Analysis on Manifolds
Lecture notes for the 2009/2010 Master Class

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LECTURE 6
Appendix: A special map in symbol space

6.4. The exponential of a differential operator

In these notes we assume that A is a symmetric $n \times n$ matrix with complex entries and with $\mathrm{Re}\,\langle A\xi, \xi\rangle \geq 0$ for all $\xi \in \mathbb{R}^n$. Here $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the standard bilinear pairing $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$. The function

$$x \mapsto e^{-\langle A\xi,\xi\rangle} \tag{6.1}$$

is bounded on $\mathbb{R}^n$. Moreover, every derivative of (6.1) is polynomially bounded. Hence, multiplication by the function (6.1) defines a continuous linear endomorphism $M(A)$ of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. As the operator $M(A)$ is symmetric with respect to the usual pairing $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ defined by integration, it follows that $M(A)$ has a unique extension to a continuous linear endomorphism $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$.

Clearly, $M(A)$ leaves each subspace $L^2_s(\mathbb{R}^n)$, for $s \in \mathbb{R}$, invariant and restricts to a bounded linear endomorphism with operator norm at most 1 on it.

We define $E(A)$ to be the unique continuous linear endomorphism of $\mathcal{S}'(\mathbb{R}^n)$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{S}'(\mathbb{R}^n) & \xrightarrow{M(A)} & \mathcal{S}'(\mathbb{R}^n) \\ {\scriptstyle \mathcal{F}}\uparrow & & \uparrow{\scriptstyle \mathcal{F}} \\ \mathcal{S}'(\mathbb{R}^n) & \xrightarrow{E(A)} & \mathcal{S}'(\mathbb{R}^n) \end{array}$$

As $\mathcal{F}$ restricts to a topological automorphism of $\mathcal{S}(\mathbb{R}^n)$ and to an isometric automorphism isomorphism from $H_s(\mathbb{R}^n)$ onto $L^2_s(\mathbb{R}^n)$, it follows that $E(A)$ restricts to a bounded endomorphism of $H_s(\mathbb{R}^n)$ of operator norm at most 1. Furthermore, $E(A)$ restricts to a continuous linear endomorphism of $\mathcal{S}(\mathbb{R}^n)$.

If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then clearly $\partial_t M(tA)\varphi + \langle A\xi, \xi\rangle M(tA)\varphi = 0$. By application of the inverse Fourier transform, we see that for a given function $f \in \mathcal{S}$ the function $f_t := E(tA)f$ satisfies:

$$\partial_t f_t = -\langle AD, D\rangle f_t, \quad \text{where} \quad -\langle AD, D\rangle = \sum_{ij} A_{ij}\partial_j\partial_i.$$

We note that $f_0 = f$, so that f_t may be viewed as a solution to the associated Cauchy problem with initial datum f.

For obvious reasons, we will write

$$E(tA) = E^{-t\langle AD,D\rangle}$$

from now on. The purpose of these notes is to derive estimates for E which are needed for symbol calculus.

Lemma 6.4.1. *The operator $e^{\langle AD,D\rangle} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ commutes with the translations T^*_a translations and the partial differentiations ∂_j, for $a \in \mathbb{R}^n$ and $1 \leq j \leq n$.*

Proof This is obvious from the fact that translation and partial differentiation become multiplication with a function after Fourier transform; each such multiplication operator commutes with $M(A)$. □

Lemma 6.4.2. *Assume that A is non-singular. Then the tempered function $x \mapsto e^{-\langle Ax,x\rangle/2}$ has Fourier transform*

$$\mathcal{F}(e^{-\langle Ax,x\rangle/2}) = c(A)e^{-\langle B\xi,\xi\rangle/2}$$

with $c(A)$ a non-zero constant.

Remark 6.4.3. It can be shown that $c(A) = (\det A)^{-1/2}$, where a suitable analytic branch of the square root must be chosen. However, we shall not need this here.

Proof For $v \in \mathbb{R}^n$ let ∂_v denote the directional derivative in the direction v. Thus, $\partial_v f(x) = df(x)v$. Then the tempered distribution f given by the function $x \mapsto \exp(-\langle Ax,x\rangle/2)$ satisfies the differential equations $\partial_v f = -\langle Av,x\rangle f$. It follows that the Fourier transform $\widehat{f}$ satisfies the differential equations $\langle v,\xi\rangle\widehat{f} = -\partial_{Av}\widehat{f}$ for all $v \in \mathbb{R}^n$, or, equivalently, $\partial_v f = -\langle Bv,\xi\rangle f$. This implies that the tempered distribution

$$\varphi = e^{\langle B\xi,\xi\rangle/2}\widehat{f}$$

has all partial derivatives equal to zero, hence is the tempered distribution coming from a constant function $c(A)$. □

Proposition 6.4.4. *For each $k \in \mathbb{N}$ there exists a positive constant $C_k > 0$ such that the following holds. Let A be a complex symmetric $n \times n$-matrix with* $\operatorname{Re} A \geq 0$. *Let $f \in \mathcal{S}(\mathbb{R})$ and let $x \in \mathbb{R}^n$ be a point such that the distance $d(x)$ from x to* $\operatorname{supp} u$ *is at least one. Then*

$$|e^{-\langle AD,D\rangle}f(x)| \leq C_k d(x)^{-k}\|A\|^{s+k} \max_{|\alpha|\leq 2s+k} \sup |D^\alpha f|. \tag{6.2}$$

Proof The function $e^{-\langle A\xi,\xi\rangle}\widehat{f}$ in $\mathcal{S}(\mathbb{R}^n)$ depends continuously on A and hence, so does $e^{\langle AD,D\rangle}f$. We may therefore assume that A is non-singular.

