## Exercise HW1

(a) Show that $\mathrm{GL}(n,\mathbb{R}))$ is dense in $\mathrm{Mat}(\mathbb{R}^n)$. Hint: let $A \in \mathrm{End}(\mathbb{R}^n)$ and consider the function $t \mapsto \det(A+tI)$.

(b) We consider the function $R : \mathrm{Mat}(n,\mathbb{R}) \to \mathbb{R}$ given by

$$R(H) = \det(I+H) - 1 - \mathrm{trace}(H).$$

Show that there exists a $C > 0$ such that $|R(H)| \leq C\|H\|^2$ for all $H \in \mathrm{Mat}(n,\mathbb{R})$ with $\|H\| \leq 1$.

(c) Show that the function $f : \mathrm{Mat}(n,\mathbb{R}) \to \mathbb{R}$ given by $f(X) = \det X$ is differentiable at $I$. Show that the associated derivative $Df(I)$ is equal to the linear map $\mathrm{Mat}(n,\mathbb{R}) \to \mathbb{R}$ given by

$$Df(I) : H \mapsto \mathrm{trace}(H).$$

(d) If $A \in \mathrm{GL}(n,\mathbb{R})$ show that $f$ is differentiable at $A$ and that

$$Df(A)(H) = \mathrm{tr}\,(A^{\#}H), \qquad (H \in \mathrm{Mat}(n,\mathbb{R})).$$

Here $A^{\#}$ is the complementary matrix which appears in Cramer's rule.

(e) Show that the result of (d) is true for any $A \in \mathrm{Mat}(n,\mathbb{R})$.