Analysis on Manifolds
Lecture notes for the 2009/2010 Master Class

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LECTURE 5
Pseudo-differential operators, local theory

5.1. The space of symbols

We consider a differential operator P on $\mathbb{R}^n$ of the form

$$P = p(x, D_x) = \sum_{|\alpha| \le d} c_\alpha(x) D_x^\alpha; \tag{5.1}$$

here we recall that $D_x^\alpha = (-i\partial_x)^\alpha$. The coefficients c_α are assumed to be smooth functions on $\mathbb{R}^n$. The (full) symbol of P is the function $p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ given by

$$p(x, \xi) = \sum_{|\alpha| \le d} c_\alpha(x) \xi^\alpha.$$

If $f \in C_c^\infty(\mathbb{R}^n)$, then $\mathcal{F}(D^\alpha f) = \xi^\alpha \mathcal{F} f$, so that by the Fourier inversion formula we have

$$D^\alpha f(x) = \int_{\mathbb{R}^n} \xi^\alpha \widehat{f}(\xi)\, e^{i\xi x}\, d\xi,$$

where we have written $\widehat{f} = \mathcal{F} f$. It follows that the action of P on $C_c^\infty(\mathbb{R}^n)$ can be described by

$$Pf(x) = \int_{\mathbb{R}^n} p(x, \xi) \widehat{f}(\xi)\, e^{i\xi x}\, d\xi.$$

Pseudo-differential operators are going to be defined by the same formula, but with p from a larger class of spaces of functions, the so-called symbol spaces. The idea is to make the class large enough to allow a kind of division. This in turn will allow us to construct inverses to elliptic operators modulo smoothing operators, the so-called parametrices.

We return to the full symbol p of the differential operator P of degree at most d considered above. By the polynomial nature of the symbol p in the ξ-variable, there exists, for every compact subset $K \subset \mathbb{R}^n$ and all $\alpha, \beta \in \mathbb{N}^n$, a constant $C = C_{K,\alpha,\beta} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \le C(1 + \|\xi\|)^{d - |\beta|}, \qquad ((x, \xi) \in K \times \mathbb{R}^n).$$

Exercise 5.1.1. Prove this.

These observations motivate the following definition of the space of symbols of order d, for d a real number.

Definition 5.1.2. Let $U \subset \mathbb{R}^n$ be an open subset and let $d \in \mathbb{R}$. The space of symbols on U of order at most d is defined to be the space of functions $q \in C^\infty(U \times \mathbb{R}^n)$ such that for each compact subset $K \subset U$ and all multi-indices α, β, there exists a constant $C = C_{K,\alpha,\beta}$ such that

$$|\partial_x^\alpha \partial_\xi^\beta q(x, \xi)| \le C(1 + \|\xi\|)^{d - |\beta|}, \qquad ((x, \xi) \in K \times \mathbb{R}^n).$$

This space is denoted by $S^d(U)$.

We note that $S^d(U)$ can be equipped with the locally convex topology induced by the seminorms

$$\mu^d_{K,k}(q) := \max_{|\alpha|,|\beta|\leq k} \sup_{K\times\mathbb{R}^n} (1+\|\xi\|)^{|\beta|-d}\,|\partial_x^\alpha\partial_\xi^\beta q(x,\xi)|,$$

for $K \subset U$ compact and $k \in \mathbb{N}$. Moreover, $S^m(U)$ is a Fréchet space for this topology.

Exercise 5.1.3. Show that $d_1 \leq d_2$ implies $S^{d_1}(U) \subset S^{d_2}(U)$ with continuous inclusion map.

We agree to write

$$S^\infty(U) = \cup_{d\in\mathbb{R}} S^d(U), \qquad S^{-\infty}(U) = \cap_{d\in\mathbb{R}} S^d(U).$$

Then $S^{-\infty}(U)$ equals the space of smooth functions $f : U \times \mathbb{R}^n \to \mathbb{C}$ such that for all $K \subset U$ compact, $N \in \mathbb{N}$ and $k \in \mathbb{N}$

$$\nu_{K,k,N}(f) := \max_{|\alpha|,|\beta|\leq k} \sup_{K\times\mathbb{R}^n} (1+\|\xi\|)^N |\partial_x\partial_\xi^\beta f(x,\xi)| < \infty.$$

Moreover, the norms $\nu_{K,k,N}$ induce a locally convex topology on $S^{-\infty}(U)$, which turn this space into a Fréchet space. Here we note that a function φ in the usual Schwartz space $\mathcal{S}(\mathbb{R}^n)$ can be viewed as the function $(x,\xi) \mapsto \varphi(\xi)$ in $S^{-\infty}(U)$. The corresponding natural linear map $\mathcal{S}(\mathbb{R}^n) \to S^{-\infty}(U)$ is a topological linear isomorphism onto the closed subspace of functions in $S^{-\infty}(U)$ that are constant in the x-variable. More generally, if $f \in C^\infty(U)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then the function

$$f \otimes \varphi : (x,\xi) \mapsto f(x)\varphi(\xi)$$

belongs to $S^{-\infty}(U)$. It can be shown that $S^{-\infty}(U)$ is the closure of the subspace $C^\infty(U) \otimes \mathcal{S}(\mathbb{R}^n)$ generated by these elements. Accordingly, we may view $S^{-\infty}(U)$ as a topological tensor product; this is expressed by the notation

$$S^{-\infty}(U) = C^\infty(U)\,\widehat{\otimes}\,\mathcal{S}(\mathbb{R}^n).$$

Exercise 5.1.4. Show that the following functions are symbols on $U = \mathbb{R}^n$. What can be said about their orders?

(a) $p(x,\xi) = \|x\|^2(1+\|\xi\|^2)^s$, for $s \in \mathbb{R}$.
(b) $p(x,\xi) = (1+\|x\|^2+\|\xi\|^2)^s$, for $s \in \mathbb{R}$.

Exercise 5.1.5. Let $U \subset \mathbb{R}^n$ be an open subset.