As $e^{-\langle AD,D\rangle}$ commutes with translation, we may as well assume that $x = 0$. We assume that f has support outside the unit ball B in $\mathbb{R}^n$.

For each j let Ω_j denote the set points y on the unit sphere $S = \partial B$ with $|\langle y,e_j\rangle| > 1/2\sqrt{n}$. Then the U_j form an open cover of S. Let $\{\psi_j\}$ be a partition of unity subordinate to this covering and define $\chi_j : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ by $\chi_j(y) = \psi_j(y/\|y\|)$. Then each of the functions $f_j = \chi_j f$ satisfies the same hypotheses as f and in addition, $|\langle y,e_j\rangle| \geq |y|/2\sqrt{n}$ for $y \in \operatorname{supp} f_j$. As $f = \sum_j f_j$, it suffices to prove the estimate for each of the f_j. Thus, without loss of generality, we may assume from the start that there exists a unit vector $v \in \mathbb{R}^n$ such that $|\langle y,v\rangle| \geq |y|/2\sqrt{n}$ for all $y \in \operatorname{supp} f$.

We now observe that

$$e^{-\langle AD,D\rangle}f(0) = \int e^{-\langle A\xi,\xi\rangle}\widehat{f}(\xi)\,d\xi = c\int e^{-\langle By,y\rangle/4}f(y)\,dy,$$

where $B = A^{-1}$. The idea is to apply partial differentiation with the directional derivative ∂_{Av} to this formula. For this we note that

$$e^{-\langle By,y\rangle/4} = -\frac{2}{\langle v,y\rangle}\partial_{Av}e^{-\langle By,y\rangle/4}$$

on supp f, so that, for each $j \geq 0$,

$$\begin{aligned} e^{-\langle AD,D\rangle}f(0) &= c\,2^j\int e^{-\langle By,y\rangle/4}\,[\langle v,y\rangle^{-1}\partial_{Av}]^j f(y)\,dy \\ &= [e^{-\langle AD,D\rangle}(\langle v,\,\cdot\,\rangle^{-1}\partial_{Av})^j f](0). \end{aligned}$$

By using the Sobolev lemma, we find, for each natural number $s > n/2$, that

$$\begin{aligned} |e^{-\langle AD,D\rangle}f(0)| &\leq C'\max_{|\alpha|\leq s}\|D^\alpha e^{-\langle AD,D\rangle}(\langle v,\,\cdot\,\rangle^{-1}\partial_{Av})^j f\|_{L^2} \\ &= C'\max_{|\alpha|\leq s}\|e^{-\langle AD,D\rangle}D^\alpha(\langle v,\,\cdot\,\rangle^{-1}\partial_{Av})^j f\|_{L^2} \\ &\leq C'\max_{|\alpha|\leq s}\|D^\alpha(\langle v,\,\cdot\,\rangle^{-1}\partial_{Av})^j f\|_{L^2}. \end{aligned}$$

By application of the Leibniz rule and using that $|\langle v,y\rangle| \geq \|y\|/2\sqrt{n}$ and $\|y\| \geq d \geq 1$ for $y \in \operatorname{supp} f$, we see that, for $j > 2n$,

$$|e^{-\langle AD,D\rangle}f(0)| \leq C_j'\|A\|^j d^{n/2-j}\max_{|\alpha|\leq s+j}\sup|D^\alpha f|.$$

We now take $j = s + k$ to obtain the desired estimate. □

Our next estimate is independent of supports.

Lemma 6.4.5. *Let $s > n/2$ be an integer. Then there exists a positive constant with the following property. Let $A \in \mathrm{M}_n(\mathbb{C})$ be symmetric with $\mathrm{Re}\,A \geq 0$. Then for all $f \in \mathcal{S}(\mathbb{R}^n)$ and all $x \in \mathbb{R}^n$,*

$$|e^{-\langle AD,D\rangle}f(x)| \leq C\max_{|\alpha|\leq s}\|D^\alpha f\|_{L^2}.$$

Proof By the Sobolev lemma we have

$$\begin{aligned} |e^{-\langle AD,D\rangle}f(x)| &\leq C\max_{|\alpha|\leq s}\|D^\alpha e^{-\langle AD,D\rangle}f\|_{L^2} \\ &= C\max_{|\alpha|\leq s}\|e^{-\langle AD,D\rangle}D^\alpha f\|_{L^2} \\ &\leq C\max_{|\alpha|\leq s}\|D^\alpha f\|_{L^2} \end{aligned}$$

