# LECTURE 3
# Functional spaces on manifolds

The aim of this section is to introduce Sobolev spaces on manifolds (or on vector bundles over manifolds). These will be the Banach spaces of sections we were after (see the previous lectures). To define them, we will take advantage of the fact that we have already introduced the very general spaces of sections (the generalized sections, or distributions), and our Banach spaces of sections will be defined as subspaces of the distributional spaces.

It turns out that the Sobolev-type spaces associated to vector bundles can be built up from smaller pieces and all we need to know are the Sobolev spaces $H_r$ of the Euclidean space $\mathbb{R}^n$ and their basic properties. Of course, it is not so important that we work with the Sobolev spaces themselves, but only that they satisfy certain axioms (e.g. invariance under changes of coordinates). Here we will follow an axiomatic approach and explain that, starting with a subspace of $\mathcal{D}'(\mathbb{R}^n)$ satisfying certain axioms, we can extend it to all vector bundles over manifolds. Back to Sobolev spaces, there is a subtle point: to have good behaved spaces, we will first have to replace the standard Sobolev spaces $H_r$, by their "local versions", denoted $H_{r,\mathrm{loc}}$. Hence, strictly speaking, it will be these local versions that will be extended to manifolds. The result will deserve the name "Sobolev space" (without the adjective "local") only on manifolds which are compact.

## 3.1. General functional spaces

When working on $\mathbb{R}^n$, we shorten our notations to

$$\mathcal{E} = \mathcal{E}(\mathbb{R}^n), \mathcal{D} = \mathcal{D}(\mathbb{R}^n), \mathcal{D}' = \ldots$$

and, similarly for the Sobolev space of order $r$:

$$H_r = H_r(\mathbb{R}^n) = \{u \in \mathcal{D}' : \partial^\alpha u \in L^2 : \quad \forall\ |\alpha| \leq r\}.$$

**Definition 3.1.1.** A functional space on $\mathbb{R}^n$ is a l.c.v.s. space $\mathcal{F}$ satisfying:

1. $\mathcal{D} \subset \mathcal{F} \subset \mathcal{D}'$ and the inclusions are continuous linear maps.
2. for all $\phi \in \mathcal{D}$, multiplication by $\phi$ defines a continuous map $m_\phi : \mathcal{F} \to \mathcal{F}$.

Similarly, given a vector bundle $E$ over a manifold $M$, one talks about functional spaces on $M$ with coefficients in $E$ (or just functional spaces on $(M, E)$).

As in the case of smooth functions, one can talk about versions of $\mathcal{F}$ with supports. Given a functional space $\mathcal{F}$ on $\mathbb{R}^n$, we define for any compact $K \subset \mathbb{R}^n$,

$$\mathcal{F}_K = \{u \in \mathcal{F},\ \mathrm{supp}(u) \subset K\},$$

endowed with the topology induced from $\mathcal{F}$, and we also define

$$\mathcal{F}_{\mathrm{comp}} := \cup_K \mathcal{F}_K$$

(union over all compacts in $\mathbb{R}^n$), endowed with the inductive limit topology. In terms of convergence, that means that a sequence $(u_n)$ in $\mathcal{F}_{\mathrm{comp}}$ converges to $u \in \mathcal{F}_{\mathrm{comp}}$ if and only if there exists a compact $K$ such that

$$\mathrm{supp}(u_n) \subset K \ \ \forall n, \ \ u_n \to u \ \text{ in } \mathcal{F}.$$

Note that $\mathcal{F}_{\mathrm{comp}}$ is itself a functional space.

Finally, for any functional space $\mathcal{F}$, one has another functional space (dual in some sense to $\mathcal{F}_{\mathrm{com}}$), defined by:

$$\mathcal{F}_{\mathrm{loc}} = \{u \in \mathcal{D}'(\mathbb{R}^n) : \phi u \in \mathcal{F} \ \ \forall\ \phi \in C_c^\infty(\mathbb{R}^n)\}.$$

This has a natural l.c. topology so that all the multiplication operators

$$m_\phi : \mathcal{F}_{\mathrm{loc}} \to \mathcal{F},\ u \mapsto \phi u \quad (u \in \mathcal{D})$$

are continuous- namely the smallest topology with this property. To define it, we use a family $P$ of seminorms defining the l.c. topology on $\mathcal{F}$ and, for every $p \in P$ and $\phi \in C_c^\infty(\mathbb{R}^n)$ we consider the seminorm $q_{p,\phi}$ on $\mathcal{F}_{\mathrm{loc}}$ given by

$$q_{p,\phi}(u) = p(\phi u).$$

The l.c. topology that we use on $\mathcal{F}_{\mathrm{loc}}$ is the one induced by the family $\{q_{p,\phi} : p \in P, \phi \in \mathcal{D}$. Hence, $u_n \to u$ in this topology means $\phi u_n \to \phi u$ in $\mathcal{F}$, for all $\phi$'s.

**Exercise 3.1.2.** Show that, for any l.c.v.s. $V$, a linear map

$$A : V \to \mathcal{F}_{\mathrm{loc}}$$

is continuous if and only if , for any test function $\phi \in \mathcal{D}$, the composition with the multiplication $m_\phi$ by $\phi$ is a continuous map $m_\phi \circ A : V \to \mathcal{F}$.

**Example 3.1.3.** The four basic functional spaces $\mathcal{D}$, $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{D}'$ are functional spaces and

$$\mathcal{D}_{\mathrm{comp}} = \mathcal{E}_{\mathrm{comp}} = \mathcal{D},\ (\mathcal{D}^{'})_{\mathrm{comp}} = (\mathcal{E}^{'})_{\mathrm{comp}} = \mathcal{E}',$$

$$\mathcal{D}_{\mathrm{loc}} = \mathcal{E}_{\mathrm{loc}} = \mathcal{E},\ (\mathcal{D}^{'})_{\mathrm{loc}} = (\mathcal{E}^{'})_{\mathrm{loc}} = \mathcal{D}'.$$

The same holds in the general setting of vector bundles over manifolds.

