Solution to Exercise 3.6.3.

1. *If $\mathcal{F}$ is a local functional space on $\mathbb{R}^n$ then $\Omega \to \mathcal{F}(\Omega)$ is a sheaf of distributions.*

Recall that

$$\mathcal{F}(\Omega) = \{u \in \mathcal{D}'(\Omega) | \phi u \in \mathcal{F}, \forall \phi \in C_c^\infty(\Omega)\}, \tag{1}$$

endowed with the seminorms

$$\mathcal{P}_\Omega = \{q_{p,\phi} | \phi \in C_c^\infty(\Omega), p \in \mathcal{P}\},$$

where $q_{p,\phi}(u) = p(\phi u)$ and $\mathcal{P}$ is a set of seminorms on $\mathcal{F}$ defining the topology on $\mathcal{F}$.
We divide the proof in several steps:

Step A: The restriction map is well defined and continuous

Consider $\Omega_2 \subseteq \Omega_1$ two opens in $\mathbb{R}^n$. For $u \in \mathcal{F}(\Omega_1)$, we have to show that $u_{|\Omega_2} \in \mathcal{F}(\Omega_2)$. By (1), we have to show that $\phi \in C_c^\infty(\Omega_2)$ implies $\phi u_{|\Omega_2} \in \mathcal{F}$. Let $\widetilde{\phi} \in C_c^\infty(\Omega_1)$ be the extension by zero of ϕ to Ω_1. Then we have that

$$\phi u_{|\Omega_2} = \widetilde{\phi} u \in \mathcal{F}.$$

To prove continuity of the restriction map, consider $q_{\phi,p}$ a seminorm on $\mathcal{F}(\Omega_2)$, and denote again by $\widetilde{\phi} \in C_c^\infty(\Omega_1)$ be the extension by zero of ϕ to Ω_1. Then for all $u \in \mathcal{F}(\Omega_2)$ we have that

$$q_{\phi,p}(u_{|\Omega 2}) = p(\phi u_{|\Omega_2}) = p(\widetilde{\phi} u) = q_{\widetilde{\phi},p}(u),$$

and this proves continuity.

Step B: The map $T : \mathcal{F}(\Omega) \to \Pi_{i\in I}\mathcal{F}(\Omega_i)$ is continuous

Consider Ω an open in $\mathbb{R}^n$ and $\{\Omega_i\}_{i\in I}$ an open cover of Ω. Define $T : \mathcal{F}(\Omega) \to \Pi_{i\in I}\mathcal{F}(\Omega_i)$, $T(u) = \{u_{|\Omega_i}\}_{i\in I}$. Since $\Pi_{i\in I}\mathcal{F}(\Omega_i)$ is endowed with the product topology, continuity of T is equivalent to continuity of $pr_j \circ T$, where $pr_j : \Pi_{i\in I}\mathcal{F}(\Omega_i) \to \mathcal{F}(\Omega_j)$ is the projection onto the j component. But this composition is just the restriction map from $\mathcal{F}(\Omega) \to \mathcal{F}(\Omega_j)$, which is continuous by step A.

Step C: Construct $R : \Pi_{i\in I}\mathcal{F}(\Omega_i) \to \mathcal{F}(\Omega)$

Fix $\{\eta_i\}_{i\in I}$ a partition of unity subordinate to the cover $\{\Omega_i\}_{i\in I}$. Define $R : \Pi_{i\in I}\mathcal{F}(\Omega_i) \to \mathcal{F}(\Omega)$, by

$$R(\{u_i\}_{i\in I})(f) = \sum_{i\in I} u_i(\eta_i f), \text{ for } f \in C_c^\infty(\Omega). \tag{2}$$

We show first that the sum on the right is actually finite. For each point $x \in \mathrm{supp}(f)$ we can find a neighborhood V_x, which intersects only finitely many $\mathrm{supp}(\eta_i)$'s, call the set of such indexes $I_x \subseteq I$. Since $\mathrm{supp}(f)$ is compact, we can choose a finite subcover of the cover $\{V_x\}_{x\in\mathrm{supp}(f)}$, call it $V_{x_1}, \ldots V_{x_k}$. This shows that the only indexes in (2), which give nonzero terms are those in $I_{x_1} \cup \ldots \cup I_{x_k}$, which is a finite set.