□

Corollary 6.4.6. *Let $s > n/2$ be an integer and let $C > 0$ be the constant of Lemma 6.4.5. Let $\mathcal{K} \subset \mathbb{R}^n$ a compact subset. Let $A \in \mathrm{M}_n(\mathbb{C})$ be symmetric and $\mathrm{Re}\,A \geq 0$. Then for every $f \in C^s_{\mathcal{K}}(\mathbb{R}^n)$, the distribution $e^{-\langle AD,D\rangle}f$ is a continuous function, and*

$$|e^{-\langle AD,D\rangle}f(x)| \leq C\sqrt{\mathrm{vol}\,(\mathcal{K})}\max_{|\alpha|\leq s}\sup|D^\alpha f|, \qquad (x \in \mathbb{R}^n).$$

Proof We first assume that $f \in C^\infty_{\mathcal{K}'}(\mathbb{R}^n)$ with $\mathcal{K}'$ compact. Then by straightforward estimation,

$$\|D^\alpha f\|_{L^2} \leq \mathrm{vol}\,(\mathcal{K}') \sup |D^\alpha f|$$

and the estimate follows with $\mathcal{K}'$ instead of $\mathcal{K}$. Let now $f \in C^s_{\mathcal{K}}(\mathbb{R}^n)$. Then by regularization there is a sequence $f_n \in C^\infty_{\mathcal{K}_n}(\mathbb{R}^n)$, with $\mathcal{K}_n \to \mathcal{K}$ and $f_n \to f$ in $C^s(\mathbb{R}^n)$. By the above estimate, the sequence $e^{-\langle AD,D\rangle} f_n$ is Cauchy in $C(\mathbb{R}^n)$. By passing to a subsequence we may arrange that the sequence already converges to a limit φ in $C(\mathbb{R}^n)$. By continuity of $e^{-\langle AD,D\rangle}$ in $\mathcal{S}'(\mathbb{R}^n)$ it follows that $\varphi = e^{-\langle AD,D\rangle} f$. The required estimate for φ now follows from the similar estimates for $e^{-\langle AD,D\rangle} f_n$ by passing to the limit for $n \to \infty$. □

In the sequel we shall frequently refer to a principle that is made explicit in the following lemma.

Lemma 6.4.7. *Let $L : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be a continuous linear endomorphism. Let V, W be linear subspaces of $\mathcal{S}'(\mathbb{R}^n)$ equipped with locally convex topologies for which the inclusion maps are continuous. Assume that $C^\infty_c(\mathbb{R}^n)$ is dense in V and that W is complete. If L maps $C^\infty_c(\mathbb{R}^n)$ into W, and the restricted map $L_0 : C^\infty_c(\mathbb{R}^n) \to W$ is continuous with respect to the V-topology on the first space, then $L(V) \subset W$.*

Proof The restricted map L_0 has a unique extension to a continuous linear map $L_1 : V \to W$. Thus, it suffices to show that $L_1 = L$ on V. Fix $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then, the linear functional $\langle\,\cdot\,,\varphi\rangle$ is continuous on W. It follows that the linear functional μ on $C^\infty_c(\mathbb{R}^n)$ given by $\mu(f) = \langle L_1 f, \varphi\rangle$ is continuous linear for the V-topology.

From the assumption about the continuity of L is follows that the functional $\nu : f \mapsto \langle Lf, \varphi\rangle$ is continuous for the $\mathcal{S}'(\mathbb{R}^n)$ topology. In particular, this implies that ν is continuous for the V-topology.

As $\mu = \nu$ on $C^\infty_c(\mathbb{R}^n)$ and $C^\infty_c(\mathbb{R}^n)$ is dense in V it follows that $L_1 = L$ on V. □

If $p \in \mathbb{N}$ we denote by $C^p_b(\mathbb{R}^n)$ the Banach space of p times continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{C}$ with $\max_{|\alpha|\leq p} \sup |D^\alpha f| < \infty$.

Proposition 6.4.8. *Let $s > n/2$ be an integer. Then there exists a constant $C > 0$ with the following property. For each symmetric $A \in \mathrm{M}_n(\mathbb{C})$ with $\mathrm{Re}\, A \geq 0$ and all $f \in C^{2s}_b(\mathbb{R}^n)$ the distribution $e^{-\langle AD,D\rangle} f$ is continuous and*

$$|e^{-\langle AD,D\rangle} f(x)| \leq C\|A\|^s \max_{|\alpha|\leq 2s} \sup |D^\alpha f|.$$

For x with $d(x) := d(x, \mathrm{supp}\, f) \geq 1$ the stronger estimate (6.2) is valid.

Proof As in the proof of the previous corollary, we first prove the estimate for $f \in C^\infty_c(\mathbb{R}^n)$. By translation invariance we may as well assume that $x = 0$.

We fix a function $\chi \in C^\infty_c(\mathbb{R}^n)$ which equals 1 on the unit ball and has support contained in $\mathcal{K} = B(0;2)$ and such that $0 \leq \chi \leq 1$. Then the desired estimate follows from combining the estimate of Corollary 6.4.6 for χf with the estimate of Proposition 6.4.4 with $k = 0$ for $(1-\chi) f$.