Regarding the Sobolev spaces, they are functional spaces as well, but the inclusions

$$H_{r,\mathrm{com}} \hookrightarrow H_r \hookrightarrow H_{r,\mathrm{loc}}$$

are strict (and the same holds on any open $\Omega \subset \mathbb{R}^n$).

The local nature of the spaces $\mathcal{F}_K$ is indicated by the following partition of unity argument which will be very useful later on.

**Lemma 3.1.4.** *Assume that $K \subset \mathbb{R}^n$ is compact, and let $\{\eta_j\}_{j\in J}$ a finite partition of unity over $K$, i.e. a family of compactly supported smooth functions on $\mathbb{R}^n$ such that $\sum_j \eta_j = 1$ on $K$. Let $K_j = K \cap supp(\eta_j)$. Then the linear map*

$$I : \mathcal{F}_K \to \Pi_{j\in J}\mathcal{F}_{K_j},\ u \mapsto (\eta_j u)_{j\in J}$$

*is a continuous embedding (i.e. it is an isomorphism between the l.c.v.s. $\mathcal{F}_K$ and the image of $I$, endowed with the subspace topology) and the image of $I$ is closed.*

**Proof** The fact that $I$ is continuous follows from the fact that each component is multiplication by a compactly supported smooth function. The main observation is that there is a continuous map $R$ going backwards, namely the one which sends $(u_j)_{j\in J}$ to $\sum_j u_j$, such that $R \circ I = \mathrm{Id}$. The rest is a general fact about t.v.s.'s: if $I : X \to Y$, $R : Y \to X$ are continuous linear maps between two t.v.s.'s such that $R \circ I = \mathrm{Id}$, then $I$ is an embedding and $D(X)$ is closed in $Y$. Let's check this. First, $I$ is open from $X$ to $D(X)$: if $B \subset X$ is open then, remarking that

$$I(B) = I(X) \cap R^{-1}(B)$$

and using the continuity of $R$, we see that $I(B)$ is open in $I(X)$. Secondly, to see that $I(X)$ is closed in $Y$, one remarks that

$$I(X) = \mathrm{Ker}(\mathrm{Id} - I \circ R).$$

□

Similar to Lemma 3.1.4, we have the following.

**Lemma 3.1.5.** *Let $\mathcal{F}$ be a functional space on $\mathbb{R}^n$ and let $\{\eta_i\}_{i\in I}$ be a partition of unity, with $\eta_i \in \mathcal{D}$. Let $K_i$ be the support of $\eta_i$. Then*

$$I : \mathcal{F}_{loc} \to \Pi_{i\in I}\mathcal{F}_{K_i},\quad u \mapsto\ (\mu_i u)_{i\in I}$$

*is a continuous embedding with closed image.*

**Proof** This is similar to Lemma 3.1.4 and the argument is identical. Denoting by $X$ and $Y$ the domain and codomain of $I$, we have $I : X \to Y$. On the other hand, we can consider $R : Y \to X$ sending $(u_i)_{i\in I}$ to $\sum_i u_i$ (which clearly satisfies $R \circ I = \mathrm{Id}$). What we have to make sure is that, if $\{K_i\}_{i\in I}$ is a locally finite family of compact subsets of $\mathbb{R}^n$, then one has a well-defined continuous map

$$R : \Pi_i\mathcal{F}_{K_i} \to \mathcal{F}_{\mathrm{loc}},\quad (u_i)_{i\in I} \mapsto \sum_i u_i.$$

First of all, $u = \sum_i u_i$ makes sense as a distribution: as a linear functional on test functions,

$$u(\phi) := \sum_i u_i(\phi)$$

(this is a finite sum whenever $\phi \in \mathcal{D}$). Even more, when restricted to $\mathcal{D}_K$, one finds $I_K$ finite such that the previous sum is a sum overall $i \in I_K$ for all $\phi \in \mathcal{D}_K$. This shows that $u \in \mathcal{D}'$. To check that it is in $\mathcal{F}_{\mathrm{loc}}$, we look at $\phi u$ for $\phi \in \mathcal{D}$ (and want to check that it is in $\mathcal{F}$). But, again, we will get a finite sum of $\phi u_i$'s, hence an element in $\mathcal{F}$. Finally, to see that the map is continuous, we have to check (see Exercise 3.1.2) that $m_\phi \circ A$ is continuous as a map to $\mathcal{F}$, for all $\phi$. But, again, this is just a finite sum of the projections composed with $m_\phi$. □

## 3.2. The Banach axioms

Regarding the Sobolev spaces $H_r$ on $\mathbb{R}^n$, one of the properties that make them suitable for various problems (and also for the index theorem) is that they are Hilbert spaces. On the other hand, as we already mentioned, we will have to use variations of these spaces for which this property is lost when we deal with manifolds which are not compact. So, it is important to realize what remains of this property.

**Definition 3.2.1.** (**Banach axiom**) Let $\mathcal{F}$ be a functional space on $\mathbb{R}^n$. We say that:

1. $\mathcal{F}$ is Banach if the topology of $\mathcal{F}$ is a Banach topology.
2. $\mathcal{F}$ is locally Banach if, for each compact $K \subset \mathbb{R}^n$, the topology of $\mathcal{F}_K$ is a Banach topology.

Similarly, we talk about "Frechet", "locally Frechet", "Hilbert" and "locally Hilbert" functional spaces on $\mathbb{R}^n$, or, more generally, on a vector bundle $E$ over a manifold $M$.

Of course, if $\mathcal{F}$ is Banach then it is also locally Banach (and similarly for Frechet and Hilbert). However, the converse is not true.

**Example 3.2.2.** $\mathcal{E}$ is Frechet (but not Banach- not even locally Banach). $\mathcal{D}$ is not Frechet, but it is locally Frechet. The Sobolev spaces $H_r$ are Hilbert. Their local versions $H_{r,\text{loc}}$ are just Frechet and locally Hilbert. The same applies for the same functional spaces on opens $\Omega \subset \mathbb{R}^n$.

