LECTURE 10
Characteristic classes

At this point we know that, for an elliptic differential operator of order d,

$$P : \Gamma(E) \to \Gamma(F)$$

its analytical index

$$\text{Index}(P) = \dim(\text{Ker}(P)) - \dim(\text{Coker}(P))$$

is well-defined (finite) and only depends on the principal symbol

$$\sigma_d(P) : \pi^* E \to \pi^* F$$

(where $\pi : T^*M \to M$ is the projection). The Atiyah-Singer index theorem gives a precise formula for Index(P) in terms of topological data associated to $\sigma_d(P)$,

$$\text{Index}(P) = (-1)^n \int_{TM} ch(\sigma_d(P)) Td(TM \otimes C).$$

The right hand side, usually called the topological index, will be explained in the next two lectures. On short, the two terms "*ch*" and "*Td*" are particular characteristic classes associated to vector bundles. So our aim is to give a short introduction into the theory of characteristic classes.

The idea is to associate to vector bundles E over a manifold M certain algebraic invariants which are cohomology classes in $H^*(M)$ which "measure how non-trivial E is", and which can distinguish non-isomorphic vector bundles. There are various approaches possible. Here we will present the geometric one, which is probably also the simplest, based on the notion of connection and curvature. The price to pay is that we need to stay in the context of smooth manifolds, but that is enough for our purposes.

Conventions: Although our main interest is on complex vector bundles, for the theory of characteristic classes it does not make a difference (for a large part of the theory) whether we work with complex or real vector bundles. So, we will fix a generic ground field $\mathbb{F}$ (which is either $\mathbb{R}$ or $\mathbb{C}$) and, unless a clear specification is made, by vector bundle we will mean a vector bundle over the generic field $\mathbb{F}$.

Accordingly, when referring to $C^\infty(M)$, TM, $\Omega^p(M)$, $\mathcal{X}(M)$, $H^*(M)$ without further specifications, we mean in this lecture the versions which take into account $\mathbb{F}$; i.e., when $\mathbb{F} = \mathbb{C}$, then they denote the algebra of $\mathbb{C}$-valued smooth functions on M, the complexification of the real tangent bundle, complex-valued forms, complex vector fields (sections of the complexified real tangent bundle), DeRham cohomology with coefficients in $\mathbb{C}$. Also, when referring to linearity (of a map), we mean linearity over $\mathbb{F}$.

10.1. Connections

Throughout this section E is a vector bundle over a manifold M. Unlike the case of smooth functions on manifolds (which are sections of the trivial line bundle!), there is no canonical way of taking derivatives of sections of (an arbitrary) E along vector fields. That is where connections come in.

Definition 10.1.1. A connection on E is a bilinear map ∇

$$\mathcal{X}(M) \times \Gamma(E) \to \Gamma(E), \quad (X, s) \mapsto \nabla_X(s),$$

satisfying

$$\nabla_{fX}(s) = f\nabla_X(s), \ \nabla_X(fs) = f\nabla_X(s) + L_X(f)s,$$

for all $f \in C^\infty(M)$, $X \in \mathcal{X}(M)$, $s \in \Gamma(E)$.

Remark 10.1.2. In the case when E is trivial, with trivialization frame

$$e = \{e_1, \dots, e_r\},$$

giving a connection on E is the same thing as giving an r by r matrix whose entries are 1-forms on M:

$$\omega := (\omega_i^j)_{i,j} \in M_r(\Omega^1(U)).$$

Given ∇, ω is define by

$$\nabla_X^U(e_i) = \sum_{j=1}^r \omega_i^j(X)e_j.$$

Conversely, for any matrix ω, one has a unique connection ∇ on E for which the previous formula holds: this follows from the Leibniz identity.

Remark 10.1.3. Connections are local in the sense that, for a connection ∇ and $x \in M$,

$$\nabla_X(s)(x) = 0$$

for any $X \in \mathcal{X}(M)$, $s \in \Gamma(E)$ such that $X = 0$ or $s = 0$ in a neighborhood U of x. This can be checked directly, or can be derived from the remark that ∇ is a differential operator of order one in X and of order zero in f.

Locality implies that, for $U \subset M$ open, ∇ induces a connection ∇^U on the vector bundle $E|_U$ over U, uniquely determined by the condition

$$\nabla_X(s)|_U = \nabla^U_{X|_U}(s_U).$$

Choosing U the domain of a trivialization of E, with corresponding local frame $e = \{e_1, \dots, e_r\}$, the previous remark shows that, over U, ∇ is uniquely determined by a matrix

$$\theta := (\theta_i^j)_{i,j} \in M_r(\Omega^1(U)).$$

This matrix is called the connection matrix of ∇ over U, with respect to the local frame e (hence a more appropriate notation would be $\theta(\nabla, U, e)$).

Proposition 10.1.4. *Any vector bundle E admits a connection.*

Proof Start with a partition of unity η_i subordinated to an open cover $\{U_i\}$ such that $E|_{U_i}$ is trivializable. On each $E|_{U_i}$ we consider a connection ∇^i (e.g., in the previous remark consider the zero matrix). Define ∇ by

$$\nabla_X(s) := \sum_i \nabla_{X|_{U_i}})(\eta_i s).$$

□

Next, we point out s slightly different way of looking at connections, in terms of differential forms on M. Recall that the elements $\omega \in \Omega^p(M)$ (p-forms) can be written locally, with respect to coordinates $(x_1, \dots, x_n)$ in M, as

$$\omega = \sum_{i_1,\dots,i_p} f^{i_1,\dots,i_p} dx_{i_1} \dots dx_{i_p}, \tag{10.1}$$

with $f^{i_1,\dots,i_p}$-smooth functions while globally, they are the same thing as $C^\infty(M)$-multilinear, antisymmetric maps

$$\omega : \underbrace{\mathcal{X}(M) \times \dots \times \mathcal{X}(M)}_{p \text{ times}} \to C^\infty(M),$$

where $\mathcal{X}(M)$ is the space of vector fields on M.

