Extra exercises Analysis in several variables

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Exercise 1.

(a) Show that $\mathrm{GL}(n, \mathbb{R}))$ is dense in $\mathrm{M}(n, \mathbb{R}) := \mathrm{M}(n \times n, \mathbb{R})$. Hint: let $A \in \mathrm{End}\,(\mathbb{R}^n)$ and consider the function $t \mapsto \det\,(A + tI)$.

(b) We consider the function $R : \mathrm{M}(n, \mathbb{R}) \to \mathbb{R}$ given by

$$R(H) = \det\,(I + H) - 1 - \mathrm{trace}(H).$$

Show that there exists a $C > 0$ such that $|R(H)| \leq C\|H\|^2$ for all $H \in \mathrm{M}(n, \mathbb{R})$ with $\|H\| \leq 1$.

(c) Show that the function $f : \mathrm{M}(n, \mathbb{R}) \to \mathbb{R}$ given by $f(X) = \det X$ is differentiable at I. Show that the associated derivative $Df(I)$ is equal to the linear map $\mathrm{M}(n, \mathbb{R}) \to \mathbb{R}$ given by

$$Df(I) : H \mapsto \mathrm{trace}(H).$$

(d) If $A \in \mathrm{GL}(n, \mathbb{R})$ show that f is differentiable at A and that

$$Df(A)(H) = \mathrm{tr}\,(A^{\#} H), \qquad (H \in \mathrm{M}(n, \mathbb{R})).$$

Here $A^{\#}$ is the complementary matrix which appears in Cramer's rule.

(e) Show that the result of (d) is true for any $A \in \mathrm{M}(n, \mathbb{R})$.

Exercise 2.

(a) Let $\beta : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^N$ be a bilinear map, i.e., $\beta(\,\cdot\,, v) \in \mathrm{Lin}(\mathbb{R}^p, \mathbb{R}^N)$ and $\beta(u, \,\cdot\,) \in \mathrm{Lin}(\mathbb{R}^q, \mathbb{R}^N)$ for all $u \in \mathbb{R}^p$ and $v \in \mathbb{R}^q$. Show that there exists a constant $C > 0$ such that

$$\|\beta(u, v)\| \leq C\|u\|\|v\|$$

for all $u \in \mathbb{R}^p$ and $v \in \mathbb{R}^q$.

(b) Let $\mu : \mathrm{M}(n,\mathbb{R}) \times \mathrm{M}(n,\mathbb{R}) \to \mathrm{M}(n,\mathbb{R})$ be a bilinear map, let $U \subset \mathrm{M}(n,\mathbb{R})$ be open, and let $A \in U$. Let $f, g : U \to \mathrm{M}(n,\mathbb{R})$ be maps which are differentiable at A. Show that the map

$$M : U \to \mathrm{M}(n,\mathbb{R}),\ X \mapsto \mu(f(X), g(X))$$

is differentiable at A, with derivative given by

$$DM(A)(H) = \mu(Df(A)(H), g(A)) + \mu(f(A), Dg(A)(H)), \quad (H \in \mathrm{M}(n,\mathbb{R})).$$

(c) We consider $\mathrm{GL}(n,\mathbb{R}) := \{X \in \mathrm{M}(n,\mathbb{R}) \mid \det(X) \neq 0\}$. Show that $\mathrm{GL}(n,\mathbb{R})$ is an open subset of $\mathrm{M}(n,\mathbb{R})$.

(d) Show that the map $F : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R}), \quad X \mapsto X^{-1}$ is C^1.

(e) For each $A \in \mathrm{GL}(n,\mathbb{R})$ show that the map F is differentiable at A, with derivative $DF(A) : \mathrm{M}(n,\mathbb{R}) \to \mathrm{M}(n,\mathbb{R})$ given by

$$DF(A)H = -A^{-1}HA^{-1}.$$

Hint: use the equality $F(X)X = I$.

Exercise 3. Let $U \subset \mathbb{R}^n$ be an open subset.

(a) If $g : \mathbb{R}^n \to \mathbb{R}$ is differentiable, and attains a local minimum at a point $x^0 \in U$, show that $Dg(x^0) = 0$. Hint: consider partial derivatives.