By density of $C_c^\infty(\mathbb{R}^n)$ in $C_c^s(\mathbb{R}^n)$ it follows that $e^{-\langle AD,D\rangle}$ maps $C_c^s(\mathbb{R}^n)$ continuously into $C_b(\mathbb{R}^n)$, with the desired estimate (apply Lemma 6.4.7). As $C_c^s(\mathbb{R}^n)$ is not dense in $C_b^s(\mathbb{R}^n)$ we need an additional argument to pass to the latter space.

Let χ be as above, and put $\chi_n(x) = \chi(x/n)$. Then it is readily seen that $\chi_n f \to f$ in $\mathcal{S}'(\mathbb{R}^n)$. Hence $e^{-\langle AD,D\rangle} f_n \to e^{-\langle AD,D\rangle} f$ in $\mathcal{S}'(\mathbb{R}^n)$. It follows by application of Proposition 6.4.4 that for each compact subset $\mathcal{K} \subset \mathbb{R}^n$ the sequence $e^{-\langle AD,D\rangle} f_n|_{\mathcal{K}}$ is Cauchy in $C(\mathcal{K})$. This implies that $e^{-\langle AD,D\rangle} f_n$ converges to a limit φ in the Fréchet space $C(\mathbb{R}^n)$. In particular, φ is also the limit in $\mathcal{S}'(\mathbb{R}^n)$ so that $e^{-\langle AD,D\rangle} f = \varphi$ is a continuous function.

We now note that by application of the Leibniz rule,

$$\sup |D^\alpha f_n| \leq \sup |D^\alpha f| + \mathcal{O}(1/n).$$

Hence the desired estimate for f follows from the similar estimate for f_n by passing to the limit. □

Theorem 6.4.9. *Let $s > n/2$ be an integer and let $k \in \mathbb{N}$. Then there exists a constant $C_k > 0$ with the following property. For each symmetric $A \in \mathrm{M}_n(\mathbb{C})$ with $\mathrm{Re}\, A \geq 0$ and all $f \in C_b^{2s+2k}(\mathbb{R}^n)$ the function $e^{-\langle AD,D\rangle} f$ is continuous, and*

$$|e^{-\langle AD,D\rangle} f(x) - \sum_{j<k} \frac{1}{j!}(-\langle AD, D\rangle)^j f(x)| \leq C_k \|A\|^s \max_{|\alpha|\leq 2s} \sup |D^\alpha \langle AD, D\rangle^k f|.$$

Proof Let $R_k(A)f(x)$ denote the expression between absolute value signs on the left-hand side of the above estimate. We first prove the estimate for a function $f \in C_c^\infty(\mathbb{R}^n)$. The function

$$f_t(x) := e^{-\langle tAD,D\rangle}(x)$$

is smooth in $(t, x) \in [0, \infty) \times \mathbb{R}^n$ and satisfies the differential equation

$$\partial_t f_t(x) = -\langle AD, D\rangle f_t(x).$$

By application of Taylor's formula with remainder term with respect to the variable t at $t = 0$, we find that

$$f_1(x) = \sum_{j<k} \partial_t^j f_t(x) - \frac{1}{(k-1)!} \int_0^1 (1-t)^{k-1}\, \partial_t^k f_t(x)\, dt.$$

This leads to

$$\begin{aligned} R_k(A)f(x) &= \frac{1}{(k-1)!} \int_0^1 (1-t)^{k-1}\, (-\langle AD, D\rangle)^k\, f_t(x)\, dt \\ &= \frac{1}{(k-1)!} \int_0^1 (1-t)^{k-1}\, e^{-t\langle AD,D\rangle}(-\langle AD, D\rangle)^k\, f(x)\, dt. \end{aligned}$$

By estimation under the integral sign, making use of Proposition 6.4.8, we now obtain the desired estimate for $f \in C_c^\infty(\mathbb{R}^n)$. For the extension of the estimate to $C_c^{2s+2k}(\mathbb{R}^n)$ and finally to $C_b^{2s+2k}(\mathbb{R}^n)$ we proceed as in the proof of Proposition 6.4.8. □

6.5. The exponential of a differential operator in symbol space

Let $\mathcal{K}$ be a compact subset of $\mathbb{R}^n$ and let $d \in \mathbb{R}$. Then the space of symbols $S^d_{\mathcal{K}}(\mathbb{R}^n)$ is a subspace of the space of tempered distributions $\mathcal{S}'(\mathbb{R}^{2n})$ with continuous inclusion map. Indeed, if $p \in S^d_{\mathcal{K}}(\mathbb{R}^n)$, then for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\begin{aligned}
\langle p, \varphi \rangle &= \int_{\mathbb{R}^{2n}} p(x,\xi)\, \varphi(x,\xi)\, dx\, d\xi \\
&\leq \int_{R^{2n}} (1 + \|\xi\|)^{-d-n-1} |p(x,\xi)| (1 + |(x,\xi)|)^{|d|+n+1} |\varphi(x,\xi)|\, dx\, d\xi \\
&\leq C\, \mu^d_{\mathcal{K},0}(p)\, \nu_{|d|+n+1,0}(\varphi),
\end{aligned}$$

with $C > 0$ only depending on $n, \mathcal{K}$ and d.