We show now that $R(\{u_i\}_{i\in I})$ is an element of $\mathcal{D}'(\Omega)$. By Corollary 2.2.1, for every compact $K \subseteq \Omega$ we have to find a constant $C > 0$ and an integer $r \geq 0$, such that

$$|R(\{u_i\}_{i\in I})(f)| \leq C||f||_{K,r}, \text{ for all } f \in C_K^\infty(\Omega).$$

So let $K \subseteq \Omega$ be a compact. By the argument given above, only finitely many $\mathrm{supp}(\eta_i)$'s intersect K nontrivially. Denote this finite set of indexes by I_K. Since each $u_i \in \mathcal{D}'(\Omega_i)$, there exist constants $C_i > 0$ and integers $r_i \geq 0$ such that

$$|u_i(f_i)| \leq C_i||f_i||_{\mathrm{supp}(\eta_i)\cap K,r_i}, \text{ for all } f_i \in C^\infty_{\mathrm{supp}(\eta_i)\cap K}(\Omega_i).$$

On the other hand, by an application of the Leibniz rule, one can find constants C_i' (depending on η_i) such that

$$||\eta_i f||_{\mathrm{supp}(\eta_i)\cap K,r_i} \leq C_i'||f||_{K,r_i}, \text{ for all } f \in C_K^\infty(\Omega).$$

For $f \in C_K^\infty(\Omega)$, we have that

$$\begin{aligned} |R(\{u_i\}_{i_I})(f)| &= |\sum_{i\in I_K} u_i(\eta_i f)| \leq \sum_{i\in I_K} |u_i(\eta_i f)| \leq \sum_{i\in I_K} C_i||\eta_i f||_{\mathrm{supp}(\eta_i)\cap K,r_i} \leq \\ &\leq \sum_{i\in I_K} C_iC_i'||f||_{K,r_i} \leq C||f||_{K,r}, \end{aligned}$$

where $r = max\{r_i | i \in I_K\}$, $C = \sum_{i\in I_K} C_iC_i'$. This proves that $R(\{u_i\}_{i\in I})$ is continuous.
Finally we show that $R(\{u_i\}_{i\in I}) \in \mathcal{F}(\Omega)$. Let $\phi \in C_c^\infty(\Omega)$, and denote I_ϕ the set of indexes i, for which $\mathrm{supp}(\eta_i) \cap \mathrm{supp}(\phi) \neq \emptyset$, which is finite. Then we have that

$$\phi R(\{u_i\}_{i\in I}) = \sum_{i\in I_\phi} \phi\eta_i u_i \in \mathcal{F},$$

since $(\phi\eta_i)u_i \in \mathcal{F}$, by (1). Again (1), implies that $R(\{u_i\}_{i\in I}) \in \mathcal{F}(\Omega)$.

Step D: R is continuous

Recall that the product topology on $\Pi_{\alpha\in A}V_\alpha$, where the V_α's are l.c.v.s with corresponding seminorms $\mathcal{P}_\alpha$, is generated by the family of seminorms

$$\{\sum_{\alpha\in A'} p_\alpha \circ pr_\alpha | A' \subseteq A, |A'| < \infty, p_\alpha \in \mathcal{P}_\alpha\},$$

where $pr_\alpha : \Pi_{\alpha\in A} V_\alpha \to V_\alpha$ are the natural projections.
Let $q_{\phi,p}$ be a seminorm on $\mathcal{F}(\Omega)$, where $p \in \mathcal{P}$ and $\phi \in C_c^\infty(\Omega)$. Then we have that

$$\begin{aligned} q_{\phi,p}(R(\{u_i\}_{i\in I})) &= p(\phi R(\{u_i\}_{i\in I})) = \sum_{i\in I_\phi} p((\phi\eta_i)u_i) = \\ &= (\sum_{i\in I_\phi} q_{\phi\eta_i} \circ pr_i)(\{u_j\}_{j\in I}). \end{aligned}$$

So, by the previous remark, R is continuous.

Step E: $R \circ T = Id_{\mathcal{F}(\Omega)}$

Let $u \in \mathcal{F}(\Omega)$, $f \in C_c^\infty(\Omega)$ and denote I_f the finite set of indexes i for which $\mathrm{supp}(f) \cap \mathrm{supp}(\eta_i) \neq \emptyset$. Then we have that

$$R \circ T(u)(f) = R(\{u_{|\Omega_i}\}_{i\in I})(f) = \sum_{i\in I_f} u_{|\Omega_i}(\eta_i f) = u(\sum_{i\in I_f} \eta_i f) = u(f).$$

Step F: $\mathrm{Im}(T) = \{\{u_i\}_{i\in I} \in \Pi_{i\in I}\mathcal{F}(\Omega_i) | u_{i|\Omega_i\cap\Omega_j} = u_{j|\Omega_i\cap\Omega_j} \forall i,j \in I\}$

Clearly we have that

$$\mathrm{Im}(T) \subseteq \{\{u_i\}_{i\in I} \in \Pi_{i\in I}\mathcal{F}(\Omega_i) | u_{i|\Omega_i\cap\Omega_j} = u_{j|\Omega_i\cap\Omega_j} \forall i,j \in I\}$$

