Exercise HW3

Let $p, q, n \geq 1$ be positive integers, such that $n = p + q$. Let $a \in \mathbb{R}^n$. Let $g : \mathbb{R}^n \to \mathbb{R}^p$ be a C^1-function such that $g(a) = 0$ and $Dg(a) : \mathbb{R}^n \to \mathbb{R}^p$ is surjective. Using the identification $\mathbb{R}^p \simeq \mathbb{R}^p \times \{0\} \subset \mathbb{R}^n$ we will view $Dg(a)$ as a linear map $\mathbb{R}^n \to \mathbb{R}^n$ with image $\mathbb{R}^p$.

(a) Show that there exist indices $j_1 < \ldots < j_p$ such that $Dg(a)(e_{j_k})$, for $k = 1, \ldots, p$, span $\mathbb{R}^p$.

After a permutation of coordinates in $\mathbb{R}^n$ we may as well assume that $j_k = k$, for $1 \leq k \leq p$. We then define the map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Phi(x) = (g(x), x_{p+1} - a_{p+1}, \ldots, x_n - a_n)^{\mathrm{T}},$$

where the superscript T indicates that the transposed of the row vector has been taken.

(b) Show that the Jacobian matrix of Φ at a takes the form

$$\operatorname{mat} D\Phi(a) = \begin{pmatrix} D'g(a) & D''g(a) \\ 0 & I_q \end{pmatrix}$$

Here $D'g(a)$ stands for the matrix consisting of the first p columns of the Jacobian matrix $\operatorname{mat} Dg(a)$, and $D''g(a)$ stands for the matrix consisting of the remaining q columns of $\operatorname{mat} Dg(a)$. Finally, I_q stands for the $q \times q$ identity matrix.

(c) Show that there exists an open neighborhood U of a in $\mathbb{R}^n$ such that Φ maps U diffeomorphically onto an open subset V of $\mathbb{R}^n$.

(d) Let $S := \{x \in \mathbb{R}^n \mid g_1(x) = g_2(x) = \cdots = g_p(x) = 0\}$. Show that $\Phi(a) = 0$ and that

$$\Phi(S \cap U) = V \cap (\{0\} \times \mathbb{R}^q).$$

Now assume that $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function such that $f(a) \leq f(x)$ for all $x \in S$. Let $\Psi : V \to U$ be the inverse of Φ.

(e) Show that $D_j(f \circ \Psi)(0) = 0$ for $j \geq p + 1$.

(f) Show that $D\Psi(0)(\{0\} \times \mathbb{R}^q) = \ker Dg(a)$.

(g) Show that $Df(a) = 0$ on $\ker Dg(a)$.