We consider a differentiable map $\Phi : U \to \mathbb{R}^n$ such that $D\Phi(x)$ is invertible for every $x \in U$. Let $a \in U$ and put $b := \Phi(a)$.

(b) Show that there exists a constant $C > 0$ such that $\|D\Phi(a)v\| \geq C\|v\|$ for all $v \in \mathbb{R}^n$.

(c) Show that there exists a $\delta > 0$ such that $\bar{B}(a;\delta) \subset U$ and

$$\|\Phi(x) - b\| \geq \frac{2}{3}\, C\, \|x - a\|$$

for all $x \in \bar{B}(a;\delta)$ (the bar indicates that the closed ball is taken).

For $y \in B(b; \frac{1}{3}\, C\, \delta)$ we consider the function

$$f_y : \bar{B}(a;\delta) \to \mathbb{R},\ x \mapsto \|\Phi(x) - y\|^2.$$

(d) Show that the function f_y attains a minimum value $m(y)$ at a point $x(y) \in \bar{B}(a;\delta)$.

(e) Show that $\sqrt{m(y)} < \frac{1}{3}C\delta$ and that $x(y) \in B(a;\delta)$.

(f) Show that $\Phi(x(y)) = y$.

(g) By using the previous items, prove that $\Phi(U)$ is open.

Exercise 4. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^p$ be open subsets and let $\pi : U \to V$ be a C^k map which is a surjective submersion. By a local C^k section of π we mean a C^k-map $s : V_0 \to U$ with V_0 open in U, such that $\pi \circ s = \mathrm{id}_{V_0}$.

(a) Let $b \in V$ and $a \in U$ be such that $\pi(a) = b$. Show that there exists an open neighborhood V_0 of b in V and a local C^k section $s : V_0 \to U$ such that $s(b) = a$.

(b) Let $f : V \to \mathbb{R}^q$ be a map. Show that f is C^k if and only if $f \circ \pi$ is C^k.

Exercise 5. We consider the determinant map $f : \mathrm{M}(n \times n, \mathbb{R}) \to \mathbb{R}, X \mapsto \det X$.

(a) Show that the set $\mathrm{SL}(n, \mathbb{R}) := f^{-1}(\{1\})$ is a subgroup of $\mathrm{GL}(n, \mathbb{R})$.

(b) Show that $\mathrm{SL}(n, \mathbb{R})$ is a C^∞ submanifold of $\mathrm{M}(n \times n, \mathbb{R})$ at its point I.

(c) Let now $A \in \mathrm{SL}(n, \mathbb{R})$. Show that the map $L_A : \mathrm{M}(n \times n, \mathbb{R}) \to \mathrm{M}(n \times n, \mathbb{R}), X \mapsto AX$ is a C^∞ diffeomorphism.

(d) Show that $\mathrm{SL}(n, \mathbb{R})$ is a C^∞ submanifold of $\mathrm{M}(n \times n, \mathbb{R})$. What is its dimension?

Exercise 6.

(a) Let $\mathrm{S}_n(\mathbb{R})$ denote the set of symmetric $n \times n$-matrices. Show that this set is a linear subspace of $\mathrm{M}_n(\mathbb{R})$ which is linearly isomorphic to $\mathbb{R}^{n(n+1)/2}$.

(b) Let $\mathrm{O}(n)$ be the set of matrices $A \in M_n(\mathbb{R})$ such that $A^{\mathrm{T}}A = I$. We consider the map $g : \mathrm{M}_n(\mathbb{R}) \to \mathrm{S}_n(\mathbb{R}), X \mapsto X^{\mathrm{T}}X - I$. Show that the total derivative of g at $A \in \mathrm{M}_n(\mathbb{R})$ equals the linear map given by

$$Dg(A) : \mathrm{M}_n(\mathbb{R}) \to \mathrm{S}_n(\mathbb{R}), \quad H \mapsto A^{\mathrm{T}}H + H^{\mathrm{T}}A.$$

Hint: use the definition.

(c) Show that g is a submersion at $I \in \mathrm{O}(n)$.

(d) Show that $\mathrm{O}(n)$ is a submanifold of $\mathrm{M}_n(\mathbb{R})$ at the point I. Determine the tangent space $T_I\mathrm{O}_n(\mathbb{R})$.

