Analysis on Manifolds
lecture notes for the 2009/2010 Master Class

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LECTURE 1
Differential operators

1.1. Differential operators I: trivial coefficients

In this section we discuss differential operators acting on spaces of functions on a manifold, while in the next section we will move to those acting on spaces of sections of vector bundles. We first discuss differential operators on an open subset $U \subset \mathbb{R}^n$.

We use the following notation for multi-indices $\alpha \in \mathbb{N}^n$:

$$|\alpha| = \sum_{j=1}^{n} \alpha_j; \quad \alpha! = \prod_{j=1}^{n} \alpha_j!.$$

Moreover, if $\beta \in \mathbb{N}^n$ we write $\alpha \leq \beta$ if and only if $\alpha_j \leq \beta_j$ for all $1 \leq j \leq n$. If $\alpha \leq \beta$ we put

$$\left(\begin{array}{c} \beta \\ \alpha \end{array} \right) := \prod_{j=1}^{n} \left(\begin{array}{c} \beta_j \\ \alpha_j \end{array} \right).$$

Finally, we put $\partial_j = \partial / \partial x_j$ and

$$x^\alpha = \prod_{j=1}^{n} x_j^{\alpha_j}, \qquad \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}. \tag{1.1}$$

Lemma 1.1.1. (Leibniz' rule) *Let $f, g \in C^\infty(U)$ and $\alpha \in \mathbb{N}^n$. Then*

$$\partial^\alpha (fg) = \sum_{\beta \leq \alpha} \left(\begin{array}{c} \alpha \\ \beta \end{array} \right) \partial^\beta f \, \partial^{\alpha - \beta} g.$$

Proof Exercise. □

Definition 1.1.2. A differential operator of order at most $k \in \mathbb{N}$ on U is an endomorphism $P \in \mathrm{End}(C^\infty(U))$ of the form

$$P = \sum_{|\alpha| \leq k} c_\alpha(x) \, \partial^\alpha, \tag{1.2}$$

with $c_\alpha \in C^\infty(U)$ for all α.

The linear space of differential operators on U of order at most k is denoted by $\mathcal{D}_k(U)$. The union of these, for $k \in \mathbb{N}$, is denoted by $\mathcal{D}(U)$. Via Leibniz' rule one easily verifies that the composition of two differential operators from $\mathcal{D}_k(U)$ and $\mathcal{D}_l(U)$ is again a differential operator, in $\mathcal{D}_{k+l}(U)$. Accordingly, the set $\mathcal{D}(U)$ of differential operators is a (filtered) algebra with unit.

Next, we look at the effect of coordinate changes. More precisely, let h be a diffeomorphism from U onto a second open subset $U' \subset \mathbb{R}^n$. Then by pull-back, h induces the bijection $h^* : C^\infty(U') \to C^\infty(U)$. Thus, $h^* f(x) = f(h(x))$.

Accordingly, we have an induced map $h_* : \mathrm{End}(C^\infty(U)) \to \mathrm{End}(C^\infty(U'))$, given by

$$h_*(T) = h^{*-1} \circ T \circ h^*.$$

Lemma 1.1.3. *The map h_* maps $\mathcal{D}(U)$ bijectively onto $\mathcal{D}(U')$.*

Proof It follows by repeated application of the chain rule for differentiation, in combination with Leibniz' rule. □

We now move to arbitrary manifolds now.

Definition 1.1.4. Let M be a smooth manifold. A linear operator $P \in \mathrm{End}(C^\infty(M))$ is called *local* if

$$\mathrm{supp}(P(f)) \subset \mathrm{supp}(f) \quad \forall\, f \in C^\infty(M).$$

Since the complement of the support of a function is the largest open on which the function vanishes, the previous condition is equivalent to: for any open $U \subset M$, $f \in C^\infty(M)$, one has the implication:

$$f|_U = 0 \Longrightarrow P(f)|_U = 0.$$

Lemma 1.1.5. *There is a unique way to associate to any local operator $P \in \mathrm{End}(C^\infty(M))$ on a manifold M and any open $U \subset M$, a "restricted operator"*

$$P_U = P|_U \in \mathrm{End}(C^\infty(U))$$

such that, if $V \subset U$, then $(P|_U)|_V = P|_V$.

Proof For $f \in C^\infty(U)$, let's look at what the value of $P_U(f) \in C^\infty(U)$ at an arbitrary point $x \in U$ can be. We choose a function $f_x \in C^\infty(M)$ which coincides with f in an open neighborhood $V_x \subset U$ of x. From the condition in the statement, we must have

$$P_U(f)(x) = P(f_x)(x).$$

We are left with checking that this can be taken as definition of P_U. All we have to check is the independence of the choice of f_x. But if g_x is another one, then $f_x - g_x$ vanishes on a neighborhood of x; since P is local, we deduce that $P(f_x) - P(g_x) = P(f_x - g_x)$ vanishes on that neighborhood, hence also at x. □

Local operators can be represented in local charts: if (U, κ) is a coordinate chart, then $P|_U$ can be moved to $\kappa(U)$ using the pull-back map $k^* : C^\infty(\kappa(U)) \to C^\infty(U)$, to obtain an operator

$$P_\kappa : C^\infty(\kappa(U)) \to C^\infty(\kappa(U)),\ P_\kappa = \kappa_*(P|_U) = (\kappa^*)^{-1} \circ P|_U \circ \kappa^*.$$

Definition 1.1.6. Let M be a smooth manifold. A **differential operator** of order at most k on M is a local linear operator $P \in \mathrm{End}(C^\infty(M))$ with the property that, for any coordinate chart (U, κ), $P_\kappa \in \mathcal{D}_k(\kappa(U))$.