We consider the second order differential operator

$$\langle D_x, \partial_\xi \rangle = i \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \frac{\partial}{\partial \xi_j}.$$

Thus, with notation as in the previous section, $\langle D_x, \partial_\xi \rangle = -\langle AD, D \rangle$, where

$$A = i \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix},$$

with I_n the $n \times n$ identity matrix. The matrix A is complex, symmetric, and has real part equal to zero, hence fulfills all conditions of the previous section. Moreover, its operator norm $\|A\|$ equals 1.

In the rest of this section we will discuss the action of $e^{\langle D_x, \partial_\xi \rangle}$ on $S^d_{\mathcal{K}}(\mathbb{R}^n)$. The following lemma is obvious.

Lemma 6.5.1. *For each $k \in \mathbb{N}$,*

$$\frac{1}{k!} \langle D_x, \partial_\xi \rangle = \sum_{|\alpha|=k} \frac{1}{\alpha!} D_x^\alpha \partial_\xi^\alpha.$$

In particular, $\langle D_x, \partial_\xi \rangle$ defines a continuous linear map $S^d(\mathbb{R}^n) \to S^{d-k}(\mathbb{R}^n)$, preserving supports.

Theorem 6.5.2. *Let $k \in \mathbb{N}$. Then*

$$e^{\langle D_x, \partial_\xi \rangle} - \sum_{|\alpha|<k} \frac{1}{\alpha!} D_x^\alpha \partial_\xi^\alpha, \tag{6.3}$$

originally defined as an endomorphism of $\mathcal{S}'(\mathbb{R}^n)$, maps $S^d_{\mathcal{K}}(\mathbb{R}^n)$ continuous linearly into $S^{d-k}(\mathbb{R}^n)$. In particular, $e^{\langle D_x, \partial_\xi \rangle}$ restricts to a continuous linear map $S^d_{\mathcal{K}}(\mathbb{R}^n) \to S^d(\mathbb{R}^n)$.

Before turning to the proof of the theorem, we list a corollary that will be important for applications.

Corollary 6.5.3. *Let $p \in S^d_{\mathcal{K}}(\mathbb{R}^n)$. Then $e^{\langle D_x, \partial_\xi \rangle} p \in S^d(\mathbb{R}^n)$ and*

$$e^{\langle D_x, \partial_\xi \rangle} p \sim \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} D_x^\alpha \partial_\xi^\alpha p.$$

We will prove Theorem 6.5.2 through a number of lemmas of a technical nature. The next lemma will be used frequently for extension purposes.

Lemma 6.5.4. *Let $\mathcal{K} \subset U$ be compact and let $d < d'$. Then the space $C^\infty_{\mathcal{K},c}(\mathbb{R}^n)$ is dense in $S^d_{\mathcal{K}}(U)$ for the topology of $S^{d'}_{\mathcal{K}}(U)$.*

Proof Let $p \in S^d_{\mathcal{K}}(U)$. Select $\psi \in C^\infty_c(\mathbb{R}^n)$ such that $\psi = 1$ on a neighborhood of 0. Put $\psi_n(\xi) = \psi(\xi/n)$ and

$$p_n(x,\xi) = \psi_n(\xi)p(x,\xi).$$

Then by an application of the Leibniz rule in a similar fashion as in the proof of Lemma 4.1.9, it follows that $\nu^{d'}_{\mathcal{K},k}(p_n - r) \to 0$ as $n \to \infty$, for each $k \in \mathbb{N}$. $\square$

The expression (6.3) is abbreviated by $R_k(D)$. It will be convenient to use the notation

$$C^\infty_{\mathcal{K},c}(\mathbb{R}^{2n}) := \{f \in C^\infty_c(\mathbb{R}^{2n}) \mid \operatorname{supp} f \subset \mathcal{K} \times \mathbb{R}^n\}.$$

Lemma 6.5.5. *Let $k \in \mathbb{N}$. Then for each $d < k$ the map $R_k(D)$ maps $S^d_{\mathcal{K}}(\mathbb{R}^n)$ continuous linearly into $C_b(\mathbb{R}^{2n})$.*

Proof Let $s > n/2$ be an integer. Let $f \in C^\infty_{\mathcal{K},c}(\mathbb{R}^{2n})$. Then by Theorem 6.4.9,

$$\begin{aligned}
|R_k(D)f(x,\xi)| &\leq C_k \max_{|\alpha|+|\beta|\leq 2s} \sup_{\mathcal{K}\times\mathbb{R}^n} |D^\alpha_x \partial^\beta_\xi \langle D_x, \partial_\xi\rangle^k f(x,\xi)| \\
&\leq C'_k \max_{|\alpha|+|\beta|\leq 2s, |\gamma|=k} \sup_{\mathcal{K}\times\mathbb{R}^n} |D^{\alpha+\gamma}_x \partial^{\beta+\gamma}_\xi f(x,\xi)| \\
&\leq C'_k \max_{|\alpha|+|\beta|\leq 2s, |\gamma|=k} \sup_{\mathcal{K}\times\mathbb{R}^n} (1+\|\xi\|)^{d-k}\, \nu^d_{\mathcal{K},2s+2k}(f) \\
&\leq C'_k\, \nu^d_{\mathcal{K},2s+2k}(f).
\end{aligned}$$

It follows that the map $R_k(D)$ is continuous $C^\infty_{\mathcal{K},c}(\mathbb{R}^{2n}) \to C_b(\mathbb{R}^{2n})$, with respect to the $S^d_{\mathcal{K}}(\mathbb{R}^n)$-topology on the first space, for *each* $d < k$.

