarrow} \nu_0$ *in* $\mathcal{P}(S)$*. Then* (ν_n) *is* τ_ρ*-tight.*

Proof. S is Suslin, so any probability measure in $\mathcal{P}(S)$ is a Radon measure for both τ and τ_ρ [35, III.69]. Hence, the result follows from [28, Theorem 8, Appendix III]. QED

The above sufficient condition for τ_ρ-tightness of a sequence will play a role further on. It seems to have no analogue for τ-tightness when τ is nonmetrizable. The following result, also to be used later, is [27, Proposition 7.19]:

Proposition 2.9 (countable determination of $\Rightarrow$) *There exists a countable set* $\mathcal{C}_0 \subset \{c \in \mathcal{C}_u(S,\rho) : \sup_S |c| = 1\}$ *such that for every* (ν_n) *and* ν_0 *in* $\mathcal{P}(S)$ *one has* $\nu_n \overset{\rho}{\Rightarrow} \nu_0$ *if and only if* $\lim_n \int_S c\, d\nu_n = \int_S c\, d\nu_0$ *for every* $c \in \mathcal{C}_0$*. In particular,* $\mathcal{C}_0$ *separates the points of* $\mathcal{P}(S)$*.*

3 K-convergence of Young measures

A *Young measure* is a function $\delta : \Omega \to \mathcal{P}(S)$ that is measurable with respect to $\mathcal{A}$ and $\mathcal{B}(\mathcal{P}(S))$ The set of all such Young measures is denoted by $\mathcal{R}(\Omega; S)$. By $\mathcal{B}(S) = \mathcal{B}(S, \tau_\rho)$ of the previous section it is not hard to see that Young measures are precisely the transition probabilities from $(\Omega, \mathcal{A})$ into $(S, \mathcal{B}(S))$ [45, III.2], i.e., $\delta : \Omega \to \mathcal{P}(S)$ belongs to $\mathcal{R}(\Omega; S)$ if and only if $\omega \mapsto \delta(\omega)(B)$ is $\mathcal{A}$-measurable for every $B \in \mathcal{B}(S)$. For some elementary measure-theoretical properties of Young measures the reader is referred to [45, III.2] or [2, 2.6]. In particular, the product measure induced on $(\Omega \times S, \mathcal{A} \times \mathcal{B}(S))$ by μ and any $\delta \in \mathcal{R}(\Omega; S)$ (cf. [45, III.2]) is denoted by $\mu \otimes \delta$. Let $\mathcal{L}^0(\Omega; S)$ be the set of all measurable functions from $(\Omega, \mathcal{A})$ into $(S, \mathcal{B}(S))$. A Young measure $\delta \in \mathcal{R}(\Omega; S)$ is said to be *Dirac* if it is a degenerate transition probability [45, III.2], i.e., if there exists a function $f \in \mathcal{L}^0(\Omega; S)$ such that for every ω in Ω

$$\delta(\omega) = \epsilon_{f(\omega)} := \text{Dirac measure at the point } f(\omega).$$

Conversely, δ is also called the Young measure *relaxation* of f. In this special case δ is denoted by $\delta = \epsilon_f$. The set of all Dirac Young measures in $\mathcal{R}(\Omega; S)$ is denoted by $\mathcal{R}_{Dirac}(\Omega; S)$.

The fundamental idea behind Young measure theory is that, in some sense, $\mathcal{R}(\Omega; S)$ forms a *completion* of $\mathcal{L}^0(\Omega; S)$, when the latter is identified with $\mathcal{R}_{Dirac}(\Omega; S)$.

Let us agree to the following terminology: an *integrand* on $\Omega \times S$ is a function $g : \Omega \times S \to (-\infty, +\infty]$ such that for every $\omega \in \Omega$ the function $g(\omega, \cdot)$ on S is $\mathcal{B}(S)$-measurable. A function $g : \Omega \times S \to (-\infty, +\infty]$ is said to be a (sequentially) *τ-lower semicontinuous [τ-continuous] [[τ-inf-compact]] integrand* on $\Omega \times S$ if for every $\omega \in \Omega$ the function $g(\omega, \cdot)$ on S is (sequentially) τ-lower semicontinuous [τ-continuous] [[τ-inf-compact]] respectively. Let g be an integrand on $\Omega \times S$. The following expression is meaningful for any $\delta \in \mathcal{R}(\Omega; S)$:

$$I_g(\delta) := \int_\Omega^* [\int_S g(\omega, x)\delta(\omega)(dx)]\mu(d\omega),$$

provided that the two integral signs are interpreted as follows: (1) for every fixed ω the integral over the set S of the function $g(\omega, \cdot)$, which is $\mathcal{B}(S)$-measurable by definition of the term integrand, is a *quasi-integral* in the sense of [45, p. 41], (2) the integral over Ω is interpreted as an outer integral (note that outer integration comes down to quasi-integration when measurable functions are involved – cf. [9, Appendix A] or [22, Appendix B]). The resulting functional $I_g : \mathcal{R}(\Omega; S) \to [-\infty, +\infty]$ is called the *Young measure integral functional* associated to g. Another integral functional associated to g, this time on the set $\mathcal{L}^0(\Omega; S)$ of all measurable functions from Ω into S, is given by the formula

$$J_g(f) := \int_\Omega^* g(\omega, f(\omega))\mu(d\omega) = I_g(\epsilon_f).$$

The following notion of convergence was introduced and studied in a more abstract context in [12, 13].

Definition 3.1 (K-convergence in $\mathcal{R}(\Omega; S)$) A sequence (δ_n) in $\mathcal{R}(\Omega; S)$ *K-conver-ges* with respect to the topology τ to $\delta_0 \in \mathcal{R}(\Omega; S)$ (notation: $\delta_n \xrightarrow{K,\tau} \delta_0$) if for every subsequence $(\delta_{n'})$ of (δ_n)

$$\frac{1}{N}\sum_{n'=1}^{N} \delta_{n'}(\omega) \overset{\tau}{\Rightarrow} \delta_0(\omega) \text{ as } N \to \infty \text{ for a.e. } \omega \text{ in } \Omega.$$

Note that in the expression above the exceptional null set is allowed to vary with the subsequence $(\delta_{n'})$. We remark that K-convergence is *nontopological.* If in the above definition τ is replaced by τ_ρ and " $\overset{\tau}{\Rightarrow}$ " by " $\overset{\rho}{\Rightarrow}$ ", we obtain the weaker notion of K-convergence with respect to τ_ρ. This is denoted by " $\xrightarrow{K,\rho}$ ". We shall occasionally use " $\xrightarrow{K}$ " in situations where we need not distinguish between the two at all.

Example 3.2 Let $(\Omega, \mathcal{A}, \mu)$ be $([0,1], \mathcal{L}([0,1]), \lambda_1)$ (i.e., the Lebesgue unit interval). Let (f_n) be the sequence of *Rademacher functions*, defined by $f_n(\omega) := \text{sign} \sin(2^n\pi\omega)$ (here $S := \mathbb{R}$, of course). Then $\epsilon_{f_n} \xrightarrow{K} \delta_0$, where $\delta_0 \in \mathcal{R}([0,1]; \mathbb{R})$ is the constant function $\delta_0(\omega) \equiv \frac{1}{2}\epsilon_1 + \frac{1}{2}\epsilon_{-1}$. This can be proven by the (scalar) strong law of large numbers, analogous to the proof of Theorem 3.8.

Definition 3.3 (tightness in $\mathcal{R}(\Omega; S)$) A sequence (δ_n) in $\mathcal{R}(\Omega; S)$ is *τ-tight* if there exists a non-negative, sequentially τ-inf-compact integrand h on $\Omega \times S$ such that $\sup_n I_h(\delta_n) < +\infty$.

This definition comes from [26]; clearly this extends Definition 2.2. Recall from the previously given definition of integrands that a sequentially τ-inf-compact integrand h is simply a function on $\Omega \times S$ with the following property: for every $\omega \in \Omega$ the function $h(\omega, \cdot)$ is sequentially τ-inf-compact.

Remark 3.4 *Similar to Remark 2.3, Definition 3.3 can easily be shown to be equivalent to the following one [39]: (δ_n) is τ-tight if and only if for every $\epsilon > 0$ there exists a multifunction $\Gamma_\epsilon : \Omega \to 2^S$, with $\Gamma_\epsilon(\omega)$ sequentially τ-compact for every $\omega \in \Omega$, such that*

$$\sup_n \int_\Omega^* \delta_n(\omega)(S \backslash \Gamma_\epsilon(\omega))\mu(d\omega) \leq \epsilon.$$

Example 3.5 (a) Let E be a separable reflexive Banach space with norm $\|\cdot\|$. Let E' be the dual space of E. Suppose that $(f_n) \subset \mathcal{L}^1(\Omega; E)$ is $\mathcal{L}^1$-bounded: $\sup_n \int_\Omega \|f_n(\omega)\|\mu(d\omega) < +\infty$. Then (ϵ_{f_n}) is $\sigma(E, E')$-tight in $\mathcal{R}(\Omega; S)$: simply set $h(\omega, x) := \|x\|$ in Definition 3.3.

(b) Let E be a separable Banach space with norm $\|\cdot\|$. Suppose that $(f_n) \subset \mathcal{L}^1(\Omega; E)$ is $\mathcal{L}^1$-bounded and that there exists a multifunction $R : \Omega \to 2^S$ such that for a.e. ω both $\{f_n(\omega) : n \in \mathbb{N}\} \subset R(\omega)$ and $R(\omega)$ is $\sigma(E, E')$-ball-compact [i.e., the intersection of $R(\omega)$ with every closed ball in E is $\sigma(E, E')$-compact]. Then (ϵ_{f_n}) is $\sigma(E, E')$-tight: now we set $h_R(\omega, x) := \|x\|$ if $x \in R(\omega)$, and $h_R(\omega, x) := +\infty$ otherwise. Then for every $\omega \in \Omega$ and $\beta \in \mathbb{R}_+$ the set of all $x \in E$ such that $h_R(\omega, x) \leq \beta$ is the intersection of $R(\omega)$ and the closed ball with radius β around 0. By the Eberlein-Šmulian theorem it is sequentially $\sigma(E, E')$-compact as well.

