LECTURE 2
Distributions on manifolds

As explained in the previous lecture, to show that an elliptic operator between sections of two vector bundles E and F,

$$P : \Gamma(M, E) \to \Gamma(M, F)$$

has finite index, we plan to use the general theory of Fredholm operators between Banach spaces. In doing so, we first have to interpret our P's as operators between certain "Banach spaces of sections". The problem is that the usual spaces of smooth sections $\Gamma(M, E)$ have no satisfactory Banach space structure. Given a vector bundle E over M, by a "Banach space of sections of E", $\mathbb{B}(M, E)$, one should understand (some) Banach space which contains the space $\Gamma(M, E)$ of all the smooth sections of E as a (dense) subspace. One way to introduce such Banach spaces is to consider the completion of $\Gamma(M, E)$ with respect to various norms of interest. This can be carried out in detail, but the price to pay is the fact that the resulting "Banach spaces of sections" have a rather abstract meaning (being defined as completions). We will follow a different path, which is based on the following remark: there is a very general (and natural!!) notion of "generalized sections of a vector bundle E over M", hence a space $\Gamma_{\text{gen}}(M; E)$ of such generalized sections (namely the space $\mathcal{D}'(M, E)$ of distributions, discussed in this lecture), so general that all the other "Banach spaces of sections" are subspaces of $\Gamma_{\text{gen}}(M; E)$. The space $\Gamma_{\text{gen}}(M; E)$ itself will not be a Banach space, but all the Banach spaces of sections which will be of interest for us can be described as subspaces of $\Gamma_{\text{gen}}(M; E)$ satisfying certain conditions (and that is how we will define them).

Implicit in our discussion is the fact that all the spaces we will be looking at will be vector spaces endowed with a topology (t.v.s.'s= topological vector spaces). Although our final aim is to deal with Banach spaces, the general t.v.s.'s will be needed along the way (however, all the spaces we will be looking at will be l.c.v.s.'s= locally convex vector spaces, i.e., similarly to Banach spaces, they can be defined using certain seminorms).

In this lecture, after recalling the notion of t.v.s. (topological vector space) and the special case of l.c.v.s. (locally convex vector space), we will discuss the space of generalized functions (distributions) on opens in $\mathbb{R}^n$ and then their generalizations to functions on manifolds or, more generally, to sections of vector bundles over manifolds. Since t.v.s.'s, l.c.v.s.' and the local theory of generalized functions (distributions) on opens in $\mathbb{R}^n$ have already been discussed in the intensive reminder, our job will be to pass from local (functions on opens in $\mathbb{R}^n$) to global (sections of vector bundles over arbitrary manifolds). However, these lecture notes also contain some of the local theory that has been discussed in the "intensive reminder".

2.1. Locally convex vector spaces

We start by recalling some of the standard notions from functional analysis (which have been discussed in the intensive reminder).

Topological vector spaces

First of all, a **t.v.s.** (**topological vector space**) is a vector space V (over $\mathbb{C}$) together with a topology $\mathcal{T}$, such that the two structures are compatible, i.e. the vector space operations

$$V \times V, (v, w) \mapsto v + w, \ \mathbb{C} \times V \to V, (\lambda, v) \mapsto \lambda v$$

are continuous. Recall that associated to the topology $\mathcal{T}$ and to the origin $0 \in V$, one has the family of all open neighborhoods of 0:

$$\mathcal{T}(0) = \{D \in \mathcal{T} : 0 \in D\}.$$

Since the translations $\tau_x : V \to V$, $y \mapsto y + x$ are continuous, the topology $\mathcal{T}$ is uniquely determined by by $\mathcal{T}(0)$: for $D \subset V$, we have

$$D \in \mathcal{T} \iff \forall\, x \in D \quad \exists\, B \in \mathcal{T}(0) \text{ such that } \ x + B \subset D. \tag{2.1}$$

In this characterization of the opens inside V, one can replace $\mathcal{T}(0)$ by any basis of neighborhoods of 0, i.e. by any family $\mathcal{B}(0) \subset \mathcal{T}(0)$ with the property that

$$D \in \mathcal{T}(0) \Longrightarrow \ \exists\, B \in \mathcal{B} \text{ such that } \ B \subset D.$$

In other words, if we knows a basis of neighborhoods $\mathcal{B}(0)$ of $0 \in V$, then we know the topology $\mathcal{T}$.

Exercise 2.1.1. Given a family $\mathcal{B}(0)$ of subsets of V containing the origin, what axioms should it satisfy to ensure that the resulting topology (defined by (2.1)) is indeed a topology which makes $(V, \mathcal{T})$ into a t.v.s.?

Note that, in a t.v.s. $(V, \mathcal{T})$, also the convergence can be spelled out in terms of a (any) basis if neighborhoods $\mathcal{B}(0)$ of 0: a sequence $(v_n)_{n \geq 1}$ of elements of V **converges to** $v \in V$, written $v_n \to v$, if and only if:

$$\forall\, B \in \mathcal{B}(0), \quad \exists\, n_B \in \mathbb{Z}_+ \ \text{ such that } \ v_n - v \in B \ \ \forall\, n \geq n_B.$$

Of course, this criterion can be used for $\mathcal{B}(0) = \mathcal{T}(0)$, but often there are smaller bases of neighborhoods $\mathcal{B}(0)$ at hand (after all, "b" is just the first letter of the word "ball"). For instance, if $(V, || - ||)$ is a normed space, then the resulting t.v.s. has as basis of neighborhoods

$$\mathcal{B}(0) = \{B(0, r) : r \geq 0\},$$

where

$$B(0, r) = \{v \in V : ||v|| < r\}.$$

In a t.v.s. $(V, \mathcal{T})$, one can also talk about the notion of Cauchy sequence: a sequence $(v_n)_{n\geq 1}$ in V is called a **Cauchy sequence** if:

$$\forall\ D \in \mathcal{T}(0)\ \ \exists\ \ n_D \in \mathbb{Z}_+ \ \text{ such that }\ v_n - v_m \in D\ \ \forall\ n, m \geq n_D.$$

Again, if we have a basis of neighborhoods $\mathcal{B}(0)$ at our disposal, it suffices to require this condition for $D = B \in \mathcal{B}(0)$.

In particular, one can talk about completeness of a t.v.s: one says that $(V, \mathcal{T})$ is (sequentially) **complete** if any Cauchy sequence in V converges to some $v \in V$.

Locally convex vector spaces

Recall also that a **l.c.v.s.** (**locally convex vector space**) is a t.v.s. $(V, \mathcal{T})$ with the property that "there are enough convex neighborhoods of the origin". That means that

$$\mathcal{T}_{\text{convex}}(0) := \{C \in \mathcal{T}(0) : C \text{ is convex}\}$$

is a basis of neighborhoods of $0 \in V$ or, equivalently:

$$\forall\ D \in \mathcal{T}(0)\ \ \exists\ \ C \in \mathcal{T}(0) \text{ convex, such that }\ C \subset D.$$

In general, l.c.v.s.'s are associated to families of seminorms (and sometimes this is taken as "working definition" for locally convex vector spaces). First recall that a **seminorm** on a vector space V is a map $p : V \to [0, \infty)$ satisfying

$$p(v + w) \leq p(v) + p(w), \quad p(\lambda v) = |\lambda| p(v),$$

for all $v, w \in V$, $\lambda \in \mathbb{C}$ (and it is called a norm if $p(v) = 0$ happens only for $v = 0$).

