Analysis on Manifolds
Lecture notes for the 2009/2010 Master Class

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Received by the editors September, 2009.

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LECTURE 6
Pseudo-differential operators, continued

6.1. The distribution kernel of a pseudo-differential operator

Let $U \subset \mathbb{R}^n$ be an open subset. If $p \in S^d(U)$, then the associated pseudo-differential operator $\Psi_p : C_c^\infty(U) \to C^\infty(U)$ may be viewed as a continuous linear operator $C_c^\infty(U) \to \mathcal{D}'(U)$. Hence, by the Schwartz kernel theorem, the operator Ψ_p has a (uniquely determined) distribution kernel $K_p \in \mathcal{D}'(U \times U)$. By this we mean that the pseudo-differential operator is given by

$$\langle \Psi_p f, g \rangle = \langle K_p, g \otimes f \rangle, \qquad (f, g \in C_c^\infty(U)).$$

It turns out that the kernel K_p can be quite easily determined, without reference to the Schwartz kernel theorem.

The idea is to extend the formula of Lemma 5.2.4 to the more general setting with $p \in S^m(U)$. Fourier transform with respect to the second variable defines a linear map

$$\mathcal{F}_2 : C_c^\infty(U \times \mathbb{R}^n) \to C_c^\infty(U) \,\widehat{\otimes}\, \mathcal{S}(\mathbb{R}^n)$$

which is readily verified to be continuous. The transposed of this map is a continuous linear map $[C_c^\infty(U) \,\widehat{\otimes}\, \mathcal{S}(\mathbb{R}^n)]' \to \mathcal{D}'(U \times \mathbb{R}^n)$ which we denote by $\mathcal{F}_2$ as well. Via the bilinear pairing

$$S^m(U) \times [C_c^\infty(U) \,\widehat{\otimes}\, \mathcal{S}(\mathbb{R}^n)] \to \mathbb{C}$$

defined by integration, the space $S^m(U)$ is mapped continuously and injectively into $[C_c^\infty(U) \,\widehat{\otimes}\, \mathcal{S}(\mathbb{R}^n)]'$. Via composition with this injection, the Fourier transform $\mathcal{F}_2$ gives a a continuous linear map

$$\mathcal{F}_2 : S^m(U) \to \mathcal{D}'(U \times \mathbb{R}^n).$$

We note that on $C_c^\infty(U \times \mathbb{R})$ this map coincides with the usual Fourier transform in the second variable. Of course, this justifies the use of the notation $\mathcal{F}_2$ for it.

Proposition 6.1.1. *Let $p \in S^d(U)$. The distribution kernel $K_p \in \mathcal{D}'(U \times U)$ of Ψ_p is given by the formula*

$$K_p(x, y) = \mathcal{F}_2 p(x, y - x) \quad \text{(in distribution sense)}. \tag{6.1}$$

Remark 6.1.2. The phrase 'in distribution sense' should be interpreted as follows. The formula is well defined for $p \in C_c^\infty(U \times \mathbb{R}^n)$ and can be rewritten in such a way that it extends continuously to the space of distributions. Indeed, since for a compactly supported smooth function p the transform $\mathcal{F}_2 p$ is a smooth function, $\mathcal{F}_2 p(x, y - x) = L^*(\mathcal{F}_2 p)(x, y)$, with L the map $(x, y) \mapsto (x, y - x)$ from $U \times \mathbb{R}^n$ to $U \times \mathbb{R}^n$. The map L is smooth with smooth inverse $L^{-1} : (x, y) \mapsto (x, y + x)$, hence is a diffeomorphism.