Similarly, for a vector bundle E over M, we define the space of E-valued p-differential forms on M

$$\Omega^p(M; E) = \Gamma(\Lambda^p T^* M \otimes E).$$

As before, its elements can be written locally, with respect to coordinates $(x_1, \dots, x_n)$ in M,

$$\eta = \sum_{i_1,\dots,i_p} dx_{i_1} \dots dx_{i_p} \otimes e^{i_1,\dots,i_p}. \tag{10.2}$$

with $e^{i_1,\dots,i_p}$ local sections of E. Using also a local frame $e = \{e_1, \dots, e_r\}$ for E, we obtain expressions of type

$$\sum_{i_1,\dots,i_p,i} f_i^{i_1,\dots,i_p} dx_{i_1} \dots dx_{i_p} \otimes e_i.$$

Globally, such an η is a $C^\infty(M)$-multilinear antisymmetric maps

$$\omega : \underbrace{\mathcal{X}(M) \times \dots \times \mathcal{X}(M)}_{p \text{ times}} \to \Gamma(E).$$

Recall also that

$$\Omega(M) = \bigoplus_p \Omega^p(M)$$

is an algebra with respect to the wedge product: given $\omega \in \Omega^p(M)$, $\eta \in \Omega^q(M)$, their wedge product $\omega \wedge \eta \in \Omega^{p+q}(M)$, also denoted $\omega\eta$, is given by
(10.3)

$$(\omega\wedge\eta)(X_1, \ldots, X_{p+q}) = \sum_{\sigma} sign(\sigma)\omega(X_{\sigma(1)}, \ldots, X_{\sigma(p)})\cdot\eta(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)}),$$

where the sum is over all (p, q)-shuffles σ, i.e. all permutations σ with $\sigma(1) < \ldots < \sigma(p)$ and $\sigma(p+1) < \ldots < \sigma(p+q)$. Although this formula no longer makes sense when ω and η are both E-valued differential forms, it does make sense when one of them is E-valued and the other one is a usual form. The resulting operation makes

$$\Omega(M, E) = \bigoplus_{p} \Omega^p(M, E)$$

into a (left and right) module over $\Omega(M)$. Keeping in mind the fact that the spaces Ω are graded (i.e are direct sums indexed by integers) and the fact that the wedge products involved are compatible with the grading (i.e. $\Omega^p \wedge \Omega^q \subset \Omega^{p+q}$), we say that $\Omega(M)$ is a graded algebra and $\Omega(M, E)$ is a graded bimodule over $\Omega(M)$. As for the usual wedge product of forms, the left and right actions are related by[1]

$$\omega \wedge \eta = (-1)^{pq}\eta \wedge \omega \quad \forall\ \omega \in \Omega^p(M), \eta \in \Omega^q(M, E).$$

In what follows we will be mainly using the left action.

Finally, recall that $\Omega(M)$ also comes with DeRham differential d, which increases the degree by one, satisfies the Leibniz identity

$$d(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^{|\omega|}\omega \wedge d(\eta),$$

where $|\omega|$ is the degree of ω[2], and is a differential (i.e. $d \circ d = 0$). We say that $(\Omega(M)$ is a DGA (differential graded algebra). However, in the case of $\Omega(M, E)$ there is no analogue of the DeRham operator.

Proposition 10.1.5. *Given a vector bundle E over M, a connection ∇ on E induces a linear operator which increases the degree by one,*

$$d_{\nabla} : \Omega^{\bullet}(M, E) \to \Omega^{\bullet+1}(M, E)$$

which satisfies the Leibniz identity

$$d_{\nabla}(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^{|\omega|}\omega \wedge d_{\nabla}(\eta)$$

for all $\omega \in \Omega(M)$, $\eta \in \Omega(M, E)$. The operator ∇ is uniquely determined by these conditions and

$$d_{\nabla}(s)(X) = \nabla_X(s)$$

for all $s \in \Omega^0(M, E) = \Gamma(E)$, $X \in \mathcal{X}(M)$.

Moreover, the correspondence $\nabla \leftrightarrow d_{\nabla}$ is a bijection between connections on E and operators d_{∇} as above.

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[1]Important: this is the first manifestation of what is known as the "graded sign rule": in an formula that involves graded elements, if two elements a and b of degrees p and q are interchanged, then the sign $(-1)^{pq}$ is introduced

[2]Note: the sign in the formula agrees with the graded sign rule: we interchange d which has degree 1 and ω

Instead of giving a formal proof (which is completely analogous to the proof of the basic properties of the DeRham differential), let us point out the explicit formulas for d_∇, both global and local. The global one is completely similar to the global description of the DeRham differential- the so called Koszul formula: for $\omega \in \Omega^p(M)$, $d\omega \in \Omega^{p+1}(M)$ is given by

$$\begin{aligned} d(\eta)(X_1, \ldots, X_{p+1}) &= \sum_{i<j} (-1)^{i+j} \eta([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots X_{p+1})) \\ &+ \sum_{i=1}^{p+1} (-1)^{i+1} L_{X_i}(\eta(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1})). \end{aligned} \tag{10.4}$$

Replacing the Lie derivatives L_{X_i} by ∇_{X_i}, the same formula makes sense for $\eta \in \Omega(M, E)$; the outcome is precisely $d_\nabla(\eta) \in \Omega^{p+1}(M, E)$. For the local description we fix coordinates $(x_1, \ldots, x_n)$ in M and a local frame $e = \{e_1, \ldots, e_r\}$ for E. We have to look at elements of form (10.2). The Leibniz rule for d_∇ implies that

$$d_\nabla(\eta) = \sum_{i_1, \ldots, i_p} (-1)^p dx_{i_1} \ldots dx_{i_p} \otimes d_\nabla(e^{i_1, \ldots, i_p})$$

hence it suffices to describe d_∇ on sections of E. The same Leibniz formula implies that it suffices to describe d_∇ on the frame e. Unraveling the last equation in the proposition, we find

$$d_\nabla(e_i) = \sum_{j=1}^{r} \omega_i^j e_j, \tag{10.5}$$

where $\theta = (\theta_j^i)_{i,j}$ is the connection matrix of ∇ with respect to e.

Exercise 10.1.6. Let ∇ be a connection on E, $X \in \mathcal{X}(M)$, $s \in \Gamma(E)$, $x \in M$ and $\gamma : (-\epsilon, \epsilon) \to M$ a curve with $\gamma(0) = x$, $\gamma'(0) = X_x$. Show that if $X_x = 0$ or $s = 0$ along γ, then

$$\nabla_X(s)(x) = 0.$$

Deduce that, for any $X_x \in T_xM$ and any section s defined around x, it makes sense to talk about $\nabla_{X_x}(s)(x) \in E_x$.