To prove the converse, we show that for each $\{u_i\}_{i\in I} \in \Pi_{i\in I}\mathcal{F}(\Omega_i)$ which satisfies $u_{i|\Omega_i\cap\Omega_j} = u_{j|\Omega_i\cap\Omega_j}$ for all $i,j \in I$, we have that $\{u_i\}_{i\in I} = T\circ R(\{u_i\}_{i\in I})$, which is equivalent to showing that $R(\{u_i\}_{i\in I})_{|\Omega_j} = u_j$. For $f \in C_c^\infty(\Omega_j)$ we have that

$$R(\{u_i\}_{i\in I})_{|\Omega_j}(f) = \sum_{i\in I_f} u_i(\eta_i f) = u_j(\sum_{i\in I_f} \eta_i f) = u_j(f),$$

where we have used the fact that $\mathrm{supp}(\eta_i f) \subseteq \Omega_i \cap \Omega_j$, and so $u_i(\eta_i f) = u_j(\eta_i f)$.

Step G: Conclusion

By Step E it follows that $\mathrm{Im}(T) = \mathrm{Ker}(T \circ R - Id)$ which is closed, since R is continuous (Step D). Step E implies that T is an embedding.

2. *If $\hat{\mathcal{F}}$ is a sheaf of distributions on $\mathbb{R}^n$ then $\mathcal{F} = \hat{\mathcal{F}}(\mathbb{R}^n)$ is a local functional space on $\mathbb{R}^n$ and $\hat{\mathcal{F}}(\Omega) = \mathcal{F}(\Omega)$ for all Ω's*

Step A: $\mathcal{F} = \mathcal{F}_{\mathrm{loc}}$ *as sets*

Consider $u \in \mathcal{F}_{\mathrm{loc}}$. Then $\phi u \in \mathcal{F}$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$, and $(\phi u)_{|\Omega} \in \hat{\mathcal{F}}(\Omega)$ for all opens Ω. Choosing Ω with compact closure and ϕ compactly supported with $\phi_{|\Omega} = 1$, it follows that

$$(\phi u)_{|\Omega} = u_{|\Omega} \in \hat{\mathcal{F}}(\Omega).$$

Choosing now $\{\Omega_i\}_{i\in I}$ an open cover of $\mathbb{R}^n$ such that $\bar{\Omega}_i$ is compact it follows that

$$\{u_{|\Omega_i}\}_{i\in I} \in \{\{u_i\}_{i\in I} \in \Pi_{i\in I}\hat{\mathcal{F}}(\Omega_i) | u_{i|\Omega_i\cap\Omega_j} = u_{j|\Omega_i\cap\Omega_j} \forall i,j \in I\}.$$

Since $\hat{\mathcal{F}}$ is a sheaf of distribution on $\mathbb{R}^n$ we have that $u \in \mathcal{F}$, and since $\mathcal{F} \subseteq \mathcal{F}_{\text{loc}}$ always holds, it follows that $\mathcal{F} = \mathcal{F}_{\text{loc}}$.

Step B: $\mathcal{F} = \mathcal{F}_{\text{loc}}$ as l.c.v.s.

The inclusion $\mathcal{F} \hookrightarrow \mathcal{F}_{\text{loc}}$ is continuous for any functional space.
Choose $\{\Omega_i\}_{i\in I}$ an open cover of $\mathbb{R}^n$ with $\bar{\Omega}_i$ compact for all $i \in I$ and choose $\{\eta_i\}_{i\in I}$ compactly supported functions with $\eta_{i|\Omega_i} = 1$. The maps

$$\mathcal{F}_{\text{loc}} \to \hat{\mathcal{F}}(\Omega_i) \qquad u \to u_{|\Omega_i}$$

are continuous since they can be written as the composition of the continuous maps

$$\mathcal{F}_{\text{loc}} \to \mathcal{F} \to \hat{\mathcal{F}}(\Omega_i) \qquad u \to \eta_i u \to (\eta_i u)_{|\Omega_i} = u_{|\Omega_i}$$

(the first map is continuous for any functional space and the second is continuous because $\hat{\mathcal{F}}$ is a sheaf of distributions). This shows that the map

$$S : \mathcal{F}_{\text{loc}} \to X, \qquad u \to \{u_{|\Omega_i}\}_{i\in I}$$

is continuous, where

$$X = \{\{u_i\}_{i\in I} \in \Pi_{i\in I}\hat{\mathcal{F}}(\Omega_i) | u_{i|\Omega_i\cap\Omega_j} = u_{j|\Omega_i\cap\Omega_j}\}.$$

Since $\hat{\mathcal{F}}$ is a sheaf of distributions it follows that S is a homeomorphism, when viewed as a map from $\mathcal{F} \to X$, with the corresponding topologies. Therefore $Id = S^{-1} \circ S : \mathcal{F}_{\text{loc}} \to \mathcal{F}$ is continuous.