For all $f, g \in C^\infty(U \times \mathbb{R}^n)$ with $\operatorname{supp} g$ compact, we have

$$\langle L^* f, g \rangle = \langle f, L^{-1*} g \rangle,$$

by substitution of variables, so that $\langle L^* f, \cdot\rangle = L^{-1*t}(\langle f, \cdot\rangle)$. We thus see that the operator $L_*^{-1} := L^{-1*t} \in \mathrm{End}(\mathcal{D}'(U \times \mathbb{R}^n))$ is the unique continuous linear extension of the operator $L^* \in \mathrm{End}(C^\infty(U \times \mathbb{R}^n))$. Accordingly, the above formula (6.1) should be read as:

$$K_p = L_*^{-1}(\mathcal{F}_2(p))|_{U\times U}. \tag{6.2}$$

Proof Let $g \in C_c^\infty(U)$ and $f \in C_c^\infty(U)$. Then

$$\begin{aligned}
\langle g, \Psi_p(f)\rangle &= \int_U g(x)\Psi_p f(x)\, dx \\
&= \int_U\int_{\mathbb{R}^n} g(x)e^{i\xi x}p(x,\xi)\widehat{f}(\xi)\, d\xi\, dx \\
&= \int_U\int_{\mathbb{R}^n} g(x)p(x,\xi)\mathcal{F}(T_x^* f)(\xi)\, d\xi\, dx \\
&= \int_U\int_{\mathbb{R}^n} g(x)\mathcal{F}_2 p(x,y)(T_x^* f)(y)\, d\xi\, dx \\
&= \langle \mathcal{F}_2 p, h\rangle,
\end{aligned}$$

where $h(x, y) = g(x)f(y + x) = L^{-1*}(g \otimes f)(x, y)$. It follows that

$$\langle g, \Psi_p(f)\rangle = \langle \mathcal{F}_2(p), L^{-1*}(g \otimes f)\rangle = \langle L_*^{-1}\mathcal{F}_2(p), g \otimes f\rangle.$$

This implies that $\Psi_p = T_K$, with $K = L_*^{-1}\mathcal{F}_2(p)|_{U\times U}$. □

Proposition 6.1.3. *Let $p \in S^m(U)$. Then the distribution kernel K_p of Ψ_p is smooth on the complement of the diagonal* diag (U) *in $U \times U$.*

Proof The diagonal diag (U) is the pre-image of $\mathbb{R}^n \times \{0\}$ under the map $L : U \times \mathbb{R}^n \to U \times \mathbb{R}^n$. Therefore, it suffices to show that $\mathcal{F}_2 p$ is smooth outside $\mathbb{R}^n \times \{0\}$. For this it suffices to show that $(1 \otimes \psi)\mathcal{F}_2 p$ is smooth for all $\psi \in C_c^\infty(\mathbb{R}^n)$ with $0 \notin \operatorname{supp}\psi$. By the usual formulas for the Fourier transform of a differentiated function, we derive that

$$(1 \otimes \psi)\mathcal{F}_2 p = [1 \otimes \|y\|^{-2N}\psi]\, \mathcal{F}_2((-\Delta_\xi)^N p), \tag{6.3}$$

for every $N \in \mathbb{N}$, where Δ_ξ denotes the Laplace operator in the ξ-variable. Now Δ_ξ^N maps $S^d(U)$ to $S^{d-2N}(U)$. If $k \in \mathbb{N}$ and $d - 2N < -n - k$, it is readily seen that $\mathcal{F}_2$ maps $S^{d-2N}(U)$ continuously to $C^k(U \times \mathbb{R}^n)$. Thus, for such N the distribution on the right-hand side of (6.3) belongs to $C^k(U \times \mathbb{R}^n)$. As the left-hand side is independent of N, it follows that $(1\otimes\psi)\mathcal{F}_2 p$ belongs to $C^k(U\times\mathbb{R}^n)$ for every $k \in \mathbb{N}$. The result follows. □

Corollary 6.1.4. *Let $P \in \Psi^d(U)$. Then the distribution kernel of the operator P, denoted $K_P \in \mathcal{D}'(U \times U)$, is smooth outside the diagonal* diag (U).

Proof By definition, there exist $p \in S^d(U)$ and $K \in C^\infty(U \times U)$ such that $P = \Psi_p + T_K$. Now $K_P = K_p + K$. □

Exercise 6.1.5. Let $k \in \mathbb{N}$ and assume that $d < -n - k$. Let $p \in S^d(U)$. Show that the distribution kernel K_p of Ψ_p belongs to $C^p(U \times U)$.