(a) Show that for each $\alpha \in \mathbb{N}^n$ the operator ∂_x^α gives a continuous linear map $S^d(U) \to S^d(U)$.
(b) Show that for each multi-index α as above the operator ∂_ξ^α restricts to a continuous linear map $S^d(U) \to S^{d-|\alpha|}(U)$, for every $d \in \mathbb{R}$.
(c) Show that the product map $(p,q) \mapsto pq$ restricts to a continuous bilinear map $S^d(U) \times S^e(U) \to S^{d+e}(U)$, for all $d, e \in \mathbb{N}$. Discuss what happens if $d = -\infty$ or $e = -\infty$.

Exercise 5.1.6. Let $U \subset \mathbb{R}^n$ be an open subset and let P be an elliptic differential operator of order d on U. This means that its principal symbol $\sigma^d(P)$ does not vanish on $U \times (\mathbb{R}^n \setminus \{0\})$. Let p the full symbol of P. The purpose of this exercise is to show that there exists a $q \in S^{-d}(U)$ such that

$pq - 1 \in S^{-\infty}(U)$. We first address the local question. Let $V \subset U$ be an open subset with compact closure in U. We write p_V for the restriction of p to $V \times \mathbb{R}^n$.

(a) Show that there exists a constant $R = R_V > 0$ such that $p(x, \xi) \neq 0$ for $x \in V$ and $\xi \in \mathbb{R}^n \setminus B(0; R)$.

(b) Show that there exists a smooth function $\chi_V \in C_c^\infty(\mathbb{R}^n)$ such that the function $q : V \times \mathbb{R}^n \to \mathbb{C}$ defined by

$$q(x, \xi) := (1 - \chi(\xi))p(x, \xi)^{-1}$$

if $p(x, \xi) \neq 0$ and by zero otherwise, is smooth.

(c) With χ and q as above, show that $q \in S^{-d}(V)$.

(d) Show that $p_V q - 1 \in \Psi^{-\infty}(V)$.

(e) Show that there exists a symbol $q \in S^{-d}(U)$ such that $pq - 1 \in \Psi^{-\infty}(U)$.

The following invariance result will allow us to extend the definition of the symbol space to an arbitrary smooth manifold. Let $\varphi : U \to V$ be diffeomorphism between open subsets of $\mathbb{R}^n$. We define the map $\varphi_* : C^\infty(U \times \mathbb{R}^{n*}) \to C^\infty(V \times \mathbb{R}^{n*})$ by

$$\varphi_* f(y, \eta) = f(\varphi^{-1}(y), \eta \circ d\varphi(\varphi^{-1}(y))).$$

Identifying $\mathbb{R}^n$ with its dual $\mathbb{R}^{n*}$ by using the standard inner product, we may view $S^d(U)$ as a subspace of $C^\infty(U \times \mathbb{R}^{n*})$.

Lemma 5.1.7. *For every $d \in \mathbb{R}$, the map φ_* restricts to a topological linear isomorphism $S^d(U) \to S^d(V)$.*

Proof Put $\psi = \varphi^{-1}$. Then, for $f \in S^d(M)$, the function $\varphi_* f$ is given by $\varphi_* f(y, \eta) = f(\psi(y), \eta \circ d\varphi(\psi(y)))$. The continuity of φ_* follows from checking that $\partial_y^\alpha \partial_\eta^\beta (\varphi_* f)$ satisfies the required estimates by a straightfoward but tedious application of the chain rule combined with the Leibniz rule. Similarly, ψ_* is seen to be continuous linear. □

We define the symbol spaces on a manifold as follows.

Definition 5.1.8. Let M be a smooth manifold and let $d \in \mathbb{R}$. A symbol of order d is defined to be smooth function $\sigma : T^*M \to \mathbb{R}^n$ such that for each $x_0 \in M$ there exists a coordinate patch U_κ containing x_0 such that the natural map $\kappa_* : C^\infty(T^*U_\kappa) \to C^\infty(\kappa(U_\kappa) \times \mathbb{R}^{n*})$ maps $\sigma|_{T^*U}$ to an element $\kappa_*\sigma$ of $S^d(\kappa(U_\kappa))$. The space of these symbols is denoted by $S^d(M)$.

Remark 5.1.9. Let $\varphi : M \to N$ be a diffeomorphism of smooth manfolds. Then it follows by application of Lemma 5.1.7 that the natural map $\varphi_* : C^\infty(T^*M) \to C^\infty(T^*N)$, given by

$$\varphi_* f(\eta_{\varphi(x)}) = \eta_{\varphi(x)} \circ T_x\varphi, \qquad (x \in M,\ \eta_{\varphi(x)} \in T^*_{\varphi(x)}N)$$

restricts to a linear isomorphism

$$\varphi_* : S^d(M) \xrightarrow{\simeq} S^d(N).$$

In this lecture we will concentrate on the local theory of symbols and the associated pseudo-differential operators. The extension to manifolds will be rather straightforward, by using invariance and partitions of unity. In particular, one needs to localize on the x-variable in the symbol space. Accordingly, given $A \subset U$ compact and $d \in \widehat{\mathbb{R}}$ we define

$$S^d_A(U) = \{p \in S^d(U) \mid \mathrm{pr}_1(\mathrm{supp}\, p) \subset A\},$$

where $\mathrm{pr}_1 : U \times \mathbb{R}^n \to \mathbb{R}^n$ is the natural projection map. The union of these spaces, for $A \subset U$ compact, is denoted by $S^d_c(U)$. Here we note that $S^d_c(U) \subset S^d_c(\mathbb{R}^n)$ naturally, by extension by zero outside $U \times \mathbb{R}^n$.

5.2. Pseudo-differential operators

In this section we will give the definition of the space $\Psi^\infty(U)$ of pseudo-differential operators on an open subset $U \subset \mathbb{R}^n$. For this, we first need to introduce the space $\Psi^{-\infty}(U)$ of smoothing operators on U. Given a smooth function $K \in C^\infty(U \times U)$, we define the integral operator T_K with integral kernel K to be the continuous linear operator $C^\infty_c(U) \to C^\infty(U)$ given by

$$T_K f(x) = \int_U K(x, y)\, f(y)\, dy.$$

We define $\Psi^{-\infty}$ to be the subspace of $\mathrm{Hom}(C^\infty_c(U), C^\infty(U))$ consisting of all operators of the form T_K, with $K \in C^\infty(U \times U)$. It is readily seen that $K \to T_K$ is injective, and thus provides a linear isomorphism from $C^\infty(U \times U)$ onto $\Psi^{-\infty}(U)$.