10.2. Curvature

Recall that, for the standard Lie derivatives of functions along vector fields,

$$L_{[X,Y]} = L_X L_Y(f) - L_Y L_X(f).$$

Of course, this can be seen just as the definition of the Lie bracket $[X, Y]$ of vector fields but, even so, it still says something: the right hand side is a derivation on f (i.e., indeed, it comes from a vector field). The similar formula for connections fails dramatically (i.e. there are few vector bundles which admit a connection for which the analogue of this formula holds). The failure is measured by the curvature of the connection.

Proposition 10.2.1. *For any connection ∇, the expression*

$$k_{\nabla}(X,Y)s = \nabla_X\nabla_Y(s) - \nabla_Y\nabla_X(s) - \nabla_{[X,Y]}(s), \tag{10.6}$$

is $C^{\infty}(M)$-linear in the entries $X, Y \in \mathcal{X}(M)$, $s \in \Gamma(E)$. Hence it defines an element

$$k_{\nabla} \in \Gamma(\Lambda^2 T^*M \otimes End(E)) = \Omega^2(M; End(E)),$$

called the curvature of ∇.

Proof It follows from the properties of ∇. For instance, we have

$$\begin{aligned}\nabla_X\nabla_Y(fs) &= \nabla_X(f\nabla_Y(s) + L_Y(f)s)\\ &= f\nabla_X\nabla_Y(s) + L_X(f)\nabla_Y(s) + L_X(f)\nabla_Y(s) + L_XL_Y(f)s,\end{aligned}$$

and the similar formula for $\nabla_X\nabla_Y(fs)$, while

$$\nabla_{[X,Y]}(fs) = f\nabla_{[X,Y]}(s) + L_{[X,Y]}(f)s.$$

Hence, using $L_{[X,Y]} = L_XL_Y - L_YL_X$, we deduce that

$$k_{\nabla}(X,Y)(fs) = fk_{\nabla}(X,Y)(s),$$

and similarly the others. $\square$

Remark 10.2.2. One can express the curvature locally, with respect to a local frame $e = \{e_1, \ldots, e_r\}$ of E over an open U, as

$$k_{\nabla}(X,Y)e_i = \sum_{j=1}^{r} k_i^j(X,Y)e_j,$$

where $k_i^j(X,Y) \in C^{\infty}(U)$ are smooth functions on U depending on $X, Y \in \mathcal{X}(M)$. The previous proposition implies that each k_i^j is a differential form (of degree two). Hence k_{∇} is locally determined by a matrix

$$k = (k_i^j)_{i,j} \in M_n(\Omega^2(U)),$$

called the curvature matrix of ∇ over U, with respect to the local frame e. Of course, we should be able to compute k in terms of the connection matrix θ. This will be done as bit later.

There is another interpretation of the curvature, in terms of forms with values in E. While ∇ defines the operator d_{∇} which is a generalization of the DeRham operator d, it is very rarely that it squares to zero (as d does). Again, k_{∇} measure this failure. To explain this, we first look more closely to elements

$$K \in \Omega^p(M, End(E)).$$

The wedge product formula (10.3) has a version when $\omega = K$ and $\eta \in \Omega^q(M,E)$:

$$(K\wedge\eta)(X_1,\ldots,X_{p+q}) = \sum_{\sigma} sign(\sigma)K(X_{\sigma(1)},\ldots,X_{\sigma(p)})(\eta(X_{\sigma(p+1)},\ldots,X_{\sigma(p+q)})),$$

Any such K induces a linear map

$$\hat{K} : \Omega^{\bullet}(M,E) \to \Omega^{\bullet+p}(M,E),\ \hat{K}(\eta) = K\wedge\eta.$$

For the later use not also that the same formula for the wedge product has an obvious version also when applied to elements $K \in \Omega^p(M, End(E))$ and $K' \in \Omega^q(M, End(E))$, giving rise to operations

$$\wedge : \Omega^p(M, End(E)) \times \Omega^q(M, End(E)) \to \Omega^{p+q}(M, End(E)) \tag{10.7}$$

which make $\Omega(M, End(E))$ into a (graded) algebra.

Exercise 10.2.3. Show that $\hat{K}$ is an endomorphism of the graded (left) $\Omega(M)$-module $\Omega(M, E)$ i.e., according to the graded sign rule (see the previous footnotes):

$$\hat{K}(\omega \wedge \eta) = (-1)^{pq} \omega \wedge K(\eta),$$

for all $\omega \in \Omega^q(M)$.

Moreover, the correspondence $K \mapsto \hat{K}$ defines a bijection

$$\Omega^p(M, End(E)) \cong End^p_{\Omega(M)}(\Omega(M, E))$$

between $\Omega^p(M, End(E))$ and the space of all endomorphisms of the graded (left) $\Omega(M)$-module $\Omega(M, E)$ which rise the degree by p.

Finally, via this bijection, the wedge operation (10.7) becomes the composition of operators, i.e.

$$\widehat{K \wedge K'} = \hat{K} \circ \hat{K}'$$

for all $K, K' \in \Omega(M, End(E))$.

Due to the previous exercise, we will tacitly identify the element K with the induced operator m_K. For curvature of connections we have

Proposition 10.2.4. *If ∇ is a connection on E, then*

$$d_\nabla^2 = d_\nabla \circ d_\nabla : \Omega^\bullet(M, E) \to \Omega^{\bullet+2}(M, E)$$

is given by

$$d_\nabla^2(\eta) = k_\nabla \wedge \eta$$

for all $\eta \in \Omega^*(M; E)$, and this determines k_∇ uniquely.

Proof Firs of all, d_∇ is $\Omega(M)$-linear: for $\omega \in \Omega^p(M)$ and $\eta \in \Omega^p(M, E)$,

$$\begin{aligned} d_\nabla^2(\omega \wedge \eta) &= d_\nabla(d(\omega) \wedge \eta + (-1)^p \omega \wedge d_\nabla(\eta)) \\ &= [d^2(\omega) \wedge \eta + (-1)^{p+1} d(\omega) \wedge d_\nabla(\eta)] + (-1)^p [d(\omega) \wedge d_\nabla(\eta) + (-1)^p \omega \wedge d_\nabla^2(\eta) \\ &= \omega \wedge d_\nabla(\eta). \end{aligned}$$

Hence, by the previous exercise, it comes from multiplication by an element $k \in \Omega^2(M)$. Using the explicit Koszul-formula for d_∇ to compute d_∇^2 on $\Gamma(E)$, we see that $d_\nabla^2(s) = k_\nabla \wedge s$ for all $s \in \Gamma(E)$. We deduce that $k = k_\nabla$. □

Exercise 10.2.5. Let θ and k be the connection and curvature matrices of ∇ with respect to a local frame e. Using the local formula (10.5) for d_∇ and the previous interpretation of the curvature, show that

$$k_i^j = d\theta_i^j - \sum_k \theta_i^k \wedge \theta_k^j,$$

or, in a more compact form,

$$\boxed{k = d\theta - \theta \wedge \theta}. \tag{10.8}$$

10.3. Characteristic classes

The local construction of characteristic classes is obtained by gluing together expressions built out of connection matrices associated to a connection. Hence it is important to understand how connection matrices change when the frame is changed.