In the sequel we shall use the notation K_P for the distribution kernel of a pseudo-differential operator P on U. The operator P is said to be *properly supported* if the maps $\mathrm{pr}_j : \mathrm{supp}\, K_P \to U$ are proper, for $j = 1, 2$. Recall that a continuous map $\Phi : X \to Y$ between locally compact Hausdorff spaces is proper if and only if the preimage $\Phi^{-1}(A)$ is compact for each compact subset $A \subset Y$. Thus, the requirement that P is properly supported is equivalent to the requirement that the intersections $(A \times U) \cap \mathrm{supp}\,(K_P)$ and $(U \times A) \cap \mathrm{supp}\,(K_P)$ are compact, for each compact subset $A \subset U$.

The following lemma asserts that modulo a smoothing operator any pseudo-differential operator may be represented by a properly supported one, with kernel supported arbitrary close to the diagonal.

Lemma 6.1.6. *Let $P \in \Psi^d(U)$ and let Ω be an open neighborhood of the diagonal in $U \times U$. Then there exists a $q \in S^d(U)$ such that Ψ_q is properly supported with* $\mathrm{supp}\, K_q \subset \Omega$ *such that $P - P_q \in \Psi^{-\infty}(U)$.*

For the proof of this lemma, we need a generality about partitions of unity.

Lemma 6.1.7. *Let X be a paracompact locally compact Hausdorff space. Let Ω be an open neighborhood of the diagonal* $\mathrm{diag}\,(X)$ *in $X \times X$. Then for every open cover $\mathcal{U}$ of X there exists a locally finite refinement $\mathcal{V}$ such that for all $V, V' \in \mathcal{V}$*

$$V \cap V' \neq \emptyset \Rightarrow V \times V' \subset \Omega.$$

Proof Let $\mathcal{U}$ be an open cover of X. Then there exists an open cover $\mathcal{W}$ of X, finer than $\mathcal{U}$, such that $W \times W \subset \Omega$ for all $W \in \mathcal{W}$. By paracompactness of X, we may assume that $\mathcal{W}$ is locally finite. For each $x \in X$ there exists an open neighborhood V_x such that V_x is contained in every $W \in \mathcal{W}$ containing x, whereas it has empty intersection with every $W \in \mathcal{W}$ not containing x. Let $\mathcal{V}$ be a locally finite refinement of $\{V_x\}_{x \in X}$. Let $V, V' \in \mathcal{V}$ have a point x in common. Then $x \in W$ for some $W \in \mathcal{W}$. Now $V \subset W$ and $V' \subset W$ so that $V \times V' \subset W \times W \subset \Omega$. □

Corollary 6.1.8. *Let X be a smooth manifold and let Ω be an open neighborhood of the diagonal* $\mathrm{diag}\,(X)$ *in $X \times X$. Then for every open covering $\mathcal{U}$ of X there exists a partition of unity $\{\psi_i\}_{i \in I}$, subordinate to $\mathcal{U}$, such that for all $i, j \in I$,*

$$\mathrm{supp}\, \psi_i \cap \mathrm{supp}\, \psi_j \neq \emptyset \Rightarrow \mathrm{supp}\, \psi_i \times \mathrm{supp}\, \psi_j \subset \Omega.$$

Proof of Lemma 6.1.6 Let $\{\psi_i\}_{i \in I}$ be a partition of unity on U as in the preceding corollary. Then the operators $P_{ij} := M_{\psi_i} \Psi_p M_{\psi_j}$ are all of the form $\Psi_{q_{ij}}$, with $q_{ij} \in S^d(U)$. The kernel of P_{ij} equals $K_{ij} := (\psi_i \otimes \psi_j) K_P$, hence has support inside $\mathrm{supp}\, \psi_i \times \mathrm{supp}\, \psi_j$. If $\mathrm{supp}\, \psi_i \cap \mathrm{supp}\, \psi_j = \emptyset$ then K_{ij} is smooth outside the diagonal and supported outside the diagonal, hence smooth everywhere. On the other hand, if $\mathrm{supp}\, \psi_i \cap \mathrm{supp}\, \psi_j \neq \emptyset$ then K_{ij} has support contained in Ω.