The space of operators on M of order at most k is denoted by $\mathcal{D}_k(M)$.

Note that, in the previous definition, it would have been enough to require the condition only for a family of coordinate charts whose domains cover M.

Note also that the condition on a coordinate chart $(U, \kappa = (x_1^\kappa, \ldots, x_n^\kappa))$ simply means that $P|_U$ is of type

$$P_U = \sum_{|\alpha|\leq k} c_\alpha(x)\, \partial_\kappa^\alpha,$$

with $c_\alpha \in C^\infty(U)$. Here ∂_κ^α act on $C^\infty(U)$ and are defined analogous to ∂^α but using the derivative along the vector fields $\partial/\partial x_j^\kappa$ induce by the chart.

Next, we discuss the symbols of differential operators.

Definition 1.1.7. Let $U \subset \mathbb{R}^n$ and let $P \in \mathcal{D}_k(U)$ be of the form (1.2). The **full symbol** of the operator P is the function $\sigma(P) : U \times \mathbb{R}^n \to \mathbb{C}$ defined by

$$\sigma(P)(x, \xi) = \sum_{|\alpha|\leq k} c_\alpha(x)(i\xi)^\alpha.$$

The **principal symbol** of order k of P is the function $\sigma_k(P) : U \times \mathbb{R}^n \to \mathbb{C}$,

$$\sigma_k(P)(x, \xi) = \sum_{|\alpha|=k} c_\alpha(x)(i\xi)^\alpha.$$

A nice property of the principal symbol has, which fails for the total one is its multiplicativity property (see Exercise 1.5.2).

It is not difficult to check the following formulas for the symbols:

$$\sigma(P)(x, \xi) = e^{-i\xi}P(e^{i\xi})(x), \sigma_k(P)(x, \xi) = \lim_{t\to\infty} t^{-k}e^{-it\xi}P(e^{it\xi})(x).$$

Here we have identified ξ with the linear functional $x \mapsto \sum \xi_j x_j$. Accordingly, $e^{i\xi}$ stands for the function $x \mapsto e^{i\xi x}$. See also below.

Although the total symbol may look more natural then the principal one, the situation is the other way around: it is the principal symbol that can be globalized to manifolds (hence expressed coordinate free). Continuing the discussion above, it is most natural to view ξ as a variable in the dual of $\mathbb{R}^n$. Accordingly, $U \times \mathbb{R}^n$ should be viewed as the cotangent bundle T^*U. In other words, the principal symbol should be viewed as the function

$$\sigma_k(P) : T^*U \to \mathbb{C}, \quad \xi_1(dx_1)_x + \ldots + \xi_n(dx_n)_x \mapsto \sum_{|\alpha|=k} c_\alpha(x)(i\xi)^\alpha.$$

The following lemma supports this interpretation: it shows that the symbol can be characterized intrinsically (without reference to the coordinates).

Lemma 1.1.8. *Let $U \subset \mathbb{R}^n$, $P \in \mathcal{D}_k(U)$.*

For $\xi_x = (x, \xi) \in T_x^*\mathbb{R}^n$ $(x \in U)$, choose $\varphi \in C^\infty(U)$ such that $(d\varphi)_x = \xi_x$. Then

$$\sigma_k(P)(x, \xi) = \lim_{t\to\infty} t^{-k}e^{-it\varphi(x)}P(e^{it\varphi})(x).$$

Proof Left to the reader. The proof follows by application of Leibniz' rule. □

Although this should be already clear from the previous lemma, let us check explicitly that the symbol behaves well under coordinate changes. More precisely, let h be a diffeomorphism from U onto a second open subset $U' \subset \mathbb{R}^n$. It induces the map $T^*h : T^*U \to T^*U'$ given by $T^*h(x, \xi) = (h(x), \xi \circ T_x h^{-1})$.

Accordingly we have the map $h_* : C^\infty(T^*U) \to C^\infty(T^*U')$ given by $h_*\sigma = \sigma \circ (T^*h)^{-1}$. Thus,

$$h_*\sigma(x, \xi) = \sigma(h^{-1}(x), \xi \circ T_x h).$$

Lemma 1.1.9. *For all $P \in \mathcal{D}_k(U)$,*

$$\sigma_k(h_*(P)) = h_*(\sigma_k(P)).$$

Proof Fix $(x, \xi) \in U \times \mathbb{R}^n \simeq T^*U$. Put $y = h(x)$ and $\eta = \xi \circ T_x h^{-1}$. Select select $\varphi \in C^\infty(U')$ with $d\varphi(y) = \eta$. Then

$$\begin{aligned} \sigma_k(h_*(P))(y, \eta) &= \lim_{t\to\infty} t^{-k} e^{-it\varphi(y)}(h_*P)(e^{it\varphi})(y) \\ &= \lim_{t\to\infty} t^{-k} e^{-ith^*\varphi(x)} P(e^{ith^*\varphi})(x) \\ &= \sigma_k(P)(x, \xi). \end{aligned}$$

This establishes the desired formula. □

Corollary 1.1.10. *Let M be a manifold and $P \in \mathcal{D}_k(M)$. Then there is a well-defined smooth function*

$$\sigma_k(P) : T^*M \to \mathbb{C}$$

such that, for any coordinate chart $(U, \kappa = (x_1^\kappa, \ldots, x_n^\kappa))$,

$$T_x^*M \ni \xi_1(dx_1^\kappa)_x + \ldots + \xi_n(dx_n^\kappa)_x \stackrel{\sigma_k(P)}{\longrightarrow} \sigma_k(P_\kappa)(x, \xi_1, \ldots, \xi_n) \in \mathbb{C}.$$

Definition 1.1.11. Let $P \in \mathcal{D}_k(M)$. The function $\sigma_k(P) : T^*M \to \mathbb{C}$ from the previous lemma is called **the principal symbol** of order k of the operator P.