Lemma 10.3.1. *Let ∇ be a connection on E. Let $e = \{e_1, \ldots, e_r\}$ be a local frame of E over an open U and let θ and k be the associated connection matrix and curvature matrix, respectively. Let $e' = \{e'_1, \ldots, e'_r\}$ be another local frame of E over some open U' and let θ' and k' be the associated connection and curvature matrix of ∇. Let*

$$g = (g_i^j) \in M_n(C^\infty(U \cap U'))$$

be the matrix of coordinate changes from e to e', i.e. defined by:

$$e_i^{'} = \sum_{j=1}^{r} g_i^j e_j$$

over $U \cap U'$. Then, on $U \cap U'$,

$$\boxed{\theta' = (dg)g^{-1} + g\theta g^{-1}.}$$

$$\boxed{k' = gkg^{-1}.}$$

Proof Using formula (10.5) for d_∇ we have:

$$\begin{aligned} d_\nabla(e'_i) &= d_\nabla(\sum_l g_i^l e_l) \\ &= \sum_l d(g_i^l)e_l + \sum_{l,m} g_i^l \theta_l^m e_m, \end{aligned}$$

where for the last equality we have used the Leibniz rule and the formulas defining θ. Using the inverse matrix $g^{-1} = (\overline{g}_j^i)_{i,j}$ we change back from the frame e to e' by $e_j = \sum_i \overline{g}_j^i \omega_i$ and we obtain

$$d_\nabla(e'_i) = \sum_{l,j} d(g_i^l)\overline{g}_l^j e'_j + \sum_{l,m,j} g_i^l \theta_l^m \overline{g}_m^j e'_j.$$

Hence

$$(\theta')_i^j = \sum_l d(g_i^l)\overline{g}_l^j + \sum_{l,m} g_i^l \theta_l^m \overline{g}_m^j,$$

i.e. the first formula in the statement. To prove the second equation, we will use the formula (refinvariance) which expresses k in terms of θ. We have

$$d\theta' = d(dg \cdot g^{-1} + g\theta g^{-1}) = -dgd(g^{-1}) + d(g)\theta g^{-1} + gd(\theta)g^{-1} - g\theta d(g^{-1}).$$

For $\theta' \wedge \theta'$ we find

$$dgg^{-1} \wedge d(g)g^{-1} + dgg^{-1} \wedge g\theta g^{-1} + g\theta g^{-1} \wedge dgg^{-1} + g\theta g^{-1} \wedge g\theta g^{-1}.$$

Since

$$g^{-1}dg = d(g^{-1}g) - d(g^{-1})g = -d(g^{-1})g,$$

the expression above equals to

$$-dgd(g^{-1}) + d(g)\theta g^{-1} - g\theta d(g^{-1}) + g\omega\omega g^{-1}.$$

Comparing with the expression for $d\theta'$, we find

$$k' = d\theta' - \theta' \wedge \theta' = g(d\theta - \theta \wedge \theta)g^{-1} = gkg^{-1}.$$

□

Since the curvature matrix stays the same "up to conjugation", it follows that any expression that is invariant under conjugation will produce a globally defined form on M. The simplest such expression is obtained by applying the trace:

$$Tr(k) = \sum_i k_i^i \in \Omega^2(U).$$

Indeed, it follows immediately that, if k' corresponds to another local frame e' over U', then $Tr(k) = Tr(k')$ on the overlap $U \cap U'$. Hence all these pieces glue to a global 2-form on M:

$$Tr(k_\nabla) \in \Omega^2(M).$$

As we will see later, this form is closed, and the induced cohomology class in $H^2(M)$ does not depend on the choice of the connection (and this will be, up to a constant, the first Chern class of E). More generally, one can use other "invariant polynomials" instead of the trace. We recall that we are working over the field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 10.3.2. We denote by $I_r(\mathbb{F})$ the space of all functions

$$P : M_r(\mathbb{F}) \to \mathbb{F}$$

which are polynomial (in the sense that $P(A)$ is a polynomial in the entries of A), and which are invariant under the conjugation, i.e.

$$P(gAg^{-1}) = P(A)$$

for all $A \in M_r(\mathbb{F})$, $g \in Gl_r(\mathbb{F})$.

Note that $I_r(\mathbb{F})$ is an algebra (the product of two invariant polynomials is invariant).

Example 10.3.3. For each $p \geq 0$,

$$\Sigma_p : M_r(\mathbb{F}) \to \mathbb{F},\ \Sigma_p(A) = Tr(A^p)$$

is invariant. One can actually show that the elements with $0 \leq p \leq r$ generate the entire algebra $I_r(\mathbb{F})$: any $P \in I_r(\mathbb{F})$ is a polynomial combination of the Σ_p's. Even more, one has an isomorphism of algebras

$$I_r(\mathbb{F}) = \mathbb{F}[\Sigma_0, \Sigma_1, \ldots, \Sigma_p].$$

Example 10.3.4. Another set of generators are obtained using the polynomial functions

$$\sigma_p : M_r(\mathbb{F}) \to \mathbb{F}$$

defined by the equation

$$det(I + tA) = \sum_{p=0}^{r} \sigma_p(A) t^p.$$

For instance, $\sigma_1 = \Sigma_1$ is just the trace while $\sigma_p(A) = det(A)$. One can also prove that

$$I_r(\mathbb{F}) = \mathbb{F}[\sigma_0, \sigma_1, \ldots, \sigma_p].$$

Remark 10.3.5. But probably the best way to think about the invariant polynomials is by interpreting them as symmetric polynomials, over the base field $\mathbb{F}$, in r variables $x_1, \ldots x_r$ which play the role of the eigenvalues of a generic matrix A. More precisely, one has an isomorphism of algebras

$$I_r(\mathbb{F}) \cong Sym_{\mathbb{F}}[x_1, \ldots, x_r]$$

which associates to a symmetric polynomial S the invariant function (still denoted by S) given by

$$S(A) = S(x_1(A), \ldots, x_r(A)),$$

where $x_i(A)$ are the eigenvalues of A. Conversely, any $P \in I_r(\mathbb{F})$ can be viewed as a symmetric polynomial by evaluating it on diagonal matrices:

$$P(x_1, \ldots, x_r) := P(diag(x_1, \ldots, x_r)).$$

For instance, via this bijection, the Σ_p's correspond to the polynomials

$$\Sigma_p(x_1, \ldots, x_r) = \sum_i (x_i)^p,$$

while the σ_p's correspond to

$$\sigma_p(x_1, \ldots, x_r) = \sum_{i_1 < \ldots < i_p} x_{i_1} \ldots x_{i_p},$$

