Take home exercise Lecture 9, to be handed in December 7
revised version: dec 2

We assume that M is a connected compact manifold **of dimension at least** 2. A differential operator $P : C^\infty(M) \to C^\infty(M)$ is said to be **real** if Pf is a real valued function whenever $f \in C^\infty(M)$ is real valued. Let $\mathcal{D}$ denote the algebra of **real** differential operators on M, and let $\mathcal{D}_k$ denote the real subspace consisting of $P \in \mathcal{D}$ of degree at most k. We will write $\sigma^k(P)$ for the k-th order principal symbol of $P \in \mathcal{D}_k$ in the sense of differential operators. Thus $\sigma^k(P)$ is a function on T^*M which restricts to a homogeneous polynomial function of degree k on each cotangent space T^*_xM, for $x \in M$. We modify the principal symbol by a factor $1/i^k$ and put

$$\underline{\sigma}^k(P) = i^{-k}\sigma^k(P).$$

(a) Show that for each $P \in \mathcal{D}_k$ the modified principal symbol $\underline{\sigma}^k(P)$ is real-valued.

(b) Let $P \in \mathcal{D}_k$ be elliptic. Show that either $\underline{\sigma}^k(P)(\xi_x) > 0$ for all $x \in M$ and $\xi_x \in T^*_xM \setminus \{0\}$, or $\underline{\sigma}^k(P)(\xi_x) < 0$ for all $x \in M$ and $\xi_x \in T^*_xM \setminus \{0\}$.

(b2) Show that $\mathcal{D}_k$ has no elliptic operators if k is odd.

(c) Let $P_0, P_1 \in \mathcal{D}_k$ be elliptic. Show that $\text{index}(P_0) = \text{index}(P_1)$. Hint: observe that we may as well assume that $\underline{\sigma}^k(P_0)$ and $\underline{\sigma}^k(P_1)$ have the same sign. Now consider the homotopy $P_t = (1-t)P_0 + tP_1$ on the level of suitable Sobolev spaces.

We will denote the common value of the indices of the elliptic operators in $\mathcal{D}_k$ by n_k.

We now consider the case that M is the 2-dimensional unit sphere in $\mathbb{R}^3$. Each tangent space T_xM may be identified with the plane $x^\perp \subset \mathbb{R}^3$ (orthocomplement relative to the standard inner product). Accordingly, T_xM is equipped with the restriction g_x of the standard inner product of $\mathbb{R}^3$. Then g_x is an inner product on T_xM, which depends smoothly on the base point $x \in M$. A general manifold with such a metric structure on its tangent bundle is said to be Riemannian. By means of a partition of unity it can be shown that any manifold has a Riemannian structure. The arguments you are asked to provide below actually work in the context of a general manifold. You may choose to work in the context you like best.

Suggestion. You may consider the approach below, in terms of general language, or you may decide to skip (d), (e), (f), and instead show directly for M the 2-sphere, that the spherical Laplacian is real elliptic of order 2. See the suggested alternative approach at the end of this exercise.

Let dV be the Riemannian volume density on M, i.e., $dV_x(f_1, f_2) = 1$ for each orthonormal basis f_1, f_2 of T_xM.

(d) Let $\mathfrak{X}(M)$ denote the space of real vector fields on M. Thus, $\mathfrak{X}(M) = \Gamma^\infty(TM)$. Show that the operator grad : $C^\infty(M) \to \mathfrak{X}(M)$ defined by

$\langle \mathrm{grad} f\, ,\, v_x \rangle = df(x)v_x$ is a differential operator from the trivial bundle $\mathbb{C}_M$ to the complexified tangent bundle $(TM)_{\mathbb{C}}$. Show that the principal symbol of grad is given by

$$g_x(\sigma^1(\mathrm{grad})(\xi_x)(1_x),\, \cdot\,) = i\xi_x, \qquad (x \in M, \xi_x \in T_x^*M).$$

Here 1_x denotes the element $(x, 1)$ of the fiber $\{x\} \times \mathbb{C}$ of the trivial bundle $\mathbb{C}_M = M \times \mathbb{C}$. Hint: Use the characterization of Lemma 1.2.2.

(e) Show that there exists a unique first order differential operator div : $\mathfrak{X}(M) \to C^\infty(M)$ such that

$$\int_M (\mathrm{div}\, v)(x)\, f(x) dV = -\int_M g_x(v(x), \mathrm{grad} f(x))\, dV,$$

Show that div is a first order differential operator from $(TM)_{\mathbb{C}}$ to the trivial bundle $\mathbb{C}_M$. Show that the principal symbol of div is given by

$$\sigma^1(\mathrm{div})(\xi_x) = i(\xi_x)_{\mathbb{C}} : (T_xM)_{\mathbb{C}} \to \mathbb{C}_M.$$

Hint: apply the characterization of Lemma 1.2.2 with uniformity in the variable x to the integrals of (e).

(f) Determine the principal symbol of the (Riemannian) Laplace operator

$$\Delta := \mathrm{div} \circ \mathrm{grad} : C^\infty(M) \to C^\infty(M).$$

Show that Δ is real elliptic of order 2.

Hint: use the dual inner product g_x^* on T_x^*M. This inner product is defined as follows. Write g_x for the (invertible) linear map $T_xM \to T_x^*M$ given by $g_x(v) = g_x(v,\, \cdot\,)$. Define $g_x^*(v^*, w^*) := g_x(g_x^{-1}(v^*), g_x^{-1}(w^*))$, for $v^*, w^* \in T_x^*M$.

(g) Show that $\langle \Delta f\, ,\, f \rangle_{L^2} < 0$ for every non-constant smooth function $f : M \to \mathbb{R}$.

(h) Show that $\dim \ker \Delta = 1$. Show that $\mathrm{index}\, \Delta = 0$. Hint: use that Δ is the transpose of Δ relative to dV, and show that $\mathrm{im}(\Delta) = \ker(\Delta)^\perp$.

(k) Show that $n_{2k} = 0$ for all $k \in \mathbb{N}$. Hint: use a general result on the index of the composition of Fredholm operators.

(x) Extra question for bonus points: discuss what can happen if M is one-dimensional (the circle).

Alternative approach. We now really work with the 2-dimensional unit sphere $M \subset \mathbb{R}^3$. The spherical Laplacian $\Delta : C^\infty(M) \to C^\infty(M)$ may be defined as follows. Let

$$L = \partial_1^2 + \partial_2^2 + \partial_3^2$$

be the usual Euclidean Laplacian on $\mathbb{R}^3$. Let $\pi : \mathbb{R}^3 \setminus \{0\} \to M$ be the projection to the sphere given by $\pi(x) = x/\|x\|$. Then the spherical Laplacian is defined by

$$\Delta f = L(f \circ \pi)|_M, \qquad (f \in C^\infty(M)).$$

By using spherical coordinates, show directly that Δ is real elliptic of second order. You may now use the above approach to also prove (g). Now continue with the other items as suggested.