You should now try to solve Exercises 1.5.3, 1.5.5 and 1.5.6.

1.2. Differential operators II: arbitrary coefficients

We shall now introduce the notion of a differential operator between smooth vector bundles E and F on a smooth manifold M, acting at the level of sections

$$P : \Gamma(E) \to \Gamma(F).$$

It is useful to have in mind that degree zero differential operators correspond to sections $C \in \Gamma(\underline{\mathrm{Hom}}\,(E, F))$ i.e. smooth maps

$$M \ni x \mapsto C_x \in \mathrm{Hom}(E_x, F_x).$$

More precisely, any such C defines an operator $C : \Gamma(E) \to \Gamma(F)$ acting on sections by

$$C(s)(x) = C_x(s(x)).$$

This construction identifies sections of $\underline{\mathrm{Hom}}\,(E, F)$ with $C^\infty(M)$-linear maps $\Gamma(E)$ to $\Gamma(F)$ (see Exercise 1.5.7).

First, we place ourselves in the following situation:

1. $M = U$ is the domain of a coordinate chart $(U, \kappa = (x_1^\kappa, \ldots, x_n^\kappa))$.
2. E is trivializable and we have a fixed trivialization $E \cong U \times \mathbb{C}^r$ with associated frame $\{s_1, \ldots\}$.

Note that, in this case, we have "higher order derivatives operators"

$$\partial_\kappa^\alpha : \Gamma(E) \to \Gamma(E), \quad f^1 s_1 + \ldots \mapsto \partial_\kappa^\alpha(f^1)s_1 + \ldots .$$

A differential operator of order at most k from E to F is a linear map $P : \Gamma(U, E) \to \Gamma(U, F)$ of the form

$$P = \sum_{|\alpha| \le k} C_\alpha \circ \partial^\alpha,$$

with $C_\alpha \in \Gamma(U, \underline{\mathrm{Hom}}\,(E, F))$. The space of such differential operators is denoted by $\mathcal{D}_k(U, E, F)$.

Note that if also F is trivialized, with trivializing frame $\{s_1', \ldots\}$, then each C_α is uniquely determined by a matrix of smooth functions on U, $c(\alpha)_j^i \in C^\infty(U)$ (characterized by $C_\alpha(s_i) = \sum_j c(\alpha)_i^j s_j'$. With respect to the identification $\Gamma(U, E) \cong C^\infty(U, \mathbb{C})^r$, $(f^1 s_1 + \ldots) \mapsto (f^1, \ldots)$, and similarly for F, P becomes

$$P(f_1, \ldots) = (\sum_{\alpha, j} \partial_\kappa^\alpha(f^j) c(\alpha)_j^1, \ldots).$$

Next, we explain that $\mathcal{D}_k(U, E, F)$ does not depend on the trivialization of E. Another trivialization (over U) is associated with a vector bundle isomorphism $\varphi_E : E \to E$. Then $\varphi_E(x, v) = (x, \Phi_E(x)v)$, with Φ_E a smooth map $U \to \mathrm{GL}\,(\mathbb{C}^r)$. The map φ_E induces a linear isomorphism φ_{E*} of $\Gamma(U, E)$ onto itself, given by $\varphi_{E*}s = \varphi_E \circ s$. Accordingly, we have an induced linear isomorphism φ_* from $\mathrm{Hom}(\Gamma(U, E), \Gamma(U, F))$ onto itself given by $\varphi_*(T) = T \circ \varphi_{E*}^{-1}$. By an easy repeated application of Leibniz' formula, we see that the map φ_* maps $\mathcal{D}_k(U, E, F)$ bijectively onto $\mathcal{D}_k(U, E, F)$. Hence the space $\mathcal{D}_k(U, E, F)$ defined above only depends on the coordinate patch (U, κ) and on the fact that E is trivializable (in turn, one can already proceed as in the previous section and show that it does not depend on the choice of the coordinates on U either).

Proceeding as in the previous section, we say that a linear operator between spaces of sections of two vector bundles E and$F$ over a manifold M,

$$P : \Gamma(E) \to \Gamma(F)$$

is local if, for any $s \in \Gamma(E)$,

$$\mathrm{supp}(P(s)) \subset \mathrm{supp}(s).$$

By the same arguments as before, any such local operator can be restricted to arbitrary opens $U \subset M$, obtaining new operators $P|_U$ from $E|_U$ to $F|_U$. Also, for a coordinate chart (U, κ), we obtain an operator P_κ acting on the resulting bundles over $\kappa(U)$: from $E_\kappa := \kappa_*(E|U) = (\kappa^{-1})^*(E|U)$ to F_κ defined similarly.

Definition 1.2.1. Let E, F be smooth vector bundles over a smooth manifold M. A **differential operator of order at most** k **from** E **to** F is a linear local operator $P : \Gamma(E) \to \Gamma(F)$ with the property that, for any coordinate chart (U, κ) with the property that $E|_U$ is trivializable, $P_\kappa \in \mathcal{D}_k(E_\kappa, F_\kappa)$.