**Proposition 3.2.3.** *A functional space $\mathcal{F}$ is locally Banach if and only if each $x \in \mathbb{R}^n$ admits a compact neighborhood $K_x$ such that $\mathcal{F}_{K_x}$ has a Banach topology (similarly for Frechet and Hilbert).*

**Proof** From the hypothesis it follows that we can find an open cover $\{U_i : i \in I\}$ of $\mathbb{R}^n$ such that each $\overline{U}_i$ is compact and $\mathcal{F}_{\overline{U}_i}$ is Banach. It follows that, for each compact $K$ inside one of these opens, $\mathcal{F}_K$ is Banach. We choose a partition of unity $\{\eta_i\}_{i\in I}$ subordinated to this cover. Hence each $\text{supp}(\eta)_i$ is compact inside $U_i$, $\{\text{supp}(\eta)_i\}_{i\in I}$ is locally finite and $\sum_i \eta_i = 1$.

Now, for an arbitrary compact $K$, $J := \{j \in I : \eta_j|_K \neq 0\}$ will be finite and then $\{\eta_j\}_{j\in J}$ will be a finite partition of unity over $K$ hence we can apply Lemma 3.1.4. There $K_j$ will be inside $U_j$, hence the spaces $\mathcal{F}_{K_j}$ have Banach topologies. The assertion follows from the fact that a closed subspace of a Banach space (with the induced topology) is Banach. □

**Remark 3.2.4.** If $\mathcal{F}$ is of locally Banach (or just locally Frechet), then $\mathcal{F}_{\text{comp}}$ (with its l.c. topology) is a complete l.c.v.s. which is not Frechet (hint: Theorem 2.1.11 ).

## 3.3. Invariance axiom

In general, a diffeomorphism $\chi : \mathbb{R}^n \to \mathbb{R}^n$ (i.e. a change of coordinates) induces a topological isomorphism $\chi_*$ on $\mathcal{D}'$:

$$\chi_*(u)(\phi) := u(\phi \circ \chi^{-1}).$$

To be able to pass to manifolds, we need invariance of $\mathcal{F}$ under changes of coordinates. In order to have a notion of local nature, we also consider a local version of invariance.

**Definition 3.3.1.** Let $\mathcal{F}$ be a functional space on $\mathbb{R}^n$. We say that:

1. $\mathcal{F}$ is invariant if for any diffeomorphism $\chi$ of $\mathbb{R}^n$, $\chi_*$ restricts to a topological isomorphism
$$\chi_* : \mathcal{F} \xrightarrow{\sim} \mathcal{F}.$$
2. $\mathcal{F}$ is locally invariant if for any diffeomorphism $\chi$ of $\mathbb{R}^n$ and any compact $K \subset \mathbb{R}^n$, $\chi_*$ restricts to a topological isomorphism
$$\chi_* : \mathcal{F}_K \xrightarrow{\sim} \mathcal{F}_{\chi(K)}.$$

Similarly, we talk about invariance and local invariance of functional spaces on vector bundles over manifolds.

Clearly, invariant implies locally invariant (but not the other way around).

**Example 3.3.2.** Of course, the standard spaces $\mathcal{D}$, $\mathcal{E}$, $\mathcal{D}'$, $\mathcal{E}'$ are all invariant (in general for vector bundles over manifolds). However, $H_r$ is not invariant but, fortunately, it is locally invariant (this is a non-trivial result which will be proved later on using pseudo-differential operators). As a consequence (see also below), the spaces $H_{r,\mathrm{loc}}$ are invariant.

**Proposition 3.3.3.** *A functional space $\mathcal{F}$ is locally invariant if and only if for any diffeomorphism $\chi$ of $\mathbb{R}^n$, any $x \in \mathbb{R}^n$ admits a compact neighborhood $K_x$ such that $\chi_*$ restricts to a topological isomorphism*

$$\chi_* : \mathcal{F}_{K_x} \xrightarrow{\sim} \mathcal{F}_{\chi(K_x)}.$$

**Proof** We will use Lemma 3.1.4, in a way similar to the proof of Proposition 3.2.3. Let $K$ be an arbitrary compact and $\chi$ diffeomorphism. We will check the condition for $K$ and $\chi$. As in the proof of Proposition 3.2.3, we find a finite partition of unity $\{\eta_j\}_{j\in J}$ over $K$ such that each $K_j = K \cap \mathrm{supp}(\eta_j)$ has the property that

$$\chi_* : \mathcal{F}_{K_j} \xrightarrow{\sim} \mathcal{F}_{\chi(K_j)}.$$

We apply Lemma 3.1.4 to $K$ and the partition $\{\eta_j\}$ (with map denoted by $I$) and also to $\chi(K)$ and the partition $\{\chi_*(\eta_i) = \eta_j \circ \chi^{-1}\}$ (with the map denoted $I_\chi$). Once we show that $\chi_*(\mathcal{F}_K) = \mathcal{F}_{\chi(K)}$ set-theoretically, the lemma clearly implies that this is also a topological equality. So, let $u \in \mathcal{F}_K$. Then $\eta_j u \in \mathcal{F}_{K_j}$ hence

$$\chi_*(\eta_i) \cdot \chi_*(u) = \chi_*(\eta_j \cdot u) \in \mathcal{F}_{\chi(K_j)} \subset \mathcal{F}_K,$$

hence also

$$\chi_*(u) = \sum_j \chi_*(\eta_i) \cdot \chi_*(u) \in \mathcal{F}_K.$$

□

## 3.4. Density axioms

We briefly mention also the following density axioms.

**Definition 3.4.1.** Let $\mathcal{F}$ be a functional space on $\mathbb{R}^n$. We say that:

1. $\mathcal{F}$ is normal if $\mathcal{D}$ is dense in $\mathcal{F}$.
2. $\mathcal{F}$ is locally normal if, for any compact $K \subset \mathbb{R}^n$, $\mathcal{F}_K$ is contained in the closure of $\mathcal{D}$ in $\mathcal{F}$.

Similarly, we talk about normal and locally normal functional spaces on vector bundles over manifolds.

Again, normal implies locally normal and one can prove a characterization of local normality analogous to Proposition 3.2.3 and Proposition 3.3.3.