With this it is now easier to express the Σ's in term of the σ's and the other way around (using "Newton's formulas": $\Sigma_1 = \sigma_1$, $\Sigma_2 = (\sigma_1)^2 - 2\sigma_2$, $\Sigma_3 = (\sigma_1)^3 - 3\sigma_1\sigma_2 + 3\sigma_3$, etc.).

From the previous lemma we deduce:

Corollary 10.3.6. *Let $P \in I_r(\mathbb{F})$ be an invariant polynomial of degree p. Then for any vector bundle E over M of rank r and any connection ∇ on E, there exists a unique differential form of degree $2p$,*

$$P(E, \nabla) \in \Omega^{2p}(M)$$

with the property that, for any local frame e of E over some open U,

$$P(E, \nabla)|_U = P(k) \in \Omega^{2p}(U),$$

where $k \in M_r(\Omega^2(U))$ is the connection matrix of ∇ with respect to e.

The following summarizes the construction of the characteristic classes.

Theorem 10.3.7. *Let $P \in I_r(\mathbb{F})$ be an invariant polynomial of degree p. Then for any vector bundle E over M of rank r and any connection ∇ on E, $P(E, \nabla)$ is a closed form and the resulting cohomology class*

$$P(E) := [P(E, \nabla)] \in H^{2p}(M)$$

does not depend on the choice of the connection ∇. It is called the P-characteristic class of E.

This theorem can be proven directly, using local connection matrices. Also, it suffices to prove the theorem for the polynomials $P = \Sigma_p$. This follows from the fact that these polynomials generate $I_r(\mathbb{F})$ and the fact that the construction

$$I_r(\mathbb{F}) \ni P \mapsto P(E, \nabla) \in \Omega(M)$$

is compatible with the products. In the next lecture we will give a detailed global proof for the Σ_p's; the price we will have to pay for having a coordinate-free proof is some heavier algebraic language. What we gain is a better understanding on one hand, but also a framework that allows us to generalize the construction of characteristic classes to "virtual vector bundles with compact support". Here we mention the main properties of the resulting cohomology classes (proven at the end of this lecture).

Theorem 10.3.8. *For any $P \in I_r(\mathbb{F})$, the construction $E \mapsto P(E)$ is natural, i.e.*

1. *If two vector bundles E and F over M, of rank r, are isomorphic, then $P(E) = P(F)$.*
2. *If $f : N \to M$ is a smooth map and*

$$f^* : H^\bullet(M) \to H^\bullet(N)$$

is the pull-back map induced in cohomology, then for the pull-back vector bundle f^*E,

$$P(f^*E) = f^*P(E).$$

10.4. Particular characteristic classes

Particular characteristic classes are obtained by applying the constructions of the previous section to specific polynomials. Of course, since the polynomials Σ_p (and similarly the σ_p's) generate $I(\mathbb{F})$, we do not loose any information if we restrict ourselves to these polynomials and the resulting classes. Why don't we do that? First, one would have to make a choice between the Σ_p's or σ_p's. But, most importantly, it is the properties that we want from the resulting characteristic classes that often dictate the choice of the invariant polynomials (e.g. their behaviour with respect to the direct sum of vector bundles- see below). Sometimes the Σ_p's are better, sometimes the σ_p's, and sometimes others. On top, there are situations when the relevant characteristic classes are not even a matter of choice: they are invariants that show up by themselves in a specific context (as is the case with the Todd class which really shows up naturally when comparing the "Thom isomorphism" in DeRham cohomology with the one in K-theory- but that goes beyond this course).

Here are some of the standard characteristic classes that one considers. We first specialize to the complex case $\mathbb{F} = \mathbb{C}$.

1. Chern classes: They correspond to the invariant polynomials

$$c_p = \left(\frac{1}{2\pi i}\right)^p \sigma_p,$$

with $0 \leq p \leq r$. Hence they associate to complex vector bundle E rank r a cohomology class, called the p-th Chern class of E:

$$c_p(E) \in H^{2p}(M) \quad (0 \leq p \leq r).$$

The total Chern class of E is defined as

$$c(E) = c_0(E) + c_1(E) + \ldots + c_r(E) \in H^{\text{even}}(M);$$

it corresponds to the inhomogeneous polynomial

$$c(A) = det(I + \frac{1}{2\pi i}A).$$

For the purpose of this lecture, the rather ugly constants in front of σ_p (and the similar constants below) are not so important. Their role will be to "normalize" some formulas so that the outcome (the components of the Chern character) are real, or even integral (they come from the cohomology with integral coefficients; alternatively, one may think that they produce integrals which are integers). If you solve the following exercise you will find out precisely such constants showing up.

Exercise 10.4.1. Let $M = \mathbb{CP}^1$ be the complex projective space, consisting of complex lines in $\mathbb{C}^2$ (i.e. 1-dimensional complex vector subspaces) in $\mathbb{C}^2$. Let $L \subset \mathbb{CP}^1 \times \mathbb{C}^2$ be the tautological line bundle over M (whose fiber above $l \in \mathbb{CP}^1$ is l viewed as a complex vector space). Show that

$$c_1(L) \in H^2(\mathbb{CP}^1)$$

is non-trivial. What is its integral? (the element $a := -c_1(L) \in H^2(\mathbb{CP}^1)$ will be called the canonical generator).

Here are the main properties of the Chern classes (the proofs will be given at the end of the section).

