Analysis on Manifolds
Lecture notes for the 2009/2010 Master Class

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LECTURE 7
Pseudo-differential operators, continued

7.1. The symbol of the composition

In this section we will investigate the composition $P \circ Q$ of two pseudo-differential operators $P, Q \in \Psi(U)$; here $U \subset \mathbb{R}^n$ is an open subset. We first assume that $P = \Psi_p$ and $Q = \Psi_q$, with $p \in S^d(U)$ and $q \in S^e(U)$, where $d, e \in \mathbb{R}$. In general the operator Q maps $C_c^\infty(U)$ to $C^\infty(U)$, but not to $C_c^\infty(U)$. For the composition to exist we therefore require that the projection $\mathrm{pr}_1(\mathrm{supp}\, q)$ has compact closure A in U. Then Ψ_q maps $C_c^\infty(U)$ to $C_A^\infty(U)$ and $P \circ Q$ is a well-defined continuous linear operator $C_c^\infty(U) \to C^\infty(U)$.

Proposition 7.1.1. *Let $p \in S^d(U)$ and $q \in S_c^e(U)$. Then $\Psi_p \circ \Psi_q = \Psi_r$, with $r \in S^{d+e}(U)$ given by*

$$r(x,\xi) = e^{\langle D_y, \partial_\xi \rangle} p(x,\xi) q(y,\eta)|_{y=x, \eta=\xi}. \tag{7.1}$$

In particular, $\mathrm{pr}_1(\mathrm{supp}\, r) \subset \mathrm{pr}_1(\mathrm{supp}\, p)$ *and*

$$r \sim \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial_\xi^\alpha p(x,\xi) D_x^\alpha q(x,\xi).$$

As a preparation we prove the following result.

Lemma 7.1.2. *The Fourier transform of the function $u : \mathbb{R}^{2n} \to \mathbb{C}$, $(x,\xi) \mapsto e^{i\xi x}$ (which defines a tempered distribution) is given by $\mathcal{F}u(\hat{x}, \hat{\xi}) = e^{-i\hat{\xi}\hat{x}}$.*

Proof Let $\Omega := \{z \in \mathbb{C} \mid \mathrm{Re}\, z > 0\}$. For $z \in \Omega$ the function $v_z : x \mapsto e^{-zx^2/2}$ defines a function in $\mathcal{S}(\mathbb{R})$. By checking the Cauchy-Riemann equations relative to the variable z, using differentiation under the integral sign, we see that for $\hat{x} \in \mathbb{R}$, the Fourier transform

$$\mathcal{F}v_z(\hat{x}) = \int_{\mathbb{R}} e^{-zx^2/2}\, e^{-i\hat{x}x}\, dx$$

depends holomorphically on $z \in \Omega$. For $z \in \mathbb{R} \cap \Omega$ we find, by the substitution of variables $x \to z^{-1/2}x$, that $\mathcal{F}v_z(\hat{x}) = z^{-1/2}\mathcal{F}v_1(z^{-1/2}\hat{x})$, hence

$$\mathcal{F}v_z(\hat{x}) = z^{-1/2} e^{-\hat{x}^2/2z}. \tag{7.2}$$

By analytic continuation in z the latter formula is valid for all $z \in \Omega$, provided the branch of $z \mapsto z^{1/2}$ over Ω which is positive on $\mathbb{R} \cap \Omega$ is taken. If $f \in \mathcal{S}(\mathbb{R})$ then by continuity of $z \mapsto \langle \mathcal{F}v_z, f \rangle = \langle v_z, \mathcal{F}f \rangle$ we conclude that formula (7.2) remains valid for $z \in \overline{\Omega} \setminus \{0\}$, provided the continuous extension of the fixed branch of the square root is taken. In particular we find that

$$\mathcal{F}v_i(\hat{x}) = e^{-\pi i/4}\, e^{i\hat{x}^2/2}, \qquad \mathcal{F}v_{-i}(\hat{x}) = e^{\pi i/4}\, e^{-i\hat{x}^2/2}.$$