**Example 3.4.2.** All the four basic functional spaces $\mathcal{D}$, $\mathcal{E}$, $\mathcal{D}'$ and $\mathcal{E}'$ are normal (also with coefficients in vector bundles). Also the space $H_r$ is normal. However, for arbitrary opens $\Omega \subset \mathbb{R}^n$, the functional spaces $H_r(\Omega)$ (on $\Omega$) are in general not normal (but they are locally normal). The local spaces $H_{r,\mathrm{loc}}$ are always normal (see also the next section).

The normality axiom is important especially when we want to consider duals of functional spaces. Indeed, in this case a continuous linear functional $\xi : \mathcal{F} \to \mathbb{C}$ is zero if and only if its restriction to $\mathcal{D}$ is zero. It follows that the canonical inclusions dualize to continuous injections

$$\mathcal{D} \hookrightarrow \mathcal{F}^* \hookrightarrow \mathcal{D}'.$$

The duality between $\mathcal{F}_{\mathrm{loc}}$ and $\mathcal{F}_{\mathrm{comp}}$ can then be made more precise- one has:

$$(\mathcal{F}_{\mathrm{loc}})^* = (\mathcal{F}^*)_{\mathrm{comp}}, (\mathcal{F}_{\mathrm{comp}})^* = (\mathcal{F}^*)_{\mathrm{loc}}$$

(note: all these are viewed as vector subspaces of $\mathcal{D}'$, each one endowed with its own topology, and the equality is an equality of l.c.v.s.'s).

## 3.5. Locality axiom

In general, the invariance axiom is not enough for passing to manifolds. One also needs a locality axiom which allows us to pass to opens $\Omega \subset \mathbb{R}^n$ without loosing the properties of the functional space (e.g. invariance).

**Definition 3.5.1.** (**Locality axiom**) We say that a functional space $\mathcal{F}$ is local if, as l.c.v.s.'s,

$$\mathcal{F} = \mathcal{F}_{\mathrm{loc}}.$$

Similarly we talk about local functional spaces on vector bundles over manifolds.

Note that this condition implies that $\mathcal{F}$ is a module not only over $\mathcal{D}$ but also over $\mathcal{E}$.

**Example 3.5.2.** From the four basic examples, $\mathcal{E}$ and $\mathcal{D}'$ are local, while $\mathcal{D}$ and $\mathcal{E}'$ are not. Unfortunately, $H_r$ is not local- and we will soon replace it with $H_{r,\mathrm{loc}}$ (in general, for any functional space $\mathcal{F}$, $\mathcal{F}_{\mathrm{loc}}$ is local).

With the last example in mind, we also note that, in general, when passing to the localized space, the property of being of Banach (or Hilbert, or Frechet) type does not change.

**Exercise 3.5.3.** Show that, for any functional space $\mathcal{F}$ and for any compact $K \subset \mathbb{R}^n$

$$(\mathcal{F}_{\mathrm{loc}})_K = \mathcal{F}_K,$$

as l.c.v.s.'s. In particular, $\mathcal{F}$ is locally Banach (or locally Frechet, or locally Hilbert), or locally invariant, or locally normal if and only if $\mathcal{F}_{\mathrm{loc}}$ is.

With the previous exercise in mind, when it comes to local spaces we have the following:

**Theorem 3.5.4.** *Let $\mathcal{F}$ be a local functional space on $\mathbb{R}^n$. Then one has the following equivalences:*

1. *$\mathcal{F}$ is locally Frechet if and only if it is Frechet.*
2. *$\mathcal{F}$ is locally invariant if and only if it is invariant.*
3. *$\mathcal{F}$ is locally normal if and only if it is normal.*

Note that in the previous theorem there is no statement about locally Banach. As we have seen, this implies Frechet. However, local spaces cannot be Banach.

**Proof** (of Theorem 3.5.4) In each part, we still have to prove the direct implications. For the first part, if $\mathcal{F}$ is locally Frechet, choosing a countable partition of unity and applying the previous lemma, we find that $\mathcal{F}_{\mathrm{loc}}$ is Frechet since it is isomorphic to a closed subspace of a Frechet space (a countable product of Frechet spaces is Frechet!). For the second part, the argument is exactly as the one for the proof of Proposition 3.3.3, but using Lemma 3.1.5 instead of Lemma 3.1.4. For the last part, let us assume that $\mathcal{F}$ is locally normal. It suffices to show that $\mathcal{E}$ is dense in $\mathcal{F}$: then, for any open $U \subset \mathcal{F}$, $U \cap \mathcal{E} \neq \emptyset$; but $U \cap \mathcal{E}$ is an open in $\mathcal{E}$ (because $\mathcal{E} \hookrightarrow \mathcal{F}$ is continuous) hence, since $\mathcal{D}$ is dense in $\mathcal{E}$ (with its canonical topology), we find $U \cap \mathcal{D} \neq \emptyset$.

To show that $\mathcal{E}$ is dense in $\mathcal{F}$, we will need the following variation of Lemma 3.1.5. We choose a partition of unity $\eta_i$ as there, but with $\eta_i = \mu_i^2$, $\mu_i \in \mathcal{D}$. Let

$$A = \mathcal{F}, \quad X = \Pi_{i \in I}\mathcal{F},$$

($X$ with the product topology). We define

$$i : A \to X,\ u \mapsto (\mu_i u)_i, \qquad p : X \to A,\ (u_i)_i \mapsto \sum \mu_i u_i.$$

As in the lemma, these make $A$ into a closed subspace of $X$. Hence we can place ourselves into the setting that we have a subspace $A \subset X$ of a l.c.v.s. $X$, which has a projection into $A$, $p : X \to A$ (not that we will omit writing $i$ from now on). Consider the *subset*

$$Y = \Pi_{i \in I}\mathcal{D} \subset X.$$

Modulo the inclusion $A \hookrightarrow X$, $B = A \cap Y$ becomes $\mathcal{E}$ and $p(Y) = B$. Also, since $A$ (or its image by $i$) is inside the closed subspace of $X$ which is $\Pi_i \mathcal{F}_{K_i}$, we see that the hypothesis of local normality implies that $A \subset \overline{Y}$ (all closures are w.r.t. the topology of $X$). We have to prove that $A$ is in the closure of $B$. Let

$a \in A$, $V$ an open neighborhood of $a$ in $X$. We have to show that $V \cap B \neq \emptyset$. From $V$ we make $V' = p^{-1}(A \cap V)$- another open neighborhood of $a$ in $X$. Since $A \subset \overline{Y}$, we have $V' \cap Y \neq \emptyset$. It now suffices to remark that $p(V') \subset V \cap B$. $\square$

Finally, let us point out the following corollary which shows that, in the case of locally Frechet spaces, locality can be checked directly, using test functions and without any reference to $\mathcal{F}_{\text{loc}}$.