Proposition 10.4.2. *The Chern classes of a complex vector bundle, priory cohomology classes with coefficients in $\mathbb{C}$, are actually real (cohomology classes with coefficients in $\mathbb{R}$). Moreover,*

1. *The total Chern class has an exponential behaviour with respect to the direct sum of vector bundles i.e., for any two complex vector bundles E and F over M,*

$$c(E \oplus F) = c(E)c(F)$$

or, component-wise,

$$c_p(E \oplus F) = \sum_{i+j=p} c_i(E)c_j(F).$$

2. If $\overline{E}$ is the conjugated of the complex vector bundle E, then

$$c_k(\overline{E}) = (-1)^k c_k(E).$$

Remark 10.4.3. One can show that the following properties of the Chern classes actually determines them uniquely:

C1: Naturality (see Theorem 10.3.8).
C2: The behaviour with respect to the direct sum (see the previous proposition).
C3: For the tautological line bundle L, $c_1(L) = -a \in H^2(\mathbb{CP}^1)$ (see the previous exercise).

2. Chern character: The Chern character classes correspond to the invariant polynomials

$$Ch_p = \frac{1}{p!}\left(-\frac{1}{2\pi i}\right)^p \Sigma_p.$$

Hence Ch_p associates to a complex vector bundle E a cohomology class, called the p-th component of the Chern character of E:

$$Ch_p(E) \in H^{2p}(M).$$

They assemble together into the full Chern character of E, defined as

$$Ch(E) = \sum_{p \geq 0} Ch_p(E) \in H^{\text{even}}(M);$$

it corresponds to the expression (which, strictly speaking is a powers series and not a polynomial, but which when evaluated on a curvature matrix produces a finite sum):

$$Ch(A) = Tr(e^{-\frac{1}{2\pi i}A}).$$

Note that $Ch(E)$ are always real cohomology classes; this follows e.g. from the similar property for the Chern classes, and the fact that the relationship between the Σ's and the σ's involve only real coefficients (even rational!). Here is the main property of the Chern character.

Proposition 10.4.4. *The Chern character is additive and multiplicative i.e., for any two complex vector bundles E and F over M,*

$$Ch(E \oplus F) = Ch(E) + Ch(F), \quad Ch(E \otimes F) = Ch(E)Ch(F).$$

3. Todd class: Another important characteristic class is the Todd class of a complex vector bundle. To define it, we first expand formally

$$\frac{t}{1 - e^{-t}} = B_0 + B_1 t + B_2 t^2 + \ldots.$$

(the coefficients B_k are known as the Bernoulli numbers). For instance,

$$B_0 = 1, B_1 = \frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, \text{etc},$$

(and they have the property that $B_k = 0$ for k-odd, $k \geq 3$ and they have various other interesting interpretations). For r-variables, we expand the resulting product

$$T := \Pi_{i=1}^{r} \frac{x_i}{1 - e^{-x_i}} = T_0 + T_1(x_1, \ldots, x_r) + T_2(x_1, \ldots, x_r) + \ldots$$

where each T_k is a symmetric polynomial of degree k. We proceed as before and define

$$Td_k(E) = \left(\frac{1}{2\pi i}\right)^k T_k(E) \in H^{2k}(M),$$

The Todd class of a vector bundle E is the resulting total characteristic class

$$Td(E) = \sum_k Td_k(E) \in H^{\text{even}}(M).$$

Again, $Td(E)$ are real cohomology classes and Td is multiplicative.

4. Pontryagin classes: We now pass to the case $\mathbb{F} = \mathbb{R}$. The Pontryagin classes are the analogues for real vector bundles of the Chern classes. They correspond to the polynomials

$$p_k = \left(\frac{1}{2\pi}\right)^{2k} \sigma_{2k},$$

for $2k \leq r$. The reason we restrict to the σ's of even degree ($2k$) is simple: the odd dimensional degree produce zero forms (see below). Hence p_k associates to a real vector bundle E of rank r a cohomology class, called the k-th Pontryagin class of E:

$$p_k(E) \in H^{4k}(M) \quad (0 \leq k \leq r/2).$$

They assemble together into the full Pontryagin class of E, defined as

$$p(E) = p_0(E) + p_1(E) + \ldots + p_{[\frac{r}{2}]} \in H^{\bullet}(M).$$

The Pontryagin class has the same property as the Chern class: for any two real vector bundles,

$$p(E \oplus F) = p(E)p(F).$$

The relationship between the two is actually much stronger (which is expected due to their definitions). To make this precise, we associate to any real vector bundle E its complexification

$$E \otimes \mathbb{C} := E \otimes_{\mathbb{R}} \mathbb{C} = \{e_1 + ie_2 : e_1, e_2 \in E, \}.$$

Proposition 10.4.5. *For any real vector bundle E,*

$$c_l(E \otimes \mathbb{C}) = \begin{cases} (-1)^k p_k(E) & \text{if } l = 2k \\ 0 & \text{if } l = 2k+1 \end{cases}$$

Finally, one can go from a complex vector bundle E to a real one, denoted $E_{\mathbb{R}}$, which is just E vied as a real vector bundle. One has:

Proposition 10.4.6. *For any complex vector bundle E,*

$$p_0(E_{\mathbb{R}}) - p_1(E_{\mathbb{R}}) + p_2(E_{\mathbb{R}}) - \ldots = (c_0(E) + c_1(E) + \ldots)(c_0(E) - c_1(E) + \ldots).$$

5. <u>The Euler classs</u>: There are other characteristic classes which are, strictly speaking, not immediate application of the construction from the previous section, but are similar in spirit (but they fit into the general theory of characteristic classes with structural group smaller then GL_n). That is the case e.g. with the Euler class, which is defined for real, oriented vector bundles E of even rank $r = 2l$. The outcome is a cohomology class over the base manifold M:

$$e(E) \in H^{2l}(M).$$

To construct it, there are two key remarks:

1. For real vector bundles, connection matrices can also be achived to be antisymmetric. To see this, one choose a metric $\langle \cdot, \cdot \rangle$ on E (fiberwise an inner product) and, by a partition of unity, one can show that one can choose a connection ∇ which is compatible with the metric, in the sense that
$$L_x\langle s_1, s_2\rangle = \langle \nabla_X(s_1), s_2\rangle + \langle s_1, \nabla_X(s_2)\rangle$$
for all $s_1, s_2 \in \Gamma(E)$. Choosing orthonormal frames e, one can easily show that this compatibility implies that the connection matrix θ with respect to e is antisymmetric.
2. With the inner product and ∇ as above, concentrating on positively oriented frames, the change of frame matrix, denoted g in the previous section, has positive determinant.
3. In general, for for skew symmetric matrices A of even order $2l$, $det(A)$ is naturally a square of another expression, denoted $Pf(A)$ (polynomial of degree l):
$$det(A) = Pf(A)^2.$$
For instance,
$$det \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = (af + cd - be)^2.$$
Moreover, for g with positive determinant,
$$Pf(gAg^{-1}) = Pf(A).$$