Let J be the set of $(i, j) \in I \times I$ with $\mathrm{supp}\, \psi_i \cap \mathrm{supp}\, \psi_j \neq \emptyset$. We note that $\mathrm{pr}_1 \mathrm{supp}\, q_{ij} \subset \mathrm{supp}\, \psi_i$, so that the collection of sets $\mathrm{cl}\, \mathrm{pr}_1 \mathrm{supp}\, q_{ij}$ for $(i, j) \in J$

is locally finite. It follows that

$$q := \sum_{(i,j)\in J} q_{ij}$$

is a symbol in $S^d(U)$. We note that

$$K := \sum_{(i,j)\in I\times I\setminus J} K_{ij}$$

is a locally finite sum of smooth functions on $U \times U$, hence a smooth function of its own right. It is now straightforward to verify that $\Psi_p = \Psi_q + T_K$. Moreover, the kernel K_q is given by the locally finite sum

$$K_q = \sum_{(i,j)\in J} K_{ij}.$$

It follows that $\operatorname{supp} K \subset \Omega$. We will finish the proof by showing that Ψ_q is properly supported. Let $A \subset U$ be compact. The set J_A of $(i,j) \in J$ with $\operatorname{supp}\psi_i \cap A \neq \emptyset$ is finite. As $(A \times U) \cap \operatorname{supp} K_q$ is contained in the union of the sets $\operatorname{supp}\psi_i \times \operatorname{supp}\psi_j$ for $(i,j) \in J_A$, it follows that $(A \times U) \cap \operatorname{supp} K_q$ is compact. The compactness of $(U \times A) \cap K$ is proved in a similar manner. ☐

Exercise 6.1.9. Let $P : C_c^\infty(U) \to C^\infty(U)$ be a linear operator such that for each $\chi, \psi \in C_c^\infty(U)$ the operator $M_\chi \circ P \circ M_\psi$ is a pseudo-differential operator in $\Psi^d(U)$. Show that $P \in \Psi^d(U)$.

Exercise 6.1.10. Let $P \in \Psi^d(U)$ be properly supported.

Show that for every compact subset $\mathcal{K} \subset U$ there exists a compact subset $\mathcal{K}' \subset U$ such that P maps $C^\infty_{\mathcal{K}}(U)$ to $C^\infty_{\mathcal{K}'}(U)$.

Exercise 6.1.11. Let $P \in \Psi^d(U)$ be properly supported.

Show that the operator P has a unique extension to a continuous linear operator $C^\infty(U) \to C^\infty(U)$.

6.2. The adjoint of a pseudo-differential operator

Let $U \subset \mathbb{R}^n$ be an open subset. It is a well known fact that the space $C_c^\infty(\mathbb{R}^n)$ is reflexive. By this we mean that the natural map $j : C_c^\infty(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)'$ is a topological linear isomorphism. Here $\mathcal{D}'(R^n)$ is equipped with the strong dual topology, and so is the dual of $\mathcal{D}'(\mathbb{R}^n)$.

Accordingly, if $T : C_c^\infty(U) \to \mathcal{D}'(U)$ is a continuous linear map, then the transposed T^t may be viewed as a continuous linear map $C_c^\infty(U) \to \mathcal{D}'(U)$. According to this definition,

$$\langle Tf, g\rangle = \langle f, T^t g\rangle, \qquad (f, g \in C_c^\infty(U)).$$

Lemma 6.2.1. *The distribution kernel K_{T^t} (which exists by Schwartz' theorem) of the adjoint map can be expressed in terms of the kernel K_T by*

$$K_{T^t}(x,y) = K_T(y,x), \qquad ((x,y) \in U \times U).$$

Of course, this equality should be interpreted in the sense of distributions. Let $\tau : U \times U \to U \times U$ be defined by $\tau(x, y) = (y, x)$, then the above equality should be read as

$$K_{T^t} = \tau_*(K_T).$$

Proof Put $K = K_T$ and $K^1 = \tau_*(K_T)$. Then the integral operator T^1 with distribution kernel K^1 is given by

$$\langle T^1(f), g\rangle = \langle K^1, g \otimes f\rangle = \langle K, \tau^*(g \otimes f)\rangle = \langle K, f \otimes g\rangle = \langle f, Tg\rangle.$$