Associated to any family

$$P = \{p_i\}_{i \in I}$$

of seminorms (on a vector space V), one has a notion of balls:

$$B^r_{i_1,\ldots,i_n} := \{v \in V : p_{i_k}(v) < r,\ \forall 1 \leq k \leq n\},$$

defined for all $r > 0$, $i_1, \ldots, i_n \in I$. The collection of all such balls form a family $\mathcal{B}(0)$, which will induce a locally convex topology $\mathcal{T}_P$ on V (convex because each ball is convex). Note that, the convergence in the resulting topology is the expected one:

$$v_n \to v\ \text{ in } (V, \mathcal{T}_P) \Longleftrightarrow\ p_i(v_n - v) \to 0\ \ \forall\ i \in I.$$

(and there is a similar characterization for Cauchy sequences). The fact that, when it comes to l.c.v.s.'s it suffices to work with families of seminorms, follows from the following:

Theorem 2.1.2. *A t.v.s. $(V, \mathcal{T})$ is a l.c.v.s. if and only if there exists a family of seminorms P such that $\mathcal{T} = \mathcal{T}_P$.*

Proof Idea of the proof: to produce seminorms, one associates to any $C \subset V$ convex the functional

$$p_C(v) = \inf\{r > 0 : x \in rC\}.$$

Choosingn C "nice enough", this will be a seminorm. One then shows that one can find a basis of neighborhoods of the origin consisting of "nice enough" convex neighborhoods. □

By abuse of terminology, we also say that (V, P) is a l.c.v.s. (but one should keep in mind that all that matters is not the family of seminorms P but just the induced topology $\mathcal{T}_P$).

Remark 2.1.3. In most of the examples of l.c.v.s.'s, the seminorms come first (quite naturally), and the topology is the associated one. However, there are some examples in which the topology comes first and one may not even care of what the seminorms are (see the general construction of inductive limit topologies at the end of this section).

On the other hand, one should be aware that different sets of seminorms may induce the same l.c.v.s. (i.e. the same topology). For instance, if $P_0 \subset P$ is a smaller family of seminorms, but which has the property that, for any $p \in P$, there exists $p_0 \in P_0$ such that $p_0 \leq p$ (i.e. $p_0(v) \leq p(v)$ for all $v \in V$), then P and P_0 define the same topology. This trick will be repeatedly used in the examples.

Exercise 2.1.4. Prove the last statement.

Next, it will be useful to have a criteria for continuity of linear maps between l.c.v.s.'s in terms of the seminorms. The following is a very good exercise.

Proposition 2.1.5. *Let (V, P) and (W, Q) be two l.c.v.s.'s and let*

$$A : V \to W$$

be a linear map. Then T is continuous if and only if, for any $q \in Q$, there exist $p_1, \ldots, p_n \in P$ and a constant $C > 0$ such that

$$q(A(v)) \leq C \cdot max\{p_1(v), \ldots, p_n(v)\} \ \ \forall \ v \in V.$$

Note that we will deal only with l.c.v.s.'s which are separated (Hausdorff).

Exercise 2.1.6. Let (V, P) be a l.c.v.s., where $P = \{p_i\}_{i \in I}$ is a family of seminorms on V. Show that it is Hausdorff if and only if, for $v \in V$, one has the imlication:

$$p_i(v) = 0 \ \ \forall \ i \in I \Longrightarrow \ v = o.$$

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Finally, recall that a **Frechet space** is a t.v.s. V with the following properties:

1. it is complete.
2. its topology is induced by a countable family of semi-norms $\{p_1, p_2, \ldots\}$.

In this case, it follows that V is metrizable, i.e. the topology of V can also be induced by a (complete) metric:

$$d(v, w) := \sum_{n \geq 1} \frac{1}{2^n} \frac{p_n(v - w)}{1 + p_n(v - w)}.$$

Example 2.1.7. Of course, any Hilbert or Banach space is a l.c.v.s. This applies in particular to all the familiar Banach spaces such as the L^p-spaces on an open $\Omega \subset \mathbb{R}^n$

$$L^p(\Omega) = \{f : \Omega \to \mathbb{C} : f \text{ is measurable }, \int_\Omega |f|^p < \infty\},$$

with the norm

$$||f||_{L^p} = (\int_\Omega |f|^p)^{1/p}$$

Recall that, for $p = 2$, this is a Hilbert space with inner product

$$\langle f, g\rangle_{L^2} = \int_\Omega f\overline{g}.$$

Example 2.1.8. Another class of examples come from functions of a certain order, eventually with restrictions on their support. For instance, for an open $\Omega \subset \mathbb{R}^n$, $r \in \mathbb{Z}_+$ and $K \subset \Omega$ compact, we consider the space

$$\mathcal{C}^r_K(\Omega) = \{\phi : \Omega \to \mathbb{C} : \phi \text{ is of class } C^r \text{ and } \mathrm{supp}(\phi) \subset K\}.$$

The norm which is naturally associated to this space is $|| \cdot ||_{K,r}$ defined by

$$||\phi||_{r,K} = \sup\{|\partial^\alpha \phi(x)| : x \in K, |\alpha| \leq r\}.$$

With this norm, $\mathcal{C}^r_K(\Omega)$ becomes a Banach space. Note that convergence in this space is uniform convergence on K of all derivatives up to order r.

However, if we consider $r = \infty$, then $C^\infty_K(\Omega)$ should be considered with the family of seminorms $\{|| \cdot ||_{K,r} : r \in \mathbb{Z}_+\}$. The result is a Frechet space. Note that convergence in this space is uniform convergence on K of all derivatives.

Yet another natural space is the space of all smooth functions $C^\infty(\Omega)$. A nice topology on this space is the one induced by the family of seminorms

$$\{|| \cdot ||_{K,r} : K \subset \Omega \text{ compact}, r \in \mathbb{Z}_+\}.$$

Using an exhaustion of Ω by compacts, i.e. a sequence $(K_n)_{n\geq 0}$ of compacts with

$$\Omega = \cup_n K_n, \quad K_n \subset \mathrm{Int}(K_{n+1}),$$

we see that the original family of seminorms can be replaced by a countable one:

$$\{|| \cdot ||_{K_n,r} : n, r \in \mathbb{Z}_+\}$$

(using Remark 2.1.3, check that the resulting topology is the same!). Hence $C^\infty(\Omega)$ with this topology has the chance of being Frechet- which is actually the case.

Note that convergence in this space is uniform convergence on compacts of all derivatives.

Example 2.1.9. As a very general construction: for any t.v.s. (locally convex or not), there are (at least) two important l.c. topologies on the continuous dual:

$$V^* := \{u : V \to \mathbb{R} : u \text{ is linear and continuous}\}.$$

The first topology, denoted $\mathcal{T}_s$, is the one induced by the family of seminorms $\{p_v\}_{v\in V}$, where

$$p_v : V^* \to \mathbb{R}, \ p_v(u) = |u(v)|.$$

This topology is called the weak* topology on V^*, or the topology of simple convergence. Note that $u_n \to u$ in this topology if and only if $u_n(v) \to u(v)$ for all $v \in V$.

The second topology, denoted $\mathcal{T}_b$, called the strong topology (or of uniform convergence on bounded sets) is defined as follows. First of all, recall that a subset $B \subset V$ is called bounded if, for any neighborhood of the origin, there exists $\lambda > 0$ such that $B \subset \lambda V$. If the topology of V is generated by a family of seminorms P, this means that for any $p \in P$ there exists $\lambda_p > 0$ such that

$$B \subset B_p(r_p) = \{v \in V : p(v) < r_p\}.$$

This implies (see also Proposition 2.1.5) that for any continuous linear functional $u \in V^*$,

$$p_B(u) := \sup\{|u(v)| : v \in B\} < \infty.$$

In this way we obtain a family $\{p_B\}_B$ of seminorms (indexed by all the bounded sets B), and $\mathcal{T}_b$ is defined as the induced topology.

A related topology on V^* is the topology $\mathcal{T}_c$ of uniform convergence on compacts, induced by the family of seminorms $\{p_C : C \subset V^* \text{ compact}\}$.