**Corollary 3.5.5.** *If $\mathcal{F}$ is a functional space which is locally Frechet, then $\mathcal{F}$ is local if and only if the following two (test-)conditions are satisfied:*

1. *$u \in \mathcal{D}'(\mathbb{R}^n)$ belongs to $\mathcal{F}$ if and only if $\phi u \in \mathcal{F}$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$.*
2. *$u_n \to u$ in $\mathcal{F}$ if and only if $\phi u_n \to \phi$ in $\mathcal{F}$, for all $\phi \in C_c^\infty(\mathbb{R}^n)$.*

**Proof** The direct implication is clear. For the converse, assume that $\mathcal{F}$ is a functional space which satisfies these conditions. The first one implies that $\mathcal{F} = \mathcal{F}_{\text{loc}}$ as sets and we still have to show that the two topologies coincide. Let $\mathcal{T}$ be the original topology on $\mathcal{F}$ and let $\mathcal{T}_{\text{loc}}$ be the topology coming from $\mathcal{F}_{\text{loc}}$. Since $\mathcal{F} \hookrightarrow \mathcal{F}_{\text{loc}}$ is always continuous, in our situation, this tells us that $\mathcal{T}_{\text{loc}} \subset \mathcal{T}$. On the other hand, Id : $(\mathcal{F}, \mathcal{T}_{\text{loc}}) \to (\mathcal{F}, \mathcal{T})$ is clearly sequentially continuous hence, since $\mathcal{F}_{\text{loc}}$ is metrizable, the identity is also continuous, hence $\mathcal{T} \subset \mathcal{T}_{\text{loc}}$. This concludes the proof. $\square$

## 3.6. Restrictions to opens

The main consequence of the localization axiom is the fact that one can restrict to opens $\Omega \subset \mathbb{R}^n$. The starting remark is that, for any such open $\Omega$, there is a canonical inclusion

$$\mathcal{E}'(\Omega) \subset \mathcal{E}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$$

which should be thought of as "extension by zero outside $\Omega$", which comes from the inclusion $\mathcal{E}'(\Omega) \subset \mathcal{E}'(\mathbb{R}^n)$ (obtained by dualizing the restriction map $C^\infty(\mathbb{R}^n) \to C^\infty(\Omega)$). In other words, any compactly supported distribution on $\Omega$ can be viewed as a (compactly supported) distribution on $\mathbb{R}^n$. On the other hand,

$$\phi u \in \mathcal{E}'(\Omega), \quad \forall\, \phi \in C_c^\infty(\Omega),\ u \in \mathcal{D}'(\Omega).$$

Hence the following makes sense:

**Definition 3.6.1.** Given a <u>local</u> functional space $\mathcal{F}$, for any open $\Omega \subset \mathbb{R}^n$, we define

$$\mathcal{F}(\Omega) := \{u \in \mathcal{D}'(\Omega) : \phi u \in \mathcal{F} \ \ \forall\, \phi \ \in C_c^\infty(\Omega)\},$$

endowed with the following topology. Let $P$ be a family of seminorms defining the l.c. topology on $\mathcal{F}$ and, for every $p \in P$ and $\phi \in C_c^\infty(\Omega)$ we consider the seminorm $q_{p,\phi}$ on $\mathcal{F}$ given by

$$q_{p,\phi}(u) = p(\phi u).$$

We endow $\mathcal{F}(\Omega)$ with the topology associated to the family $\{q_{p,\phi} : p \in P, \phi \in C_c^\infty(\Omega)$.

**Theorem 3.6.2.** *For any local functional space $\mathcal{F}$ and any open $\Omega \subset \mathbb{R}^n$, $\mathcal{F}(\Omega)$ is a local functional space on $\Omega$ and, as such,*

*1. $\mathcal{F}(\Omega)$ is locally Banach (or Hilbert, or Frechet) if $\mathcal{F}$ is.*
*2. $\mathcal{F}(\Omega)$ is invariant if $\mathcal{F}$ is.*
*3. $\mathcal{F}(\Omega)$ is normal if $\mathcal{F}$ is.*

**Proof** For the first part, one remarks that $\mathcal{F}(\Omega)_K = \mathcal{F}_K$. For the second part, applying Theorem 3.5.4 to the local functional space $\mathcal{F}(\Omega)$ on $\Omega$, it suffices to show local invariance. I.e., it suffices to show that for any $\chi : \Omega \to \Omega$ diffeomorphism and $x \in \Omega$, we find a compact neighborhood $K = K_x$ such that $\chi_*$ is an isomorphism between $\mathcal{F}(\Omega)_K (= \mathcal{F}_K)$ and $\mathcal{F}(\Omega)_{\chi(K)} (= \mathcal{F}_{\chi(K)})$. The difficulty comes from the fact that $\chi$ is not defined on the entire $\mathbb{R}^n$. Fix $\chi$ and $x$. Then we can find a neighborhood $\Omega_x$ of $x \in \Omega$ and a diffeomorphism $\tilde{\chi}$ on $\mathbb{R}^n$ such that

$$\tilde{\chi}|_{\Omega_x} = \chi|_{\Omega_x}$$

(this is not completely trivial, but it can be done using flows of vector fields, on any manifold). Fix any compact neighborhood $K \subset \Omega_x$. Using the invariance of $\mathcal{F}$, it suffices to show that $\chi_*(u) = \tilde{\chi}_*(u)$ for all $u \in \mathcal{F}_K$. But