The name smoothing operator is derived from the following observation. We may extend the definition of T_K to the space $\mathcal{E}'(U)$ of compactly supported distributions on U by the formula

$$T_K u(x) = u(K(x, \cdot\,)).$$

Since $x \mapsto K(x, \cdot\,)$ is a smooth function on U with values in the Fréchet space $C^\infty(U)$, and since $u : C^\infty(U) \to \mathbb{C}$ is continuous linear, it follows that $T_K(u)$ is smooth. Moreover, the map

$$T_K : \mathcal{E}'(U) \to C^\infty(U)$$

is continuous linear. Conversely, by the Schwartz kernel theorem it follows that any continuous linear map $T : \mathcal{E}'(U) \to C^\infty(U)$ is of the form T_K, with K a uniquely determined smooth function on $U \times U$. We define $\Psi^{-\infty}(U)$ to be the space of all operators of the form T_K, for $K \in C^\infty(U \times U)$. Accordingly,

$$\Psi^{-\infty}(U) \simeq \mathrm{Hom}(\mathcal{E}'(U), C^\infty(U)).$$

We proceed to the definition of pseudo-differential operators on U. For $\varphi \in S^{-\infty}(U) = C^\infty(U)) \,\widehat{\otimes}\, \mathcal{S}(\mathbb{R}^n)$ the integral

$$W(\varphi)(x) := \int_{\mathbb{R}^n} e^{i\xi x} \varphi(x, \xi)\, dx$$

is absolutely convergent for every $x \in U$ and is readily seen to define a function $W(\varphi) \in C^\infty(U)$. More precisely, the following result holds.

Lemma 5.2.1. *For all $\varphi \in S^{-\infty}(U)$ and $\alpha \in \mathbb{N}^n$,*

(a) $\partial^\alpha W(\varphi) = W((\partial_x + i\xi)^\alpha \varphi)$;

(b) $(ix)^\alpha W(\varphi) = W(\partial_\xi^\alpha \varphi)$.

The map W is continuous linear from $S^{-\infty}(U)$ to $C^\infty(U)$.

Proof The proof is an obvious adaptation of the proof that Fourier transform maps $\mathcal{S}(\mathbb{R}^n)$ continuously to $C^\infty(\mathbb{R}^n)$. □

If $p \in S^d(U)$ and $F \in \mathcal{S}(\mathbb{R}^n)$ then it follows by a straightforward application of the Leibniz rule that the function $pF : (x, \xi) \mapsto p(x, \xi)F(\xi)$ belongs to $S^{-\infty}(U)$. Moreover, the map $(p, F) \mapsto pF$ is continuous and bilinear. These observations justify the following definition.

Definition 5.2.2. Let $p \in S^d(U)$. Then we define the operator $\Psi_p : C_c^\infty(U) \to C^\infty(U)$ by

$$\Psi_p f(x) := W(p\widehat{f})(x) = \int_{\mathbb{R}^n} e^{i\xi x} p(x, \xi)\widehat{f}(\xi)\, d\xi. \tag{5.2}$$

We note that $(p, f) \mapsto \Psi_p f$ is continuous and bilinear $S^d(U) \times C_c^\infty(U) \to C^\infty(U)$.

Lemma 5.2.3. *The (linear) map $p \mapsto \Psi_p$, $S^d(\mathbb{R}^n) \to \mathrm{Hom}(C_c^\infty(\mathbb{R}^n), C^\infty(\mathbb{R}^n))$ is injective.*

Proof Assume that $p \in S^d(\mathbb{R}^n)$ and $\Psi_p = 0$. Then for each $x \in \mathbb{R}^n$ the smooth function $e^{ix}p_x$ given by $\xi \mapsto e^{i\xi x}p(x, \xi)$ is perpendicular to all functions from $\mathcal{F}(C_c^\infty(\mathbb{R}^n)) \subset \mathcal{S}(\mathbb{R}^n)$. By density of $C_c^\infty(\mathbb{R}^n)$ in $\mathcal{S}(\mathbb{R}^n)$ and the fact that $\mathcal{F}$ is a continuous linear automorphism of $\mathcal{S}(\mathbb{R}^n)$, it follows that $\mathcal{F}(C_c^\infty(\mathbb{R}^n))$ is dense in $\mathcal{S}(\mathbb{R}^n)$. Hence, $\xi \mapsto e^{i\xi x}p(x, \xi)$ is perpendicular to all functions from $\mathcal{S}(\mathbb{R}^n)$. In particular, p_x is perpendicular to $C_c^\infty(\mathbb{R}^n)$ and it follows that $p_x = 0$. □

If P is a differential operator with smooth coefficients of order d on U then its full symbol p belongs to $S^d(U)$ and

$$P = \Psi_p.$$

We now generalize the notion of differential operator as follows.

Definition 5.2.4. Let $U \subset \mathbb{R}^n$ be an open subset and let $d \in \mathbb{R}$. A *pseudo-differential operator* of order d on U is a continuous linear operator $P : C_c^\infty(U) \to C^\infty(U)$ of the form

$$P = \Psi_p + T,$$

with $p \in S^d(U)$ and $T \in \Psi^{-\infty}(U)$. The space of these operators is denoted by $\Psi^d(U)$.

Lemma 5.2.5. *Let $A \subset U$ be compact.*

(a) *Let $p \in S^{-\infty}(U)$. Then there exists a unique $K \in C_c^\infty(U \times U)$ such that $\Psi_p = T_K$. In particular, $\Psi_p \in \Psi^{-\infty}(U)$. If Ψ_p vanishes on $C_c^\infty(U \setminus A)$ then* $\mathrm{pr}_2 \operatorname{supp} K \subset A$.