Let now $d < k$ and fix d' with $d < d' < k$. Then by density of $C^\infty_{\mathcal{K},c}(\mathbb{R}^{2n})$ in $S^d_{\mathcal{K}}(\mathbb{R}^{2n})$ for the $S^{d'}_{\mathcal{K}}(\mathbb{R}^{2n})$-topology, it follows by application of Lemma 6.4.7 that $R_k(D)$ maps $S^d_{\mathcal{K}}(\mathbb{R}^n)$ to $C_b(\mathbb{R}^n)$ with continuity relative to the $S^{d'}_{\mathcal{K}}(\mathbb{R}^n)$-topology on the domain. As this topology is weaker than the original topology on $S^d_{\mathcal{K}}(\mathbb{R}^n)$, the result follows. $\square$

Lemma 6.5.6. *Let $d \in \mathbb{R}$ and assume that $k > |d|$. Let s be an integer $> n/2$. Then there exists a constant $C > 0$ such that for all $f \in C^\infty_{\mathcal{K},c}(\mathbb{R}^n)$ and all $(x,\xi) \in \mathbb{R}^{2n}$ with $\|\xi\| \geq 4$ we have*

$$|R_k(D)f(x,\xi)| \leq C(1+\|\xi\|)^{|d|-k}\nu^d_{\mathcal{K},2s+2k}(f). \tag{6.4}$$

Proof Let $\chi \in C^\infty_c(\mathbb{R}^n)$ be a smooth function which is identically 1 on the unit ball of $\mathbb{R}^n$, and has support inside the ball $B(0;2)$. For $t > 0$ we define the function $\chi_t \in C^\infty_c(\mathbb{R}^n)$ by $\chi_t(\xi) = \chi(t^{-1}\xi)$. Then $\chi_t(\xi)$ is identically 1 on $B(0;t)$ and has support inside the ball $B(0;2t)$. We agree to write $\psi = 1-\chi$ and $\psi_t(\xi) = \psi(t^{-1}\xi)$. In the following we will frequently use the obvious equalities

$$\sup|\partial^\alpha_\xi \chi_t| = t^{-|\alpha|}\sup|\partial^\alpha_\xi \chi|, \qquad \sup|\partial^\alpha_\xi \psi_t| = t^{-|\alpha|}\sup|\partial^\alpha_\xi \psi|.$$

Let $f \in C^\infty_{\mathcal{K},c}(\mathbb{R}^n)$. Then f is a Schwartz function, hence $e^{\langle D_x, \partial_\xi\rangle} f$ is a Schwartz function as well, and therefore, so is $R_k(D)f$. For $t > 0$ we agree to write $f_t(x,\xi) = \chi_t(\xi) f(x,t)$ and $g_t(x,\xi) = \psi_t(\xi)) f(x,\xi)$. Then $f = f_t + g_t$. From now on we assume that $(x,\xi) \in \mathbb{R}^{2n}$, that $\|\xi\| \geq 4$ and $t = \frac{1}{4}\|\xi\|$.

We will complete the proof by showing that both the values $|R_k(D) f_t(x,\xi)|$ and $|R_k(D) g_t(x,\xi)|$ can be estimated by $C' \nu^d_{\mathcal{K},2s+k}(f)$ with $C' > 0$ a constant independent of f, x, ξ. We start with the first of these functions. As f_t has support inside $B(0; 2t) = B(0; \|\xi\|/2)$, it follows that $d(\xi, \operatorname{supp} f_t) \geq \|\xi\|/2 \geq 2$. In view of Proposition 6.4.4 it follows that there exists a constant $C_k > 0$, only depending on k, such that

$$\begin{aligned} |R_k(D) f(x,\xi)| &= |e^{\langle D_x, \partial_\xi\rangle} f(x,\xi)| \\ &\leq C_k (\|\xi\|/2)^{-k} \max_{|\alpha|+|\beta| \leq 2s+k} \sup |D_x^\alpha \partial_\xi^\beta (\chi_t f)| \\ &\leq C_k' (1 + \|\xi\|)^{-k} \max_{|\alpha|+|\beta_1+\beta_2| \leq 2s+k} \sup |\partial_\xi^{\beta_1} \chi_t \, D_x^\alpha \partial_\xi^{\beta_2} f|, \end{aligned}$$

with $C_k' > 0$ independent of f, x and ξ. For $\eta \in \operatorname{supp} \chi_t$ we have $\|\eta\| \leq \|\xi\|/2$, so that

$$\begin{aligned} |\partial_\xi^{\beta_1} \chi_t(\eta) \, D_x^\alpha \partial_\xi^{\beta_2} f(y,\eta)| &\leq C_k'' t^{-|\beta_1|} (1 + \|\eta\|)^{d - |\beta_2|} \nu^d_{\mathcal{K},2s+k}(f) \\ &\leq C_k'' (1 + \|\xi\|/2)^{|d|} \nu^d_{\mathcal{K},2s+k}(f) \\ &\leq C_k''' (1 + \|\xi\|)^{|d|} \nu^d_{\mathcal{K},2s+k}(f). \end{aligned}$$