The space of differential operators of order at most k from E to F is denoted by $\mathcal{D}_k(E, F)$.

We extend the definition of principal symbol as follows. We denote by $\pi : T^*M \to M$ the canonical projection. For a vector bundle E over M, let π^*E be the pull-back of E to T^*M (whose fiber above $\xi_x \in T^*_xM$ is E_x). For two vector bundles E and F over M, we consider the vector bundle $\underline{\mathrm{Hom}}\,(E, F)$ over M (whose fiber above $x \in M$ is $\mathrm{Hom}(E_x, F_x)$) and its pull-back to T^*M,

$$\pi^*\underline{\mathrm{Hom}}\,(E, F) \cong \underline{\mathrm{Hom}}\,(\pi^*E, \pi^*F)$$

whose fiber above $\xi_x \in T^*_xM$ is $\mathrm{Hom}(E_x, F_x)$.

Lemma 1.2.2. *Let E, F be smooth vector bundles on M and let $P \in \mathcal{D}_k(E, F)$. There exists a unique section $\sigma_k(P)$ of $\pi^*\underline{\mathrm{Hom}}\,(E, F)$ (called again* **the principal symbol** *of P), i.e. a smooth function*

$$T^*_x \ni \xi_x \mapsto \sigma_k(P)(\xi_x) \in \mathrm{Hom}(E_x, F_x),$$

with the following property: for each $x_0 \in M$ and all $s \in \Gamma(E)$ and $\varphi \in C^\infty(M)$,

$$\sigma_k(P)((d\varphi)_{x_0})(s(x_0)) = \lim_{t\to\infty} t^{-k} e^{-it\varphi(x_0)} P(e^{it\varphi}s)(x_0). \tag{1.3}$$

Moreover, for each $x \in M$ the function $\xi \mapsto \sigma_k(P)(x, \xi)$ is a degree k homogeneous polynomial function $T^*_xM \to \mathrm{Hom}(E_x, F_x)$.

Proof Uniqueness follows from the fact that for every $(x, \xi) \in T^*M$ and $v \in E_x$ there exists a $s \in \Gamma^\infty(E)$ such that $s(x) = v$ and a function $\varphi \in C^\infty(M)$ such that $d\varphi(x) = \xi$. Let $x_0 \in M$ be given and select a coordinate patch $U \ni x_0$ over which E and F admit trivializations $\tau_E : E|_U \to U \times E_0$ and $\tau_F : F|_U \to U \times F_0$. trivialize. Let $(x_1, \dots, x_n)$ be a system of local coordinates on U. Then

$$\tau_*(P) = \sum_{|\alpha|\le k} C_\alpha \partial^\alpha,$$

with $C_\alpha \in C^\infty(U, \underline{\mathrm{Hom}}\,(E_0, F_0))$. It follows by application of Lemma 1.1.8 that the limit on the right-hand side of (1.3) is given by

$$\sum_{|\alpha|\le k} [\tau_F^{-1}] \circ C_\alpha(x_0)\tau_E[s(x_0)]\eta^\alpha.$$

Here we have used the multi-index notation with $\eta_j = \partial_j\varphi(x_0)$, $\partial_1, \dots, \partial_n$ being the derivations induced by the choice of coordinates. It follows that the limit on the right-hand side of (1.3) depends on s and ϕ through the values $s(x_0)$ and $d\phi(x_0)$. This implies the existence of a section $\sigma_k(P)$ of $\underline{\mathrm{Hom}}\,(\pi^*E, \pi^*F)$ with the property (1.3). The local computation just given also implies the final assertions about smoothness and homogeneity. □

Example 1.2.3. We consider the complexified version of the DeRham complex. I.e., we define $\Omega^k(M)_\mathbb{C} = \Omega^k(M) \otimes_\mathbb{R} \mathbb{C}$, which should be interpreted as the space of sections of the complex vector bundle $\Lambda_\mathbb{C}T^*M$ whose fiber at $x \in M$ consists of antisymmetric, k-multilinear maps from T_xM to $\mathbb{C}$. The exterior differentiation clearly extends to a $\mathbb{C}$-linear map $d = d_k : \Omega^k(M)_\mathbb{C} \to \Omega^{k+1}(M)_\mathbb{C}$. Let U be a coordinate patch of M with local coordinates $x_1, \dots, x_n$. Then for each $a \in U$, the one forms $dx_1(a), \dots, dx_n(a)$ span the cotangent space T_a^*M. Thus, $\wedge^k T_a^*M$ has the basis

$$dx_{j_1}(a) \wedge \cdots \wedge dx_{j_k}(a), \quad \text{with} j_1 < \cdots < j_k.$$

With respect to this basis, the restriction of a section $s \in \Omega^k(M)$ to U may be expressed as

$$s|_U = \sum_{j_1 < \cdots < j_k} s_{j_1,\dots,j_k} dx_{j_1} \wedge \cdots \wedge dx_{j_k}.$$

Exterior differentiation is given by

$$ds|_U = \sum_{j_1 \cdots < j_k} d(s_{j_1,\dots,j_k}) \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k},$$

where $ds_{j_1,\dots,j_k} = \sum_i \partial_i s_{j_1,\dots,j_k}$. From this we see that d is a differential operator of order one from $\wedge^k T^*M$ to $\wedge^{k+1} T^*M$. For its principal symbol, see Exercise 1.5.9.