Some explanations (for your curiosity): In this course, when dealing with a particular l.c.v.s. V, what will be of interest to us is to understand the convergence in V, understand continuity of linear maps defined on V or the continuity of maps with values in V (i.e., in practical terms, one may forget the l.c. topology and just keep in mind convergence and continuity). From this point of view, in almost all the cases in which we consider the dual V^* of a l.c.v.s. V (e.g. the space of distributions), in this course we will be in the fortunate situation that it does not make a difference if we use $\mathcal{T}_s$ or $\mathcal{T}_b$ on V^* (note: this does not mean that the two topologies coincide- it just means that the specific topological aspects we are interested in are the same for the two).

What happens is that the spaces we will be dealing with in this course have some very special properties. Axiomatising these properties, one ends up with particular classes of l.c.v.s.'s which can be understood as part of the general theory of l.c.v.s.'s. Here we give a few more details of what is really going on (the references below send you to the book "Topological vector spaces, distributions and kernels" by F. Treves).

First of all, as a very general fact: for any t.v.s. V, $\mathcal{T}_s$ and $\mathcal{T}_c$ induce the same topology on any equicontinuous subset $H \subset V^*$ (Prop. 32.5, pp 340). Recall that H is called equicontinuous if, for every $\epsilon > 0$, there exists a neighborhood B of the origin such that

$$|u(v)| \leq \epsilon, \quad \forall\ v \in B, \quad \forall\ u \in H.$$

An important class of t.v.s.'s is the one of barreled space, which we now recall. A barrel in a t.v.s. V is a non-empty closed subset $A \subset V$ with the following properties:

1. A is absolutely convex: $|\alpha|A + |\beta|A \subset A$ for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| + |\beta| = 1$.
2. A is absorbing: $\forall\ \ v \in V, \ \ \exists\ \ r > 0$ such that $v \in rA$.

A t.v.s. V is said to be barreled if any barrel in V is a neighborhood of zero. For instance, all Frechet spaces are barreled.

For a barreled space V, given $H \subset V^*$, the following are equivalent (Theorem 33.2, pp.349):

1. H is weakly bounded (i.e. bounded in the l.c.v.s. $(V^*, \mathcal{T}_s)$).
2. H is strongly bounded (i.e. bounded in the l.c.v.s. $(V^*, \mathcal{T}_b)$).

3. H is relatively compact in the weak topology (i.e. the closure of H in $(V^*, \mathcal{T}_s)$ is compact there).
4. H is equicontinuous.

Hence, for such spaces, the notion of "bounded" is the same in $(V^*, \mathcal{T}_s)$ and $(V^*, \mathcal{T}_b)$, and we talk simply about "bounded subsets of V^*. However, the notion of convergence of sequences may still be different; of course, strong convergence implies weak convergence, but all we can say about a weakly convergent sequence is that it is bounded in the strong topology. More can be said for a more special class of t.v.s.'s.

A t.v.s. is called a Montel space if V is barreled and every closed bounded subset of E is compact. Note that this notion is much more restrictive then that of barreled space. For instance, while all Banach spaces are barreled, the only Banach spaces which are Montel are the finite dimensional ones (because the unit ball is compact only in the finite dimensinal Banach spaces). On the other hand, while all Frechet spaces are barreled, there are Frechet spaces which are Montel, but also others which are not Montel. The main examples of Montel spaces which are of interest for us are: the space of smooth functions, and the space of test functions (discussed below).

For a Montel space V, it follows that the topologies $\mathcal{T}_c$ and $\mathcal{T}_b$ are the same (Prop. 34.5, pp. 357). From the general property of equicontinuous subsets H mentioned above, we deduce that on such H's,

$$\mathcal{T}_s|_H = \mathcal{T}_b|_H.$$

(also, by the last result we mentioned, H being equicontinuous is equivalent to being bounded). Taking for H to be the elements of a weakly convergent sequence and its weak limit (clearly weakly bounded!), it follows that the sequence is also strongly convergent; hence convergence w.r.t. $\mathcal{T}_s$ and w.r.t. $\mathcal{T}_b$ is the same. Note that this not implies that the two topologies are the same: we know from point-set topology that the notion of convergence w.r.t. a topology $\mathcal{T}$ does not determine the topology uniquely unless the topology satisfies the first axiom of countability (e.g. if it is metrizable).

As a summary, for Montel spaces V,

1. the notion of boundedness in $(V^*, \mathcal{T}_s)$ and in $(V^*, \mathcal{T}_b)$ is the same (and coincides with equicontinuity).
2. $\mathcal{T}_s$ and $\mathcal{T}_b$ induce the same topology on any bounded $H \subset V^*$.
3. a sequence in V^* is weakly convergent if and only it is strongly convergent.

Inductive limits

As we saw in all examples (and we will see in almost all the other examples), l.c.v.s.'s usually come with naturally associated seminorms and the topology is just the induced one. However, there is an important example in which the topology comes first (and one usually doesn't even bother to find seminorms inducing it): the space of test functions (see next section). This example fits into a general construction of l.c. topologies, known as "the inductive limit". The general framework is the following. Start with

$$X = \text{vector space }, X_\alpha \subset X \text{ vector subspaces such that } \; X = \cup_\alpha X_\alpha,$$

where α runs in an indexing set I. We also assume that, for each α, we have given:

$$\mathcal{T}_\alpha - \text{locally convex topology on } X_\alpha.$$

One wants to associate to this data a topology $\mathcal{T}$ on X, so that

1. $(X, \mathcal{T})$ is a l.c.v.s.

2. all inclusions $i_\alpha : X_\alpha \to X$ become continuous.

There are many such topologies (usually the "very small" ones, e.g. the one containing just $\emptyset$ and X itself) and, in general, if $\mathcal{T}$ works, then any $\mathcal{T}' \subset \mathcal{T}$ works as well. The question is: is there "the best one" (i.e. the smallest one)? The answer is yes, and that is what the inductive limit topology on X (associated to the initial data) is. On short, this is induced by the following basis of neighborhoods:

$$\mathcal{B}(0) := \{B \subset X : B - \text{convex such that } B \cap X_\alpha \in \mathcal{T}_\alpha(0) \text{ for all } \ \alpha \in I\},$$

(show that one gets a l.c. topology and it is the largest one!). One should keep in mind that what is important about $(X, \mathcal{T})$ is to recognize when a function on X is continuous, and when a sequence in X converges. The first part is a rather easy exercise with the following conclusion:

Proposition 2.1.10. *Let X be endowed with the inductive limit topology $\mathcal{T}$, let Y be another l.c.v.s. and let*

$$A : X \to Y$$

be a linear map. Then A is continuous if and only if each

$$A_\alpha := A|_{X_\alpha} : X_\alpha \to Y$$

is.

The recognition of convergent subsequences is a bit more subtle and, in order to have a more elegant statement, we place ourselves in the following situation: the indexing set I is the set $\mathbb{N}$ of positive integers,

$$X_1 \subset X_2 \subset X_3 \subset \ldots, \quad , X_n - \text{closed in } X_{n+1}, \quad \mathcal{T}_n = \mathcal{T}_{n+1}|_{X_n}$$

(i.e. each $(X_n, \mathcal{T}_n)$ is embedded in $(X_{n+1}, \mathcal{T}_{n+1})$ as a closed subspace). We assume that all the inclusions are strict. The following is a quite difficult exercise.

Theorem 2.1.11. *In the case above, a sequence$(x_n)_{n\geq 1}$ of elements in X converges to $x \in X$ (in the inductive limit topology) if and only if the following two conditions hold:*

1. *$\exists\ n_0$ such that $x, x_m \in X_{n_0}$ for all m.*
2. *$x_m \to x$ in X_{n_0}.*

(note: one can also show that $(X, \mathcal{T})$ cannot be metrizable).