Part (b) in the above example generalizes part (a): simply observe that in part (a) E itself is $\sigma(E, E')$-ball-compact by reflexivity, so there we can take $R \equiv E$. A very important property of K-convergence of Young measures is as follows [13, 12, 18]:

Proposition 3.6 (Fatou-Vitali for $\xrightarrow{K}$) (i) *Let $\delta_n \xrightarrow{K,\rho} \delta_0$ in $\mathcal{R}(\Omega; S)$. Then $\liminf_n I_g(\delta_n) \geq I_g(\delta_0)$ for every τ_ρ-lower semicontinuous integrand g on $\Omega \times S$ such that*

$$s(\alpha) := \sup_n \int_\Omega^* [\int_{\{g \leq -\alpha\}_\omega} g^-(\omega, x)\delta_n(\omega)(dx)]\mu(d\omega) \to 0 \text{ for } \alpha \to \infty. \tag{3.1}$$

(ii) Moreover, if (δ_n) is τ-tight, then also $\liminf_n I_g(\delta_n) \geq I_g(\delta_0)$ for every sequentially τ-lower semicontinuous integrand g on $\Omega \times S$ such that (3.1) holds.

Here, as usual, $g^- := \max(-g, 0)$ and $\{g \leq -\alpha\}_\omega$ denotes $\{x \in S : g(\omega, x) \leq -\alpha\}$. Note that footnote 1 applies to each $g(\omega, \cdot)$ in part (ii).

Remark 3.7 *If $\delta_n = \epsilon_{f_n}$ for all $n \in \mathbb{N}$, then (3.1) runs as follows:*

$$\lim_{\alpha \to \infty} \sup_n \int_\Omega^* 1_{\{g(\cdot, f_n(\cdot)) \leq -\alpha\}} g^-(\omega, f_n(\omega))\mu(d\omega) = 0.$$

Since $g(\omega, f_n(\omega)) \leq -\alpha$ if and only if $g^-(\omega, f_n(\omega)) \geq \alpha$, (3.1) comes down to uniform (outer) integrability of the sequence $(g^-(\cdot, f_n(\cdot)))$ in the case of a Dirac sequence, in agreement with standard formulations; cf. [37, 5].

Proof of Proposition 3.6. The proof of (i) will be given in two steps.

Step 1: $g \geq 0$. Set $\beta := \liminf_n I_g(\delta_n)$; then there is a subsequence $(\delta_{n'})$ such that $\beta = \lim_{n'} I_g(\delta_{n'})$. Define $\psi_N(\omega) := \frac{1}{N}\sum_{n'=1}^N \int_S g(\omega, x)\delta_{n'}(\omega)(dx)$ and $\psi_0(\omega) := \int_S g(\omega, x)\delta_0(\omega)(dx)$. Then $\liminf_N \psi_N \geq \psi_0$ a.e. by Theorem 2.4(c), because by Definition 3.1 $\frac{1}{N}\sum_{n'=1}^N \delta_{n'}(\omega) \stackrel{\rho}{\Rightarrow} \delta_0(\omega)$ in $\mathcal{P}(S)$ for a.e. ω. Thus, Fatou's lemma can be applied (it remains valid for outer integration in the direction that suits us; cf. [22, Appendix B]). This gives $\beta \geq \liminf_{N \to \infty} \int_\Omega^* \psi_N \, d\mu \geq \int_\Omega^* \psi_0 \, d\mu = I_g(\delta_0)$ by subadditivity of outer integration.

Step 2: general case. We essentially follow Ioffe [37] by pointing out that

$$\int_S g(\omega,x)\delta_n(\omega)(dx) + \int_S 1_{\{g\le -\alpha\}}(\omega,x)g^-(\omega,x)\delta_n(\omega)(dx) \ge \int_S g_\alpha(\omega,x)\delta_n(\omega)(dx),$$

by $g + 1_{\{g\le-\alpha\}}g^- \ge g_\alpha := \max(g,-\alpha)$. One more (outer) integration gives, in the notation of (3.1), $I_g(\delta_n)+s(\alpha) \ge I_{g_\alpha}(\delta_n)$, where we use subadditivity of outer integration. Now step 1 trivially extends to any g that is bounded from below, such as g_α. This gives

$$\liminf_n I_g(\delta_n) + s(\alpha) \ge \liminf_n I_{g_\alpha}(\delta_n) \ge I_{g_\alpha}(\delta_0) \ge I_g(\delta_0),$$

where we use $g_\alpha \ge g$. The proof of (i) is finished by letting α go to infinity.

(ii) Let h be as in Definition 3.3 and denote $s := \sup_n I_h(\delta_n)$. We augment g, similar to the proof of Theorem 2.4(ii): For $\epsilon > 0$ define $g^\epsilon := g + \epsilon h$. Then $g^\epsilon \ge g$ and $g^\epsilon(\omega,\cdot)$ is τ_ρ-lower semicontinuous on S for every $\omega \in \Omega$ (see the proof of Theorem 2.4(ii)). Thus, part (i) gives $\liminf_n I_g(\delta_n) + \epsilon s \ge \liminf_n I_{g^\epsilon}(\delta_n) \ge I_{g^\epsilon}(\delta_0) \ge I_g(\delta_0)$ for any $\epsilon > 0$. Letting ϵ go to zero gives the desired inequality. QED

The following important Prohorov-type relative compactness criterion for K-conver-gence is [13, Theorem 5.1]. It was obtained as a specialization to Young measures of an abstract version of Komlós' theorem [41]; see also [14].

Theorem 3.8 (Prohorov's theorem for $\xrightarrow{K}$) *Let (δ_n) in $\mathcal{R}(\Omega;S)$ be τ-tight. Then there exist a subsequence $(\delta_{n'})$ of (δ_n) and $\delta_* \in \mathcal{R}(\Omega;S)$ such that $\delta_{n'} \xrightarrow{K,\tau} \delta_*$.*

To prove Theorem 3.8 we use the following theorem, due to Komlós [41].

Theorem 3.9 (Komlós) *Let (ϕ_n) be a sequence in $\mathcal{L}^1(\Omega;\mathbb{R})$ such that $\sup_n \int_\Omega |\phi_n| d\mu < +\infty$. Then there exist a subsequence $(\phi_{n'})$ of (ϕ_n) and a function $\phi_* \in \mathcal{L}^1(\Omega;\mathbb{R})$ such that for every further subsequence $(\phi_{n''})$ of $(\phi_{n'})$*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n''=1}^{N} \phi_{n''}(\omega) = \phi_*(\omega) \text{ for a.e. } \omega \text{ in } \Omega.$$

Lemma 3.10 *Let (ν_n) in $\mathcal{P}(S)$ be τ-tight and let $\mathcal{C}_0$ be a subset of $\{c \in \mathcal{C}_b(S,\tau) : \sup_S |c| = 1\}$ that separates the points of $\mathcal{P}(S)$. If*

$$\lim_n \int_S c d\nu_n \text{ exists for every } c \in \mathcal{C}_0,$$

then there exists $\nu_* \in \mathcal{P}(S)$ such that $\nu_n \overset{\tau}{\Rightarrow} \nu_*$.

This lemma is a direct result of Theorem 2.5 and the point separating property of $\mathcal{C}_0$; cf. Proposition 2.9.

Proof of Theorem 3.8. Let $\mathcal{C}_0 = \{c_i : i \in \mathbb{N}\}$ be as in Lemma 3.10. Define $\phi_{i,n}(\omega) := \int_S c_i(x)\delta_n(\omega)(dx)$; then $\sup_n \int_\Omega |\phi_{i,n}| d\mu < +\infty$ for every $i \in \mathbb{N}$. Let h be as in Definition 3.3. By definition of outer integration, there exists for each $n \in \mathbb{N}$ a function $\phi_{0,n} \in \mathcal{L}^1(\Omega;\mathbb{R})$ such that $\phi_{0,n}(\omega) \ge \int_S h(\omega,x)\delta_n(\omega)(dx)$ for a.e. $\omega \in \Omega$ and $\int_\Omega \phi_{0,n}\, d\mu = I_h(\delta_n)$. Applying Theorem 3.9 in a diagonal extraction procedure, we obtain a subsequence $(\delta_{n'})$ of (δ_n) and functions $\phi_{i,*} \in \mathcal{L}^1(\Omega;\mathbb{R})$, $i \in \mathbb{N}\cup\{0\}$, such that $\lim_N \frac{1}{N}\sum_{n''=1}^{N} \phi_{i,n''} = \phi_{i,*}$ a.e. for every further subsequence $(\delta_{n''})$ and for all $i \in \mathbb{N}\cup\{0\}$. Explicitly, we have every such $(\delta_{n''})$ for a.e. ω in Ω

$$\lim_N \int_S h(\omega,x)\frac{1}{N}\sum_{n''=1}^{N} \delta_{n''}(\omega)(dx) = \phi_{0,*}(\omega) < +\infty, \tag{3.2}$$

$$\lim_N \int_S c_i(x) \frac{1}{N} \sum_{n''=1}^{N} \delta_{n''}(\omega)(dx) = \phi_{i,*}(\omega) \text{ for all } i \in \mathbb{N}. \tag{3.3}$$

Let us first consider $(\delta_{n'})$ itself as the subsequence in question. Fix ω outside the exceptional null set M, associated with this particular choice of a subsequence in (3.2)–(3.3). Then (3.2) implies that the sequence (ν_N) in $\mathcal{P}(S)$, defined by $\nu_N := \frac{1}{N}\sum_{n'=1}^{N}\delta_{n'}(\omega)$, is τ-tight in the classical sense of Definition 2.2. Also, (3.3) implies that $\lim_N \int_S c_i d\nu_N$ exists for every i. By Lemma 3.10 there exists $\nu_{\omega,*}$ in $\mathcal{P}(S)$ such that $\nu_N \overset{\tau}{\Rightarrow} \nu_{\omega,*}$. Define $\delta_*(\omega) := \nu_{\omega,*}$ for $\omega \in \Omega\backslash M$. Also, on M we define δ_* to be equal to an arbitrary fixed element of $\mathcal{P}(S)$. Then it is elementary to show that δ_* belongs to $\mathcal{R}(\Omega; S)$. The argument following (3.3) can be repeated with a change of the null set M (for which Definition 3.1 leaves room) if one starts out with an arbitrary subsequence $(\delta_{n''})$ of $(\delta_{n'})$. QED

The next example extends Example 3.2 and demonstrates the power of Theorem 3.8, which brings K-convergence (for subsequences!) to settings where Kolmogorov's law of large numbers, used in the special Example 3.2, stands no chance at all.