This shows that $T^1 = T^t$ and that $K^1 = K_{T^t}$. □

It follows from the above proof that if K^t and K are distributions in $\mathcal{D}'(U \times U)$ related by $K^t = \tau_*(K)$, then the associated operator T_{K^t} is the adjoint of T_K. As each pseudo-differential operator $P \in \Psi(U)$ has a distribution kernel, we do not have to invoke the reflexivity of the space $C_c^\infty(U)$ to establish the existence of the adjoint operator P^t. However, the above reasoning puts the existence of an adjoint operator in a general framework.

In the following proposition an important role is played by the continuous linear map $e^{\langle D_x, \partial_\xi\rangle} \in \mathrm{End}(\mathcal{S}'(\mathbb{R}^{2n}))$, defined by

$$\mathcal{F} \circ e^{\langle D_x, \partial_\xi\rangle} = e^{i\hat{x}\hat{\xi}} \circ \mathcal{F},$$

where $\mathcal{F}$ denotes the Fourier transform $\mathcal{S}'(\mathbb{R}^{2n}) \to \mathcal{S}'(\mathbb{R}^{2n})$. For details we refer to the appendix to this lecture. For a proper understanding of the proposition below, we mention that $e^{\langle D_x, \partial_\xi\rangle}$ maps $S_c^d(U)$ continuously into $S^d(U)$. Moreover, for each $p \in S_c^d(U)$,

$$e^{\langle D_x, \partial_\xi\rangle} p \sim \sum_{k \geq 0} \frac{1}{k!} \langle D_x, \partial_\xi\rangle^k = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} D_x^\alpha \partial_\xi^\alpha.$$

Proposition 6.2.2.

(a) *Let* $p \in S_c^d(U)$. *Then*

$$\Psi_p^t = \Psi_q, \qquad \text{with} \quad q = e^{\langle D_x, \partial_\xi\rangle} p^\vee, \quad p^\vee(x, \xi) = p(x, -\xi).$$

(b) *Let* $P \in \Psi^d(U)$. *Then* $P^t \in \Psi(U)$. *If* $p \in S^d(U)$ *represents the full symbol of* P, *then the full symbol of* P^t *is given by the expansion*

$$\sigma(P^t)(x, \xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{(-1)^{|\alpha|}}{\alpha!} D_x^\alpha \, \partial_\xi^\alpha p(x, -\xi).$$

Remark 6.2.3. Here we recall that the full symbol of P^t is defined modulo $S^{-\infty}(U)$ hence is completely determined by the given expansion.

Proof Since $\mathrm{pr}_1(p)$ has compact closure, p defines a tempered distribution on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$. We denote the variables in $\mathbb{R}^{2n}$ by $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$. The Fourier transform of $\mathbb{R}^{2n}$ defines a continuous linear isomorphism $\mathcal{S}(\mathbb{R}^{2n}) \to \mathcal{S}(\mathbb{R}^{2n})$. We agree to denote the (dual) variables on the Fourier transform side by $(\hat{x}, \hat{\xi})$. Let $\mathcal{F}_1$ denote the Fourier transform with respect to the first variable x (or $\hat{x}$) and $\mathcal{F}_2$ the Fourier transform with respect to the second variable ξ (or $\hat{\xi}$). Then both partial Fourier transforms $\mathcal{F}_1, \mathcal{F}_2$ are readily seen to be topological linear automorphisms of $\mathcal{S}(\mathbb{R}^{2n})$ and we have

$$\mathcal{F} = \mathcal{F}_1 \circ \mathcal{F}_2. \tag{6.4}$$

By transposition we see that all these remarks extend to tempered distributions. In particular, $\mathcal{F}_2 p \in \mathcal{S}'(\mathbb{R}^{2n})$ and we see that the kernel $K_p = L_*^{-1}\mathcal{F}_2 p$ of Ψ_p is a tempered distribution as well. Let

$$K_p^t = \tau_* K_p, \quad q = \mathcal{F}_2^{-1} L_*(K_p^t).$$

Then q is a tempered distribution on $\mathbb{R}^{2n}$. We will show that actually $q \in S^d(U)$. Then by (6.2) it follows that Ψ_q has kernel K_p^t hence is the transposed of Ψ_p.