(b) *Let $K \in C^\infty(U \times U)$ be such that* $\mathrm{pr}_2(\operatorname{supp} K) \subset A$. *Then there exists a $p \in S^{-\infty}(U)$ such that the integral operator $T_K : C_c^\infty(U) \to C^\infty(U)$ equals Ψ_p.*

Proof Let $\mathcal{F}_2 F$ denote the Fourier transform of a function $F \in C^\infty(U) \widehat{\otimes} \mathcal{S}(\mathbb{R}^n)$ with respect to the second component. By straightforward estimation it follows that $\mathcal{F}_2$ is a continuous linear endomorphism of $C^\infty(U) \widehat{\otimes} \mathcal{S}(\mathbb{R}^n)$. By application of Theorem 4.1.14 with respect to the second variable it follows that in fact $\mathcal{F}_2$ is a topogical linear automorphism.

We can now prove (a). Let p be as asserted and define

$$\widetilde{K}(x, y) = \mathcal{F}_2 p(x, y - x).$$

Then $\widetilde{K} \in C^\infty(U) \widehat{\otimes} \mathcal{S}(\mathbb{R}^n)$. Moreover, by the Fourier inversion theorem we see that $p(x, \xi) = e^{-i\xi x} \mathcal{F}_2(\tilde{K})(x, -\xi)$. We put $K = \tilde{K}|_{U \times U}$. Then for all $f \in C_c^\infty(U)$ and $x \in U$ we have

$$T_K f(x) = T_{\tilde{K}} f(x) = \int_{\mathbb{R}^n} \mathcal{F}_2 K(x, -\xi) \widehat{f}(\xi)\, d\xi = \Psi_p f(x).$$

Uniqueness of K is obvious.

Now assume that Ψ_p vanishes on $C_c^\infty(U \setminus A)$. Then T_K vanishes on $C_c^\infty(U \setminus A)$. This implies that K is zero when tested with functions from $C_c^\infty(U \times (U \setminus A))$. Hence, $\operatorname{supp} K \subset U \times A$.

We turn to (b). Let $K \in C^\infty(U \times U)$ and assume $\operatorname{supp} K \subset U \times A$. Then $K \in C^\infty(U) \widehat{\otimes} \mathcal{S}(\mathbb{R}^n)$. Applying Proposition 4.2.5 with respect to the second variable, we see that

$$T_K = \Psi_p, \qquad p(x, \xi) = e^{-i\xi x} \mathcal{F}_2(K)(x, -\xi).$$

It is clear that $p \in S^{-\infty}(U) = S^{-\infty}(U)$.

□

Exercise 5.2.6. Let $U \subset \mathbb{R}^n$ be an open subset, and let $K \in C^\infty(U \times U)$. Show that T_K is a local operator if and only if $K = 0$.

In particular, it follows that pseudo-differential operators are not local in general, in contrast to differential operators. In fact, in view of the following result, differential operators are precisely those pseudo-differential operators that are local.

Theorem 5.2.7. (Peetre's theorem) *Let $P : C_c^\infty(U) \to C^\infty(U)$ be a linear map such that* $\operatorname{supp} Pf \subset \operatorname{supp} f$ *for all $f \in C_c^\infty(\mathbb{R}^n)$. Then P is a differential operator (with bounded degree on every compact subset of U).*

It is remarkable that this result is true without any assumption of continuity for P. The analogous result is much easier to prove if P is required to be continuous. From this the above characterization of differential operators among the pseudo-differential operators already follows. A proof is suggested in the following exercise.

Exercise 5.2.8. Let $P : C_c^\infty(U) \to C^\infty(U)$ be a continuous linear operator such that for all $f \in C_c^\infty(U)$ we have $\operatorname{supp} Pf \subset \operatorname{supp} f$.

(a) Show that for every $a \in U$ the map $u_a : f \mapsto Pf(a)$ is a distribution supported by $\{a\}$. Hint: use a suitable cut off function.

(b) Let $V \subset U$ be an open subset whose closure is compact in U. Show that there exists a constant $d = d_V \in \mathbb{N}$ such that for every $a \in V$ the distribution u_a has order at most d. This means that for every compact neighborhood K of a there exists a constant $C_K > 0$ such that

$$|u_a(f)| \leq C_K \max_{|\alpha| \leq d} \sup_K |\partial^\alpha f|.$$

(c) Show that there exist uniquely determined constants $c_\alpha(a) \in \mathbb{C}$, for $|\alpha| \leq d$ such that

$$u_a = \sum_{|\alpha| \leq d} c_\alpha(a)(-\partial)^\alpha \delta_a,$$

where δ_a denotes the Dirac measure at a (see the exercise below).

(d) Show that the functions c_α, for $|\alpha| \leq k$, are smooth on V.

(e) Show that the restriction of P to V is a differential operator of order at most d_V.

Exercise 5.2.9. The purpose of this exercise is to show the following. Let $a \in \mathbb{R}^n$ and let $u \in \mathcal{E}'(\mathbb{R}^n)$ be a distribution with $\operatorname{supp} u \subset \{a\}$. Then there exists a constant $k \in \mathbb{N}$ and constants $c_\alpha \in \mathbb{C}$, for $\alpha \in \mathbb{N}^n$, such that

$$u = \sum_{|\alpha| \leq k} c_\alpha \partial^\alpha \delta_\alpha.$$

(a) Show that without loss of generality we may assume that $a = 0$.

(b) Let $\chi \in C_c^\infty(\mathbb{R}^n)$ be identically 1 in a neighborhood of 0. Show that

$$u(\chi f) = u(f)$$

for all $f \in C^\infty(\mathbb{R}^n)$.

(c) Let $f \in C^\infty(\mathbb{R}^n)$. Let p be the k-th order (multivariable) Taylor polynomial of f at 0 and let R be the remainder term, so that $f = p + R$. Show that for all α with $|\alpha| \leq k$, we have

$$\lim_{x \to 0} \|x\|^{|\alpha| - k - 1/2} |\partial^\alpha R(x)| = 0.$$

(d) For $\epsilon > 0$ define $\chi_\epsilon(x) = \chi(x/\epsilon)$. Show that

$$u(\chi_\epsilon R) \to 0 \ (\epsilon \to 0).$$

(e) Show that $u(f) = u(p)$.

(f) Conclude the proof.