$$\chi_*(u),\ \tilde{\chi}_*(u) \in \mathcal{F} \subset \mathcal{D}'$$

are two distributions whose restriction to $\chi(\Omega_x)$ is the same and whose restrictions to $\mathbb{R}^n - \chi(K)$ are both zero. Hence they must coincide. For the last part, since we deal with local spaces, it suffices to show that $\mathcal{F}(\Omega)$ is locally normal, i.e that for any compact $K \subset \Omega$, $\mathcal{F}_K(\Omega) = \mathcal{F}_K \subset \mathcal{F}(\Omega)$ is contained in the closure of $\mathcal{D}(\Omega)$. I.e., for any $u \in \mathcal{F}_K$ and any open $U \subset \mathcal{F}(\Omega)$ containing $u$, $U \cap \mathcal{D}(\Omega) =\neq \emptyset$. But since $\mathcal{F}$ is locally normal and the restriction map $r : \mathcal{F} \to \mathcal{F}(\Omega)$ is continuous, we have $r^{-1}(U) \cap \mathcal{D} \neq \emptyset$ and the claim follows. $\square$

**Exercise 3.6.3.** By a sheaf of distributions $\hat{\mathcal{F}}$ on $\mathbb{R}^n$ we mean an assignment

$$\Omega \mapsto \hat{\mathcal{F}}(\Omega)$$

which associates to an open $\Omega \subset \mathbb{R}^n$ a functional space $\hat{\mathcal{F}}(\Omega)$ on $\Omega$ (local or not) such that:

1. if $\Omega_2 \subset \Omega_1$ and $u \in \hat{\mathcal{F}}(\Omega_1)$, then $u|_{\Omega_2}$ is in $\hat{\mathcal{F}}(\Omega_2)$. Moreover, the map
$$\hat{\mathcal{F}}(\Omega_1) \to \hat{\mathcal{F}}(\Omega_2),\ u \mapsto u|_{\Omega_1}.$$
is continuous.
2. If $\Omega = \cup_{i\in I}\Omega_i$ with $\Omega_i \subset \mathbb{R}^n$ opens ($I$ some index set), then the map
$$\hat{\mathcal{F}}(\Omega) \to \Pi_{i\in I}\hat{\mathcal{F}}(\Omega_i),\ u \mapsto (u|_{\Omega_i})_{i\in I}$$
is a topological embedding which identifies the l.c.v.s. on the left with the closed subspace of the product space consisting of elements $(u_i)_{i\in I}$ with the property that $u_i|_{\Omega_i\cap\Omega_j} = u_j|_{\Omega_i\cap\Omega_j}$ for all $i$ and $j$.

Show that

1. If $\mathcal{F}$ is a local functional space on $\mathbb{R}^n$ then $\Omega \mapsto \mathcal{F}(\Omega)$ is a sheaf of distributions.
2. Conversely, if $\hat{\mathcal{F}}$ is a sheaf of distributions on $\mathbb{R}^n$ then
$$\mathcal{F} := \hat{\mathcal{F}}(\mathbb{R}^n)$$
is a local functional space on $\mathbb{R}^n$ and $\hat{\mathcal{F}}(\Omega) = \mathcal{F}(\Omega)$ for all $\Omega$'s.

Below, for diffeomorphisms $\chi : \Omega_1 \to \Omega_2$ between two opens, we consider the induced $\chi_* : \mathcal{D}'(\Omega_1) \to \mathcal{D}'(\Omega_2)$.

**Corollary 3.6.4.** *Let $\mathcal{F}$ be a local functional space. If $\mathcal{F}$ is invariant then, for any diffeomorphism $\chi : \Omega_1 \to \Omega_2$ between two opens in $\mathbb{R}^n$, $\chi_*$ induces a topological isomorphism*

$$\chi_* : \mathcal{F}(\Omega_1) \to \mathcal{F}(\Omega_2).$$

**Proof** The proof of 2. of Theorem 3.6.2, when showing invariance under diffeomorphisms $\chi : \Omega \to \Omega$ clearly applies to general diffeomorphism between any two opens. □

## 3.7. Passing to manifolds

Throughout this section we fix

$$\mathcal{F} = \underline{\text{local, invariant functional space on } \mathbb{R}^n}.$$

and we explain how to induce functional spaces $\mathcal{F}(M, E)$ (of "generalized sections of $E$ of type $\mathcal{F}$") for any vector bundle $E$ over an $n$-dimensional manifold $M$.

To define them, we will use local total trivializations of $E$, i.e. triples $(U, \kappa, \tau)$ consisting of a local chart $(U, \kappa)$ for $M$ and a trivialization $\tau : E|_U \to U \times \mathbb{C}^p$ of $E$ over $U$. Recall (see Example 2.3.7) that any such total trivialization induces an isomorphism

$$h_{\kappa,\tau} : \mathcal{D}'(U, E|_U) \to \mathcal{D}'(\Omega_\kappa)^p \quad (\text{where } \Omega_\kappa = \kappa(U) \subset \mathbb{R}^n)^1.$$

**Definition 3.7.1.** We define $\mathcal{F}(M, E)$ as the space of all $u \in \mathcal{D}'(M, E)$ with the property that for any domain $U$ of a total trivialization of $E$, $h_{\kappa,\tau}(u|_U) \in \mathcal{F}(\Omega_\kappa)^p$.

We still have to define the topology on $\mathcal{F}(M, E)$, but we make a few remarks first. Since the previous definition applies to all $n$-dimensional manifolds:

[1] Let us make this more explicit. The total trivialization induces

1. a local frame $s_1, \ldots, s_p$ for $E$ over $U$. Then, for any $s \in \Gamma(E)$ we find (local) coefficients $f_s^i \in C^\infty(\Omega_\kappa)$, i.e. satisfying
$$s(x) = \sum_i f_s^i(\kappa(x)) s_i(x) \quad \text{for } x \in U.$$
2. the local dual frame $s^1, \ldots, s^p$ of $E^*$ and a local frame (i.e. non-zero section on $U$) of the density bundle of $M$, $|dx_\kappa^1 \wedge \ldots \wedge x_\kappa^n|$. Then, for any $\xi \in \Gamma(E^\vee)$ we find (local) coefficients $\xi_i \in C^\infty(\Omega_\kappa)$, i.e. satisfying
$$\xi(x) = \sum_i \xi_i(\kappa(x)) s^i(x) |dx_\kappa^1 \wedge \ldots \wedge x_\kappa^n|_x \quad (x \in U).$$
3. any $u \in \mathcal{D}'(M, E)$ has coefficients $u^i \in \mathcal{D}'(\Omega_\kappa)$, i.e. satisfying
$$u(\xi) = \sum_i u_i(\xi_i) \quad (\xi \in \Gamma_c(U, E^\vee)).$$

The map $h_{\kappa,\tau}$ sends $u$ to $(u_1, \ldots, u_n)$.