(e) Show that for $B \in \mathrm{O}(n)$ the map $L_B : \mathrm{M}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})$ given by $L_B(X) = BX$ is a linear automorphism of $\mathrm{M}_n(\mathbb{R})$, which preserves $\mathrm{O}(n)$.

(f) Show that $\mathrm{O}(n)$ is a submanifold of $\mathrm{M}_n(\mathbb{R})$. Determine the dimension of this submanifold. Determine the tangent space $T_A\mathrm{O}(n)$ for every $A \in \mathrm{O}(n)$.

Exercise 7. Let $A \in \mathrm{M}_n(\mathbb{R})$ be a symmetric matrix, i.e., $A_{ij} = A_{ji}$ for all $1 \leq i, j \leq n$.

(a) Show that
$$\langle Ax, y\rangle = \langle x, Ay\rangle$$
for all $x, y \in \mathbb{R}^n$.

(b) Consider the C^∞-function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = \langle Ax, x\rangle$. Determine the total derivative $Df(x) \in \mathrm{Lin}(\mathbb{R}^n, \mathbb{R})$, for every $x \in \mathbb{R}^n$. Determine the gradient $\mathrm{grad} f(x) \in \mathbb{R}^n$ for every $x \in \mathbb{R}^n$.

(c) Consider the unit sphere $S = S^{n-1}$ in $\mathbb{R}^n$ given by the equation $g(x) = 0$, where $g : \mathbb{R}^n \to \mathbb{R}$, $\|x\|^2 - 1$. Show that $f|_S$ attains a maximal value M at a suitable $x^0 \in S$.

(d) By using the multiplier method, show that $Ax^0 = Mx^0$. Thus, M is an eigenvalue for A.

(e) Formulate and prove a similar result with the minimal value m of $f|_S$.

(f) Show that all eigenvalues of A are contained in $[m, M]$.

Exercise 8. In this exercise we assume that $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ are C^k submanifolds of dimensions d and e, respectively. In addition, we assume that $f : M \to N$ is a C^k map, that $a \in M$ and $b = f(a) \in N$. Here we do not assume that f is defined on an neighborhood of M in $\mathbb{R}^m$.

(a) Let I be an open interval in $\mathbb{R}$ and $\gamma : I \to M$ a C^1 curve (just differentiable would be enough). Show that $f \circ \gamma : I \to N$ is a C^1-curve.

(b) (U, κ) be a chart of M with $a \in U$. Assume $0 \in I$ and let $\gamma_1, \gamma_2 : I \to M$ be two C^1-curves such that $\gamma_j(0) = a$, for $j = 1, 2$. Show that $\kappa \circ \gamma_j$ define C^1 curves with domain a suitable open interval containing 0. Furthermore, show that
$$\gamma_1'(0) = \gamma_2'(0) \iff (\kappa \circ \gamma_1)'(0) = (\kappa \circ \gamma_2)'(0)$$

(c) Show that there exists a unique map $L : T_aM \to T_bN$ such that
$$L(\gamma'(0)) = (f \circ \gamma)'(0)$$
for every C^1-curve $\gamma : I \to M$ with $\gamma(0) = a$.

(d) Let (V, λ) be a chart of N with $b \in V$ and (U, κ) a chart of M with $a \in U$ and $f(U) \subset V$. Argue that the following diagram makes sense and commutes:
$$\begin{array}{ccc} T_aM & \xrightarrow{L} & T_bN \\ {\scriptstyle D(\kappa^{-1})(a')} \uparrow & & \uparrow {\scriptstyle D(\lambda^{-1})(b')} \\ \mathbb{R}^d & \xrightarrow{L'} & \mathbb{R}^e \end{array}$$

Here we have written $a' = \kappa(a)$, $b' = \lambda(b)$ and $L' = D(\lambda \circ f \circ \kappa^{-1})(a')$.

(e) Show that the map L in (c) is linear. We will denote it by $T_a f$ and call it the tangent map of f at a.

(f) Let $g : N \to R$ be a C^k map from N to a C^k submanifold R of $\mathbb{R}^r$. Show that $g \circ f : M \to R$ is C^k and that

$$T_a(g \circ f) = T_b g \circ T_a f.$$

This formula is known as the chain rule for tangent maps.

Exercise 9.