5.3. Localization of pseudo-differential operators

We turn to the problem of localizing a pseudo-differential operator $P \in \Psi^d(U)$, for $U \subset \mathbb{R}^n$ open. More precisely, if $\chi \in C_c^\infty(U)$ we denote by M_χ or simply χ, the operator in $\operatorname{End}(C_c^\infty(U))$ given by multiplication by χ, i.e., $M_\chi(f) = \chi f$. The problem is whether $M_\chi \circ P \circ M_\psi$ is a pseudo-differential operator again, for $\chi, \psi \in C_c^\infty(U)$.

It is immediate from the definitions that

$$M_\chi \circ \Psi_p = \Psi_{(\chi \otimes 1)p}, \quad M_\chi \circ T_K = T_{(\chi \otimes 1)K},$$

for $p \in S^d(U)$ and $K \in C^\infty(U \times U)$. Moreover, it is also clear that $T_K \circ M_\psi = T_{K(1\otimes\psi)}$. The answer to the question whether $\Psi_p \circ M_\psi$ is pseudo-differential is provided by the following result.

Proposition 5.3.1. *Let $p \in S^d(U)$ and let $\psi \in C_c^\infty(U)$. Then there exists a $q \in C^d(U)$ such that*

$$\Psi_p \circ M_\psi = \Psi_q.$$

Proof Let $f \in C_c^\infty(U)$. Then $\mathcal{F}(\psi f) = \mathcal{F}(\psi) * \mathcal{F}(f)$ so that

$$\begin{aligned}
\Psi_p \circ M_\psi f(x) &= \int e^{i\xi x} p(x,\xi)(\widehat{\psi} * \widehat{f})(\xi)\, d\xi \\
&= \int\int e^{i\xi x} p(x,\xi)\widehat{\psi}(\xi-\zeta)\widehat{f}(\zeta)\, d\zeta\, d\xi \\
&= \int\int e^{i\xi x} p(x,\xi)\widehat{\psi}(\xi-\zeta)\widehat{f}(\zeta)\, d\xi\, d\zeta \\
&= \int\int e^{i\zeta x} e^{i\xi x}\, p(x,\xi+\zeta)\, \widehat{\psi}(\xi)\widehat{f}(\zeta)\, d\xi\, d\zeta \\
&= \int e^{i\zeta x} q(x,\zeta)\widehat{f}(\zeta)\, d\zeta,
\end{aligned}$$

where

$$q(x,\zeta) = \int_{\mathbb{R}^n} e^{i\xi x}\, p(x,\xi+\zeta)\, \widehat{\psi}(\xi)\, d\xi. \tag{5.3}$$

Note that all of the above integrals are absolutely convergent, because $\widehat{\psi}$ and $\widehat{f}$ are Schwartz functions. We will finish the proof by showing that q belongs to $S^d(U)$. More precisely, we will show that the map $(p,\psi) \mapsto q$ defined by (5.3) is continuous bilinear $S^d(U) \times \mathcal{S}(\mathbb{R}^n) \to S^d(U)$.

Let $K \subset U$ be compact, and $k \in \mathbb{N}$. Then for multi-indices α, β of order at most k, for $x \in K$ and $\xi, \zeta \in \mathbb{R}^n$ we have

$$\begin{aligned}
|\partial_x^\alpha \partial_\zeta^\beta [p(x,\xi+\zeta)\widehat{\psi}(\xi)]| &= |(\partial_x^\alpha \partial_\xi^\beta p)(x,\xi+\zeta)||\widehat{\psi}(\xi)| \\
&\leq (1+\|\xi+\zeta\|)^{d-|\beta|}(1+\|\xi\|)^{-N}\mu_{K,k}^d(p)\nu_{N,0}(\widehat{\psi}) \\
&\leq (1+\|\zeta\|)^{d-|\beta|}(1+\|\xi\|)^{k+|d|-N}\mu_{K,k}^d(p)\nu_{N,0}(\widehat{\psi}).
\end{aligned}$$

By application of the Leibniz rule, we now see that there exists a constant $C > 0$, only depending on K, k, N, such that

$$\begin{aligned}
&|\partial_x^\alpha \partial_\zeta^\beta [e^{i\xi x}\, p(x,\xi+\zeta)\widehat{\psi}(\xi)]| \\
&\quad\leq C\,(1+\|\zeta\|)^{d-|\beta|}(1+\|\xi\|)^{2k+|d|-N}\mu_{K,k}^d(p)\nu_{N,0}(\widehat{\psi}).
\end{aligned}$$

Choosing N such that $2k + |d| - N < -n$, we see that in (5.3) differentiation under the integral sign is allowed, and leads to the estimate

$$\mu_{K,k}^d(q) \leq C' \mu_{K,k}^d(p)\nu_{N,0}(\widehat{\psi}), \qquad ((x,\zeta) \in K \times \mathbb{R}^n),$$

with C' a constant only depending on K, k and N. As $\psi \mapsto \widehat{\psi}$ is continuous in $\mathcal{S}(\mathbb{R}^n)$, the result follows. □

5.4. The full symbol

The formula (5.3) gives rise to an interesting characterization of q which we shall now describe. We recall that U is an open subset of $\mathbb{R}^n$.

If $\mathcal{N}$ is a countable set and $\nu \mapsto d_\nu$ a real-valued function on $\mathcal{N}$, then by $\lim_{\nu\to\infty} d_\nu = -\infty$ we mean that for every $m \in \mathbb{R}$ there exists a finite subset $F \subset \mathcal{N}$ such that $\nu \in \mathcal{N} \setminus F \Rightarrow d_\nu < m$.

Definition 5.4.1. Let $\mathcal{N}$ be a countable set, and $\nu \mapsto d_\nu$ a real-valued function on $\mathcal{N}$ with $d_\nu \to -\infty$ for $\nu \to \infty$. Let $p_\nu \in S^{d_\nu}(U)$, for each $\nu \in \mathcal{N}$, and let $p \in S^{d'}(U)$. Then

$$p \sim \sum_{\nu\in\mathcal{N}} p_\nu \tag{5.4}$$

means that for every $d \in \mathbb{R}$ there exists a finite subset $F_0 \subset \mathcal{N}$ such that for every finite subset $F \subset \mathcal{N}$ with $F \supset F_0$,

$$p - \sum_{\nu\in F} p_\nu \in S^d(U).$$

We observe that if a symbol $p' \in S^\infty(U)$ has the same expansion, then it follows that $p - p' \in S^d(U)$ for all d, hence $p - p' \in S^{-\infty}(U)$. We thus see that the asymptotic expansion (5.4) determines p modulo $S^{-\infty}(U)$.