For E_1, E_2 smooth vector bundles on M and $P \in \mathcal{D}_k(E_1, E_2)$, the principal symbol $\sigma_k(P)$ is a section of the bundle $\underline{\text{Hom}}\,(\pi^*E_1, \pi^*E_2)$. Equivalently, the symbol may be viewed as a homomorphism from the bundle π^*E_1 to π^*E_2. Thus, if E_3 is a third vector bundle and $Q \in \mathcal{D}_l(E_2, E_3)$ then the composition $\sigma_l(Q) \circ \sigma_k(P)$ is a vector bundle homomorphism from E_1 to E_3.

Lemma 1.2.4. *Let E_1, E_2, E_3 be smooth vector bundles on M. Let $P \in \mathcal{D}_k(E_1, E_2)$ and $Q \in \mathcal{D}_l(E_2, E_3)$. Then the composition $Q \circ P$ belongs to $\mathcal{D}_{k+l}(E_1, E_3)$ and*

$$\sigma_{k+l}(Q \circ P) = \sigma_l(Q) \circ \sigma_k(P).$$

Finally, we discuss the notion of formal adjoint. Assume now that E and F are equipped with hermitian inner products $\langle -, - \rangle^{E}$[1] and $\langle -, - \rangle^F$. We also choose a strictly positive density on M, call it $d\mu$. One has an induced inner-product on the space $\Gamma_c(E)$ of compactly supported sections of E given by

$$\langle s, s' \rangle^E := \int_M \langle s(x), s' \rangle_x^E \, d\mu,$$

and similarly an inner product on $\Gamma_c(F)$. Given $P \in \mathcal{D}_k(E, F)$, a **formal adjoint** of P (with respect to the hermitian metrics and the density) is an operator $P^* \in \mathcal{D}_k(F, E)$ with the property that

$$\langle P(s_1), s_2 \rangle^F = \langle s_1, P^*(s_2) \rangle^E, \quad \forall\ s_1 \in \Gamma_c(E), s_2 \in \Gamma_c(F).$$

Proposition 1.2.5. *For any $P \in \mathcal{D}_k(E, F)$, the formal adjoint $P^* \in \mathcal{D}_k(F, E)$ exists and is unique. Moreover, the principal symbol of P^* is $\sigma_k(P^*) = \sigma_k(P)^*$, where $\sigma_k(P)^*(\xi_x)$ is the adjoint of the linear map*

$$\sigma_k(P)(\xi_x) : E_x \to F_x$$

(with respect to the inner products $\langle -, - \rangle_x^E$ and $\langle -, - \rangle_x^F$).[2]

[1]hence $\langle -, - \rangle^E$ is a family $\{\langle -, - \rangle_x^E : x \in M\}$ of inner products on the vector spaces E_x, which "varies smoothly with respect to x". The last part means, e.g., that for any $s, s' \in \Gamma(E)$, the function $\langle s, s' \rangle^E$ on M, sending x to $\langle s(x), s'(x) \rangle_x^E$ is smooth; equivalently, it has the obvious meaning in local trivializations.

[2]note that P^* depends both on the hermitian metrics on E and F as well as on the density, while it principal symbol does not depend on the density.

Proof Due to the local property of differential operators (or, more precisely to its sheaf property, cf. Exercise 1.5.8), it suffices to prove the statement (both the existence as well as the uniqueness) locally. So assume that $M = U \subset \mathbb{R}^n$, where we can write $P = \sum_{|\alpha|\leq k} C_\alpha \circ \partial^\alpha$. We have $d\mu = \rho|dx|$ for some smooth function ρ on U. Writing out $\langle P(s_1), s_2\rangle^F$ and integrating by parts $|\alpha|$ times (to move ∂^α from s to s'), we find the operator P^* which does the job:

$$P^*(s') = \sum_{|\alpha|\leq k} \frac{1}{\rho}\partial^\alpha(\rho C_\alpha^* s').$$

Clearly, this is a differential operator of order at most k. For the principal symbol, we see that the only terms in this sum which matter are:

$$\sum_{|\alpha|=k} (-1)^{|\alpha|}\frac{1}{\rho}\rho C_\alpha^*\partial^\alpha(s') = \sum_{|\alpha|=k} (-1)^{|\alpha|} C_\alpha^*\partial^\alpha(s'),$$

i.e. the symbol is given by

$$\sum_{|\alpha|=k} (-1)^{|\alpha|} C_\alpha^*(i\xi)^\alpha = (\sum_{|\alpha|=k} C_\alpha^*(i\xi)^\alpha)^*.$$

The uniqueness follows from the non-degeneracy property of the integral: if $\int_U fg = 0$ for all compactly supported smooth functions, then $f = 0$. □

1.3. Ellipticity and a preliminary version of the Atiyah-Singer index theorem

Definition 1.3.1. Let $P \in \mathcal{D}_k(E, F)$ be a differential operator between two vector bundles E and F over a manifold M. We say that P is an **elliptic operator** of order k if, for any $\xi_x \in T_x^*M$ non-zero,

$$\sigma_k(P)(\xi_x) : E_x \to F_x$$

is an isomorphism.

The aim of these lectures is to explain and complete the following theorem (a preliminary version of the Atiyah-Singer index theorem).

Theorem 1.3.2. *Let M be a compact manifold and let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic differential operator. Then $Ker(P)$ and $Coker(P)$ are finite dimensional,*

$$Index(P) := dim(ker(P)) - dim(Coker(P))$$

depends only on the principal symbol $\sigma_k(P)$, and $Index(P)$ can be expressed in terms of (precise) topological data associated to $\sigma_k(P)$.