(a) Show that the following result is a particular case of [DK2, Thm. 6.4.5]: Let $B = \prod_{j=1}^n [a_j, b_j]$ be a block in $\mathbb{R}^n$ and $f : B \to \mathbb{R}$ a continuous function. Then

$$\int_B f(x)\, dx = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \dots, x_n)\, dx_n \dots dx_1.$$

(b) Write $B = [a_1, b_1] \times C$ with C a rectangle in $\mathbb{R}^{n-1}$. Inspect the proof of [DK2, Thm. 6.4.5] and show that the function

$$F : x_1 \mapsto \int_C f(x_1, y)\, dy$$

is continuous $[a_1, b_1] \to \mathbb{R}$.

Exercise 10. Let $A \subset \mathbb{R}^n$ be a bounded set. Prove that the following assertions are equivalent.

(a) the set A is (Jordan) negligable.

(b) for every $\epsilon > 0$ there exists a finite collection of rectangles $R_1, \dots, R_k$ such that

$$A \subset \cup_{j=1}^k R_j \quad \text{and} \quad \sum_{j=1}^n \mathrm{vol}_n(R_j) < \epsilon.$$

(c) for every $\epsilon > 0$ there exists a finite collection of rectangles $R_1, \dots, R_k$ such that

$$A \subset \cup_{j=1}^k \mathrm{int}(R_j) \quad \text{and} \quad \sum_{j=1}^n \mathrm{vol}_n(R_j) < \epsilon.$$

Exercise 11. For the purpose of this exercise, by a semi-rectangle in $\mathbb{R}^n$ we shall mean a subset R for which there exists an n-dimensional rectangle $B \subset \mathbb{R}^n$ such that $\mathrm{int}(B) \subset R \subset B$.

(a) Argue that a semi-rectangle R is Jordan measurable, with volume given by

$$\mathrm{vol}_n(R) = \mathrm{vol}_n(\bar{R}).$$

(b) Let $B \subset \mathbb{R}^n$ be a rectangle and $\mathcal{B} = \{B_1, B_2, \ldots, B_k\}$ a partition of B. Show that there exist semi-rectangles $R_1, \ldots, R_k$ with $R_j \subset B_j$ and

$$\sum_{j=1}^{k} 1_{R_j} = 1_B.$$

By a step function on $\mathbb{R}^n$ we mean a finite linear combination of functions of the form 1_R, with R a semi-rectangle in $\mathbb{R}^n$. The linear space of these step functions is denoted by $\Sigma(\mathbb{R}^n)$.

(c) Let $f : B \to \mathbb{R}$ be a bounded function. Show that there exist step functions $s_\pm$ with

$$s_- \leq f \leq s_+, \qquad \overline{S}(f, \mathcal{B}) = \int s_+(x)\, dx, \qquad \underline{S}(f, \mathcal{B}) = \int s_-(x)\, dx.$$

(d) We denote by $\Sigma_+(f)$ the set of step functions $s : \mathbb{R}^n \to \mathbb{R}$ with $f \leq s$. Show that

$$\overline{\int_B} f(x)\, dx = \inf_{s \in \Sigma_+(f)} \int s(x)\, dx.$$

Give a similar characterisation of the lower integral of f over B.

Exercise 12. We define $\Sigma(\mathbb{R}^n)$ as above.

(a) Let B be a rectangle, and $S \subset \partial B$. Show that 1_S is a step function.

(b) Let $\mathcal{B}$ be a partition of B. Let $s : \mathbb{R}^n \to \mathbb{R}$ be a function such that

(1) $s(\mathbb{R}^n)$ has finitely many values;

(2) for all $B' \in \mathcal{B}$, s is constant on $\mathrm{int}(B')$;

(3) $s = 0$ outside B.

(c) Let $s \in \Sigma(\mathbb{R}^n)$ vanish outside the rectangle B. Show that there exists a partition $\mathcal{B}$ of B such that the above condition (2) is fulifilled.

(d) If s, t are step functions, show that both $\min(s, t)$ and $\max(s, t)$ are step functions.

Exercise 13. Let $U \subset \mathbb{R}^n$ be an open subset. We denote by $\mathcal{J}(U)$ the collection of compact subsets of U which are Jordan measurable.