The symbol q of (5.3) may now be characterized modulo $S^{-\infty}(U)$ as follows. We note that the operator ∂_ξ^α maps $S^d(U)$ continuous linearly to $S^{d-|\alpha|}(U)$.

Lemma 5.4.2. *Let $p \in S^d(U)$ let $\psi \in C_c^\infty(U)$ and let the symbol $q \in S^d(U)$ be defined as in (5.3). Then*

$$q \sim \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\, D_x^\alpha\psi\, \partial_\xi^\alpha p.$$

Proof By the multi-variable Taylor formula with remainder term, we have, for $M \in \mathbb{N}$,

$$p(x, \xi + \zeta) = \sum_{|\alpha|\le M} \frac{\xi^\alpha}{\alpha!}\partial_\xi^\alpha p(x, \zeta) + R_M(x, \xi, \zeta),$$

with remainder term given by

$$R_M(x, \xi, \zeta) = -\frac{1}{M!}\int_0^1 (1-t)^M \partial_t^{M+1}[p(x, \zeta + t\xi)]\, dt. \tag{5.5}$$

This leads to

$$q(x, \zeta) = \sum_{|\alpha|\le M} q_\alpha(x, \zeta) + q_M(x, \zeta),$$

where

$$q_M(x, \zeta) = \int_{\mathbb{R}^n} R_M(x, \xi, \zeta)\widehat{\psi}(\xi)\, d\xi,$$

and where

$$\begin{aligned} \alpha!\, q_\alpha(x,\zeta) &= \int e^{i\xi x}\, \xi^\alpha \partial_\xi^\alpha p(x,\zeta)\, \widehat{\psi}(\xi)\; d\xi \\ &= \partial_\xi^\alpha p(x,\zeta)\, \mathcal{F}^{-1}(\xi^\alpha \mathcal{F}(\psi))(x) \\ &= \partial_\xi^\alpha p(x,\zeta)\, D_x^\alpha \psi(x). \end{aligned}$$

Thus, to complete the proof, it suffices to show that

$$q_M \in S^{d-(M+1)}(U). \tag{5.6}$$

Let $K \subset U$ be compact. Then by differentiation under the integral sign in (5.5), application of the Leibniz rule, and a straightforward estimation of the resulting integrals we see that there exists a constant $C > 0$ only depending on K, M, k such that for all multi-indices α, β with $|\alpha|, |\beta| \leq k$ and all $x \in K <$ $\xi, \zeta \in \mathbb{R}^n$, we have

$$\begin{aligned} &|\partial_x^\alpha \partial_\zeta^\beta R_M(x,\xi,\zeta)| \\ &\leq \; C(1+\|\xi\|)^{M+1} \sup_{0\leq t\leq 1} (1+\|\zeta + t\xi\|)^{d-(M+1)-|\beta|} \mu^d_{K,k+M+1}(p). \end{aligned}$$

We now observe that for $0 \leq t \leq 1$ we have

$$(1+\|\zeta + t\xi\|)^{d-(M+1)-|\beta|} \leq (1+\|\zeta\|)^{d-(M+1)-|\beta|}(1+\|\xi\|)^{|d|+M+1+|\beta|},$$

so that

$$\begin{aligned} &|\partial_x^\alpha \partial_\zeta^\beta R_M(x,\xi,\zeta)| \\ &\leq \; C(1+\|\zeta\|)^{d-(M+1)-|\beta|}\,(1+\|\xi\|)^{|d|+2(M+1)+k} \mu_{K,k+M+1}(p). \end{aligned}$$

Now put

$$q_M(x,\zeta) = \int_{\mathbb{R}^n} R_M(x,\xi,\zeta)\widehat{\psi}(\xi)\; d\xi$$

Fix $N \in \mathbb{N}$ such that $|d| + 2(M+1) + k - N < -n$. Then by the usual method we infer that there exists a constant $C' > 0$, only depending on K, k, N such that

$$\mu^{d-(M+1)}_{K,k}(q_M) \leq C'\, \mu^d_{K,k+M+1}(p)\, \nu_{N,0}(\widehat{\psi})$$

The result follows.

Remark 5.4.3. From the estimate at the end of the proof we see that the map $(p, \psi) \mapsto q_M$ is continuous bilinear from $S^d(U) \times C_c^\infty(U)$ to $S^{d-(M+1)}(U)$. In this sense, the asymptotic expansion for q depends on (p, ψ) in a continuous bilinear fashion.

Using the asymptotic expansion above, we can now derive an important theorem. We begin by a useful sharpening of Proposition 5.3.1 Given an open subsets $V \subset U$ of $\mathbb{R}^n$, we define the restriction map $p \mapsto p_V$ from $S^d(U)$ to $S^d(V)$ by $p_V(x,\xi) = p(x,\xi)$, for $x \in V$ and $\xi \in \mathbb{R}^n$. Thus,

$$p_V := p|_{V\times\mathbb{R}^n}.$$

In the following we will use the notation $V \Subset U$ to indicate that V has compact closure in U.

Proposition 5.4.4. *Let $p \in S^d(U)$. Let $U' \Subset U$ be an open subset and let $\psi \in C_c^\infty(U)$ be equal to 1 on an open neighborhood of $\mathrm{cl}\,(U')$. Then there exists a symbol $q \in S^d(U)$ such that*

(a) $\Psi_q = \Psi_p \circ M_\psi$;

(b) $q_{U'} - p_{U'} \in S^{-\infty}(U')$.