Due to the the way that elliptic operators arise in geometry (via "elliptic complexes"), it is worth giving a slightly different dress to this theorem.

Definition 1.3.3. A **differential complex** over a manifold M,

$$\mathcal{E} : \Gamma(E^0) \xrightarrow{P_0} \Gamma(E^1) \xrightarrow{P_1} \Gamma(E^2) \xrightarrow{P_2} \dots$$

consists of:

1. For each $k \geq 0$, a vector bundle E_k over M, with $E_k = 0$ for k large enough.
2. For each $k \geq 0$, a differential operator P_k from E_k to E_{k+1}, of some order d independent of k

such that, for all k, $P_{k+1} \circ P_k = 0$.

Example 1.3.4. Let $d_k : \Omega^k(M) \to \Omega^{k+1}(M)$ be exterior differentiation. Then $d_{k+1} \circ d_k = 0$ for all k. Therefore, the sequence of differential operators $d_k \in \mathcal{D}_1(\wedge^k T^*M, \wedge^{k+1} T^*M)$ forms a complex; it is called the de Rham complex.

Note that, from Lemma 1.2.4 it follows that for a complex of differential operators as above, the associated sequence $\sigma_{d_k}(P_k)$ of principal symbols is a complex of homomorphisms of the vector bundles $\pi^* E_k$ on M, i.e., for any $\xi_x \in T_x^* M$, the sequence

$$E_x^0 \stackrel{\sigma_d(P_0)(\xi_x)}{\longrightarrow} E_x^1 \stackrel{\sigma_d(P_1)(\xi_x)}{\longrightarrow} E_x^2 \stackrel{\sigma_d(P_2)(\xi_x)}{\longrightarrow} \dots$$

is a complex of vector space. In turn, this means that the composition of any two consecutive maps in this sequence is zero. Equivalently,

$$\mathrm{Ker}(\sigma_d(P_{k+1})(\xi_x)) \subset \mathrm{Im}(\sigma_d(P_k)(\xi_x)).$$

Definition 1.3.5. A differential complex $\mathcal{E}$ is called an **elliptic complex** if, for any $\xi_x \in T_x^* M$ non-zero, the sequence

$$E_x^0 \stackrel{\sigma_d(P_0)(\xi_x)}{\longrightarrow} E_x^1 \stackrel{\sigma_d(P_1)(\xi_x)}{\longrightarrow} E_x^2 \stackrel{\sigma_d(P_2)(\xi_x)}{\longrightarrow} \dots$$

is exact, i.e.

$$\mathrm{Ker}(\sigma_d(P_{k+1})(\xi_x)) = \mathrm{Im}(\sigma_d(P_k)(\xi_x)).$$

For a general differential complex $\mathcal{E}$, one can define

$$Z^k(\mathcal{E}) = \mathrm{Ker}(P_k), \quad B^k(\mathcal{E}) = \mathrm{Im}(P_{k_1}),$$

and the k-th cohomology groups

$$H^k(\mathcal{E}) = Z^k(\mathcal{E})/B^k(\mathcal{E}).$$

The space $H^k(M, P_*)$ defined as above, is called the k-th cohomology group of the elliptic complex.

Theorem 1.3.6. *If $\mathcal{E}$ is an elliptic complex over a compact manifold M, then all the cohomology groups $H^k(\mathcal{E})$ are finite dimensional and the resulting Euler characteristic*

$$\chi(\mathcal{E}) := \sum_k (-1)^k dim(H^k(\mathcal{E}))$$

can be expressed in terms of topological invariants of the principal symbols associated to $\mathcal{E}$.

Example 1.3.7. The De Rham complex of a manifold M is elliptic (see Exercise 1.5.10). Recall that the resulting cohomology in a degree k, i called the k-th de Rham cohomology of M, denoted $H^k_{\mathrm{dR}}(M)$. is defined to be the k-th cohomology of the de Rham complex. According to the above result, the de Rham cohomology of a compact manifold is finite dimensional. For a simpler proof of this result, involving Meyer-Vietoris sequences, we refer the reader to the book by Thornehaeve-Madsen or the book by Bott and Tu.

Example 1.3.8. Any elliptic operator $P \in \mathcal{D}_k(E, F)$ can be seen as an elliptic complex with $E^0 = E$, $E^1 = F$ and $E^k = 0$ for other k's, $P_0 = P$. Moreover, its Euler characteristic is just the index of P. Hence the last theorem seems to be a generalization of Theorem 1.3.1. However, there is a simple trick to go the other way around. This is explained in the last three exercises of this lecture.

1.4. General tool: Fredholm operators

As we have already mentioned, the aim of these lectures is to understand Theorem 1.3.2. The first few lectures will be devoted to proving that the index of any elliptic operator (over compact manifolds) is finite; after that we will spend some lectures to explain the precise meaning of "topological data associated to the symbol" (and the last lectures will be devoted to some examples). The nature of these three parts is Analysis- Topology- Geometry.

For the first part- on the finiteness of the index, we will rely on the fact that indices of operators are well behaved in the framework of Banach spaces. This is some very general theory that belongs to Functional Analysis, which we recall in this section. In the next few lectures we will show how this theory applies to our problem (on short, we have to pass from spaces of sections of vector bundles to appropriate "Banach spaces of sections" and show that our operators have the desired compactness properties).