Example 3.11 Let $(\Omega, \mathcal{A}, \mu)$ be $([0,1], \mathcal{L}([0,1]), \lambda_1)$ (i.e., the Lebesgue unit interval). Let $f_1 \in \mathcal{L}^1([0,1];\mathbb{R})$ be arbitrary; it can be extended periodically from $[0,1]$ to all of $\mathbb{R}$. We define $f_{n+1}(\omega) := f_1(2^n\omega)$. Clearly, the sequence (ϵ_{f_n}) is tight in the sense of Definition 3.3 (see Example 3.5(a)). By Theorem 3.8 there exist a subsequence $(f_{n'})$ of (f_n) and some $\delta_* \in \mathcal{R}([0,1];\mathbb{R})$ such that $\epsilon_{f_{n'}} \overset{K}{\longrightarrow} \delta_*$. The precise nature of δ_* can now be determined by means of Proposition 3.6, but we shall defer this to Example 4.4 later on.

The following are direct consequences of Corollary 2.6 and Theorem 2.7 for K-convergence of Young measures (by their application pointwise):

Corollary 3.12 *Let (δ_n) and δ_0 be in $\mathcal{R}(\Omega; S)$. The following are equivalent:*

(a) $\delta_n \overset{K,\rho}{\longrightarrow} \delta_0$ in $\mathcal{R}(T; S)$

(b) $\delta_n \times \epsilon_n \overset{K,\tilde{\rho}}{\longrightarrow} \delta_0 \times \epsilon_\infty$ in $\mathcal{R}(T; \tilde{S})$.

Theorem 3.13 (support theorem for $\overset{K}{\longrightarrow}$) *$(i)$ Let $\delta_n \overset{K,\rho}{\longrightarrow} \delta_0$ in $\mathcal{R}(\Omega; S)$. Then*

$$\tau_\rho\text{-supp } \delta_0(\omega) \subset \tau_\rho\text{-Ls}_n \tau_\rho\text{-supp } \delta_n(\omega) \textit{ for a.e. } \omega \textit{ in } \Omega.$$

(ii) Moreover, if $\delta_n \overset{K,\tau}{\longrightarrow} \delta_0$, then also

$$\delta_0(\omega)(\tau\text{-seq-cl } \tau\text{-Ls}_n \tau\text{-supp } \delta_n(\omega)) = 1,$$

$$\tau\text{-supp } \delta_0(\omega) \subset \tau\text{-cl } \tau\text{-Ls}_n \tau\text{-supp } \delta_n(\omega) \textit{ for a.e. } \omega \textit{ in } \Omega.$$

4 Narrow convergence of Young measures

In this section our program to transfer narrow convergence results for probability measures (section 2) to Young measures comes is completed. We use the same fundamental hypotheses as in the previous section: $(\Omega, \mathcal{A}, \mu)$ is a finite measure space and (S, τ) is a completely regular Suslin space, on which we also consider the weak metric topology $\tau_\rho \subset \tau$. We start out by giving the definition of narrow convergence for Young measures [3, 4, 10] (see also [38]).

Definition 4.1 (narrow convergence in $\mathcal{R}(T; S)$) A sequence (δ_n) in $\mathcal{R}(\Omega; S)$ converges τ-*narrowly* to δ_0 in $\mathcal{R}(\Omega; S)$ (this is denoted by $\delta_n \overset{\tau}{\Longrightarrow} \delta_0$) if for every $A \in \mathcal{A}$ and c in $\mathcal{C}_b(S, \tau)$

$$\lim_n \int_A [\int_S c(x)\delta_n(\omega)(dx)]\mu(d\omega) = \int_A [\int_S c(x)\delta_0(\omega)(dx)]\mu(d\omega).$$

The weaker notion of τ_ρ*-narrow* convergence is defined by replacing τ by τ_ρ; this is denoted by "$\stackrel{\rho}{\Longrightarrow}$". We shall occasionally use "$\Longrightarrow$" in situations where we need not distinguish between the two at all. We shall see that on τ-tight sets of Young measures these two modes actually coincide (note the complete analogy to section 2). For further benefit, note carefully the distinct notation used for narrow convergence for probability measures (indicated by *short* arrows) and Young measure convergence (indicated by *long* arrows).

Remark 4.2 ($\stackrel{K}{\longrightarrow}$ **implies** $\Longrightarrow$**)** *Let* (δ_n) *and* δ_0 *be in* $\mathcal{R}(\Omega; S)$*. The following hold: (a) If* $\delta_n \stackrel{K,\rho}{\longrightarrow} \delta_0$*, then* $\delta_n \stackrel{\rho}{\Longrightarrow} \delta_0$*. (b) If* $\delta_n \stackrel{K,\rho}{\longrightarrow} \delta_0$ *and if* (δ_n) *is* τ*-tight, then* $\delta_n \stackrel{\tau}{\Longrightarrow} \delta_0$*. (c) If* $\delta_n \stackrel{K,\tau}{\longrightarrow} \delta_0$*, then* $\delta_n \stackrel{\tau}{\Longrightarrow} \delta_0$*.*

Definition 4.1 obviously extends to a definition of the τ- and τ_ρ-narrow *topologies.* In the form given above, the definition of narrow convergence is classical in statistical decision theory [58, 43]. It merges two completely different classical modes of convergence:

Remark 4.3 *Let* (δ_n) *and* δ_0 *be in* $\mathcal{R}(\Omega; S)$*. The following are obviously equivalent:*
(a) $\delta_n \stackrel{\tau}{\Longrightarrow} \delta_0$ *in* $\mathcal{R}(\Omega; S)$*.*
(b) $[\mu \otimes \delta_n](A \times \cdot)/\mu(A) \stackrel{\tau}{\Rightarrow} [\mu \otimes \delta_0](A \times \cdot)/\mu(A)$ *in* $\mathcal{P}(S)$ *for every* $A \in \mathcal{A}$*,* $\mu(A) > 0$*.*
(c) $\int_S c(x)\delta_n(\cdot)(dx) \stackrel{*}{\rightharpoonup} \int_S c(x)\delta_0(\cdot)(dx)$ *in* $\mathcal{L}^\infty(\Omega; \mathbb{R})$ *for every* $c \in \mathcal{C}_b(S, \tau)$*. Here "*$\stackrel{*}{\rightharpoonup}$*" denotes convergence in the topology* $\sigma(\mathcal{L}^\infty(\Omega; \mathbb{R}), \mathcal{L}^1(\Omega; \mathbb{R}))$*.*

The following example continues the previous Examples 3.2 and 3.11.

Example 4.4 Let $(\Omega, \mathcal{A}, \mu)$ be $([0, 1], \mathcal{L}([0, 1]), \lambda_1)$ (cf. Example 3.2). As in Example 3.11, let $f_1 \in \mathcal{L}^1([0, 1]; \mathbb{R})$ be arbitrary and extended periodically from $[0, 1]$ to all of $\mathbb{R}$. We define $f_{n+1}(\omega) := f_1(2^n\omega)$. Then $\epsilon_{f_n} \Longrightarrow \delta_0$, where $\delta_0 \in \mathcal{R}([0, 1], \mathbb{R})$ is the constant function given by $\delta_0(\omega) \equiv \lambda_1^{f_1}$. Here $\lambda^{f_1} \in \mathcal{P}(\mathbb{R})$ is the image of λ_1 under the mapping f_1; i.e., $\lambda^{f_1}(B) := \lambda(f_1^{-1}(B))$. To prove the above convergence statement, let $c \in \mathcal{C}_b(\mathbb{R})$ be arbitrary, and let A be first of the form $A = [0, \beta]$ with $\beta > 0$. Then a simple change of variable gives $\lim_{n\to\infty} \int_A c(f_n) d\lambda_1 = \int_A [\int_{\mathbb{R}} c(x)\delta_0(\omega)(dx)] d\omega$ for $A = [0, \beta]$. By standard methods this can then be extended to all A in $\mathcal{A}$.