Proof Define q as in (5.3). Then (a) is valid, and we have the asymptotic expansion from Lemma 5.4.2. Since $D_x^\alpha \psi = 0$ on U', except when $\alpha = 0$, we see that $q_{U'} \sim p_{U'}$, or, equivalently, that (b) holds. □

Theorem 5.4.5. *The map $p \mapsto \Psi_p$ induces a linear isomorphism*

$$S^d(U)/S^{-\infty}(U) \xrightarrow{\simeq} \Psi^d(U)/\Psi^{-\infty}(U).$$

Proof From the definition of $\Psi^d(U)$ it follows that $p \mapsto \Psi_p$ induces a surjective linear map $S^d(U) \to \Psi^d(U)/\Psi^{-\infty}(U)$. We must show that its kernel equals $S^{-\infty}(U)$. Thus, let $p \in S^d(U)$ and assume that $\Psi_p \in \Psi^{-\infty}(U)$. Then $\Psi_p = T_K$ for a smooth function $K \in C^\infty(U \times U)$. We must show that this implies that $p \in S^{-\infty}(\mathbb{R}^n)$. By using a partition of unity we see that it suffices to show that $\chi p \in S^{-\infty}(\mathbb{R}^n)$ for all $\chi \in C_c^\infty(U)$. Now $\Psi_{\chi p} = M_\chi \circ \Psi_p = T_{(\chi \otimes 1)K}$. Thus, to prove the theorem, we may assume from the start that there exists a compact subset $A \subset U$ such that $\mathrm{supp}\, p \subset A \times U$ and $\mathrm{supp}\, K \subset A \times U$.

We now have that $p \in S_c^d(U) \subset S_c^d(R^n)$, so that p also defines a pseudo-differential operator $\widetilde{\Psi}_p : C_c^\infty(\mathbb{R}^n) \to C_A^\infty(\mathbb{R}^n)$. Of course, $\widetilde{\Psi}_p$ restricts to Ψ_p on $C_c^\infty(U)$.

Fix an open subset $U' \Subset U$. Then it suffices to show that $p_{U'} \in S^{-\infty}(U')$. To prove this, we select a cut off function $\psi \in C_c^\infty(U)$ which equals 1 on an open neighborhood of $\mathrm{cl}\,(U')$. Then

$$\widetilde{\Psi}_p \circ M_\psi = \Psi_p \circ M_\psi = T_{K(1\otimes\psi)} \in \Psi^{-\infty}(\mathbb{R}^n). \tag{5.7}$$

As $K(1 \otimes \psi)$ is compactly supported with support in $U \times U$, it follows that

$$T_{K(1\otimes\psi)} = \widetilde{\Psi}_r \tag{5.8}$$

for a symbol $r \in S^{-\infty}(\mathbb{R}^n)$.

On the other hand, by the above proposition applied with $\mathbb{R}^n$ in place of U, we see that

$$\widetilde{\Psi}_p \circ M_\psi = \widetilde{\Psi}_q, \tag{5.9}$$

with a symbol $q \in S^d(\mathbb{R}^n)$ that has the property that $q_{U'} - p_{U'} \in S^{-\infty}(U')$.

From (5.7), (5.7) and (5.9) we see that $\widetilde{\Psi}_r = \widetilde{\Psi}_q$, so that $r - q = 0$, by Lemma 5.2.5. This implies that

$$p_{U'} - r_{U'} = p_{U'} - q_{U'} \in S^{-\infty}(U').$$

Hence $p_{U'} \in S^{-\infty}(U')$ and the proof is complete. □

Definition 5.4.6. The inverse of the linear isomorphism of Theorem 5.4.5, denoted

$$\sigma : \Psi^d(U)/\Psi^{-\infty}(U) \to S^d(U)/S^{-\infty}(U),$$

is called the (full) *symbol map.*

This symbol map is the appropriate generalization of the symbol map for differential operators. Just as the latter symbol map cannot be extended naturally to differential operators on manifolds, the present symbol map does not allow a coordinate invariant extension to manifolds either. Just as in the case of differential operators, there is an appropriate notion of principal symbol of order d, which can be extended to the setting of manifolds.

Definition 5.4.7. The principal symbol map σ^d of order d is defined to be the following map induced by the symbol map:

$$\sigma^d : \Psi^d(U)/\Psi^{d-1}(U) \longrightarrow S^d(U)/S^{d-1}(U). \tag{5.10}$$

Corollary 5.4.8. *The principal symbol map (5.10) is a linear isomorphism.*

5.5. Expansions in symbol space

The construction of parametrices for elliptic pseudo-differential operators will make use of a recurrence that is based on the following remarkable lemma.

Lemma 5.5.1. *Let $U \subset \mathbb{R}^n$ be open. Let $\{d_j\}_{j\geq 0}$ be a sequence of real numbers with $d_j \to -\infty$ as $j \to \infty$. Assume that for each j a symbol $p_j \in S^{d_j}(U)$ is given. Then there exists a symbol $p \in S^d(U)$, where $d = \max d_j$, such that $\mathrm{pr}_1(suppp)$ is contained in the closure of the union of the sets $\mathrm{pr}_1(\mathrm{supp}\, p_j)$, for $j \geq 0$, and such that*

$$p \sim \sum_{j=0}^{\infty} p_j \qquad \text{in} \quad S^d(U). \tag{5.11}$$

The symbol p is uniquely determined modulo $S^{-\infty}(U)$.

Proof The uniqueness assertion is an immediate consequence of the meaning of (5.11).

For the existence, we note that by using partitions of unity on U we can reduce to the local situation where a compact subset $A \subset U$ is given such that $p_j \in S^{d_j}_A(U)$. It then suffices to establish the existence of a symbol $p \in S^d_A(U)$ such that (5.11) is valid.

Taking suitable groups of terms we readily see that it suffices to consider the case that the sequence d_j is strictly decreasing. Then $d = d_0$.

Fix $\chi \in C^\infty_c(\mathbb{R}^n)$ with the property that $\chi(\xi) = 0$ for $\|\xi\| \leq 1$ and $\chi(\xi) = 1$ for $\|\xi\| \geq 2$. For $t > 0$ we define the function $\chi_t : (x, \xi) \mapsto \chi(t\xi)$ (constant in the x-variable). Note that $\mathrm{supp}\, \chi_t \cap (U \times B(0; t^{-1})) = \emptyset$.