Step C: $\mathcal{F}(\Omega) = \hat{\mathcal{F}}(\Omega)$ as sets

Observe first that the argument given above when applied to Ω instead of $\mathbb{R}^n$ shows that $\hat{\mathcal{F}}(\Omega)$ is a local functional space.
Consider $u \in \mathcal{F}(\Omega)$. By (1) this means that for all $\phi \in C_c^\infty(\Omega)$ we have that $\phi u \in \mathcal{F}$. Here (and also in (1)) ϕu is viewed as a distribution on $\mathbb{R}^n$, by extension by zero outside of Ω (this can be done since ϕu is supported inside Ω). Thus $(\phi u)_{|\Omega} = \phi u \in \hat{\mathcal{F}}(\Omega)$. Since $\hat{\mathcal{F}}(\Omega)$ is local it follows that $u \in \hat{\mathcal{F}}(\Omega)$.

Conversely, consider $u \in \hat{\mathcal{F}}(\Omega)$. We have to show that $\phi u \in \mathcal{F}$, for all $\phi \in C_c^\infty(\Omega)$. Choose such a ϕ and write $\mathbb{R}^n = \Omega \cup (\mathbb{R}^n \backslash \text{supp}(\phi))$. The element

$$(u_1, u_2) = (\phi u, 0) \in \hat{\mathcal{F}}(\Omega) \times \hat{\mathcal{F}}(\mathbb{R}^n \backslash \text{supp}(\phi))$$

satisfies $u_{1|\Omega\cap(\mathbb{R}^n\backslash\text{supp}(\phi))} = u_{2|\Omega\cap(\mathbb{R}^n\backslash\text{supp}(\phi))}$. Therefore, since $\hat{\mathcal{F}}$ is a sheaf of distributions, it follows that $\phi u \in \hat{\mathcal{F}}(\mathbb{R}^n) = \mathcal{F}$.

Step D: $\mathcal{F}(\Omega) = \hat{\mathcal{F}}(\Omega)$ as l.c.v.s.

Denote $\mathcal{P}$ a set of seminorms defining the topology on $\mathcal{F}$. Consider $\phi \in C_c^\infty(\Omega)$. Then the map

$$\mathcal{F}(\Omega) \to \mathcal{F}, \qquad u \to \phi u,$$

is continuous, because for every $p \in \mathcal{P}$, we have that $p(\phi u) = q_{\phi,p}(u)$, and the $q_{\phi,p}$'s are the seminorms defining the topology on $\mathcal{F}(\Omega)$. Since also the restriction map $\mathcal{F} \to \hat{\mathcal{F}}(\Omega)$ is continuous, it follows that the map $\mathcal{F}(\Omega) \to \hat{\mathcal{F}}(\Omega)$, $u \to \phi u$ is continuous. Exercise 3.1.2 implies that the map $\mathcal{F}(\Omega) \to \hat{\mathcal{F}}(\Omega)_{\text{loc}}$, $u \to u$ is continuous, and since $\hat{\mathcal{F}}(\Omega)$ is local, we have that $Id : \mathcal{F}(\Omega) \to \hat{\mathcal{F}}(\Omega)$ is continuous.

For the converse, denote

$$X = \{(u_1, u_2) \in \hat{\mathcal{F}}(\Omega) \times \hat{\mathcal{F}}(\mathbb{R}^n \backslash \text{supp}(\phi)) | u_{1|\Omega\backslash\text{supp}(\phi)} = u_{2|\Omega\backslash\text{supp}(\phi)}\},$$

and $A : X \to \mathcal{F}$ the gluing homeomorphism. Fix $\phi \in C_c^\infty(\Omega)$. Then the maps

$$\hat{\mathcal{F}}(\Omega) \to \hat{\mathcal{F}}(\Omega) \to X \xrightarrow{A} \mathcal{F}, \qquad u \to \phi u \to (\phi u, 0) \to \phi u$$

are continuous, and their composition sends u to ϕu viewed as an element of $\mathcal{F}$, by extending by zero outside Ω. So for every $p \in \mathcal{P}$ there exist $p_1, \ldots, p_k$, seminorms on $\hat{\mathcal{F}}(\Omega)$ and a constant $C > 0$ such that

$$p(\phi u) \leq Cmax\{p_1(u), \ldots, p_k(u)\}.$$

Since $p(\phi u) = q_{\phi,p}(u)$, and the $q_{\phi,p}$'s define the topology on $\mathcal{F}(\Omega)$, it follows that $Id : \hat{\mathcal{F}}(\Omega) \to \mathcal{F}(\Omega)$ is continuous.