It follows that δ_* in Example 3.11 is equal to the above δ_0, modulo a λ_1-null set. The proviso of an exceptional null-set is indispensible, because the narrow limits in $\mathcal{R}(\Omega; S)$ are only unique modulo a μ-null set:

Proposition 4.5 *For every* δ*,* δ' *in* $\mathcal{R}(\Omega; S)$ *the following are equivalent:*
(a) $\int_A [\int_S c(x)\delta(\omega)(dx)]\mu(d\omega) = \int_A [\int_S c(x)\delta'(\omega)(dx)]\mu(d\omega)$ *for every* $A \in \mathcal{A}$ *and* $c \in \mathcal{C}_0$*.*
(b) $\delta(\omega) = \delta'(\omega)$ *for a.e.* ω *in* Ω*.*

Theorem 4.6 *Suppose that the* σ*-algebra* $\mathcal{A}$ *on* Ω *is countably generated. Then there exists a semimetric* $d_{\mathcal{R}}$ *on* $\mathcal{R}(\Omega; S)$ *such that for every* (δ_n) *and* δ_0 *in* $\mathcal{R}(\Omega; S)$ *the following are equivalent:*
(a) $\delta_n \stackrel{\rho}{\Longrightarrow} \delta_0$*.*
(b) $\lim_n d_{\mathcal{R}}(\delta_n, \delta_0) = 0$*.*

Proof. Let (c_i) enumerate $\mathcal{C}_0$ of Proposition 2.9, and let (A_j) be the at most countable algebra generating $\mathcal{A}$. Denote $q_i(A, \delta) := \int_A [\int_S c_i \delta(\cdot)(dx)] d\mu$ and define a semimetric on $\mathcal{R}(\Omega; S)$ by $d_{\mathcal{R}}(\delta, \delta') := \sum_{i,j} 2^{-i-j} |q_i(A_j, \delta) - q_i(A_j, \delta')|$. To prove $(a) \Rightarrow (b)$ we note that standard arguments [2, 1.3.11] give $\lim_n q_i(A, \delta_n) = q_i(A, \delta_0)$ for every $A \in \mathcal{A}$ and i. By Proposition 2.9 and Remark 4.3 this implies $\delta_n \stackrel{\rho}{\Longrightarrow} \delta_0$. Conversely, $(a) \Rightarrow (b)$ is simple. QED

Proposition 3.6 and Theorem 3.8 imply the following transfer of the earlier portmanteau Theorem 2.4 to the domain of Young measures [10].

Theorem 4.7 (portmanteau theorem for $\Longrightarrow$) *(i) Let (δ_n) and δ_0 be in $\mathcal{R}(\Omega;S)$. The following are equivalent:*

(a) $\delta_n \overset{\rho}{\Longrightarrow} \delta_0$.

(b) $\lim_n \int_A [\int_S c(x)\delta_n(\omega)(dx)]\mu(d\omega) = \int_A [\int_S c(x)\delta_0(\omega)(dx)]\mu(d\omega)$ *for every* $A \in \mathcal{A}$, $c \in \mathcal{C}_u(S,\rho)$.

(c) $\liminf_n I_g(\delta_n) \geq I_g(\delta_0)$ *for every τ_ρ-lower semicontinuous integrand g on $\Omega \times S$ such that*

$$\lim_{\alpha\to\infty} \sup_n \int_\Omega^* [\int_{\{g\leq -\alpha\}_\omega} g^-(\omega,x)\delta_n(\omega)(dx)]\mu(d\omega) = 0.$$

(ii) Moreover, if (δ_n) is τ-tight, then the above are also equivalent to

(d) $\delta_n \overset{\tau}{\Longrightarrow} \delta_0$.

(e) $\liminf_n I_g(\delta_n) \geq I_g(\delta_0)$ *for every sequentially τ-lower semicontinuous integrand g on $\Omega \times S$ such that*

$$\lim_{\alpha\to\infty} \sup_n \int_\Omega^* [\int_{\{g\leq -\alpha\}_\omega} g^-(\omega,x)\delta_n(\omega)(dx)]\mu(d\omega) = 0.$$

Proof. By Remark 4.3 $(a) \Leftrightarrow (b)$ follows by $(a) \Leftrightarrow (b)$ in Theorem 2.4. $(c) \Rightarrow (b)$ is obvious: apply (c) to $g(\omega,x) := \pm 1_A(\omega)c(x)$. $(a) \Rightarrow (c)$: By Remark 4.3 $\nu_n \overset{\rho}{\Rightarrow} \nu_0$, where $\nu_n := [\mu \otimes \delta_n](\Omega \times \cdot)/\mu(\Omega)$. So by Theorem 2.8 (ν_n) is τ_ρ-tight in $\mathcal{P}(S)$: there exists a τ_ρ-inf-compact $h' : S \to [0,+\infty]$ such that $\sup_n \int_S h' d\nu_n < +\infty$. So (δ_n) is τ_ρ-tight, since $\int_S h' d\nu_n = I_h(\delta_n)/\mu(\Omega)$ for $h(\omega,x) := h'(x)$, Therefore, Theorem 3.8 applies to (δ_n). For g as stated, let $\beta := \liminf_n I_g(\delta_n)$. Then $\beta = \lim_{n'} I_g(\delta_{n'})$ for a suitable subsequence $(\delta_{n'})$ and, by Theorem 3.8, we may suppose without loss of generality that $\delta_{n'} \overset{K,\rho}{\longrightarrow} \delta_*$ for some δ_* in $\mathcal{R}(\Omega;S)$. But in combination with (a) this implies $\delta_*(\omega) = \delta_0(\omega)$ a.e. (Proposition 4.5), so in fact $\delta_{n'} \overset{K,\rho}{\longrightarrow} \delta_0$. Now $\beta \geq I_g(\delta_0)$ follows from Proposition 3.6. Next, $(d) \Rightarrow (a)$ holds *a fortiori* and $(a) \Rightarrow (e)$ is proven similarly to $(a) \Rightarrow (c)$, but now τ-tightness holds *ab initio*; let h be as in Definition 3.3. In the remainder of the proof of $(a) \Rightarrow (c)$ we now substitute $g^\epsilon := g + \epsilon h$, which is a τ_ρ-lower semicontinuous integrand. Letting $\epsilon \to 0$ gives (e). Finally, $(e) \Rightarrow (d)$ is obvious. QED

Results of this kind (but less general) are usually obtained by means of approximation procedures for the lower semicontinuous integrands [31, 26, 3, 38, 5, 10, 56, 57], that are completely avoided here. Another difference is that the present approach directly produces results for *sequential* Young measure convergence.

Theorem 4.8 (characterization of $\Longrightarrow$ by $\overset{K}{\longrightarrow}$) *(i) Let (δ_n) and δ_0 be in $\mathcal{R}(\Omega;S)$. The following are equivalent:*

(a) $\delta_n \overset{\rho}{\Longrightarrow} \delta_0$.

(b) Every subsequence $(\delta_{n'})$ of (δ_n) contains a further subsequence $(\delta_{n''})$ such that $\delta_{n''} \overset{K,\rho}{\longrightarrow} \delta_0$.

(ii) Moreover, if (δ_n) is τ-tight, then the above are also equivalent to

(c) $\delta_n \overset{\tau}{\Longrightarrow} \delta_0$.

(d) Every subsequence $(\delta_{n'})$ of (δ_n) contains a further subsequence $(\delta_{n''})$ such that $\delta_{n''} \overset{K,\tau}{\longrightarrow} \delta_0$.

In parts (b) and (d) the use of subsequences cannot be replaced by the use of the entire sequence (δ_n) itself, simply because a narrowly convergent sequence does not have to K-converge as a whole [18, Example 3.17].

Corollary 4.9 *(i) Let (δ_n) and δ_0 be in $\mathcal{R}(\Omega;S)$. The following are equivalent:*

(a) $\delta_n \overset{\rho}{\Longrightarrow} \delta_0$ *in* $\mathcal{R}(\Omega;S)$.

(b) $\delta_n \times \epsilon_n \overset{\tilde{\rho}}{\Longrightarrow} \delta_0 \times \epsilon_\infty$ *in* $\mathcal{R}(\Omega;\tilde{S})$.

(ii) Moreover, if (δ_n) is τ-tight, then the above are also equivalent to

(c) $\delta_n \overset{\tau}{\Longrightarrow} \delta_0$ *in* $\mathcal{R}(\Omega;S)$.

(d) $\delta_n \times \epsilon_n \overset{\tilde{\tau}}{\Longrightarrow} \delta_0 \times \epsilon_\infty$ *in* $\mathcal{R}(\Omega;\tilde{S})$.

Proof. $(a) \Leftrightarrow (b)$ is immediate by Theorem 4.8 and Corollary 3.12. $(a) \Leftrightarrow (c)$ is contained in Theorems 4.7 and 4.8. $(b) \Leftrightarrow (d)$ is contained in Theorems 4.7 and 4.8, since $(\delta_n \times \epsilon_n)$ is $\tilde{\tau}$-tight if and only if (δ_n) is τ-tight (by compactness of $\hat{\mathbb{N}}$). QED

Transfers of Prohorov's theorem and of the support theorem to Young measure convergence are immediate because of the intermediate results obtained in section 3. The following result is evident by combining Theorem 3.8 and Remark 4.2. See [11] for the topological (i.e., nonsequential) version of this result in precisely the setting of this paper.

Theorem 4.10 (Prohorov's theorem for $\Longrightarrow$) *(i) Let (δ_n) in $\mathcal{R}(\Omega; S)$ be τ_ρ-tight. Then there exist a subsequence $(\delta_{n'})$ of (δ_n) and $\delta_* \in \mathcal{R}(\Omega; S)$ such that $\delta_{n'} \overset{\rho}{\Longrightarrow} \delta_*$.*
(ii) Let (δ_n) in $\mathcal{R}(\Omega; S)$ be τ-tight. Then there exist a subsequence $(\delta_{n'})$ of (δ_n) and $\delta_* \in \mathcal{R}(\Omega; S)$ such that $\delta_{n'} \overset{\tau}{\Longrightarrow} \delta_*$.