We will select a sequence t_j of positive real numbers with $t_j \to 0$ and define

$$p := \sum_{j=0}^{\infty} \chi_{t_j} p_j.$$

The sum is locally finite with respect to the variable ξ, hence defines a smooth function $U \times \mathbb{R}^n \to \mathbb{C}$. Moreover, $\mathrm{pr}_1(\mathrm{supp}\, p)$ is contained in A. We claim that it is possible to select a sequence $\{t_j\}$ such that for every $l \in \mathbb{N}$ and all $\alpha, \beta \in \mathbb{N}^n$ the series

$$\sum_{j\geq l} (1 + \|\xi\|)^{|\beta| - d_l} \partial_x^\alpha \partial_\xi^\beta [\chi_{t_j} p_j]$$

converges uniformly on $U \times \mathbb{R}^n$. The proof of this claim is deferred to two lemmas below. Let $\{t_j\}$ be as claimed, then the proof can be finished as follows. From the above claim about convergence, it follows that the series

$$r_l := \sum_{j \geq l} \chi_{t_j} p_j$$

convergences absolutely in the symbol space $S^{d_l}_A(U)$, relative to its continuous seminorms. By completeness, this implies that $r_l \in S^{d_l}_A(U)$. In particular it follows that $p = r_0 \in S^d_A(U)$. Now

$$p - \sum_{j=0}^{l-1} p_j = \sum_{j=0}^{l-1} (\chi_{t_j} - 1) p_j + r_l. \tag{5.12}$$

The second sum defines a function with compact ξ-support, hence belongs to $S^{-\infty}(U)$. Since $r_l \in S^{d_l}(U)$, it follows that the difference on the left-hand side of (5.12) belongs to $S^{d_l}(U)$. This implies (5.11). □

To establish the claim of the above proof we need the following.

Lemma 5.5.2. *Let $j \geq 0$, $\alpha, \beta \in \mathbb{N}^n$. Then there exists a constant $C_{j,\alpha,\beta} > 0$ such that*

$$|\partial_x^\alpha \partial_\xi^\beta [\chi_t p_j](x, \xi)| \leq C_{j,\alpha,\beta} (1 + \|\xi\|)^{d_j - \|\beta\|}, \tag{5.13}$$

for all $(x, \xi) \in U' \times \mathbb{R}^n$ and all $0 < t \leq 1$.

Proof The estimate is trivially valid in the area $\|\xi\| \leq t^{-1}$, where $\chi_t = 0$. By definition of the symbol space, it is also valid in the area $\|\xi\| \geq 2t^{-1}$, where $\chi_t = 1$. Thus, it remains to establish an estimate of the above type in the area $t^{-1} \leq \|\xi\| \leq 2t^{-1}$, with a constant independent of t, x, ξ.

The estimate then follows by application of Leibniz' formula and bookkeeping, as follows. The expression on the left-hand side of (5.13) may be estimated by a sum of binomial coefficients times the expression $|\partial_x^\alpha (\partial_\xi^\gamma \chi_t \partial_\xi^{\beta-\gamma} p_j)(x, \xi)|$. With the notation $(\partial_\xi^\gamma \chi)_t(\xi) = (\partial_\xi^\gamma \chi)(t\xi)$ the mentioned expression becomes, by application of the chain rule,

$$\begin{aligned} |\partial_x^\alpha (\partial_\xi^\gamma \chi_t \partial_\xi^{\beta-\gamma} p_j)(x, \xi)| &= t^{|\gamma|} \, | [(\partial_\xi^\gamma \chi)_t \partial_x^\alpha \partial_\xi^{\beta-\gamma} p_j](x, \xi) | \\ &\leq C'_{j,\alpha,\gamma} \, t^{|\gamma|} \, (1 + \|\xi\|)^{d_j - |\beta| + |\gamma|}, \end{aligned} \tag{5.14}$$

with a constant independent of t, x, ξ. From $t \leq 1$ and $\|\xi\| \leq 2t^{-1}$ it follows that

$$t^{|\gamma|} = 3^{|\gamma|} (3t^{-1})^{-|\gamma|} \leq 3^{|\gamma|} (1 + 2t^{-1})^{-|\gamma|} \leq 3^{|\gamma|} (1 + \|\xi\|)^{-|\gamma|}.$$

Substituting this in (5.14) we infer that the estimate (5.13) is valid in the area considered. □

Lemma 5.5.3. (Claim) *There exists a sequence $\{t_j\}$ of real numbers with $t_j \to 0$ such that for every $l \in \mathbb{N}$ and all $\alpha, \beta \in \mathbb{N}^n$ the series*

$$\sum_{j \geq l} (1 + \|\xi\|)^{|\beta| - d_l} \partial_x^\alpha \partial_\xi^\beta [\chi_{t_j} p_j]$$

converges uniformly on $U \times \mathbb{R}^n$.

Proof Let $j \geq 0$. Then we may select $t_j > 0$ such that

$$C_{j,\alpha,\beta}(1 + t_j^{-1})^{d_j - d_l} < 2^{-j}$$

for all α, β, l with $|\alpha| + |\beta| + l < j$. It follows that, for all such α, β, l and all $(x, \xi) \in U' \times \mathbb{R}^n$ with $\|\xi\| \geq t_j^{-1}$,

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\beta [\chi_t p_j](x, \xi)| &\leq C_{j,\alpha,\beta}(1 + \|\xi\|)^{d_j - \|\beta|} \\ &\leq C_{j,\alpha,\beta}(1 + \|\xi\|)^{d_j - d_l}(1 + \|\xi\|)^{d_l - \|\beta|} \\ &\leq C_{j,\alpha,\beta}(1 + t_j^{-1})^{d_j - d_l}(1 + \|\xi\|)^{d_l - \|\beta|} \\ &\leq 2^{-j}(1 + \|\xi\|)^{d_l - \|\beta|}. \end{aligned}$$

On the other hand, if $\|\xi\| \leq t_j^{-1}$, then $\chi_{t_j}(x, \xi) = \chi(t_j \xi) = 0$. We conclude that for all α, β, l with $|\alpha| + |\beta| + l < j$ and all $(x, \xi) \in U \times \mathbb{R}^n$ the following estimate is valid

$$|\partial_x^\alpha \partial_\xi^\beta [\chi_t p_j](x, \xi)| \leq 2^{-j}(1 + \|\xi\|)^{d_l - \|\beta|}.$$

This implies the claim. $\square$