It follows that, evaluating Pf on connection matrices associated to positive orthonormal frames, we obtain a globvally defined form

$$Pf(E, \nabla) \in \Omega^{2l}(M).$$

As before, this form is closed and the resulting cohomology class does not depend on the choice of the connection. The Euler class is

$$e(E) = \left[\left(\frac{1}{2\pi} \right)^l Pf(E, \nabla) \right] \in H^{2l}(M).$$

Note that, since *det* is used in the construction of $p_l(E)$, it follows that

$$p_l(E) = e(E)^2.$$

10.5. Proofs of the main properties

The proofs of the main properties of the characteristic classes (Theorem 10.3.8, Proposition 10.4.2, Proposition 10.4.4 and Proposition 10.4.5) are based on some basic constructions of connections: pull-back, direct sum, dual, tensor product.

Pullback of connections and the proof of Theorem 10.3.8: For Theorem 10.3.8 we need the construction of pull-back of connections: given a vector bundle E over M and a smooth map $f : N \to M$ then for any connection ∇ on E, there is an induced connection $f^*\nabla$ on f^*E. This can be described locally as follows: if e is a local frame e of E over U and θ is determined by the connection matrix (with respect to e) θ then, using the induced local frame f^*e of f^*E over $f^{-1}(U)$, the resulting connection matrix of $f^*\nabla$ is $f^*\theta$ (pull back all the one-forms which are the entries in the matrix θ). Of course, one has to check that these connection matrices glue together (which should be quite clear due to the naturality of the construction); alternatively, one can describe $f^*\nabla$ globally, by requiring

$$(f^*\nabla)_X(f^*s)(x) = \nabla_{(df)_x(X_x)}(s)(f(x)), \quad x \in N \tag{10.9}$$

where, for $s \in \Gamma(E)$, we denoted by $f^*s \in \Gamma(f^*E)$ the section $x \mapsto s(f(x))$; for the right hand side, see also Exercise 10.1.6.

Exercise 10.5.1. Show that there is a unique connection $f^*\nabla$ on f^*E which has the property (10.9). Then show that its connection matrices can be computed as indicated above.

With this construction, the second part of Theorem 10.3.8 is immediate: locally, the connection matrix of $f^*\nabla$ is just the pull-back of the one of ∇, hence we obtain $P(E, f^*\nabla) = f^*P(E, \nabla)$ as differential forms. The first part of the theorem is easier (exercise!).

Direct sum of connections and the proof of Proposition 10.4.2: For the proof of the behaviour of c with respect to direct sums (and similarly for the Pontryagin class) we need the construction of the direct sum of two connections: given connections ∇^0 and ∇^1 on E and F, respectively, we can form a new connection ∇ on $E \oplus F$:

$$\nabla_X(s^0, s^1) = (\nabla^0_X(s^0), \nabla^1_X(s^1)).$$

To compute its connection matrices, we will use a local frame of $E \otimes F$ which comes by putting together a local frame e for E and a local frame f for F (over the same open). It is then clear that the connection matrix θ for ∇ with respect to this frame, and similarly the curvature matrix, can be written in terms as the curvature matrices θ^i, $i \in \{0,1\}$ for E and F (with respect to e and f) as

$$\theta = \begin{pmatrix} \theta^0 & 0 \\ 0 & \theta^1 \end{pmatrix}, k = \begin{pmatrix} k^0 & 0 \\ 0 & k^1 \end{pmatrix}.$$

Combined with the remark that

$$det(I + t\begin{pmatrix} A^0 & 0 \\ 0 & A^1 \end{pmatrix}) = det(I + tA^0)\ det(I + tA^1)$$

for any two matrices A^0 and A^1, we find that

$$c(E \oplus F, \nabla) = c(E, \nabla^0)c(F, \nabla^1)$$

(as differential forms!) from which the statement follows.

Duals/conjugations of connections and end proof of Prop. 10.4.2: For the rest of the proposition we need the dual and the conjugate of a connection. First of all, a connection ∇ on E induces a connection ∇^* on E^* by:

$$\nabla_X(s^*)(s) := Ł_X(s^*(s)) - s^*(\nabla_X(s)), \quad \forall\ s \in \Gamma(E), s^* \in \Gamma(E^*).$$

(why this formula?).

Exercise 10.5.2. Show that this is, indeed, a connection on E^*.

Starting with a local frame e of E, it is not difficult to compute the connection matrix of ∇^* with respect to the induced dual frame θ^*, in terms of the connection matrix θ of ∇ with respect to e:

$$\theta^* = -\theta^t$$

(minus the transpose of θ). Hence the same holds for the curvature matrix. Since for any matrix A

$$det(I + t(-A^t)) = det(I - tA) = \sum (-1)^k t^k \sigma_k(A),$$

we have $\sigma_k(-A^t) = (-1)^k \sigma_k(A)$ from which we deduce

$$c_k(E^*) = (-1)^k c_k(E).$$

Similarly, any connection ∇ on E induces a conjugated connection $\overline{\nabla}$ on $\overline{E}$. While $\overline{E}$ is really just $\overline{E}$ but with the structure of complex multiplication changed to:

$$z \cdot v := \overline{z}v, \quad (z \in \mathbb{C}, v \in E)$$

$\overline{\nabla}$ is just ∇ but interpreted as a connection on $\overline{E}$. It is then easy to see that the resulting connection matrix of $\overline{\nabla}$ is precisely $\overline{\theta}$. Since for any matrix A,

$$det(I + \frac{1}{2\pi i}\overline{A}) = \overline{det(I - \frac{1}{2\pi i}A)},$$

we have $c_k(\overline{A}) = (-1)^k \overline{c_k(A)}$ and then

$$c_k(\overline{E}) = (-1)^k \overline{c_k(E)}.$$

Finally, note that for any complex vector bundle E, E^* and $\overline{E}$ are isomorphic (the isomorphism is not canonical- one uses a hermitian metric on E to produce one). Hence $c_k(E^*) = c_k(\overline{E})$. Comparing with the previous two formulas, we obtain

$$c_k(\overline{E}) = (-1)^k \overline{c_k(E)} = (-1)^k c_k(E)$$

which shows both that $c_k(E)$ is real as well as the last formula in the proposition.