2.2. Distributions: the local theory

In this section we recall the main functional spaces on $\mathbb{R}^n$ or, more generally, on any open $\Omega \subset \mathbb{R}^n$. Recall that, for $K \subset \mathbb{R}^n$ and $r \in \mathbb{N}$, one has the seminorm $|| \cdot ||_{K,r}$ on $C^\infty(\Omega)$ given by:

$$||f||_{r,K} = \sup\{|\partial^\alpha f(x)| : x \in K, |\alpha| \leq r\}.$$

$\mathcal{E}(\Omega)$: smooth functions:

One defines

$$\mathcal{E}(\Omega) := C^{\infty}(\Omega),$$

endowed with the locally convex topology induced by the family of seminorms $\{||\cdot||_{K,r}\}_{K\subset\Omega \text{ compact}, r\in\mathbb{Z}_+}$. This was already mentioned in Example 2.1.8. Hence, in this space, convergence means: $f_n \to f$ if and only if for each multi-index α and each compact $K \subset \Omega$, $\partial^{\alpha} f_n \to \partial^{\alpha} f$ uniformly on K.

As a l.c.v.s, it is a Frechet space (and is also a Montel space).

Algebraically, $\mathcal{E}(\Omega)$ is also a ring (or even an algebra over $\mathbb{C}$), with respect to the usual multiplication of functions. Note that this algebraic operation is continuous.

$\mathcal{D}(\Omega)$: compactly supported smooth functions (test functions):

One defines

$$\mathcal{D}(\Omega) := C_c^{\infty}(\Omega),$$

the space of smooth functions with compact support, with the following topology. First of all, for each $K \subset \Omega$, we consider

$$\mathcal{E}_K(\Omega) := C_K^{\infty}(\Omega),$$

the space of smooth functions with support inside K, endowed with the topology induced from the topology of $\mathcal{E}(\Omega)$ (which is the same as the topology discussed in Example 2.1.8., i.e. induced by the family of seminorms $\{||\cdot||_{K,r}\}_{r\in\mathbb{Z}_+}$. While, set theoretically (or as vector spaces),

$$\mathcal{D}(\Omega) = \cup_K \mathcal{E}_K(\Omega)$$

(union over all compacts $K \subset \Omega$), we consider the inductive limit topology on $\mathcal{D}(\Omega)$ (see the end of the previous section).

Convergent sequences are easy to recognize here: $f_n \to f$ in $\mathcal{D}(\Omega)$ if and only if there exist a compact K such that $f_n \in \mathcal{E}_K$ for all n, and $f_n \to f$ in $\mathcal{E}_K$ (indeed, using an exhaustion of Ω by compacts (see again Example 2.1.8), we see that we can place ourselves under the conditions which allow us to apply Theorem 2.1.11).

As a l.c.v.s., $\mathcal{D}(\Omega)$ is complete but it is not Frechet (see the end of Theorem 2.1.11). (However, it is a Montel space).

Algebraically, $\mathcal{D}(\Omega)$ is also an algebra over $\mathbb{C}$ (with respect to pointwise multiplication), which is actually an ideal in $\mathcal{E}(\Omega)$ (the product between a compactly supported smooth function and an arbitrary smooth function is again compactly supported).

$\mathcal{D}'(\Omega)$: distributions:

The space of distributions on $\mathbb{R}^n$ is defined as the (topological) dual of the space of test functions:

$$\mathcal{D}'(\Omega) := (\mathcal{D}(\Omega))^*$$

(see also Example 2.1.9). An element of this space is called a distribution on Ω. Unraveling the inductive limit topology on $\mathcal{D}(\Omega)$, one gets a more explicit description of these space. More precisely, using Proposition 2.1.10 to recognize the continuous linear maps by restricting to compacts, and using Proposition 2.1.5 to rewrite the resulting continuity conditions in terms of seminorms, one finds the following:

Corollary 2.2.1. *A distribution on* Ω *is a linear map*

$$u : C_c^\infty(\Omega) \to \mathbb{C}$$

with the following property: for any compact $K \subset \Omega$*, there exists* $C = C_K > 0$*,* $r = r_K \in \mathbb{N}$ *such that*

$$|u(\phi)| \leq C||\phi||_{K,r} \quad \forall\ \phi \in C_K^\infty(\Omega).$$

As a l.c.v.s., $\mathcal{D}'(U)$ will be endowed with the strong topology (the topology of uniform convergence on bounded subsets- see Example 2.1.9)). Note however, when it comes to convergence of sequences (u_n) of distributions, the strong convergence is equivalent to simple (pointwise) convergence. [1]

In general, any smooth function f induces a distribution u_f

$$\phi \mapsto \int_{\mathbb{R}^n} f\phi,$$

and this correspondence defines a continuous inclusion of

$$i : \mathcal{E}(\Omega) \hookrightarrow \mathcal{D}'(\Omega).$$

For this reason, distributions are often called "generalized functions", and one often identifies f with the induced distribution u_f.

Algebraically, the multiplication on $\mathcal{E}(\Omega)$ extends to a $\mathcal{E}(\Omega)$-module structure on $\mathcal{D}'(\Omega)$

$$\mathcal{E}(\Omega) \times \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega),\ (f, u) \mapsto fu,$$

where

$$(fu)(\phi) = u(f\phi).$$

[1]Explanation (for your curiosity): When it comes to the following notions:

1. bounded subsets of $\mathcal{D}'(\Omega)$,
2. convergence of sequences in $\mathcal{D}'(\Omega)$,
3. continuity of a linear map $A : V \to \mathcal{D}'(\Omega)$ defined on a Frechet space V (e.g. $V = \mathcal{E}(\Omega')$).
4. continuity of a linear map $A : V \to \mathcal{D}'(\Omega)$ defined on a l.c.v.s. V which is the inductive limit of Frechet spaces (e.g. $V = \mathcal{D}(\Omega')$).

(notions which depend on what topology we use on $\mathcal{D}'(\Omega)$), it does not matter whether we use the strong topology $\mathcal{T}_b$ or the weak topology $\mathcal{T}_s$ on $\mathcal{D}'(\Omega)$: the priory different resulting notions will actually coincide.

For boundedness and convergence this follows from the fact that $\mathcal{D}(\Omega)$ is a Montel space (Theorem 34.4, pp. 357 in the book by Treves). For continuity of linear maps defined on a Frechet space, one just uses that, because V is metrizable, continuity is equivalent to sequential continuity (i.e. the property of sending convergent sequences to convergent sequences) and the previous part. If V is an inductive limit of Frechet spaces one uses the characterization of continuity of linear maps defined on inductive limits (Proposition 2.1.10).

$\mathcal{E}'(\Omega)$: compactly supported distributions:

The space of compactly supported distributions on Ω is defined as the (topological) dual of the space of all smooth functions

$$\mathcal{E}'(\Omega) := (\mathcal{E}(\Omega))^*.$$

Using Proposition 2.1.5 to rewrite the continuity condition, we find:

Corollary 2.2.2. *A compactly supported distribution on Ω is a linear map*

$$u : C^\infty(\Omega) \to \mathbb{C}$$

with the following property: there exists a compact $K \subset \Omega$, $C > 0$ and $r \in \mathbb{N}$ such that

$$|u(\phi)| \leq C||\phi||_{K,r} \quad \forall\ \phi \in C^\infty(\Omega).$$

Again, as in the case of $\mathcal{D}'(\Omega)$, we endow $\mathcal{E}'(\Omega)$ with the strong topology.[2]

Note that the dual of the inclusion $\mathcal{D}(\Omega) \hookrightarrow \mathcal{E}$ induces a continuous inclusion

$$\mathcal{E}'(\Omega) \hookrightarrow \mathcal{D}'(\Omega).$$

Explicitly, any linear functional on $C^\infty(\Omega)$ can be restricted to a linear functional on $C_c^\infty(\Omega)$, and the estimates for the compactly supported distributions imply the ones for distributions.

Hence the four distributional spaces fit into a diagram

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{E}' & \longrightarrow & \mathcal{D}' \end{array},$$

in which all the arrows are (algebraic) inclusions which are continuous, and the spaces on the left are (topologically) the compactly supported version of the spaces on the right.