It follows that

$$|R_k(D) f_t(x,\xi)| \leq C' (1 + \|\xi\|)^{|d|-k} \nu^d_{\mathcal{K},2s+2k}(f).$$

We now turn to g_t. By application of Theorem 6.4.9 it follows that

$$\begin{aligned} &|R_k(D)(g_t)(x,\xi)| \\ &\quad\leq D_k \max_{|\alpha|+|\beta| \leq 2s} \sup |D_x^\alpha \partial_\xi^\beta \langle D_x, \partial_\xi\rangle^k (\psi_t f)| \\ &\quad\leq D_k' \max_{|\alpha|+\|\beta\| \leq 2s, |\gamma| = k} \sup |\partial_\xi^{\gamma+\beta} (\psi_t D_x^{\alpha+\gamma} f)| \end{aligned}$$

To estimate the latter expression, we concentrate on

$$|\partial_\xi^{\gamma+\beta} (\psi_t D_x^{\alpha+\gamma} f)(y,\eta)|, \tag{6.5}$$

for $y \in \mathcal{K}$ and $\eta \in \mathbb{R}^n$. Since $\psi_t(\eta)$ equals zero for $\|\eta\| \leq t = \|\xi\|/4$ and equals 1 for $\|\eta\| \geq 2t = \|\xi\|/2$, we distinguish two cases: (a) $\|\xi\|/4 \leq \|\eta\| \leq \|\xi\|/2$ and (b) $\|\eta\| \geq \|\xi\|/2$.

Case (a): the expression (6.5) can be estimated by a sum of derivatives of the form

$$|(\partial_\xi^{\gamma_1} \psi_t) D_x^{\alpha+\gamma} \partial_\xi^{\gamma_2} f(y,\eta)|, \qquad (\gamma_1 + \gamma_2 = \gamma + \beta),$$

with suitable binomial coefficients. Now

$$\begin{aligned} |(\partial_\xi^{\gamma_1} \psi_t) D_x^{\alpha+\gamma} \partial_\xi^{\gamma_2} f(y,\eta)| &\leq D_k'' t^{-|\gamma_1|} (1 + \|\eta\|)^{d - \|\gamma_2\|} \nu^d_{\mathcal{K},2s+2k}(f) \\ &\leq D_k''' (1 + \|\xi\|)^{-|\gamma_1|} (1 + \|\xi\|)^{d - |\gamma_2|} \nu^d_{\mathcal{K},2s+2k}(f) \\ &\leq D_k''' (1 + \|\xi\|)^{d-k} \nu^d_{\mathcal{K},2s+2k}(f). \end{aligned}$$

Case (b): we now have that (6.5) equals $|D_x^{\alpha+\gamma}\partial_\xi^{\gamma+\beta} f(y,\eta)|$, and can be estimated by

$$\begin{aligned} |D_x^{\alpha+\gamma}\partial_\xi^{\gamma+\beta} f)(y,\eta)| &\leq (1+\|\eta\|)^{d-|\gamma+\beta|}\nu^d_{\mathcal{K},2s+2k}(f) \\ &\leq (1+\|\eta\|)^{|d|-k}\nu^d_{\mathcal{K},2s+2k}(f) \\ &\leq (1+\|\xi\|/2)^{d-k}\nu^d_{\mathcal{K},2s+2k}(f) \\ &\leq D(1+\|\xi\|)^{d-k}\nu^d_{\mathcal{K},2s+2k}(f). \end{aligned}$$

Collecting these estimates we see that

$$|R_k(D)g_t(x,\xi)| \leq D'(1+\|\xi\|)^{|d|-k}\,\nu^d_{2s+2k}(f),$$

with $D' > 0$ a constant independent of f, x and ξ. □

Corollary 6.5.7. *Let d, k and s be as in the above lemma. With a suitable adaptation of the constant $C > 0$, the estimate (6.4) holds for all $(x,\xi) \in \mathbb{R}^{2n}$.*

Proof It follows from Lemma 6.5.5 and its proof that there exists a constant $C_1 > 0$ such that $|R_k(D)f(x,\xi)| \leq C_1\nu^d_{\mathcal{K},2s+2k}(f)$. We now use that

$$(1+\|\xi\|)^{|d|-k} \geq 5^{|d|-k}$$

for all ξ with $|\xi\| \leq 4$. Hence, the estimate (6.4) holds with $C = 5^{k-|d|}C_1$ for $\|\xi\| \leq 4$. □