So, for this section[3] we fix two Banach spaces $\mathbb{E}$ and $\mathbb{F}$ and we discuss Fredholm operators between them- i.e. operators which have a well-defined index. More formally, we denote by $\mathcal{L}(\mathbb{E}, \mathbb{F})$ the space of bounded (i.e. continuous) linear operators from $\mathbb{E}$ to $\mathbb{F}$ and we take the following:

Definition 1.4.1. A bounded operator $T : \mathbb{E} \to \mathbb{F}$ is called Fredholm if $\text{Ker}(A)$ and $\text{Coker}(A)$ are finite dimensional. We denote by $\mathcal{F}(\mathbb{E}, \mathbb{F})$ the space of all Fredholm operators from $\mathbb{E}$ to $\mathbb{F}$.

The index of a Fredholm operator A is defined by

$$\text{Index}(A) := \dim(\text{Ker}(A)) - \dim(\text{Coker}(A)).$$

Note that a consequence of the Fredholmness is the fact that $R(A) = \text{Im}(A)$ is closed. Here are the first properties of Fredholm operators.

Theorem 1.4.2. *Let $\mathbb{E}$, $\mathbb{F}$, $\mathbb{G}$ be Banach spaces.*

(i) If $B : \mathbb{E} \to \mathbb{F}$ and $A : \mathbb{F} \to \mathbb{G}$ are bounded, and two out of the three operators A, B and AB are Fredholm, then so is the third, and

$$\text{Index}(A \circ B) = \text{Index}(A) + \text{Index}(B).$$

[3] all the theorem stated in this section are proved in the auxiliary set of notes handed out to you

(ii) If $A : \mathbb{E} \to \mathbb{F}$ *is Fredholm, then so is* $A^* : \mathbb{F}^* \to \mathbb{E}^*$ *and* [4]

$$\mathrm{Index}(A^*) = -\mathrm{Index}(A).$$

(iii) $\mathcal{F}(\mathbb{E}, \mathbb{F})$ *is an open subset of* $\mathcal{L}(\mathbb{E}, \mathbb{F})$*, and*

$$\mathrm{Index} : \mathcal{F}(\mathbb{E}, \mathbb{F}) \to \mathbb{Z}$$

is locally constant.

What will be important for us is an equivalent description of Fredholm operators, in terms of compact operators. First we recall the following:

Definition 1.4.3. A linear map $T : \mathbb{E} \to \mathbb{F}$ is said to be compact if for any bounded sequence $\{x_n\}$ in $\mathbb{E}$, $\{T(x_n)\}$ has a convergent subsequence.

Equivalently, compact operators are those linear maps $T : \mathbb{E} \to \mathbb{F}$ with the property that $T(B_{\mathbb{E}}) \subset \mathbb{F}$ is relatively compact, where $B_{\mathbb{E}}$ is the unit ball of $\mathbb{E}$. Here are the first properties of compact operators.

We point out the following improvement/consequence of the Fredholm alternative for compact operators (discussed in the appendix- see Theorem **??** there).

Theorem 1.4.4. *Compact perturbations do not change Fredholmness and do not change the index, and zero index is achieved only by compact perturbations of invertible operators.*

More precisely:

(i) If $K \in \mathcal{K}(\mathbb{E}, \mathbb{F})$ *and* $A \in \mathcal{F}(\mathbb{E}, \mathbb{F})$*, then* $A + K \in \mathcal{F}(\mathbb{E}, \mathbb{F})$ *and* $\mathrm{Index}(A + K) = \mathrm{Index}(A)$.

(ii) If $A \in \mathcal{F}(\mathbb{E}, \mathbb{F})$*, then* $\mathrm{Index}(A) = 0$ *if and only if* $A = A_0 + K$ *for some invertible operator* A_0 *and some compact operator* K*.*

Finally, there is yet another relation between Fredholm and compact operators, know as the Atkinson characterization of Fredholm operators:

Theorem 1.4.5. *Fredholmness = invertible modulo compact operators.*

More precisely, given a bounded operator $A : \mathbb{E} \to \mathbb{F}$*, the following are equivalent:*

(i) A *is Fredholm.*

(ii) A *is invertible modulo compact operators, i.e. there exist and operator* $B \in \mathcal{L}(\mathbb{F}, \mathbb{E})$ *and compact operators* K_1 *and* K_2 *such that*

$$BA = 1 + K_1, AB = 1 + K_2.$$

[4] here, $A^*(\xi)(e) = \xi(A(e)))$

1.5. Some exercises

We advise you to do the following exercises (in this order): Exercise 1.5.2, 1.5.3, 1.5.9, 1.5.7 , 1.5.12, 1.5.4, 1.5.8 (you should do at least three of them!). The take home exercise is Exercise 1.5.1.

Exercise 1.5.1. This exercise provides another possible (inductive) definition of the spaces $\mathcal{D}_k(M)$. For each $f \in C^\infty(M)$, let $m_f \in \mathrm{End}(C^\infty(M))$ be the "multiplication by f" operator. The commutator of two operators P and Q is the new operator $[P, Q] = P \circ Q - Q \circ P$.

Starting with $\mathcal{D}_{-1}(M) = 0$, show that $\mathcal{D}_k(M)$ is the space of linear operators P with the property that

$$[P, m_f] \in \mathcal{D}_{k-1}(M) \quad \forall f \in C^\infty(M).$$

Exercise 1.5.2. Let $P \in \mathcal{D}_k(U)$ and $Q \in \mathcal{D}_l(U)$. Then the composition QP belongs to $\mathcal{D}_{k+l}$ and

$$\sigma_{k+l}(QP) = \sigma_l(Q)\sigma_k(P).$$

Exercise 1.5.3. Let V be a vector field on M. Show that $\partial_V : f \mapsto Vf := df{\cdot}V$ is a first order differential operator on M. Show that its principal symbol is given by $\sigma_1(P)(x, \xi) = \xi(v_x)$.