Supports of distributions

Next, we recall why $\mathcal{E}'(\Omega)$ is called the space of *compactly supported* distributions. The main remark is that the assignment

$$\Omega \mapsto \mathcal{D}'(\Omega)$$

defines a sheaf and, as for any sheaf, one can talk about sections with compact support. What happens is that the elements in $\mathcal{D}'(\Omega)$ which have compact support in this sense, are precisely the ones in the image of the inclusion $\mathcal{E}'(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$.

Here are some details. First of all, for any two opens $\Omega \subset \Omega'$, one has an inclusion ("extension by zero")

$$\mathcal{D}(\Omega) \hookrightarrow \mathcal{D}(\Omega'),\ f \mapsto \tilde{f},$$

[2]Explanation (for your curiosity): The same discussion as in the case of $\mathcal{D}'(\Omega)$ applies also to $\mathcal{E}'(\Omega)$. This is due to the fact that also $\mathcal{E}(\Omega)$ is a Montel space (with the same reference as for $\mathcal{D}(\Omega)$). Hence, when it comes to bounded subsets, convergent sequences, continuity of linear maps from a (inductive limit of) Frechet space(s) to $\mathcal{E}'(\Omega)$, it does not matter whether we use the strong topology $\mathcal{T}_b$ or the simple topology $\mathcal{T}_s$ on $\mathcal{E}'(\Omega)$.

where $\tilde{f}$ is f on Ω and zero outside. Dualizing, we get a "restriction map",

$$\mathcal{D}'(\Omega') \to \mathcal{D}'(\Omega),\ u \mapsto u|_{\Omega}.$$

The sheaf property of the distributions is the following property which follows immediately from a partition of unity argument:

Lemma 2.2.3. *Assume that $\Omega = \cup_i \Omega_i$, with $\Omega_i \subset \mathbb{R}^n$ opens, and that u_i are distributions on Ω_i such that, for all i and j,*

$$u_i|_{\Omega_i \cap \Omega_j} = u_j|_{\Omega_i \cap \Omega_j}.$$

Then there exists a unique distribution u on Ω such that

$$u|_{\Omega_i} = u_i$$

for all i.

Proof See your notes from the last lecture in the intensive reminder. □

From this it follows that, for any $u \in \mathcal{D}'(\Omega)$, there is a largest open $\Omega_u \subset \Omega$ on which u vanishes (i.e. $u|_{\Omega_u} = 0$).

Definition 2.2.4. For $u \in \mathcal{D}'(\Omega)$, define its support

$$\text{supp}(u) = \Omega - \Omega_u = \{x \in \Omega : u|_{V_x} = 0 \text{ for any neighborhood } V_x \subset \Omega \text{ of } x\}.$$

We say that u is compactly supported if supp(f) is compact.

Example 2.2.5. For any $x \in \Omega$, one has the distribution δ_x defined by

$$\delta_x(\phi) = \phi(x).$$

It is not difficult to check that its support is precisely $\{x\}$.

Exercise 2.2.6. Show that $u \in \mathcal{D}'(\Omega)$ has compact support if and only if it is in the image of the inclusion

$$\mathcal{E}'(\Omega) \hookrightarrow \mathcal{D}'(\Omega).$$

Derivatives of distributions and Sobolev spaces

Finally, we discuss one last property of distributions which is of capital importance: one can talk about the partial derivatives of any distribution! The key (motivating) remark is the following, which follows easily from integration by parts.

Lemma 2.2.7. *Let $f \in C^{\infty}(\Omega)$ and let u_f be the associated distribution.*

Let $\partial^{\alpha} f \in C^{\infty}(\Omega)$ be the higher derivative of f associated to a multi-index α, and let $u_{\partial^{\alpha} f}$ be the associated distribution.

Then $u_{\partial^{\alpha} f}$ can be expressed in terms of u_f by:

$$u_{\partial^{\alpha} f}(\phi) = (-1)^{|\alpha|} u_f(\partial^{\alpha} f).$$

This shows that the action of the operator ∂^{α} on smooth functions can be extended to distributions.

Definition 2.2.8. Fora distribution u on Ω and a multi-index α, one defines the new distribution $\partial^{\alpha} u$ on Ω, by

$$(\partial^{\alpha} u)(\phi) = (-1)^{\alpha} u(\partial^{\alpha} \phi),\ \ \forall\ \phi \in C_c^{\infty}(\Omega).$$

Example 2.2.9. The distribution u_f makes sense not only for smooth functions on Ω, but also for functions $f : \Omega \to \mathbb{C}$ with the property that $\phi f \in L^1(\Omega)$ for all $\phi \in C_c^\infty(\Omega)$ (so that the integral defining u_f) is absolutely convergent. In particular it makes sense for any $f \in L^2(\Omega)$ and, as before, this defines an inclusion

$$L^2(\Omega) \hookrightarrow \mathcal{D}'(\Omega).$$

We now see one of the advantages of the distributions: any $f \in L^2$, although it may even not be continuous, has derivatives $\partial^\alpha f$ of any order! Of course, they may fail to be functions, but they are distributions. In particular, it is interesting to consider the following spaces.

Definition 2.2.10. For any $r \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open, we define the Sobolev space on Ω of order r as:

$$H_r(\Omega) := \{u \in \mathcal{D}'(\Omega) : \partial^\alpha(u) \in L^2(\Omega) \text{ whenever } |\alpha| \le r\},$$

endowed with the inner product

$$\langle u, u' \rangle_{H_r} = \sum_{|\alpha| \le r} \langle \partial^\alpha u, \partial^\alpha u' \rangle_{L^2}.$$

In this way, $H_r(\Omega)$ becomes a Hilbert space.

2.3. Distributions: the global theory

The l.c.v.s.'s $\mathcal{E}(\Omega)$, $\mathcal{D}(\Omega)$, $\mathcal{D}'(\Omega)$ and $\mathcal{E}'(\Omega)$ can be extended from opens $\Omega \subset \mathbb{R}^n$ to arbitrary manifold M (allowing us to talk about distributions on M, or generalized functions on M) and, more generally, to arbitrary vector bundles E over a manifold M (allowing us to talk about distributional sections of E, or generalized sections of E). To explain this extension, we fix M to be an n-dimensional manifold, and let E be a complex vector bundle over M of rank p.

$\underline{\mathcal{E}(M; E)}$ **(smooth sections):**

One defines

$$\mathcal{E}(M; E) := \Gamma(E),$$

the space of all smooth sections of E endowed with the following local convex topology. To define it, we choose a cover $\mathcal{U} = \{U_i\}_{i \in I}$ of M by opens which are domains of "total trivializations" of E, i.e. both of charts (U_i, κ_i) for M as well as of trivializations $\tau_i : E|_{U_i} \to U_i \times \mathbb{C}^p$ for E. This data clearly induce an isomorphism of vector spaces

$$\phi_i : \Gamma(E|_{U_i}) \to C^\infty(\kappa_i(U_i))^p$$

(see also subsection 2.3 below). Altogether, and after restricting sections of E to the various $U_i's$, these define an injection

$$\phi : \Gamma(E) \to \Pi_i C^\infty(\kappa_i(U_i))^p = \Pi_i \mathcal{E}(\kappa_i(U_i))^p.$$

Endowing the right hand side with the product topology, the topology on $\Gamma(E)$ is the induced topology (via this inclusion). Equivalently, considering as indices $\gamma = (i, l, K, r)$ consisting of $i \in I$ (to index the open U_i), $1 \le l \le p$ (to index the

l-th component of $\phi(s|_{U_i})$), $K \subset \kappa(U_i)$ compact and r- non-negative integer, one has seminorms $||\cdot||_\gamma$ on $\Gamma(E)$ as follows: for $s \in \Gamma(E)$, restrict it to U_i, move it to $\mathcal{E}(\kappa_i(U_i))^p$ via ϕ_i, take its l-th component, and apply the seminorm $||\cdot||_{K,r}$ of $\mathcal{E}(\kappa_i(U_i))$:

$$||s||_\gamma = ||\phi(s|_{U_i})^l||_{r,K}.$$

Putting together all these seminorms will define the desired l.c. topology on $\Gamma(E)$.