Example 4.11 We continue with Example 3.5(b). By $\sigma(E, E')$-tightness of (ϵ_{f_n}) we get from Theorem 4.10 that there exist a subsequence $(f_{n'})$ of (f_n) and $\delta_* \in \mathcal{R}(\Omega; E)$ such that $\epsilon_{f_{n'}} \overset{\tau}{\Longrightarrow} \delta_*$.

(a) We now introduce a function $f_* \in \mathcal{L}^1_E$ that is "barycentrically" associated to δ_*, simply by inspecting the consequences of the tightness inequality $s := \sup_n I_{h_R}(\epsilon_{f_n}) < +\infty$ that was established there. For h_R is *a fortiori* a $\sigma(E, E')$-lower semicontinuous integrand, so Theorem 4.7(e) gives $I_{h_R}(\delta_*) \leq s < +\infty$, which implies $\int_S h_R(\omega, x)\delta_*(\omega)(dx) < +\infty$ for a.e. ω. So by the definition of h_R it follows that both $\delta_*(\omega)(R(\omega)) = 1$ and $\int_E \|x\|\delta_*(\omega)(dx) < +\infty$ for a.e. ω. By standard facts of Bochner integration it follows that the barycenter $f_*(\omega) := \text{bar } \delta_*(\omega)$ of the probability measure $\delta_*(\omega)$ is defined for a.e. ω. Thus, if we set $f_* := 0$ on the exceptional null set, we obtain a function $f_* \in \mathcal{L}^0(\Omega; E)$. Finally we notice that, as announced, f_* is μ-integrable, i.e., $f_* \in \mathcal{L}^1(\Omega; E)$. This follows simply from $I_{h_R}(\delta_*) < +\infty$ by use of Jensen's inequality and the inequality $h_R(\omega, x) \geq \|x\|$.

(b) Suppose that in part (a) one has in addition that $(\|f_{n'}\|)$ is uniformly integrable in $\mathcal{L}^1(\Omega; \mathbb{R})$. Then $f_{n'} \overset{w}{\to} f_* \in \mathcal{L}^1(\Omega; E)$ (weak convergence in $\mathcal{L}^1(\Omega; E)$). This follows directly from another application of Theorem 4.7(e), namely, to all integrands g of the type $g(\omega, x) = \pm < x, b(\omega) >$, $b \in \mathcal{L}^\infty(\Omega, E')[E]$. The latter symbol denotes the set of all scalarly measurable bounded E'-valued functions on Ω; it forms the prequotient dual of $\mathcal{L}^1(\Omega; E)$. This yields $\lim_{n'} I_g(\epsilon_{f_{n'}}) = I_g(\delta_*)$, with $I_g(\epsilon_{f_{n'}}) = J_g(f_{n'}) = \int_\Omega < f_{n'}, b(\omega) > \, d\mu$ and $I_g(\delta_*) = \int_\Omega < f_*, b(\omega) > \, d\mu$.

Part (b) in the above example implies that $f_n \overset{w}{\to} f_0$ in Example 4.4, where f_0 is the constant function given by $f_0(\omega) := \text{bar } \delta_0(\omega) = \int_\mathbb{R} f_1 \, d\lambda_1$ (apply [35, II.12]). Concatenation of Theorem 3.13 and Theorem 4.8 gives immediately the following result:

Theorem 4.12 (support theorem for $\Longrightarrow$) *(i) Let $\delta_n \overset{\rho}{\Longrightarrow} \delta_0$ in $\mathcal{R}(\Omega; S)$. Then*

$$\tau_\rho\text{-supp } \delta_0(\omega) \subset \tau_\rho\text{-Ls}_n \tau_\rho\text{-supp } \delta_n(\omega) \text{ for a.e. } \omega \text{ in } \Omega.$$

(ii) Moreover, if (δ_n) is τ-tight, then $\delta_n \overset{\tau}{\Longrightarrow} \delta_0$ in $\mathcal{R}(\Omega; S)$ and

$$\delta_0(\omega)(\tau\text{-seq-cl } \tau\text{-Ls}_n \tau\text{-supp } \delta_n(\omega)) = 1,$$

$$\tau\text{-supp } \delta_0(\omega) \subset \tau\text{-cl } \tau\text{-Ls}_n \tau\text{-supp } \delta_n(\omega) \text{ for a.e. } \omega \text{ in } \Omega.$$

The following so-called *lower closure* theorem for Young measures forms a combination of the main relative compactness, lower semicontinuity and support results of the present section. Let (D, d_D) be a metric space.

Theorem 4.13 (fundamental lower closure theorem) *Let (δ_n) in $\mathcal{R}(\Omega; S)$ be τ-tight and let $d_n \overset{\mu}{\to} d_0$ in $\mathcal{L}^0(\Omega; D)$ (convergence in measure). Then there exist a subsequence $(\delta_{n'})$ of (δ_n) and δ_* in $\mathcal{R}(\Omega; S)$ such that*

$$\liminf_{n'} \int_\Omega^* [\int_S \ell(\omega, x, d_{n'}(\omega))\delta_{n'}(\omega)(dx)]\mu(d\omega) \geq \int_\Omega^* [\int_S \ell(\omega, x, d_0(\omega))\delta_*(\omega)(dx)]\mu(d\omega)$$

for every sequentially $\tilde{\tau}$-sequentially lower semicontinuous integrand ℓ on $\Omega \times S \times D$) such that

$$s'(\alpha) := \sup_n \int_\Omega^* [\int_{\{\ell \leq -\alpha\}_{\omega,n}} \ell^-(\omega, x, d_n(\omega))\delta_n(\omega)(dx)]\mu(d\omega) \to 0 \text{ for } \alpha \to \infty. \tag{4.1}$$

More precisely, we have $\delta_{n'} \xrightarrow{K,\tau} \delta_*$, causing δ_* to be supported as follows

$$\delta_*(\omega)(\tau\text{-seq-cl } \tau\text{-Ls}_n \tau\text{-supp } \delta_n(\omega)) = 1 \text{ for a.e. } \omega \text{ in } \Omega.$$

Here $\{\ell \leq -\alpha\}_{\omega,n}$ stands for the set of all $x \in S$ for which $\ell(\omega, x, d_n(\omega)) \leq -\alpha$.

Proof. Theorem 3.8 and well-known facts about convergence in measure [28, Theorem 20.5] imply existence of a subsequence $(\delta_{n'}, d_{n'})$ of (δ_n, d_n) and existence of $\delta_* \in \mathcal{R}(T; S)$ such that $\delta_{n'} \xrightarrow{K,\tau} \delta_*$ in $\mathcal{R}(\Omega; S)$ and $d_D(d_{n'}(\omega), d_0(\omega)) \to 0$ for a.e. ω. *A fortiori* this gives $\delta_{n'} \overset{\tau}{\Longrightarrow} \delta_*$ (by Remark 4.2). By Theorem 4.12 this gives the desired pointwise support property for δ_*. By Corollary 4.9, we also have $\tilde{\delta}_{n'} \overset{\tilde{\tau}}{\Longrightarrow} \tilde{\delta}_*$ in $\mathcal{R}(\Omega; \tilde{S})$, with $\tilde{\delta}_n := \delta_n \times \epsilon_n$ and $\tilde{\delta}_* := \delta_* \times \epsilon_\infty$ Without loss of generality we discard renumbering and pretend that (n') enumerates *all* the numbers in $\mathbb{N}$. For ℓ as stated we observe that for each $n' \in \mathbb{N}$ the following identity holds

$$\int_{\tilde{S}} g_\ell(\omega, \tilde{x})\tilde{\delta}_{n'}(\omega)(d\tilde{x}) = \int_S \ell(\omega, x, d_{n'}(\omega))\delta_{n'}(\omega)(dx),$$

and it continues to hold for $n' = \infty$ if we write $d_\infty := d_0$ and $\delta_\infty := \delta_*$. Here $g_\ell(\omega, (x, k)) := \ell(\omega, x, d_k(\omega))$ defines a $\tilde{\tau}$-lower semicontinuous integrand g_ℓ on $\Omega \times \tilde{S}$ (modulo an insignificant null set). Note in particular that for $k = \infty$ lower semicontinuity of $g_\ell(\omega, \cdot)$ at (x, ∞) follows from $d_{n'}(\omega) \to d_0(\omega)$ and lower semicontinuity of $\ell(\omega, \cdot, \cdot)$ at $(x, d_0(\omega))$. Thus, the desired inequality is contained in $\liminf_{n'} I_{g_\ell}(\tilde{\delta}_{n'}) \geq I_{g_\ell}(\tilde{\delta}_*)$, a result that follows by applying Theorem 4.7 to g_ℓ (observe here that (4.1) coincides with (3.1) for $g = g_\ell$). QED

Remark 4.14 *Let h be the nonnegative, sequentially τ-inf-compact integrand h on $\Omega \times S$ that corresponds as in Definition 3.3 to the τ-tight sequence (δ_n) in Theorem 4.13; i.e., with $s := \sup_n I_h(\delta_n) < +\infty$. Then the uniform integrability condition (4.1) applies whenever the integrand ℓ has the following growth property with respect to h: for every $\epsilon > 0$ there exists $\phi_\epsilon \in \mathcal{L}^1(\Omega; \mathbb{R})$ such that for every $n \in \mathbb{N}$*

$$\ell^-(\omega, x, d_n(\omega)) \leq \epsilon h(\omega, x) + \phi_\epsilon(\omega) \text{ on } \Omega \times \times S.$$

Indeed, we can observe that the set $\{\ell \leq -\alpha\}_{\omega,n}$ in (4.1) is contained in the union of $\{\phi_\epsilon < \epsilon h\}$ and $\{\phi_\epsilon \geq \alpha/2\}$, which gives $s'(\alpha) \leq 3\epsilon s + \int_{\{\phi_\epsilon \geq \alpha/2\}} \phi_\epsilon \, d\mu$, whence $s'(\alpha) \to 0$ for $\alpha \to \infty$, as claimed.