We note that $\mathcal{F}_2 q = L_* \tau_* L_*^{-1} \mathcal{F}_2 p$. Applying $\mathcal{F}_1$ to both sides, and using (6.4) we see that

$$\mathcal{F}q = \mathcal{F}_1 L_* \tau_* L_*^{-1} \mathcal{F}_1^{-1} \mathcal{F}p.$$

To understand the composition of operators acting on $\mathcal{F}p$, we note that

$$\mathcal{F}_1 L_* \tau_* L_*^{-1} \mathcal{F}_1^{-1} = \mathcal{F}_1 (L \circ \tau \circ L^{-1})_* \mathcal{F}_1^{-1} = [\mathcal{F}_1^{-1} (L \circ \tau \circ L^{-1})^* \mathcal{F}_1]^t.$$

Now $L \circ \tau \circ L^{-1}(x, y) = (x + y, -y)$. It follows that for $f \in \mathcal{S}(\mathbb{R}^{2n})$ we have

$$\begin{aligned}
\mathcal{F}_1^{-1}(L \circ \tau \circ L^{-1})^* \mathcal{F}_1 f(\hat{x}, \hat{\xi}) &= \int_{\mathbb{R}^n} e^{i\hat{x}x} \mathcal{F}_1 f(x + \hat{\xi}, -\hat{\xi}) \, d\hat{x} \\
&= \int_{\mathbb{R}^n} e^{i\hat{x}x - i\hat{\xi}\hat{x}} \mathcal{F}_1 f(x, -\hat{\xi}) \, dx \\
&= e^{-i\hat{x}\hat{\xi}} f(\hat{x}, -\hat{\xi}) \\
&= e^{-i\hat{x}\hat{\xi}} S_2^* f(\hat{x}, \hat{\xi}),
\end{aligned}$$

where we have used the notation S_2 for the map $(x, \xi) \mapsto (x, -\xi)$. By transposition we now see that

$$[\mathcal{F}_1^{-1}(L \circ \tau \circ L^{-1})^* \mathcal{F}_1]^t = e^{i\hat{x}\hat{\xi}} S_{2*},$$

so that

$$\mathcal{F}q = e^{i\hat{x}\hat{\xi}} S_{2*} \mathcal{F}p = \mathcal{F}[e^{D_x \partial_\xi} p^\vee],$$

see appendix to this lecture. This proves the identity (a).

We turn to (b). Let $P \in \Psi^d(U)$ and let $p \in S^d(U)$ represent its symbol. There exists a symbol $p' \in S^d(U)$ such that $P' = \Psi_{p'}$ is properly supported and such that $P - P' = T \in \Psi^{-\infty}$. It follows that $p - p' \in S^{-\infty}(U)$, so that p' represents the symbol of P as well. The adjoint of T has a smooth kernel, hence belongs to $\Psi^{-\infty}$. Thus, it suffices to show that P' has an adjoint which is a pseudo-differential operator with the required symbol. We thus see that without loss of generality we may assume that P equals Ψ_p and is properly supported. Let $\{\psi_j\}$ be a partition of unity on U. Then each $P_j := M_{\psi_j} \circ \Psi_p$ is properly supported and of the form $P_j = \Psi_{p_j}$ with symbol $p_j(x, \xi) = \psi_j(x) p(x, \xi)$. Let K_j be the distribution kernel of P_j, then $K_j = (\psi_j \otimes 1) K_P$. From the fact that P is properly supported, it follows that the collection of subsets $\mathrm{pr}_2(\mathrm{supp}\, K_j)$ is locally finite in U. We also note that $p = \sum_j p_j$, with locally finite sum.