Exercise 1.5.4. Show that any differential operator $P \in \mathcal{D}_1(M)$ can be written as

$$P(\phi) = f\phi + \partial_V(\phi)$$

for some unique function $f \in C^\infty(M)$ and vector field V on M.

Exercise 1.5.5. Let $P \in \mathcal{D}_k(M)$ and $Q \in \mathcal{D}_l(M)$. Show that $QP \in \mathcal{D}_{k+l}(M)$. Moreover, $\sigma_{k+l}(QP) = \sigma_l(Q)\sigma_k(P)$. Hint: use reduction to charts.

Exercise 1.5.6. Lemma 1.1.8 gives a different description of the principal symbol without reference to charts. Show that, actually, for any $P \in \mathcal{D}_k(M)$ and any $f \in C^\infty(M)$ and all $\varphi \in C^\infty(M)$

$$f(x)\sigma_k(P)((d\varphi)_x) = \lim_{t\to\infty} t^{-k} e^{-it\varphi(x)} P(e^{it\varphi} f)(x). \tag{1.4}$$ [5]

Exercise 1.5.7. Recall that at the beginning of section 1.2, we associated to a section $C \in \Gamma(\underline{\mathrm{Hom}}(E, F))$ an operator (denoted by the same letter) $C : \Gamma(E) \to \Gamma(F)$. Show that this construction defines a 1-1 correspondence between sections of $\underline{\mathrm{Hom}}(E, F)$ and maps from $\Gamma(E)$ to $\Gamma(F)$ which are $C^\infty(M)$-linear.

Exercise 1.5.8. Show that, for any two vector bundles E and F over a manifold M, the assignment

$$U \mapsto \mathcal{D}_k(E|_U, F|_U)$$

($U \subset M$ open) is a sheaf on M.

Exercise 1.5.9. Show that the principal symbol of exterior differentiation $d : \Gamma(\Lambda^k T^*M) \to \Gamma(\Lambda^{k+1} T^*M)$ is given by

$$\sigma_1(d)(x, \xi) : \wedge^k T^*_x M \to \wedge^{k+1} T^*_x M, \;\; \omega \mapsto i\xi \wedge \omega.$$

[5] recall that $(d\varphi)_x \in T^*_x M$ sends $X_x \in T_x M$ to $T_x\varphi(X_x)$

Exercise 1.5.10. Let V be a finite dimensional complex vector space. Let $v \in V \setminus \{0\}$. Show that the complex of linear maps $T_k : \wedge^k V \to \wedge^{k+1} V$, $x \mapsto v \wedge x$, is exact.

Deduce that the DeRham complex a manifold is an elliptic complex.

In the following three exercises we explain how to relate ellipticity for elliptic complexes, and their Euler characteristic, to ellipticity of differential operators, and their index. We start with a differential complex $\mathcal{E}$ as before and we fix hermitian inner products on each E^k and a density on M. This allows us to talk about the adjoints of P_k and to form the Laplacians

$$\Delta_k := P_k^* P_k + P_{k-1} P_{k-1}^* : \Gamma(E^k) \to \Gamma(E^k).$$

It will be useful to put everything together and consider

$$E = \oplus_k E^k, P = \oplus_k P_k \in \mathcal{D}_d(E), \Delta = P^* P + P P^* = \oplus_k \Delta_k \in \mathcal{D}_{2d}(E).$$

Exercise 1.5.11.

1. Show that Δ is self-adjoint (i.e. it coincides with its formal adjoint), and then deduce that

$$\mathrm{Ker}(\Delta) \oplus \mathrm{Im}(\Delta) \subset \Gamma(E). \tag{1.5}$$

   (the only thing you have to show here is that the sum inside $\Gamma(E)$ is direct).
2. Then show that

$$\mathrm{Ker}(\Delta) = \mathrm{Ker}(P) \cap \mathrm{Ker}(P^*), \mathrm{Im}(\Delta) \subset \mathrm{Im}(P) \oplus \mathrm{Im}(P^*).$$

   and that the map

$$\mathrm{Ker}(\Delta_k) \ni s \mapsto s \bmod B^k(\mathcal{E}) \in H^k(\mathcal{E})$$

   is an injection.
3. Finally, show that if (1.5) becomes equality, then also the last inclusion becomes equality, and the last map becomes an isomorphism.

Exercise 1.5.12. Show that $\mathcal{E}$ is an elliptic complex if and only if Δ is an elliptic operator.

From the general properties of elliptic operators (to be developed later in the course), we will deduce that the inclusion (1.5) is actually an equality for any self-adjoint elliptic operator Δ. Hence, in this case, $\chi((E))$ will be equal to

$$\sum_k (-1)^k \mathrm{dim}(\mathrm{Ker}(\Delta_k)).$$

To relate this to the index of an operator, we introduce

$$E^+ = \oplus_{k-\mathrm{even}} E^k, \quad E^- = \oplus_{k-\mathrm{odd}} E^k, \quad \Delta_+ = \oplus_{k-\mathrm{even}} \Delta_k \in \mathcal{D}_{2d}(E^+, E^-).$$

Exercise 1.5.13. Show that, if the inclusion (1.5) becomes equality, then

$$\chi(\mathcal{E}) = \mathrm{Index}(\Delta_+).$$