5 Some applications to lower closure and denseness

We illustrate the power of the above apparatus by some applications to a variety of problems; we refer to [18, 22] for more extensive expositions.

As our first application, we derive an extension of the so-called fundamental theorem for Young measures in [25]. Here L is a locally compact space that is countable at infinity; its usual Alexandrov compactification is denoted by $\hat{L} := L \cup \{\infty\}$. The space $\hat{L}$ is metrizable, and its metric is denoted by $\hat{d}$. On L we use the natural restriction of $\hat{d}$, and denote it by d. Let $\mathcal{C}_0(L)$ be the usual space of continuous functions on L that converge to zero at infinity. Although it could be avoided by the additional introduction of transition subprobabilities (see the comments below), the Alexandrov compactification $\hat{L}$ of L figures explicitly in the result. Also, below ν denotes a σ-finite measure on $(\Omega, \mathcal{A})$.

Corollary 5.1 *(i) Let (f_n) in $\mathcal{L}^0(\Omega; L)$ and the closed set $C \subset L$ be such that $\lim_n \nu(f_n^{-1}(L \backslash G))$ $= 0$ for every open G, $C \subset G \subset L$. Then there exist a subsequence $(f_{n'})$ of (f_n) and δ_* in $\mathcal{R}(\Omega; \hat{L})$ such that $\delta_*(\omega)(L \backslash C) = 0$ for a.e. ω in Ω and*

$$\lim_n \int_\Omega \phi(\omega) c(f_{n'}(\omega)) \nu(d\omega) = \int_\Omega [\int_L \phi(\omega) c(x) \delta_*(\omega)(dx)] \nu(d\omega)$$

for every $\phi \in \mathcal{L}^1(\Omega; \mathbb{R})$ and every $c \in \mathcal{C}_0(L)$.
(ii) Moreover, if for that subsequence there exists a sequence (K_r) of compact sets in L such that $\lim_{r\to\infty} \sup_{n'} \nu(\{\omega \in \Omega : f_{n'}(\omega) \notin K_r\} = 0$ then $\delta_*(\omega)(\{\infty\}) = 0$ for a.e. ω in Ω and

$$\lim_n \int_A \phi(\omega) c(f_{n'}(\omega)) \nu(d\omega) = \int_A [\int_L \phi(\omega) c(x) \delta_*(\omega)(dx)] \nu(d\omega)$$

for every $A \in \mathcal{A}$, $\phi \in \mathcal{L}^1(A; \mathbb{R})$ and $c \in \mathcal{C}(L)$ for which $(1_A c(f_{n'}))$ is relatively weakly compact in $\mathcal{L}^1(A; \mathbb{R})$.

In [25] both L and Ω are Euclidean, and the K_r's are closed balls around the origin with radius r. As was done in [25], the result could be equivalently restated in terms of the transition *sub*probability δ'_* from $(\Omega, \mathcal{A})$ into $(L, \mathcal{B}(L))$, defined by obvious restriction to L, i.e., $\delta'_*(\omega)(B) := \delta_*(\omega)(B \cup \{\infty\})$, $B \in \mathcal{B}(L)$. In this connection the tightness condition in part *(ii)* guarantees that δ_* is an authentic transition probability (Young measure). Rather than via *(i)*, part *(ii)* could also have been derived directly from Theorem 3.8 or 4.13.
Proof. *(i)* By σ-finiteness of ν, there exists a finite measure μ that is equivalent to ν. Let $\tilde{\phi}$ be a version of the Radon-Nikodym density $d\nu/d\mu$. Now (δ_n), defined by $\delta_n := \epsilon_{f_n} \in \mathcal{R}(\Omega, ; \hat{L})$, is trivially tight by compactness of $\hat{L}$ (set $h \equiv 0$). By Theorem 3.8 or 4.13 (with $S := \hat{L}$, $\rho := \hat{d}$), there exist a subsequence $(f_{n'})$ of (f_n) and δ_* in $\mathcal{R}(\Omega; \hat{L})$ for which $\epsilon_{f_{n'}} \stackrel{\rho}{\Longrightarrow} \delta_*$ (and even $\epsilon_{f_{n'}} \stackrel{K,\rho}{\longrightarrow} \delta_*$). Every $c \in \mathcal{C}_0(L)$ has a canonical extension $\hat{c} \in \mathcal{C}_b(S)$ by setting $\hat{c}(\infty) = 0$. Now $\phi\tilde{\phi}$ is μ-integrable for any $\phi \in \mathcal{L}^1(\Omega, \mathcal{A}, \nu; \mathbb{R})$, and Theorem 4.7(*c*) (or 4.13) can be applied to $g : \Omega \times \hat{L} \to \mathbb{R}$ given by $g(\omega, x) :=$ $\pm\phi(\omega)\tilde{\phi}(\omega)\hat{c}(x)$. This gives the desired equality, because of the identity $\int_\Omega \phi\tilde{\phi} \int_L \hat{c}(x)\delta_*(\cdot)(dx)d\mu =$ $\int_\Omega \phi \int_L c(x)\delta_*(\cdot)(dx)d\nu$.

Next, let C be as stated. For any $i \in \mathbb{N}$ the set F_i, consisting of all $x \in L$ with d-dist$(x, C) \leq i^{-1}$, is closed in L. Note already that $\cap_i F_i = C$, by the given τ_d-closedness of C in L. Further, $\hat{F}_i :=$ $F_i \cup \{\infty\}$ is closed in $\hat{L}$. Set $\hat{g}_i(\omega, x) := \tilde{\phi}(\omega) 1_{S \backslash \hat{F}_i}(x)$. This defines a nonnegative lower semicontinuous integrand $\hat{g}_i$ on $\Omega \times \hat{L}$. Hence, $I_{\hat{g}_i}(\delta_*) \leq \beta_i := \liminf_{n'} I_{\hat{g}_i}(\epsilon_{f_{n'}})$ by Theorem 4.7(*c*). By $S \backslash \hat{F}_i =$ $L \backslash F_i$, the definitions of $\hat{g}_i$ and $\epsilon_{f_{n'}}$ give $I_{\hat{g}_i}(\epsilon_{f_{n'}}) = \nu(f_{n'}^{-1}(L \backslash F_i))$. So $\beta_i = \liminf_{n'} \nu(f_{n'}^{-1}(L \backslash F_i))$ $\leq \nu(f_{n'}^{-1}(L \backslash G_i))$, where G_i, $G_i \subset F_i$, is the τ_d-open set of all $x \in L$ with d-dist$(x, C) < i^{-1}$. Since $G_i \supset C$, the hypotheses imply $0 = \beta_i \geq I_{\hat{g}_i}(\delta_*) = \int_\Omega \delta_*(\cdot)(L \backslash F_i) d\nu$. Hence $\delta_*(\omega)(L \backslash C) = 0$ ν-a.e. because of $\cap_i F_i = C$, which was demonstrated above.

(ii) The additional condition is a tightness condition for (ϵ_{f_n}), when viewed as a subset of $\mathcal{R}(\Omega; L)$ (take $\Gamma_\epsilon \equiv K_r$ in Remark 3.4). Hence, there is a τ_ρ-inf-compact integrand h on $\Omega \times L$ with $\sup_n I_h(\delta_n) < +\infty$. Now define the inf-compact integrand $\hat{h}$ on $\Omega \times \hat{L}$ by $\hat{h}(\omega, x) := h(\omega, x)$ if $x \in L$ and $\hat{h}(\omega, \infty) := +\infty$. Then $I_{\hat{h}}(\delta_*) \leq \liminf_{n'} I_{\hat{h}}(\epsilon_{f_{n'}}) < +\infty$ by Theorem 4.7(*c*). Hence, $\delta_*(\cdot)(\{\infty\}) = 0$ μ-a.e., whence ν-a.e. Finally, for any $A \in \mathcal{A}$ with $\nu(A) < +\infty$ Theorem 4.7(*c*) applies to $g(\omega, x) := \pm 1_A(\omega)\phi(\omega)\tilde{\phi}(\omega)\hat{c}(x)$. This gives the desired limit statement. If $\nu(A) = +\infty$ and A is as stated, there exists a sequence (A_j) of subsets of A with finite ν-measure, with $A_j \uparrow A$. The previous result applies to each of the A_j and the weak relative compactness hypothesis implies uniform σ-additivity [30], so also in this case the desired limit statement follows. QED

Next, let E and F be separable Banach spaces, each equipped with a locally convex Hausdorff topology, respectively denoted by τ_E and τ_F, that is not weaker than the weak topology and not stronger than the norm topology. Let (D, d_D) be a metric space. Functions that are "barycentrically" associated to Young measures can play a special role in lower closure and existence results. This is demonstrated by our proof of the following result.