It follows from (a) that $P_j^t = \Psi_{q_j}$, where $q_j = e^{\langle D_x, \partial_\xi \rangle} p_j^\vee$. This implies that each q_j has an asymptotic expansion of the form

$$q_j = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} D_x^\alpha \partial_\xi^\alpha [p_j(x, -\xi)] = \sum_{\alpha \in \mathbb{N}^n} \frac{(-1)^{|\alpha|}}{\alpha!} D_x^\alpha \partial_\xi^\alpha p_j(x, -\xi)$$

We note that $\mathrm{pr}_1(\mathrm{supp}\, q_j) \subset \mathrm{pr}_2(K_j)$; hence the collection $\mathrm{pr}_1(\mathrm{supp}\,(q_j))$ is locally finite.

We define

$$q_\alpha(x, \xi) := \frac{(-1)^{|\alpha|}}{\alpha!} D_x^\alpha \partial_\xi^\alpha p_j(x, -\xi)$$

This sum is locally finite in the x-variable, hence defines an element of $S^d(U)$. Let $q \in S^d(U)$ be a symbol with

$$q \sim \sum_{\alpha \in \mathbb{N}^n} q_\alpha.$$

Then $\Psi_q \in \Psi^d(U)$. We claim that $\Psi_q - \Psi_p^t \in \Psi^{-\infty}$. To see this, let $\varphi \in C_c^\infty(U)$. Then $M_\varphi \circ \Psi_q$ has symbol φq and

$$M_\varphi \circ \Psi_p^t = \sum_j M_\varphi \circ \Psi_p^t \circ M_{\psi_j} = \sum_j M_\varphi \Psi_{q_j} = \Psi_{\sum_j \varphi q_j},$$

which is a finite sum. Since $\varphi q \sim \sum_j \varphi q_j$ by construction, it follows that $M_\varphi \circ \Psi_q - M_\psi \circ \Psi_p^t \in \Psi^{-\infty}(U)$ for all $\varphi \in C_c^\infty(U)$. This implies that $\Psi_q - \Psi_p^t \in \Psi^{-\infty}(U)$ and the proof is complete. □

Exercise 6.2.4. Let P be a differential operator of order d on U with full symbol p. Show by direct calculation that the full symbol q of the transposed operator P^t is given by the formula

$$q(x, \xi) = \sum_{\alpha \in \mathbb{N}^n} \frac{(-1)^{|\alpha|}}{\alpha!} D_x^\alpha \partial_\xi^\alpha p(x, -\xi).$$

Show that the terms in the above series are zero for $|\alpha| > d$.

Let $P \in \Psi^d(U)$. We define the conjugate $\bar{P}$ of P by the formula

$$\bar{P}(f) = \overline{P(\bar{f})}, \qquad (f \in C^\infty(U)).$$

It is readily seen that the kernel of $\bar{P}$ is given by $K_{\bar{P}} = \bar{K}_P$ (which should be interpreted in the sense of distributions). If $p \in S^d(U)$ then it is readily verified that

$$\overline{\Psi}_p = \Psi_{\bar{p}^\vee}.$$

Exercise 6.2.5. Verify this.

Thus, $\overline{\Psi}_p$ is a pseudo-differential operator. In general it follows that the conjugate of any pseudo-differential operator P is a pseudo-differential operator again. Moreover, its full symbol is given by

$$\sigma(\bar{P})(x, \xi) = \overline{\sigma(P)(x, -\xi)}.$$

We now define the adjoint P^* of P by $P^* = \bar{P}^t$. Then it is readily checked that

$$\langle Pf, g\rangle_{L^2} = \langle f, P^* g\rangle_{L^2}, \qquad (f, g \in C_c^\infty(U)).$$

where we have used the notation $\langle f, g\rangle_{L^2} = \langle f, \bar{g}\rangle$ for the usual sesquilinear pairing.