(a) If $f : U \to \mathbb{R}$ is absolutely Riemann integrable, show that there exists a unique real number $I \in \mathbb{R}$ such that for every $\epsilon > 0$ there exists a $K_0 \in \mathcal{J}(U)$ such that for all $K \in \mathcal{J}(U)$ with $K \supset K_0$ we have

$$\left| \int_K f(x)\, dx - I \right| < \epsilon.$$

(b) Show that $I = \int_U f(x)\, dx$.

Exercise 14.

(a) Determine the collection of all $s \in \mathbb{R}$ for which the integral

$$\int_{\mathbb{R}^2} (1 + \|x\|)^{-s} \, dx$$

is absolutely convergent. Hint: use polar coordinates. Prove the correctness of your answer.

(b) Answer the same question for

$$\int_{\mathbb{R}^3} (1 + \|x\|)^{-s} \, dx.$$

(c) Wat is your guess for the similar integral over $\mathbb{R}^n$, for $n \geq 4$. ? We will return to this question at a later stage.

Exercise 15. We consider the cone $C : x_2^2 + x_3^2 = mx_1^2$, with $m > 0$ a constant.

(a) Show that $M := C \setminus \{0\}$ is a C^∞ submanifold of dimension 2 of $\mathbb{R}^3$.

(b) Determine the area of the subset M_h of M given by

$$M_h = \{x \in C \mid 0 < x_1 < h\}, \qquad (h > 0).$$

Exercise 16. We consider a C^1-function $f : (a, b) \to (0, \infty)$. Consider the graph G of f in $\mathbb{R}^2 \simeq \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. Let S be the surface arising from G by rotating G about the x_1-axis over all angles from $[0, 2\pi]$.

(a) Guess a formula for the area of S. Explain the heuristics.

(b) Show that S is a C^1-submanifold of $\mathbb{R}^3$.

(c) Prove that the conjectured formula is correct.

Exercise 17. Let $v_1, \ldots, v_{n-1}$ be $n - 1$ vectors in $\mathbb{R}^n$.

(a) Show that $\xi : v \mapsto \det(v, v_1, v_2, \ldots, v_{n-1})$ defines a map in $\mathrm{Lin}(\mathbb{R}^n, \mathbb{R})$.

(b) Show that there exists a unique vector $v \in \mathbb{R}^n$ such that

$$\xi(u) = \langle u, v \rangle \qquad (\forall u \in \mathbb{R}^n).$$

The uniquely determined element v of (c) is called the exterior product of $v_1, \ldots, v_{n-1}$ and denoted by $v_1 \times \cdots v_{n-1}$.

(c) Show that the map $(v_1, \ldots, v_{n-1}) \mapsto v_1 \times \cdots v_{n-1}$ is alternating multilinear $(\mathbb{R}^n)^{\times(n-1)} \to \mathbb{R}^n$. Furthermore, show that $v_1 \times \cdots \times v_{n-1} \perp v_j$ for every $j = 1, \ldots, n-1$. Finally, show that $v_1 \times \cdots \times v_{n-1} = 0$ if and only if $v_1, \ldots, v_{n-1}$ are linearly independent.

(d) Show that for $n = 3$, the above corresponds to the usual exterior product.

We consider an injective linear map $L : \mathbb{R}^{n-1} \to \mathbb{R}^n$. Let $V = L(\mathbb{R}^n)$ and let d_V be the Euclidean density on V. Let $\mathbf{n}$ be a unit vector in $V^\perp$ such that $\det(\mathbf{n} \mid L) > 0$.

(e) Show that for every $v \in \mathbb{R}^n$ we have

$$L^*(\langle v, \mathbf{n} \rangle\, d_V) = \det(v \mid L) \qquad v \in \mathbb{R}^n$$

Hint: first show this for $v = \mathbf{n}$.