Theorem 5.2 *Let $d_n \xrightarrow{\mu} d_0$ in $\mathcal{L}^0(\Omega; D)$ (convergence in measure), $e_n \xrightarrow{w} e_0$ in $\mathcal{L}^1(\Omega; E)$ (weak convergence), and let (f_n) in $\mathcal{L}^1(\Omega; F)$ satisfy $\sup_n \int_\Omega \|f_n\|_F \, d\mu < +\infty$. Suppose that there exist τ_E- and τ_F-ball-compact multifunctions $R_E : \Omega \to 2^E$ and $R_F : \Omega \to 2^F$ such that*

$$\{(e_n(\omega), f_n(\omega)) : n \in \mathbb{N}\} \subset R_E(\omega) \times R_F(\omega)\mu\text{-a.e.}$$

Then there exist a subsequence $(d_{n'}, e_{n'}, f_{n'})$ of (d_n, e_n, f_n) and $f_* \in \mathcal{L}^1(\Omega; F)$ such that

$$\liminf_{n'} \int_\Omega^* \ell(\omega, e_{n'}(\omega), f_{n'}(\omega), d_{n'}(\omega))\mu(d\omega) \geq \int_\Omega^* \ell(\omega, e_0(\omega), f_*(\omega), d_0(\omega))\mu(d\omega)$$

for every sequentially $\tau_E \times \tau_F \times \tau_D$-lower semicontinuous integrand ℓ on $\Omega \times (E \times F \times D)$ such that the following hold:

$$(\ell^-(\cdot, e_n(\cdot), f_n(\cdot), d_n(\cdot))) \text{ is uniformly (outer) integrable,}$$

$$\ell(\omega, \cdot, \cdot, d_0(\omega)) \text{ is convex on } E \times F \text{ for a.e. } \omega.$$

Moreover, the functions e_0 and f_* can be localized as follows: [2]

$$(e_0(\omega), f_*(\omega)) \in \text{cl co-w-Ls}_n\{(e_n(\omega), f_n(\omega))\} \text{ for a.e. } \omega \text{ in } \Omega.$$

Observe, as was already done following Example 3.5, that the ball-compactness condition involving R_E and R_F is automatically satisfied in case the Banach spaces E and F are reflexive.

Proof. To apply Theorem 4.13 we set $S := E \times F$, $\tau := \tau_E \times \tau_F$ and $\delta_n := \epsilon_{(e_n, f_n)}$. Then S is completely regular (by the Hahn-Banach theorem) and Suslin. Note that $(\|e_n\|)$ in $\mathcal{L}^1(\Omega; \mathbb{R})$ is uniformly integrable by [30, Theorem 1] and [45, Proposition II.5.2]. In particular, this implies $\sup_n \int_\Omega \|(e_n, f_n)\|_S \, d\mu < +\infty$. This proves that (δ_n) is τ-tight, in view of Example 3.5(b). We can now apply Theorem 4.13: let the subsequence $(\delta_{n'}, d_{n'})$ of (δ_n, d_n) and δ_* in $\mathcal{R}(\Omega; S)$ be as guaranteed by that theorem, i.e., with $\delta_{n'} \stackrel{\tau}{\Longrightarrow} \delta_*$ (and even $\delta_{n'} \xrightarrow{K,\tau} \delta_*$). Then it is elementary by Definition 4.1 that, "E-marginally", $\epsilon_{e_{n'}} \stackrel{\tau}{\Longrightarrow} \delta_*^E$ and, "F-marginally", $\epsilon_{f_{n'}} \stackrel{\tau}{\Longrightarrow} \delta_*^F$. Here $\delta_*^E(\omega) := \delta_*(\omega)(\cdot \times F)$, etc. Then E-marginally Example 4.11(b) applies, which gives that bar $\delta_*^E = e_0$ a.e. Also, F-marginally Example 4.11(a) applies, giving the existence of $f_* \in \mathcal{L}^1(\Omega; F)$ such that $f_* =$ bar δ_*^F a.e. (note that τ_E- and τ_F -ball-compactness imply $\sigma(E, E')$- and $\sigma(F, F')$-ball-compactness respectively). Recombining the above two marginal cases, we find bar $\delta_* = (e_0, f_*)$ a.e. (note that barycenters decompose marginally).

We now finish the proof. For an integrand ℓ of the stated variety Theorem 4.13 gives

$$\beta \geq \int_\Omega^* [\int_{E \times F} \ell(\omega, x, y, d_0(\omega))\delta_*(\omega)(d(x,y))]\mu(d\omega),$$

where $\beta := \liminf_{n'} \int_\Omega^* \ell(\omega, e_{n'}(\omega), f_{n'}(\omega), d_{n'}(\omega))\mu(d\omega)$. In the inner integral of the above inequality the convexity of $\ell(\omega, \cdot, \cdot, d_0(\omega))$ gives

$$\int_{E \times F} \ell(\omega, x, y, d_0(\omega))\delta_*(\omega) \geq g(\omega, \text{bar } \delta_*(\omega), d_0(\omega)) = g(\omega, e_0(\omega), f_*(\omega), d_0(\omega))$$

for a.e. ω, by Jensen's inequality and our previous identity bar $\delta_* = (e_0, f_*)$ a.e. The desired inequality thus follows. QED

The above lower closure result "with convexity" is quite general: it further extends the results in [5, 8], which in turn already generalize several lower closure results in the literature, including those for orientor fields (cf. [33]). See [15] for another development, not covered by the above result. Results of this kind are very useful in the existence theory for optimal control and optimal growth

[2] In case E and F are finite-dimensional one may replace here "cl co" by "co".

theory; e.g., see [33, 15]. Recently, similar-spirited versions that employ quasi-convexity in the sense of Morrey have been given in [42, 52] (these have for e_n the gradient function of d_n and depend on a characterization of so-called gradient Young measures [40, 48]). Corollaries of Theorem 5.2 are so-called weak-strong lower semicontinuity results for integral functionals in the calculus of variations and optimal growth theory; cf. [29, 33, 37]. Another immediate corollary would be [1, Proposition C], which is obtained by activating the footnote in the statement of Theorem 5.2.

Next, we give a lower closure result "without convexity". As in previous sections, (S, τ) is a completely regular Suslin space.[3]

Theorem 5.3 (Lyapunov's theorem for Young measures) *Suppose that $(\Omega, \mathcal{A}, \mu)$ is non-atomic. Let $g := (g_1, \ldots, g_d) : \Omega \times S \to \mathbb{R}^d$ be $\mathcal{A} \times \mathcal{B}(S)$-measurable and let $\delta \in \mathcal{R}(\Omega; S)$ be such that $I_{|g|}(\delta) < +\infty$. Then there exists $f \in \mathcal{L}^1(\Omega; S)$ such that $J_{g_i}(f) = I_{g_i}(\delta)$ for $i = 1, \ldots, d$ and $f(\omega) \in \text{supp } \delta(\omega)$ for a.e. ω in Ω.*

Corollary 5.4 *Suppose that $(\Omega, \mathcal{A}, \mu)$ is nonatomic. Let $\delta \in \mathcal{R}(\Omega; \mathbb{R}^d)$ be such that $I_{|\cdot|}(\delta) < +\infty$. Then there exists $f \in \mathcal{L}^1(\Omega; \mathbb{R}^d)$ such that $\int_\Omega f \, d\mu = \int_\Omega \text{bar } \delta \, d\mu$ and $f(\omega) \in \text{supp } \delta(\omega)$ for a.e. ω in Ω.*

Here $I_{|\cdot|}(\delta) := \int_\Omega [\int_{\mathbb{R}^d} |x| \delta(\omega)(dx)] \mu(d\omega) < +\infty$. The corollary follows by applying Theorem 5.3 to $S := \mathbb{R}^d$ and $g_i(\omega, x) := x^i$ (i-th coordinate function).

Proof of Theorem 5.3. Denote $\Gamma(\omega) := \text{supp } \delta(\omega)$. By [47, Lemma] we have $p(\omega) := \int_S (|g(\omega, x)|, g(\omega, x)) \delta(\omega)(dx) \in \text{co } \{(|g(\omega, x)|, g(\omega, x)) : x \in \Gamma(\omega)\}$ for a.e. ω in Ω. The closed-valued multifunction $\Gamma : \Omega \to 2^S$ is measurable in the standard sense [32, III.9, III.10]), because for any open $U \subset S$ the set of all ω with $\Gamma(\omega) \cap U \neq \emptyset$ is precisely $\{\omega \in \Omega : \delta(\omega)(U) \neq 0\} \in \mathcal{A}$. So by Carathéodory's theorem and an obvious application of the implicit measurable selection theorem [32, Theorem III.38] there exist $\mathcal{A}$-measurable functions $\alpha_1, \ldots, \alpha_{d+2} : \Omega \to [0, 1]$, with $\sum_{i=1}^{d+2} \alpha_i(\omega) = 1$ for all ω, and $\mathcal{A}$-measurable selections $s_1, \ldots, s_{d+2} : \Omega \to S$ of Γ such that $p(\omega) = \sum_{i=1}^{d+2} \alpha_i(\omega)(|g(\omega, s_i(\omega))|, g(\omega, s_i(\omega)))$ for a.e. ω in Ω. Integration over ω in the first component of this identity gives $\int_\Omega \sum_i \alpha_i |g(\cdot, s_i(\cdot))| < +\infty$. Hence, by an extension of Lyapunov's theorem [17, Proposition 3.2] (see also [18, 22] for a much simpler proof), there exists a measurable partition $B_1, \ldots, B_{d+2}$ of Ω such that each $g(\cdot, s_i(\cdot))$ is integrable over B_i and $\int_\Omega \sum_i \alpha_i (|g(\cdot, s_i(\cdot))|, g(\cdot, s_i(\cdot))) = \sum_i \int_{B_i} (|g(\cdot, s_i(\cdot))|, g(\cdot, s_i(\cdot)))$. We define $f \in \mathcal{L}^1(\Omega; S)$ by setting $f := s_i$ on B_i, $i = 1, \ldots, d+2$. Then, f is evidently an a.e. selection of Γ and if we integrate over ω in the last d coordinates of the above identity for $p(\omega)$ we obtain $\int_\Omega \sum_i \alpha_i g(\cdot, s_i(\cdot)) = \sum_i \int_{B_i} g(\cdot, s_i(\cdot)) = \int_\Omega g(\cdot, f(\cdot))$. This is the desired identity, for the right hand side equals $(J_{g_1}(f), \ldots, J_{g_d}(f))$ and by the definition of $p(\omega)$ the left hand side is equal to $(I_{g_1}(\delta), \ldots, I_{g_d}(\delta))$. QED

The following lower closure result "without convexity" comes from [5]; it subsumes the result given in [1] and the original "Fatou lemma in several dimensions" that is due to Schmeidler [50]. See [23] for further generalizations which involve multifunctions with unbounded values and associated asymptotic correction terms.

