## Exercise HW4

Let $U := (0,\infty) \times (0,2\pi)$ and $V := \mathbb{R}^2 \setminus L$, where $L$ is the halfline $[0,\infty) \times \{0\}$.

(a) Show that $U$ and $V$ are open subsets of $\mathbb{R}^2$ and that the map $\Phi : (r,\varphi) \mapsto (r\cos\varphi, r\sin\varphi)$ is a diffeomorphism from $U$ onto $V$.

(b) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous with support contained in $V$. For $R > 0$ let $\bar{D}(0;R)$ be the closed disk of center 0 and radius $R$. Use the change of variables formula for Riemann integrable functions with compact support in $V$ to show that

$$\int_{\bar{D}(0;R)} g(x)\,dx = \int_0^R \int_0^{2\pi} g(\Phi(r,\varphi))r\,d\varphi\,dr.$$

Hint: consider the function $1_{\bar{D}(0;R)} \cdot g$.

We define the continuous function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ by

$$\varphi(x_1,x_2) = \begin{cases} 0 & \text{if } |x_2| \le 1 \\ |x_2| - 1 & \text{if } 1 < |x_2| < 2 \\ 1 & \text{if } |x_2| \ge 2. \end{cases}$$

Furthermore, for $\varepsilon > 0$ we define the function $\varphi_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ by $\varphi_\varepsilon(x) = \varphi(\varepsilon^{-1}x)$.

(c) For $R > 0$ show that

$$\lim_{\varepsilon \downarrow 0} \int_{D(0;R)} |1 - \varphi_\varepsilon(x)|dx = 0$$

and

$$\lim_{\varepsilon \downarrow 0} \int_0^R \int_0^{2\pi} |1 - \varphi_\varepsilon(\Phi(r,\varphi))|\,r\,d\varphi\,dr = 0.$$

Hint for the second integral: observe that for all $(r,\varphi) \in U$ with $|r\sin\varphi| \le 2\varepsilon$ we have $|r| \le \sqrt{2\varepsilon}$ or $|\sin\varphi| \le \sqrt{2\varepsilon}$. Use this to obtain an estimate for the second integral.

(d) For $f : \mathbb{R}^2 \to \mathbb{R}$ a continuous function, show that $\varphi_\varepsilon f$ is locally Riemann integrable for every $\varepsilon > 0$ and that

$$\lim_{\varepsilon \downarrow 0} \int_{D(0;R)} \varphi_\varepsilon(x) f(x)\,dx = \int_{D(0;R)} f(x)\,dx.$$

(e) For $f$ as in (d), show that

$$\int_{D(0;R)} f(x)\,dx = \int_0^R \int_0^{2\pi} f(\Phi(r,\varphi))\,r\,d\varphi\,dr.$$

(f) For $R > 0$ we write

$$I(R) = \int_{D(0;R)} e^{-\|x\|^2}\,dx.$$

Show that:

$$I(R) \leq \left(\int_{-R}^{R} e^{-t^2}\,dt\right)^2 \leq I(2R).$$

Show that $\int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi}$.