Exercise 2.3.1. Show that this topology does not depend on the choices involved.

Note that, since the cover $\mathcal{U}$ can be chosen to be countable (our manifolds are always assumed to satisfy the second countability axiom!), it follows that our topology can be defined by a countable family of seminorms. Using the similar local result, you can now do the following:

Exercise 2.3.2. Show that $\mathcal{E}(M; E)$ is a Frechet space.

Finally, note that a sequence $(s_m)_{m\geq 1}$ converges to s in this topology if and only if, for any open U which is the domain of a local chart κ for M and of a local frame $\{s_1, \ldots, s_p\}$ for E, and for any compact $K \subset U$, writing $s_m = (f_m^1, \ldots, f_m^p)$, $s = (f^1, \ldots, f^p)$ with respect to the frame, all the derivatives $\partial_\kappa^\alpha(f_m^i)$ converge uniformly on K to $\partial_\kappa^\alpha(f^i$ (when $m \to \infty$).

When $E = \mathbb{C}_M$ is the trivial line bundle over M, we simplify the notation to $\mathcal{E}(M)$. As in the local theory, this is an algebra (with continuous multiplication). Also, the multiplication of sections by functions make $\mathcal{E}(M, E)$ into a module over $\mathcal{E}(M)$.

$\underline{\mathcal{D}(M, E)}$ (compactly supported smooth sections)

One defines

$$\mathcal{D}(M, E) := \Gamma_c(E),$$

the space of all compactly supported smooth sections endowed with the following l.c. topology defined exactly as in the local case: one writes

$$\mathcal{D}(M, E) = \cup_K \mathcal{E}_K(M; E),$$

where the union is over all compacts $K \subset M$, and $\mathcal{E}_K(M; E) \subset \mathcal{E}(M; E)$ is the space of smooth sections supported in K, endowed with the topology induced from $\mathcal{E}(M; E)$; on $\mathcal{D}(M, E)$ we consider the inductive limit topology.

Exercise 2.3.3. Describe more explicitly the convergence in $\mathcal{D}(M, E)$.

Again, when $E = \mathbb{C}_M$ is the trivial line bundle over M, we simplify the notation to $\mathcal{D}(M)$.

$\underline{\mathcal{D}'(M; E)}$ (generalized sections):

This is the space of distributional sections of E, or the space of generalized sections of E. To define it, we do not just take the dual of $\mathcal{D}(M, E)$ as in the local case, but we first:

1. Consider the complex density line bundle $D = D_M$ on M.[3] All we need to know about D is that its compactly supported sections can be integrated

[3]This is the complex version of the density bundle from the intensive reminder. Here is the summary (the complexified version of what we have already discussed). Given an n-dimensional vector space V, one defines $D_r(V)$, the space of r-densities (for any real number $r > 0$), as the set of all maps $\omega : \Lambda^n V \to \mathbb{C}$ satisfying

$$\omega(\lambda\xi) = |\lambda|^r \omega(\xi),\ \forall\ \xi \in \Lambda^n V.$$

Equivalently (and maybe more intuitively), one can use the set $\mathrm{Fr}(V)$ of all frames of V (i.e. ordered sets $(e_1, \ldots, e_n)$ of vectors of V which form a basis of V). Then $D_r(V)$ can also be described as the set of all functions

$$\omega : Fr(V) \to \mathbb{C}$$

with the property that, for any invertible n by n matrix A, and any frame e, for the new frame $A(e)$ one has

$$\omega(A(e)) = |\det(A)|^r \omega(e).$$

Intuitively, one may think of an r-density on V as some rule of computing volumes of the hypercubes (each frame determines such a hypercube). For each r, $D_r(V)$ is one dimensional (hence isomorphic to $\mathbb{C}$), but in a non-canonical way. Choosing a frame e of V, one has an induced r-density denoted

$$\omega_e = |e^1 \wedge \ldots \wedge e^n|_r$$

uniquely determined by the condition that $\omega_e(e) = 1$ (the e^i's in the notation stand for the dual basis of V^*).

For a manifold M, we apply this construction to all the tangent spaces to obtain a line bundle $D_r(M)$ over M, whose fiber at $x \in M$ is $D_r(T_xM)$. For $r = 1$, $D_1(M)$ is simply denoted D, or D_M whenever it is necessary to remove ambiguities. The sections of D are called densities on M.

Any local chart $(U, \kappa = (x^1_\kappa, \ldots, x^n_\kappa))$ induces a frame $(\partial/\partial x^i_\kappa)_x$ for T_xM with the dual frame $(dx^i_\kappa)_x$ for T^*_xM, for all $x \in U$. Hence we obtain an induced trivialization of $D_r(M)$ over U, with trivializing section

$$|dx^1_\kappa \wedge \ldots \wedge dx^n_\kappa|_r$$

(and, as usual, the smooth structure on D is so that these sections induced by the local charts are smooth).

An r-density on M is a section ω of $D_r(M)$. Hence, locally, with respect to a coordinate chart as before, such a density can be written as

$$\omega = f_\kappa \circ \kappa \cdot |(dx^1_\kappa \wedge \ldots \wedge dx^n_\kappa|_r$$

for some smooth function defined on $\kappa(U)$. If we consider another coordinate chart κ' on the same U then, after a short (but instructive) computation, wee see that f_κ changes according to the rule:

$$f_\kappa = |\mathrm{Jac}(h)|^r f_{\kappa'} \circ h,$$

where $h = \kappa' \circ \kappa^{-1}$ is the change of coordinates, and $\mathrm{Jac}(h)$ is the Jacobian of h. The case $r = 1$ reminds us of the usual integration and the change of variable formula: the usual integration of compactly supported functions on an open $\Omega \subset \mathbb{R}^n$ defines a map

$$\int_\Omega : C^\infty_c(\Omega) \to \mathbb{C}$$

and, if we move via a diffeomorphism $h : \Omega \to \Omega'$, one has the change of variables formula

$$\int_\Omega f = \int_\Omega |\mathrm{Jac}(h)| \cdot f \circ h.$$

Hence, for 1-densities on the domain U of a coordinate chart, one has an induced integration map

$$\int_U : \Gamma_c(U, D|_U) \to \mathbb{C}$$

over M without any further choice, i.e. there is an integral

$$\int_M : \Gamma_c(D) \to \mathbb{C}.$$

If you are more familiar with integration of (top-degree) forms, you may assume that M has an orientation, $D = \Lambda^n T^*M \otimes \mathbb{C}$- the space of $\mathbb{C}$-valued n-forms (an identification induced by the orientation), and $\int_M$ is the integral that you already know. Or, if you are more familiar with integration of functions on Riemannian manifolds, you may assume that M is endowed with a metric, D is the trivial line bundle (an identification induced by the metric) and that $\int_M$ is the integral that you already know.

2. Consider the "functional dual" of E:

$$E^{\vee} := E^* \otimes D = \mathrm{Hom}(E, D),$$

the bundle whose fiber at $x \in M$ is the complex vector space consisting of all ($\mathbb{C}$-)linear maps $E_x \to D_x$.

The main point about $E^{\vee}$ is that it comes with a "pairing" (pointwise the evaluation map)

$$< -, - >: \Gamma(E^{\vee}) \times \Gamma(E) \to \Gamma(D)$$

(and its versions with supports) and then, using the integration of sections of D, we get *canonical* pairings

$$[-, -] : \Gamma_c(E^{\vee}) \times \Gamma(E) \to \mathbb{C}, \quad (s_1, s_2) \mapsto \int_M < s_1, s_2 > .$$

We now define

$$\mathcal{D}'(M; E) := (\mathcal{D}(M, E^{\vee}))^*$$

(endowed with the strong topology). Note that, it is precisely because of the way that $E^{\vee}$ was constructed, that we have canonical (i.e. independent of any choices, and completely functorial) inclusions

$$\mathcal{E}(M; E) \hookrightarrow \mathcal{D}'(M; E),$$

sending a section s to the functional $u_s :=< \cdot, s >$. And, as before, we identify s with the induced distribution u_s.