(f) Show that

$$L^*(d_V) = \|Le_1 \times \cdots \times Le_n\| \cdot d_{\mathbb{R}^{n-1}}.$$

(g) Show that

$$\|Le_1 \times \cdots \times Le_n\| = \sqrt{\det(L^{\mathrm{T}} L)}.$$

Let now $U \subset \mathbb{R}^{n-1}$ be open and $\varphi : U \to \mathbb{R}^n$ an embedding onto a codimension 1 submanifold M of $\mathbb{R}^n$. Let $\mathbf{n} : M \to \mathbb{R}^n$ be defined by $\mathbf{n}(x) \perp T_x M$ and

$$\det(\mathbf{n}(\varphi(y)), D_1\varphi(y), \ldots D_{n-1}\varphi(y)) > 0, \qquad (y \in U).$$

(h) Show that for every vector field $v : M \to \mathbb{R}^n$ we have

$$\varphi^*(\langle v, \mathbf{n} \rangle\, dx) = \det(v(\varphi(y)) \mid D\varphi(y))\, d_{\mathbb{R}^{n-1}}.$$

Exercise 18. Let $e_1, \ldots, e_n$ be the standardbasis of $\mathbb{R}^n$. Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a bijective linear map, and put $v_j = Le_j$.

(a) Show that the image of $[0,1]^n$ under L equals the parallelepiped spanned by $v_1, \ldots, v_n$, defined by

$$P(v_1, \ldots, v_n) = \{x \in \mathbb{R}^n | \exists (t_1, \ldots, t_n \in [0,1]) : \ x = \sum_{j=1}^n t_j v_j\}.$$

(b) Show that $P(v_1, \ldots, v_n)$ is Jordan measurable in $\mathbb{R}^n$ with volume given by

$$\mathrm{vol}_n(P(e_1, \ldots, e_n)) = \det(L).$$

(c) Is (b) still valid if L is not bijective?

Exercise 19. Let $V \subset \mathbb{R}^n$ be open and let $B_c(V)$ be the space of bounded functions $f : V \to \mathbb{R}$ with $\mathrm{supp} f$ compact and contained in$V$. Let $f \in B_c(V)$ and let $\mathcal{F}_+(f, V)$ be the space of functions $g \in C_c(V)$ with $g \geq f$.

(a) Show that

$$\overline{\int}_{\mathbb{R}^n} f(y)\; dy = \inf_{g \in \mathcal{F}_+(f,V)} \int_{\mathbb{R}^n} g(y)\; dy.$$

Let $\Phi : U \to V$ be a diffeomorphism between an open subset $U \subset \mathbb{R}^n$ and V. Given a bounded function $g : V \to \mathbb{R}$ with compact support we define $\Phi^{\#}(g) : U \to \mathbb{R}$ by

$$\Phi^{\#}(g)(x) = g(\Phi(x))|\det D\Phi(x)|.$$

(b) Show that $\Phi^{\#}$ is a bijective linear map from $B_c(V)$ onto $B_c(U)$.

(c) Show that $\Phi^{\#}$ maps $\mathcal{F}_+(f, V)$ bijectively onto $\mathcal{F}_+(\Phi^{\#}(f), U)$.

(d) Show that

$$\overline{\int}_{\mathbb{R}^n} f(y)\; dy = \overline{\int}_{\mathbb{R}^n} f(\Phi(x))\; |\det D\Phi(x)|\; dx.$$

Exercise 20. Let U and V be open subsets of $\mathbb{R}^n$ with closures $\bar{U}$ and $\bar{V}$ which are compact Jordan measurable in $\mathbb{R}^n$.

(a) Suppose that $f : \bar{V} \to \mathbb{R}$ is continuous. Show that f is Riemann integrable over $\bar{V}$ and that f is absolutely Riemann integrable over V with integral

$$\int_V f(x)\; dx = \int_{\bar{V}} f(x)\; dx.$$

(b) Let $\Phi : U \to V$ be a C^1-diffeomorphism such that both Φ and $D\Phi$ extend continuously to $\bar{U}$. Show that for every continuous function $f : \bar{V} \to \mathbb{R}^n$ one has that $(f \circ \Phi)|\det D\Phi|$ is Riemann integrable over $\bar{U}$ in the usual sense, and that

$$\int_{\bar{V}} f(x)\; dx = \int_{\bar{U}} f(\Phi(y))\; |\det D\Phi(y)|\; dy.$$

(c) Use (a), (b) and substitution of variables to show that for every continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ and all $R > 0$ one has

$$\int_{\bar{D}(0;R)} f(x)\; dx = \int_0^{2\pi} \int_0^R f(r\cos\varphi, r\sin\varphi)\; r\, dr\, d\varphi.$$