1. when applied to an open $\Omega \subset \mathbb{R}^n$ and to the trivial line bundle $\mathbb{C}_\Omega$ over $\Omega$, one recovers $\mathcal{F}(\Omega)$- and here we are using the invariance of $\mathcal{F}$.

2. it also applies to all opens $U \subset M$, hence we can talk about the spaces $\mathcal{F}(U, E|_U)$. From the same invariance of $\mathcal{F}$, when $U$ is the domain of a total trivialization chart $(U, \kappa, \tau)$, to check that $u \in \mathcal{D}'(U, E|_U)$ is in $\mathcal{F}(U, E|_U)$, it suffices to check that $h_{\kappa,\tau}(u) \in \mathcal{F}(\Omega_\kappa)^p$- i.e. we do not need to check the condition in the definition for <u>all</u> total trivialization charts.

3. If $\{U_i\}_{i\in I}$ is one open cover of $M$ and $u \in \mathcal{D}'(M, E)$, then
$$u \in \mathcal{F}(M, E) \Longleftrightarrow u|_{U_i} \in \mathcal{F}(U_i, E|_{U_i}) \;\; \forall\, i \in I.$$
This follows from the similar property of $\mathcal{F}$ on opens in $\mathbb{R}^n$.

**Exercise 3.7.2.** Given a vector bundle $E$ over $M$ and $U \subset M$, $\mathcal{F}$ induces two subspaces of $\mathcal{D}'(U, E|_U)$:
1. $\mathcal{F}(U, E|_U)$ just defined.
2. thinking of $\mathcal{F}(M, E)$ as a functional space on $M$ (yes, we know, we still have to define the topology, but that is irrelevant for this exercise), we have an induced space:
$$\{u \in \mathcal{D}'(U, E|_U) : \phi u \in \mathcal{F}(M, E), \;\; \forall\, \phi \in \mathcal{D}(U)\}.$$

Show that the two coincide.

Next, we take an open cover $\{U_i\}_{i\in I}$ by domains of total trivialization charts $(U_i, \kappa_i, \tau_i)$. It follows that we have an inclusion
$$h : \mathcal{F}(M, E) \to \Pi_{i\in I}\mathcal{F}(\Omega_{\kappa_i})^p.$$
We endow $\mathcal{F}(M, E)$ with the induced topology.

**Exercise 3.7.3.** Show that the topology on $\mathcal{F}(M, E)$ does not depend on the choice of the cover and of the total trivialization charts.

**Theorem 3.7.4.** *For any vector bundle $E$ over an $n$ dimensional manifold $M$, $\mathcal{F}(M, E)$ is a local functional space on $(M, E)$.*

*Moreover, if $\mathcal{F}$ is locally Banach, or locally Hilbert, or Frechet (= locally Frechet since $\mathcal{F}$ is local), or normal (= locally normal), then so is $\mathcal{F}(M, E)$.*

*Finally, if $F$ is a vector bundle over another $n$-dimensional manifold $N$ and $(h, h_0)$ is an isomorphism between the vector bundles $E$ and $F$, then $h_* : \mathcal{D}'(M, E) \to \mathcal{D}'(N, F)$ restricts to an isomorphism of l.c.v.s.'s*
$$h_* : \mathcal{F}(M, E) \to \mathcal{F}(N, F).$$

**Proof** We just have to put together the various pieces that we already know (of course, here we make use of the fact that all proofs that we have given so far work for vector bundles over manifolds). To see that $\mathcal{F}(M, E)$ is a functional space we have to check that we have continuous inclusions
$$\mathcal{D}(M, E) \hookrightarrow \mathcal{F}(M, E) \hookrightarrow \mathcal{D}'(M, E).$$
We just have to remark that the map $h$ used to define the topology of $\mathcal{F}(M, E)$ also describes the topology for $\mathcal{D}$ and $\mathcal{D}'$. To show that $\mathcal{F}(M, E)$ is local, one

uses the sheaf property of $\mathcal{F}(M,E)_{\text{loc}}$ (see Exercise 3.6.3) where the $U_i$'s there are chosen as in the construction of $h$ above. This reduces the problem to a local one, i.e. to locality of $\mathcal{F}$.

All the other properties follow from their local nature (i.e. Proposition 3.2.3 and the similar result for normality, applied to manifolds) and the fact that, for $K \subset M$ compact inside a domain $U$ of a total trivialization chart $(U, \kappa, \tau)$, $\mathcal{F}_K(M,E)$ is isomorphic to $\mathcal{F}_K(\Omega_\kappa)^p$. □

**Corollary 3.7.5.** *If $\mathcal{F}$ is locally Banach (or Hilbert) and normal then, for any vector bundle $E$ over a compact $n$ dimensional manifold $M$, $\mathcal{F}(M,E)$ is a Banach (or Hilbert) space which contains $\mathcal{D}(M,E)$ as a dense subspace.*

**Definition 3.7.6.** Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be local, invariant functional spaces on $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, and assume that $\mathcal{F}_1$ is normal. Let $M$ be an $m$-dimensional manifold and $N$ an $n$-dimensional one, and let $E$ and $F$ be vector bundles over $M$ and $N$, respectively. We say that a general operator

$$P : \mathcal{D}(M,E) \to \mathcal{D}'(N,F)$$

is of type $(\mathcal{F}_1,\ \mathcal{F}_2)$ if it takes values in $\mathcal{F}_2$ and extends to a continuous linear operator

$$P_{\mathcal{F}_1,\mathcal{F}_2} : \mathcal{F}_1(M,E) \to \mathcal{F}_2(N,E).$$

Note that, due to the normality axiom, the extension $P_{\mathcal{F}_1,\mathcal{F}_2}$ will be unique, hence the notation is un-ambiguous.