When $E = \mathbb{C}_M$, we simplify the notation to $\mathcal{D}'(M)$.

As for the algebraic structure, as in the local case, $\mathcal{D}'(M; E)$ is a module over $\mathcal{E}(M)$, with continuous multiplication

$$\mathcal{E}(M) \times \mathcal{D}'(M; E) \to \mathcal{D}'(M; E)$$

(by sending ω to $\int_{\kappa(U)} f_{\kappa}$) which does not depend on the choice of the coordinates. For the global integration map

$$\int_M : \Gamma_c(M, D) \to \mathbb{C},$$

one decomposes an arbitrary compactly supported density Ω on M as a finite sum $\sum_i \omega_i$, where each ω_i is supported in the domain of a coordinate chart U_i (e.g. use partitions of unity) and put

$$\int_M \omega = \sum_i \int_{U_i} \omega_i.$$

Of course, one has to prove that this does not depend on the way we decompose ω as such a sum, but this basically follows from the additivity of the usual integral.

defined by

$$(fu)(s) = u(fs).$$

$\underline{\mathcal{E}'(M;E)}$ **(compactly supported generalized sections):**

This is the space of compactly supported distributional sections of E, or the space compactly supported generalized sections of E. It is defined as in the local case (but making again use of $E^{\vee}$), as

$$\mathcal{E}'(M;E) := (\mathcal{E}(M,E^{\vee}))^{*}.$$

Note that, by the same pairing as before, one obtains an inclusion

$$\mathcal{D}(M,E) \hookrightarrow \mathcal{E}'(M,E).$$

Hence, as in the local case, we obtain a diagram of inclusions

$$\begin{array}{ccc} \mathcal{D}(M,E) & \longrightarrow & \mathcal{E}(M,E) \\ \downarrow & & \downarrow \\ \mathcal{E}'(M,E) & \longrightarrow & \mathcal{D}'(M,E) \end{array} .$$

Example 2.3.4.

1. when $E = \mathbb{C}_M$ is the trivial line bundle over M, we have shorten the notations to $\mathcal{D}(M)$, $\mathcal{E}(M)$ etc. Hence, as vector spaces,
$$\mathcal{D}(M) = \Gamma_c(M),\ \mathcal{E}(M) = C^{\infty}(M),$$
while the elements of $\mathcal{D}'(M)$ will be called distributions on M.
2. staying with the trivial line bundle, but assuming now that $M = \Omega$ is an open subset of $\mathbb{R}^n$, we recover the spaces discussed in the previous section. Note that, in the case of distributions, we are using the identification of the density bundle with the trivial bundle induced by the section $|dx^1 \dots dx^n|$.
3. when $E = \mathbb{C}^p_M$ is the trivial bundle over M of rank p, then clearly
$$\mathcal{D}(M,\mathbb{C}^p_M) = \mathcal{D}(M)^p,\ \mathcal{E}(M, M \times \mathbb{C}^p_M) = \mathcal{E}(M)^p.$$
On the other hand, using the canonical identification between E^* and E, we also obtain
$$\mathcal{D}'(M,\mathbb{C}^p_M) = \mathcal{D}'(M)^p,\ \mathcal{E}'(M,\mathbb{C}^p_M) = \mathcal{E}'(M)^p.$$

Note that, as in the local theory, distributions $u \in \mathcal{D}'(M,E)$ can be restricted to arbitrary opens $U \subset M$, to give distributions $u|_U \in \mathcal{D}'(U, E|_U)$. More precisely, the restriction map

$$\mathcal{D}'(M,E) \to \mathcal{D}'(U, E|_U)$$

is defined as the dual of the map

$$\mathcal{D}'(U, E^{\vee}|_U) \to \mathcal{D}(M, E^{\vee})$$

which takes a compactly supported section defined on U and extends it by zero outside U.

Exercise 2.3.5. For a vector bundle E over M,

1. Show that $U \mapsto \mathcal{D}(U, E|_U)$ forms a sheaf over M.
2. Define the support of any $u \in \mathcal{D}(M, E)$.
3. Show that the injection

$$\mathcal{E}'(M, E) \hookrightarrow \mathcal{D}'(M, E)$$

identifies $\mathcal{E}'(M, E)$ with the space of compactly supported distributional sections (as a vector space only!).

Exercise 2.3.6. Show that $\mathcal{D}(M, E)$ is dense in $\mathcal{E}(M, E)$, $\mathcal{D}'(M, E)$ and $\mathcal{E}'(M, E)$ (you are allowed to use the fact that this is known for trivial line bundles over opens in $\mathbb{R}^n$).

Invariance under isomorphisms

Given two vector bundles, E over M and F over a manifold N, an isomorphism h between E and F is a pair (h, h_0), where $h_0 : M \to N$ is a diffeomorphism and $h : E \to F$ is a map which covers h_0 (i.e. sends the fiber E_x to $F_{h_0(x)}$ or, equivalently, the diagram below is commutative) and such that, for each $x \in M$, it restricts to a linear isomorphism between E_x and $F_{h_0(x)}$.

$$\begin{array}{ccc} E & \xrightarrow{h} & F \\ \downarrow & & \downarrow \\ M & \xrightarrow{h_0} & N \end{array}\ .$$

We now explain how such an isomorphism h induces isomorphisms between the four functional spaces of E and those of F (really an isomorphism between the diagrams they fit in). The four isomorphism from the functional spaces of E to those of F will be denoted by the same letter h_*, while its inverse by h^*. At the level of smooth sections, this is simply

$$h_* : \mathcal{E}(M, E) \to \mathcal{E}(N, F),\ \ h_*(s)(y) = h(s(h_0^{-1}(y))),$$

which also restricts to the spaces $\mathcal{D}$. At the level of generalized sections, it is more natural to describe the map

$$h^* : \mathcal{D}'(N, F) \to \mathcal{D}'(M, E),$$

(and h_* will be its inverse). This will be dual of a map

$$h^\vee : \mathcal{D}(F^\vee) \to \mathcal{D}(E^\vee)$$

defined by

$$h^\vee(u)(e_x) = h_0^*(u(h(e_x))),\quad (e_x \in E_x),$$

where we have used the pull-back of densities, $h_0^* : D_{N,h_0(x)} \to D_{M,x}$.

The same formula defines h^* on the spaces $\mathcal{E}'$.

Example 2.3.7. Given a rank p vector bundle E over M, one often has to choose opens $U \subset M$ which are domains of both a coordinate chart (U, κ) for M as well as the domains of a trivialization $\tau : E|_U \to U \times \mathbb{C}^p$ for E. We say that (U, κ, τ) is a total trivialization for E over U. Note that such a data defines an isomorphism h between the vector bundle $E|_U$ over U and the trivial bundle $\kappa(U) \times \mathbb{C}^p$:

$$h_0 = \kappa, \ h(e_x) = (\kappa(x), \tau(e_x)).$$

Hence any total trivialization (U, κ, τ) induces isomorphisms

$$h_{\kappa,\tau} : \mathcal{D}(U, E|_U) \to \mathcal{D}(\kappa(U))^p, \ h_{\kappa,\tau} : \mathcal{E}(U, E|_U) \to \mathcal{E}(\kappa(U))^p, \text{ etc.}$$

(see also Example 2.3.4).

2.4. General operators and kernels

Given two vector bundles, E over a manifold M and F over a manifold N, an operator from E to F is, roughly speaking, a linear map which associates to a "section of E" a "section of F". The quotes refer to the fact that there are several different choices for the meaning of sections: ranging from smooth sections to generalized sections, or versions with compact supports (or other types of sections). The most general type of operators are is following.