Theorem 5.5 (Fatou-Vitali in several dimensions) *Let (f_n) in $\mathcal{L}^1(\Omega; \mathbb{R}^d)$ be such that both $a := \lim_n \int_\Omega f_n \, d\mu$ exists and $((f_n^i)^-))_n$ is uniformly integrable for $i = 1, \ldots, d$. Then there exists $f_* \in \mathcal{L}^1(\Omega; \mathbb{R}^d)$ such that $\int_\Omega f_* \, d\mu \leq a$ and $f_*(\omega) \in \text{Ls}_n\{f_n(\omega)\}$ for a.e. ω in Ω.*

Proof. It is easy to see from the conditions that the sequence (f_n) is bounded in $\mathcal{L}^1$-seminorm. As usual, $(\Omega, \mathcal{A}, \mu)$ can be decomposed in a nonatomic part and a purely atomic part. The latter is the union of at most countably many μ-atoms A_j. Since each of the f_n is constant a.e. on each A_j, the desired f_* follows on the purely atomic part of Ω by the obvious extraction of a diagonal

[3]Correction: As the theorem stands, S should be supposed metrizable Suslin. Only if one omits the last part of its statement (involving the support), S can be as stated above. I am grateful to F. Martins-da-Rocha (Paris) for pointing this out.

subsequence [5]. So essentially without loss of generality we can assume that $(\Omega, \mathcal{A}, \mu)$ is nonatomic. Since $(\epsilon_{f_n}))$ is tight by Example 3.5, we can apply Theorem 4.13 with $D := \hat{\mathbb{N}}$, $d_D := \hat{\rho}$, $d_n := n$ and $d_0 := \infty$. Let $(\epsilon_{f_{n'}})$ and $\delta_* \in \mathcal{R}(\Omega; \mathbb{R}^d)$ be as in that theorem. The pointwise support property for δ_* in Theorem 4.13 gives $\delta_*(\omega)(L(\omega)) = 1$ a.e., where $L(\omega) := \mathrm{Ls}_m\{f_m(\omega)\}$. Also, Theorem 4.7, applied to $g(\omega, x) := |x|$ and $g_i(\omega, x, n) := x^i$, gives $I_{|\cdot|}(\delta_*) \leq \liminf_n \int_\Omega |f_n| < +\infty$ and $\int_\Omega \mathrm{bar}\ \delta_* d\mu \leq a$. Hence, we may invoke Corollary 5.4: there exists $f_* \in \mathcal{L}^1(\Omega; \mathbb{R}^d)$ such that $\int_\Omega f_*\, d\mu = \int_\Omega \mathrm{bar}\ \delta_*\, d\mu$ and $f_*(\omega) \in \mathrm{supp}\ \delta_*(\omega) \subset L(\omega)$ a.e. QED

The above lower closure result can be used efficiently to address a number of existence problems "without convexity" in optimal control theory; e.g., cf. [6, 18, 22]. A more general approach to existence without convexity (based on the the extreme point role of Dirac young measures, not treated here) can be found in [16].

A close relationship exists between the above subject of lower closure without convexity and the classical denseness of Dirac Young measures (e.g., cf.[59]). The following very general denseness result was given in [7]:

Theorem 5.6 (denseness of Dirac Young measures) *Suppose that $(\Omega, \mathcal{A}, \mu)$ is nonatomic. Let $g := (g_1, \ldots, g_d) : \Omega \times S \to \mathbb{R}^d$ be $\mathcal{A} \times \mathcal{B}(S)$-measurable and let $\delta \in \mathcal{R}(\Omega; S)$ be such that $I_{|g|}(\delta) < +\infty$. Then there exists a sequence (f_n) in $\mathcal{L}^0(\Omega; S)$ such that $\epsilon_{f_n} \stackrel{\rho}{\Longrightarrow} \delta$ and for every n both $J_{g_k}(f_n) = I_{g_k}(\delta)$, $k = 1, \ldots, d$, and $f_n(\omega) \in \mathrm{supp}\ \delta(\omega)$ for a.e. ω in Ω.*

Proof. Recall from what followed Definition 2.1 that $\mathcal{P}(S)$ is metrizable Suslin for $\mathcal{T}_\rho$. So the σ-algebra generated on Ω by $\delta : \Omega \to \mathcal{P}(S)$ is countably generated. In conjunction with a well-known trick [32, p. 78] (see also [54, Appendix]), this shows that there exists a countably generated sub-σ-algebra $\mathcal{A}_0$ of $\mathcal{A}$ such that the given δ belongs to $\mathcal{R}_0 := \mathcal{R}(\Omega, \mathcal{A}_0; S)$ and such that $g_1, \ldots, g_d$ are $\mathcal{A}_0 \times \mathcal{B}(S)$-measurable. By Theorem 4.6 there exists a semimetric d on $\mathcal{R}_0$, given by $d_{\mathcal{R}}(\delta, \delta') := \sum_{i=1}^\infty \sum_{j=1}^\infty 2^{-i-j} | \int_{A_j} [\int_S c_i(x) \delta(\omega)(dx)] \mu(d\omega) - \int_{A_j} [\int_S c_i(x) \delta'(\omega)(dx)] \mu(d\omega)| / \mu(A_j)$. Define $g_{i,j}(\omega, x) := 1_{A_j}(\omega) c_i(x)$. For every $n \in \mathbb{N}$ there exists by Theorem 5.3 $f_n \in \mathcal{L}^0(\Omega; S)$ such that $J_{g_{i,j}}(f_n) = I_{g_{i,j}}(\delta)$ for all $1 \leq i \leq n$, $1 \leq j \leq n$ and $J_{g_k}(f_n) = I_{g_k}(\delta)$, $k = 1, \ldots, d$ and $f_n \in \mathrm{supp}\ \delta$ a.e. For the sequence (f_n) thus created we clearly have $d_{\mathcal{R}}(\epsilon_{f_n}, \delta) \to 0$. QED

The following "limiting bang-bang" result, which generalizes [54, 55], serves to underline the power of the results obtained thus far. This result is also related to L^p-Young measures; cf. [18, 20].

Corollary 5.7 *Suppose that $(\Omega, \mathcal{A}, \mu)$ is nonatomic. Let $\delta \in \mathcal{R}(\Omega; \mathbb{R}^d)$ be such that $I_{|\cdot|}(\delta) < +\infty$. Then there exists a sequence (f_n) in $\mathcal{L}^1(\Omega; \mathbb{R}^d)$ such that $f_n \stackrel{w}{\to} \mathrm{bar}\ \delta$ (weak convergence in $\mathcal{L}^1(\Omega; \mathbb{R}^d)$) and $\mathrm{Ls}_n\{f_n(\omega)\} = \mathrm{supp}\ \delta(\omega)$ for a.e. ω in Ω.*

In particular, let $f_1, \ldots, f_r$ be functions in $\mathcal{L}^0(\Omega; \mathbb{R}^d)$ and let $\alpha_1, \ldots, \alpha_r$ be nonnegative functions in $\mathcal{L}^\infty(\Omega; \mathbb{R})$, with $\sum_{i=1}^r \alpha_i = 1$, and such that $\int_\Omega \sum_{i=1}^r \alpha_i |f_i|\, d\mu < +\infty$. Then there exists a sequence (f_n) in $\mathcal{L}^1(\Omega; \mathbb{R}^d)$ such that $f_n \stackrel{w}{\to} \sum_{i=1}^r \alpha_i f_i$ and $\mathrm{Ls}_n\{f_n(\omega)\} = \{f_1(\omega), \ldots, f_r(\omega)\}$ for a.e. ω in Ω.

Proof. Let ν be the finite measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ defined by $\nu := [\mu \otimes \delta](\Omega \times \cdot)$. Then $\int_{\mathbb{R}^d} |x| \nu(dx) < +\infty$, so by de la Vallée Poussin's theorem [35, II.22] there exists $h' : \mathbb{R}^d \to \mathbb{R}_+$, continuous, convex, nondecreasing and superlinear, such that $\int_{\mathbb{R}^d} h'(|x|) \nu(dx) < +\infty$. This amounts to $I_h(\delta) < +\infty$ when we set $h(\omega, x) := h'(|x|)$. By Theorem 5.6 there exists a sequence (f_m) in $\mathcal{L}^0(\Omega; \mathbb{R}^d)$ such that $\epsilon_{f_m} \Longrightarrow \delta$, $J_h(f_m) = I_h(\delta$ and $f_m \in \mathrm{supp}\ \delta$ a.e. In particular, the latter implies $\mathrm{Ls}_n\{f_n(\omega)\} \subset \mathrm{supp}\ \delta(\omega)$ a.e. by closedness of the support. By the converse part of de la Vallée Poussin's theorem [35, II.12] and the Dunford-Pettis criterion [45, IV.2.3], the identity implies that (f_m) contains a weakly converging subsequence (f_n). It then is obvious from Theorem 4.7 that $f_n \stackrel{w}{\to} \mathrm{bar}\ \delta$ (cf. Example 4.11). Also, by the support Theorem 4.12 we get $\mathrm{supp}\ \delta(\omega) \subset \mathrm{Ls}_n\{f_n(\omega)\}$ for a.e. ω in Ω. QED

In [18] it has been shown that, following [26], the present approach can also be used to obtain some rather general results on "functional relaxation" of integral functionals by means of the method of "Young measure relaxation". In this way e.g. the principal relaxation result of [36], as improved in [55], was improved a little further in [18, ch. 9].

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