## 3.8. Back to Sobolev spaces

We apply the previous constructions to the Sobolev spaces $H_r$ on $\mathbb{R}^n$. Let us first recall some of the standard properties of these spaces:

1. they are Hilbert spaces.
2. $\mathcal{D}$ is dense in $H_r$.
3. if $s > n/2 + k$ then $H_s \subset C^k(\mathbb{R}^n)$ (with continuous injection) (Sobolev's lemma).
4. for all $r > s$ and all $K \subset \mathbb{R}^n$ compact, the inclusion $H_{s,K} \hookrightarrow H_{s,K}$ is compact (Reillich's lemma).

Also, as we shall prove later, $H_r$ are locally invariant (using pseudo-differential operators and Proposition 3.3.3). Assuming all these, we now consider the associated local spaces

$$H_{r,\text{loc}} = \{u \in \mathcal{D}' : \phi u \in H_r, \quad \forall\ \phi \in \mathcal{D}\}$$

and the theory we have developed imply that:

1. $H_{r,\text{loc}}$ is a functional space which is locally Hilbert, invariant and normal.
2. $\cap_r H_{r,\text{loc}} = \mathcal{E}$. Even better, for $r > n/2 + k$, any $s \in H_{r,\text{loc}}$ is of class $C^k$.

Hence these spaces extend to manifolds.

**Definition 3.8.1.** For a vector bundle $E$ over an $n$-dimensional manifold $M$,

1. the resulting functional spaces $H_{r,\text{loc}}(M,E)$ are called the local $r$-Sobolev spaces of $E$.

2. for $K \subset M$ compact, the resulting $K$-supported spaces are denoted $H_{r,K}(M, E)$.
3. the resulting compactly supported spaces are denoted $H_{r,\text{comp}}(M, E)$ (hence they are $\cup_K H_{r,K}(M, E)$ with the inductive limit topology).

If $M$ is compact, we define the $r$-Sobolev space of $E$ as

$$H_r(M, E) := H_{r,\text{loc}}(M, E)(= H_{r,\text{comp}}(M, E)).$$

**Corollary 3.8.2.** *For any vector bundle $E$ over a manifold $M$,*

1. *$H_{r,loc}(M, E)$ are Frechet spaces.*
2. *$\mathcal{D}(M, E)$ is dense in $H_{r,loc}(M, E)$.*
3. *if a distribution $u \in \mathcal{D}'(M, E)$ belongs to all the spaces $H_{r,loc}(M, E)$, then it is smooth.*
4. *for $K \subset M$ compact, $H_{r,K}(M, E)$ has a Hilbert topology and, for $r > s$, the inclusion*

$$H_{r,K}(M, E) \hookrightarrow H_{s,K}(M, E)$$

*is compact.*

**Proof** The only thing that may still need some explanation is the compactness of the inclusion. But this follows from the Reillich's lemma and the partition of unity argument, i.e. Lemma 3.1.4 . □

**Corollary 3.8.3.** *For any vector bundle $E$ over a compact manifold $M$, $H_r(M, E)$ has a Hilbert topology, contains $\mathcal{D}(M, E)$ as a dense subspace,*

$$\cap_r H_{r,loc}(M, E) = \Gamma(E)$$

*and, for $r > s$, the inclusion*

$$H_r(M, E) \hookrightarrow H_s(M, E)$$

*is compact.*

Finally, we point out the following immediate properties of operators. The first one says that differential operators of order $k$ are also operators of type $(H_r, H_{r-k})$.

**Proposition 3.8.4.** *For $r \geq k \geq 0$, given a differential operator $P \in \mathcal{D}_k(E, F)$ between two vector bundles over $M$, the operator*

$$P : \mathcal{D}(M, E) \to \mathcal{D}(M, F)$$

*admits a unique extension to a continuous linear operator*

$$P_r : H_{r,loc}(M, E) \to H_{r-k,loc}(M, F).$$

The second one says that smoothing operators on compact manifolds, viewed as operators of type $(H_r, H_s)$, are compact.

**Proposition 3.8.5.** *Let $E$ and $F$ be two vector bundles over a compact manifold $M$ and consider a smoothing operator $P \in \Psi^{-\infty}(E, F)$. Then for any $r$ and $s$, $P$ viewed as an operator*

$$P : H_r(M, E) \to H_s(M, F)$$

*is compact.*

**Remark 3.8.6.** Back to our strategy of proving that the index of an elliptic differential operator $P \in \mathcal{D}_k(E, F)$ (over a compact manifold) is well-defined, our plan was to use the theory of Fredholm operators between Banach spaces. We have finally produced our Banach spaces of sections on which our operator will act:

$$P_r : H_r(M, E) \to H_{r-k}(M, F).$$

To prove that $P_r$ is Fredholm, using Theorem 1.4.5 on the characterization of Fredholm operators and the fact that all smoothing operators are compact, we would need some kind of "inverse of $P_r$ modulo smoothing operators", i.e. some kind of operator "of order $-k$" going backwards, such that $PQ - \mathrm{Id}$ and $QP - \mathrm{Id}$ are smoothing operators. Such operators "of degree $-k$" cannot, of course, be differential. What we can do however is to understand what make differential operators behave well w.r.t. (e.g.) Sobolev spaces- and the outcome is: it is not important that their total symbols are polynomials (of some degree $k$) in $\xi$, but only their symbols have a certain $k$-polynomial-like behaviour (in terms of estimates). And this is a property which makes sense even for $k$-negative (and that is where we have to look for our $Q$). This brings us to pseudo-differential operators ...

**Exercise 3.8.7.** Show that if such an operator $Q : H_{r-k}(M, F) \to H_r(M, E)$ is found (i.e. with the property that $PQ - \mathrm{Id}$ and $QP - \mathrm{Id}$ are smoothing), then the kernel of the operator $P : \Gamma(E) \to \Gamma(F)$ is finite dimensional. What about the cokernel?