Definition 2.4.1. If E is a vector bundle over M and F is a vector bundle over F, a general operator from E to F is a linear continuous map

$$P : \mathcal{D}(M, E) \to \mathcal{D}'(N, F).$$

Remark 2.4.2. Note that general operators are often described with different domains and codomains. For instance, if $\mathcal{F}_1$ and $\mathcal{F}_2$ is any of the symbols $\mathcal{E}$, $\mathcal{D}$, $\mathcal{E}'$ or $\mathcal{D}'$ (or any of the other functional spaces that will be discussed in the next lecture), one can look at continuous linear operators

$$P : \mathcal{F}_1(M, E) \to \mathcal{F}_2(M, F). \tag{2.2}$$

But since in all cases $\mathcal{D} \subset \mathcal{F}_1$ and $\mathcal{F}_2 \subset \mathcal{D}'$ (with continuous inclusions), P does induce a general operator

$$P_{\text{gen}} : \mathcal{D}(M, E) \to \mathcal{D}'(N, F).$$

Conversely, since $\mathcal{D}(M, E)$ is dense in all the other functional spaces that we have discussed (Exercise 2.3.6), P_{gen} determines P uniquely. Hence, saying that we have an operator (2.2) is the same as saying that we have a general operator P_{gen} with the property that it extends to $\mathcal{F}_1(M, E)$, giving rise to a continuous operator taking values in $\mathcal{F}_2(M, F)$.

On the other extreme, one has the so called smoothing operators, i.e. operators which transforms generalized sections into smooth sections.

Definition 2.4.3. If E is a vector bundle over M and F is a vector bundle over F, a smoothing operator from E to F is a linear continuous map

$$P : \mathcal{E}'(M, E) \to \mathcal{E}(N, F).$$

We denote by $\Psi^{-\infty}(E, F)$ the space of all such smoothing operators. When E and F are the trivial line bundles, we will simplify the notation to $\Psi^{-\infty}(M)$.

In other words, a smoothing operator is a general operator $P : \mathcal{D}(M, E) \to \mathcal{D}'(N, F)$ which

1. P takes values in $\mathcal{E}(N, F)$.

2. P extends to a continuous linear map from $\mathcal{E}'(M, E)$ to $\mathcal{E}(N, F)$.

A very useful way of interpreting operators is in terms of their so called "kernels". The idea of kernel is quite simple- and to avoid (just some) notational complications, let us first briefly describe what happens when $M = U \subset \mathbb{R}^m$ and $N = V \subset \mathbb{R}^n$ are two open, and the bundles involved are the trivial line bundles. Then the idea is the following: any $K \in C^\infty(V \times U)$ induces an operator

$$P_K : \mathcal{D}(U) \to \mathcal{E}(V), \quad K(\phi)(y) = \int_U K(y, x)\phi(x)dx.$$

Even more: composing with the inclusion $\mathcal{E}(V) \hookrightarrow \mathcal{D}'(V)$, i.e. viewing P_K as an application

$$P_K : \mathcal{D}(U) \to \mathcal{D}'(V),$$

this map does not depend on K as a smooth functions, but just on K as a distribution (i.e. on $u_K \in \mathcal{D}'(V \times U)$). Indeed, for $\phi \in \mathcal{D}(U)$, $P_K(\phi)$, as a distribution on V, is

$$u_{P_K(\phi)} : \psi \mapsto \int_V P_K(\phi)\psi = \int_{V\times U} K(y, x)\psi(y)\phi(x)dydx = u_K(\psi \otimes \phi),$$

where $\psi \otimes \phi \in C^\infty(V \times U)$ is the map $(y, x) \mapsto \psi(y)\phi(x))$. In other words, any

$$K \in \mathcal{D}'(V \times U)$$

induces a linear operator

$$P_K : \mathcal{D}(U) \to \mathcal{D}'(V), \; P_K(\phi)(\psi) = K(\psi \otimes \phi)$$

which can be shown to be continuous. Moreover, this construction defines a bijection between $\mathcal{D}'(V \times U)$ and the set of all general operators (even more, when equipped with the appropriate topologies, this becomes an isomorphism of l.c.v.s.'s).

The passing from the local picture to vector bundles over manifolds works as usual, with some care to make the construction independent of any choices. Here are the details. Given the vector bundles E over M and F over N, we consider the vector bundle over $N \times M$:

$$F \boxtimes E^\vee := \mathrm{pr}_1^*(F) \otimes \mathrm{pr}_2^*(E^\vee),$$

where pr_j is the projection on the j-th component. Hence, the fiber over $(y, x) \in N \times M$ is

$$(F \boxtimes E^\vee)_{(y,x)} = F_y \otimes E_x^* \otimes D_{M,x}.$$

Note that the functional dual of this bundle is canonically identified with:

$$(F \boxtimes E^\vee)^\vee \cong F^\vee \boxtimes E.$$

Exercise 2.4.4. Work out this isomorphism. (Hints: the density bundle of $N \times M$ is canonically identified with $D_N \otimes D_M$; $D_M^* \otimes D_M \cong \underline{\mathrm{Hom}}(D_M, D_M)$ is canonically isomorphic to the trivial line bundle.)

As before, one may decide to use a fixed positive density on M and one on N, and then replace $F \boxtimes E^\vee$ by $F \boxtimes E^*$ and $F^\vee \boxtimes E$ by $F^* \boxtimes E$.

Fix now a distribution

$$K \in \mathcal{D}'(N \times M, F \boxtimes E^\vee).$$

We will associate to K a general operator

$$P_K : \mathcal{D}(M, E) \to \mathcal{D}'(N, F).$$

Due to the definition of the space of distributions, and to the identification mentioned above, K will be a continuous functions

$$K : \mathcal{D}(N \times M, F^\vee \boxtimes E) \to \mathbb{C}.$$

For $\psi \in \mathcal{D}(M, F^\vee)$ and $\phi \in \mathcal{D}(M, E)$ we denote by

$$\psi \otimes \phi \in \mathcal{D}(N \times M, F^\vee \boxtimes E)$$

the induced section $(y, x) \mapsto \psi(y) \otimes \phi(x)$. To describe P_K, let $\phi \in \mathcal{D}(M, E)$ and we have to specify $P_K(\phi) \in \mathcal{D}'(N, F)$, i.e. the continuous functional

$$P_K(\phi) : \mathcal{D}(N, F^\vee) \to \mathbb{C}.$$

We define

$$P_K(\phi)(\psi) := K(\psi \otimes \phi).$$

The general operator P_K is called the general operator associated to the kernel K. Highly non-trivial is the fact that any general operator arises in this way (and then K will be called the kernel of P_K).

Theorem 2.4.5. *The correspondence $K \mapsto P_K$ defines a 1-1 correspondence between*

1. *distributions $K \in \mathcal{D}'(N \times M; F \boxtimes E^\vee)$.*
2. *general operators $P : \mathcal{D}(M, E) \to \mathcal{D}'(N, F)$.*

Moreover, in this correspondence, one has

$$K \in \mathcal{E}(N \times M; F \boxtimes E^\vee) \Longleftrightarrow P \text{ is smoothing.}$$

Note (for your curiosity): the 1-1 correspondence actually defines an isomorphism of l.c.v.s.'s between

1. $\mathcal{D}'(N \times M; F \boxtimes E^\vee)$ with the strong topology.
2. the space $\mathcal{L}(\mathcal{D}(M, E), \mathcal{D}'(N, F))$ of all linear continuous maps, endowed with the strong topology.

Exercise 2.4.6. Let

$$P = \frac{d}{dx} : \mathcal{E}(\mathbb{R}) \to \mathcal{E}(\mathbb{R}).$$

Compute its kernel, and show that this is not a smoothing operator.