Corollary 6.5.8. *Let $d \in \mathbb{R}$ and $m \in \mathbb{N}$. Then there exist constants $C > 0$ and $l \in \mathbb{N}$ such that for all $f \in C^\infty_{\mathcal{K},c}(\mathbb{R}^n)$ and all $(x,\xi) \in \mathbb{R}^{2n}$ we have*

$$|R_m(D)f(x,\xi)| \leq C(1+\|\xi\|)^{d-m}\nu^d_{\mathcal{K},l}(f). \tag{6.6}$$

Proof Let s be as in the previous corollary. Fix $k \in \mathbb{N}$ such that $|d| - k < d - m$. Let now $C' > 0$ be constant as in the previous corollary. Then for all $f \in C^\infty_{\mathcal{K},c}(\mathbb{R}^n)$ we have

$$|R_k(D)f(x,\xi)| \leq C'(1+\|\xi\|)^{|d|-k}\nu^d_{\mathcal{K},2s+2m}(f), \qquad ((x,\xi) \in \mathbb{R}^{2n}).$$

On the other hand,

$$R_m(D) - R_k(D) = \sum_{k\leq j\leq m} \langle D_x, \partial_\xi\rangle^j$$

is a continuous linear operator $S^d_{\mathcal{K}}(\mathbb{R}^n) \to S^{d-k}_{\mathcal{K}}(\mathbb{R}^n)$. In fact, there exists a constant $C'' > 0$ such that

$$|R_m(D)f(f(x,\xi) - R_k(D)f(x,\xi)| \leq C''(1+\|\xi\|)^{d-k}\nu^d_{\mathcal{K},2m-2}(f)$$

for all $f \in S^d_{\mathcal{K}}(\mathbb{R}^n)$ and $(x,\xi) \in \mathbb{R}^{2n}$. The result now follows with $C = C' + C''$ and with $l = \max(2s+2m, 2m-2)$. □

After these technicalities we can now finally complete the proof of the main theorem of this section.

Completion of the proof of Theorem 6.5.2 Let $k \in \mathbb{N}$, let $\alpha, \beta \in \mathbb{N}^n$ and put $m = k - |\beta|$. Then by the previous corollary, applied with $d - |\beta|$ in place of d there exist constants $C > 0$ and $l \in \mathbb{N}$ such that for all $f \in C^\infty_{\mathcal{K},c}(\mathbb{R}^n)$ and all $(x, \xi) \in \mathbb{R}^{2n}$,

$$|R_k(D)f(x,\xi)| \leq (1 + \|\xi\|)^{d-|\beta|} \nu^{d-|\beta|}_{\mathcal{K},l}(f).$$

Moreover, by definition of the seminorms,

$$\nu^{d-|\beta|}_{\mathcal{K},l}(D^\alpha_x \partial^\beta_\xi f) \leq \nu^d_{\mathcal{K},l+|\alpha|+|\beta|}(f)$$

for all $f \in C^\infty_{\mathcal{K},c}(\mathbb{R}^n)$. Combining these estimates and using that $R_k(D)$ commutes with $D^\alpha_x \partial^\beta_\xi$, we find that

$$\begin{aligned} |D^\alpha_x \partial^\beta_\xi R_k(D) f(x,\xi)| &= R_k(D)[D^\alpha_x \partial^\beta_\xi f](x,\xi) \\ &\leq C\nu^d_{\mathcal{K},l+|\alpha|+|\beta|}(f), \end{aligned}$$

for all $f \in C^\infty_{\mathcal{K},c}(\mathbb{R}^n)$ and $(x,\xi) \in \mathbb{R}^{2n}$.

It follows from the above that for each $d' \in \mathbb{R}$ the map

$$R_{k+1}(D) : C^\infty_{\mathcal{K},c}(\mathbb{R}^n) \to S^{d'-(k+1)}(\mathbb{R}^n)$$

is continuous with respect to the $S^{d'}_{\mathcal{K}}(\mathbb{R}^n)$-topology on $C^\infty_{\mathcal{K},c}(\mathbb{R}^n)$. In particular, this is valid for $d' = d + 1$. As $C^\infty_{\mathcal{K},c}(\mathbb{R}^n)$ is dense in $S^d_{\mathcal{K}}(\mathbb{R}^n)$ with respect to the topology of $S^{d+1}_{\mathcal{K}}(\mathbb{R}^n)$, it follows by application of Lemma 6.4.7 that $R_{k+1}(D)$ maps $S^d_{\mathcal{K}}(\mathbb{R}^n)$ into $S^{d-k}(\mathbb{R}^n)$ with continuity relative to the $S^{d+1}_{\mathcal{K}}(\mathbb{R}^n)$-topology on the first space. As this topology is weaker than the usual one, we conclude that $R_{k+1}(D) : S^d_{\mathcal{K}}(\mathbb{R}^n) \to S^{d-k}(\mathbb{R}^n)$ is continuous. Now

$$R_{k+1}(D) - R_k(D) = \langle D_x, \partial_\xi \rangle^k$$

is continuous $S^d_{\mathcal{K}}(\mathbb{R}^n) \to S^{d-k}(\mathbb{R}^n)$ as well, and the result follows. □