We now turn to the function u. Let $a : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the linear map defined by $a(x,\xi) = (x+\xi, x-\xi)/\sqrt{2}$. Then

$$[a^*u](x,\xi) = e^{i(x^2-\xi^2)/2} = \prod_{j=1}^n v_i(\xi_j)v_{-i}(x_j).$$

Therefore,

$$\mathcal{F}[a^*u](\hat{x},\hat{\xi}) = \prod_{j=1}^n e^{i(\hat{\xi}_j^2-\hat{x}_j^2)/2} = [a^*u](-\hat{x},\hat{\xi}).$$

By orthogonality of a we have that $\mathcal{F}\circ a^* = a^*\circ\mathcal{F}$ on $\mathcal{S}(\mathbb{R})$, hence also on $\mathcal{S}'(\mathbb{R})$, and the result follows. ☐

We also need an extension of the convolution product in order to be able to convolve tempered distributions with Schwartz functions.

Lemma 7.1.3. *The convolution product* $* : \mathcal{S}(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n)$ *has a unique extension to a continuous bilinear map* $\mathcal{S}(\mathbb{R}^n)\times\mathcal{S}'(\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n)$. *For this extension, denoted by* $*$ *again,*

$$\mathcal{F}(f*u) = \mathcal{F}f\,\mathcal{F}u, \qquad (f\in\mathcal{S}(\mathbb{R}^n),\ u\in\mathcal{S}'(\mathbb{R}^n)).$$

Proof The multiplication map $(f,u)\to fu$, $\mathcal{S}(\mathbb{R}^n)\times\mathcal{S}'(\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n)$ is readily seen to be continuous bilinear. Define $\widetilde{*} = \mathcal{F}^{-1}\circ *\circ(\mathcal{F}\times\mathcal{F})$. Then $\widetilde{*}$ is continuous bilinear and extends the convolution product on $\mathcal{S}(\mathbb{R}^n)$, by Lemma 4.2.2(b). Uniqueness of the extension follows by density of $\mathcal{S}(\mathbb{R}^n)$ in $\mathcal{S}'(\mathbb{R}^n)$. ☐

As a final preparation for the proof of Proposition 7.1.1 we need the following lemma.

Lemma 7.1.4. *Let* $u\in C^\infty(\mathbb{R}^{2n})$ *be defined by* $u(x,\xi) := e^{-i\xi x}$. *Then*

$$e^{\langle D_x,\partial_\xi\rangle}f = u*f \qquad \text{for all} \qquad f\in\mathcal{S}(\mathbb{R}^n).$$

Proof In view of the definition of the operator $e^{\langle D_x,\partial_\xi\rangle}\in\mathrm{End}(\mathcal{S}(\mathbb{R}^{2n}))$ (see Section 6.4), this follows immediately by applicaton of Lemmas 7.1.3 and 7.1.2. ☐

Proof of Proposition 7.1.1 We will first prove the result under the assumption that both p and q are smooth and compactly supported. Then $p,q\in C_c^\infty(\mathbb{R}^{2n})$. Let $f\in C_c^\infty(U)$. Then

$$\begin{aligned}
[\Psi_p\Psi_q f](x) &= \int\!\!\int e^{i\xi(x-y)}p(x,\xi)\,[\Psi_q f](y)\,dy\,d\xi \\
&= \int\!\!\int\!\!\int e^{i\xi(x-y)}e^{iy\eta}\,p(x,\xi)\,q(y,\eta)\widehat{f}(\eta)\,d\eta\,dy\,d\xi \\
&= \int e^{i\eta x}r(x,\eta)\widehat{f}(\eta)\,d\eta,
\end{aligned}$$

where

$$r(x,\eta) = \int\!\!\int e^{i(\xi-\eta)(x-y)}\,p(x,\xi)\,q(y,\eta)\,d\xi dy. \tag{7.3}$$