Tensor product of connections and the proof of Proposition 10.4.2: The additivity is proven, as above, using the direct sum of connections and the fact that for any two matrices A (r by r) and A' (r' by r'),

$$Tr \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix}^k = Tr(A^k) + Tr(A'^k).$$

For the multiplicativity we need the construction of the tensor product of two connections: given ∇^0 on E and ∇^1 on F, one produces ∇ on $E \otimes F$ by requiring

$$\nabla_X(s^0 \otimes s^1) = \nabla^0_X(s^0) \otimes s^1 + s^0 \otimes \nabla^1_X(s^1)$$

for all $s^0 \in \Gamma(E)$, $s^1 \in \Gamma(F)$. Frames e and f for E and F induce a frame $e \otimes f = \{e_i \otimes f_p : 1 \leq i \leq r, 1 \leq p \leq r'\}$ for $E \otimes F$ (r is the rank of E and r' of F). After a straightforward computation, we obtain as resulting curvature matrix

$$k = k^0 \otimes I_{r'} + I_r \otimes k^1$$

where I_r is the identity matrix and, for two matrices A and B, one of size r and one of size r', $A \otimes B$ denotes the matrix of size rr' (whose columns and rows are indexed by pairs with (i, p) as before)

$$(A \otimes B)^{j,q}_{i,p} = A^j_i B^q_p.$$

Remarking that $Tr(A \otimes B) = Tr(A)Tr(B)$, we immediately find

$$Tr(e^{A \otimes I + I \otimes B}) = Tr(e^A)Tr(e^B)$$

from which the desired formula follows.

Complexifications of connections and the proof of Proposition 10.4.5: For this second part, i.e. when l is odd, it suffices to remark that, for $E \otimes \mathbb{C}$, it is isomorphic to its conjugation then just apply the last part of Proposition 10.4.2. For l even we have to go again to connections and to remark that a connection ∇ on a real vector bundle E can be complexified to give a connection $\nabla^{\mathbb{C}}$ on $E \otimes \mathbb{C}$: just extend ∇ by requiring $\mathbb{C}$-linearity. Comparing the connection matrices we immediately find that

$$c_{2k}(E \otimes \mathbb{C}) = \left(\frac{1}{i}\right)^{2k} p_k(E) = (-1)^k p_k(E).$$

Proof of Proposition 10.4.6: The main observation is that, for any complex vector bundle E, one has a canonical isomorphism

$$E_{\mathbb{R}} \otimes \mathbb{C} \cong E \oplus \overline{E}$$

defined fiberwise by

$$v + \sqrt{-1}w \mapsto (v + iw, v - iw),$$

where $\sqrt{-1}$ is "the i used to complexify $E_{\mathbb{R}}$). Hence

$$c(E_{\mathbb{R}} \otimes \mathbb{C}) = c(E)c(\overline{E})$$

and, using the formulas from Proposition 10.4.5 and from end of Proposition 10.4.2, the desired formula follows.

10.6. Some exercises

Here are some more exercises to get used with these classes but also to see some of their use (some of which are rather difficult!).

Exercise 10.6.1. Show that, for any trivial complex vector bundle T (of arbitrary rank) and any other complex vector bundle E,

$$c(E \oplus T) = c(E).$$

Exercise 10.6.2. (the normal bundle trick). Assume that a manifold N is embedded in the manifold M, with normal bundle ν. Let τ_M be the tangent bundle of M and τ_N the one of N. Show that

$$c(\tau_M)|_N = c(\tau_N)c(\nu).$$

Exercise 10.6.3. For the tangent bundle τ of S^n show that

$$p(\tau) = 1.$$

Exercise 10.6.4. Show that the the tautological line bundle L over $\mathbb{CP}^1$ (see Exercise 10.4.1) is not isomorphic to the trivial line bundle. Actually, show that over $\mathbb{CP}^1$ one can find an infinite family of non-isomorphic line bundles.

Exercise 10.6.5. Let $L \subset \mathbb{CP}^n \times \mathbb{C}^{n+1}$ be the tautological line bundle over $\mathbb{CP}^n$ (generalizing the one from Exercise 10.4.1) and let

$$a := -c_1(L) \in H^2(\mathbb{CP}^n).$$

Show that

$$\int_{\mathbb{CP}^n} a^n = 1.$$

Deduce that all the cohomology classes $a, a^2, \ldots, a^n$ are non-zero.

Exercise 10.6.6. This is a continuation of the previous exercise. Let τ be the tangent bundle of $\mathbb{CP}^n$- a complex vector bundle (why?), and let $\tau_{\mathbb{R}}$ be the underlying real vector bundle. We want to compute $c(\tau)$ and $p(\tau_{\mathbb{R}})$.

Let $L^{\perp} \subset \mathbb{CP}^n \times \mathbb{C}^{n+1}$ be the (complex) vector bundle over $\mathbb{CP}^n$ whose fiber at $l \in \mathbb{CP}^n$ is the orthogonal $l^{\perp} \subset \mathbb{C}^{n+1}$ of l (with respect to the standard hermitian metric).

1. Show that

$$Hom(L, L) \cong T^1, \ Hom(L, L^{\perp}) \oplus T^1 \cong \underbrace{L^* \oplus \ldots \oplus L^*}_{n+1},$$

   where T^k stands for the trivial complex vector bundle of rank k.
2. Show that

$$\tau \cong Hom(L, L^{\perp}).$$

3. Deduce that
$$c(\tau) = (1+a)^{n+1},\ p(\tau_{\mathbb{R}}) = (1+a^2)^{n+1}.$$
4. Compute $Ch(\tau)$.

Exercise 10.6.7. Show that $\mathbb{CP}^4$ cannot be embedded in $\mathbb{R}^{11}$.

(Hint: use the previous computation and the normal bundle trick).

Exercise 10.6.8. Show that $\mathbb{CP}^{2010}$ cannot be written as the boundary of a compact, oriented (real) manifold.

(Hint: first, using Stokes' formula, show that if a manifold M of dimension $4l$ can be written as the boundary of a compact oriented manifold then $\int_M p_l(\tau_M)) = 0$).

Exercise 10.6.9. Show that $\mathbb{CP}^{2010}$ can not be written as the product of two complex manifolds of non-zero dimension.

(Hint: for two complex manifolds M and N of complex dimensions m and n, respectively, what happens to $Ch_{m+n}(\tau_{M\times N}) \in H^{2m+2n}(M\times N)$?).