Corollary 6.2.6. *Let $P \in \Psi^d(U)$. Then the adjoint P^* belongs to $\Psi^d(U)$ and its full symbol is given by*

$$\sigma(P^*) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} D_x^\alpha \partial_\xi^\alpha \overline{\sigma(P)}$$

Proof This follows from the discussion preceding this corollary, combined with Proposition 6.2.2. □

Corollary 6.2.7. *Let $P \in \Psi^d(U)$. Then P extends uniquely to a continuous linear operator $\mathcal{E}'(U) \to \mathcal{D}'(U)$.*

Proof The transposed P^t is a pseudo-differential operator, hence defines a continuous linear operator $C_c^\infty(U) \to C^\infty(U)$. Its transposed $(P^t)^t$ is a continuous linear operator $\mathcal{E}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$. Clearly, it restricts to P on $C_c^\infty(U)$. This establishes existence. Uniqueness follows by density of $C_c^\infty(U)$ in $\mathcal{E}'(U)$. □

6.3. Pseudo-locality

In the sequel our view-point will be that a pseudo-differential operator on a manifold M is an integral operator with distribution kernel K on $M \times M$, with K smooth away from the diagonal, whereas along the diagonal the operator is described as in the local setting.

If E is a smooth vector bundle on a smooth manifold M, then by $\Gamma^{-\infty}(E)$ we denote the space of generalized sections of E. Let $u \in \Gamma^{-\infty}(E)$ be a generalized section. Then u is said to be smooth on an open subset $U \subset M$ if $\chi u \in \Gamma^\infty(E)$ for all $\chi \in C_c^\infty(U)$. The generalized section u is said to be smooth at a point $x \in M$ if their exists an open neighborhood $U \ni x$ such that u is smooth on U. The singular support of u, denoted $\mathrm{singsupp}\,(u)$, is defined to be the subset of $x \in M$ such that u is not smooth at x. Clearly, $\mathrm{singsupp}\, u$ is a closed subset of M.

Definition 6.3.1. Let E, F be vector bundles on a smooth manifold M. A linear operator $T : \Gamma_c^{-\infty}(E) \to \Gamma^{-\infty}(F)$ is said to be *pseudo-local* if

$$\mathrm{singsupp}\, Tu \subset \mathrm{singsupp}\, u, \qquad \text{for all} \quad u \in \Gamma_c^{-\infty}(E).$$

Corollary 6.3.2. *Let $P \in \Psi^d(U)$. Then the operator $P : \mathcal{E}'(U) \to \mathcal{D}'(U)$ is pseudo-local.*

Proof The adjoint operator P has a distribution kernel $K \in \mathcal{D}'(U \times U)$ which is smooth outside the diagonal of U.

Let $u \in \mathcal{E}'(\mathbb{R}^n)$ have singular support equal to A. Then A is contained in $\mathrm{supp}\, u$, hence compact. Let $x \in \mathbb{R}^n \setminus A$. Then there exist open neighborhoods U_1 of x and U_2 of A in U such that $U_1 \cap U_2 = \emptyset$. Let $\varphi \in C_c^\infty(U_1)$. Then it suffices to show that $\varphi\Psi_r(u)$ is smooth. Fix $\psi \in C_c^\infty(U_2)$ such that $\psi = 1$ on an open neighborhood of A. Then $(1-\psi)u$ is in $C_c^\infty(\mathbb{R}^n)$ hence $P((1-\psi)u) \in C^\infty(U)$. Thus, it suffices to show that $\varphi P(\psi u)$ is smooth. The function $k : (x,y) \mapsto \varphi(x)\psi(y)K(x,y)$ is smooth on $U \times U$, since $\varphi \otimes \psi$ is supported in $U_1 \times U_2 \subset U \times U \setminus \operatorname{diag}(U)$.

Moreover, for all $f \in C_c^\infty(\mathbb{R}^n)$ we have

$$\begin{aligned} \langle \varphi P(\psi u), f \rangle &= \langle P(\psi u), \varphi f \rangle \\ &= \langle \psi u, P^t(\varphi f) \rangle \\ &= \langle u, \psi P^t(\varphi f) \rangle \\ &= \langle u, T_k f \rangle \\ &= \langle T_k^t(u), f \rangle, \end{aligned}$$

where T_k denotes the integral operator with smooth kernel k. It follows that

$$\varphi P(\psi u) = T_k^t(u).$$

Since $T_k^t(u)$ is smooth, the result follows. □