Here we note that all integrands are compactly supported and continuous, so all integrals are convergent, and the order of the integrations is immaterial. For each $(x,\eta)\in\mathbb{R}^{2n}$ we define $R_{x,\eta}:\mathbb{R}^{2n}\to\mathbb{C}$ by

$$R_{x,\eta}(y,\xi)=p(x,\xi)q(y,\eta).$$

Then $R_{x,\eta}$ is smooth and compactly supported. Moreover, (7.3) can be rewritten as $r(x,\eta)=[u*R_{x,\eta}](x,\eta)$, with $u(y,\xi)=e^{-i\xi y}$. In view of Lemma 7.1.4 above, this implies that

$$r(x,\eta)=[e^{\langle D_y,\partial_\xi\rangle}R_{x,\eta}](x,\eta)=e^{\langle D_y,\partial_\xi\rangle}p(x,\xi)q(y,\eta)|_{y=x,\xi=\eta},$$

which in turn gives (7.1).

Fix a compact subset $\mathcal{K}\subset U$. Our next step is to extend the validity of (7.1) to $(p,q)\in S^d_{\mathcal{K}}(U)\times S^e_{\mathcal{K}}(U)$. We will do this by using a continuity argument. For $p,q\in S^\infty_{\mathcal{K}}(U)$ we observe that, for each $(x,\eta)\in\mathbb{R}^n\times\mathbb{R}^n$, the function $(y,\xi)\mapsto p(x,\xi)q(y,\eta)$ belongs to $S^\infty_{\mathcal{K}}(U)$ again, and we define the function $\rho(p,q):U\times\mathbb{R}^n\to\mathbb{C}$ by

$$\rho(p,q)(x,\xi)=e^{\langle D_y,\partial_\xi\rangle}p(x,\xi)q(y,\eta)|_{y=x,\eta=\xi}. \tag{7.4}$$

In view of the lemma below, ρ has values in $S^\infty(U)$ and maps $S^d_{\mathcal{K}}(U)\times S^e_{\mathcal{K}}(U)$ continuous bi-linearly to $S^{d+e}(U)$. Fix $f\in C^\infty_c(U)$. Then in view of the remark below Definition 5.2.2 follows that

$$(p,q)\mapsto\Psi_{\rho(p,q)}(f) \tag{7.5}$$

maps $S^d_{\mathcal{K}}(U)\times S^e_{\mathcal{K}}(U)$ continuous bi-linearly to $C^\infty(U)$, for all $d,e\in\mathbb{N}$.

On the other hand, for all $d\in\mathbb{N}$ the map $(r,g)\mapsto\Psi_r(g)$ is continuous bilinear from $S^d_{\mathcal{K}}(U)\times C^\infty_c(U)$ to $C^\infty_{\mathcal{K}}(U)$ and by composition it follows that for all $d,e\in\mathbb{N}$ the map

$$(p,q)\mapsto\Psi_p\Psi_q(f) \tag{7.6}$$

is continuous bilinear from $S^d_{\mathcal{K}}(U)\times S^e_{\mathcal{K}}(U)$ to $C^\infty_{\mathcal{K}}(U)$. By the first part of the proof the maps (7.5) and (7.6) are equal on $C^\infty_{\mathcal{K},c}(U)\times C^\infty_{\mathcal{K},c}(U)$. By density of $C^\infty_{\mathcal{K},c}(U)$ in $S^d_{\mathcal{K}}(U)$ for the S^{d+1}-topology and in $S^e_{\mathcal{K}}(U)$ for the S^{e+1}-topology, it follows that the equality extends to $S^d_{\mathcal{K}}(U)\times S^e_{\mathcal{K}}(U)$.

It follows that $\Psi_p\circ\Psi_q=\Psi_{\rho(p,q)}$ on $C^\infty_c(U)$, for all $p\in S^d_{\mathcal{K}}(U)$ and $q\in S^e_{\mathcal{K}}(U)$.

Now assume more generally that $p\in S^d(U)$ and that $q\in S^e_c(U)$. Then by using a partition of unity we may write p as a locally finite sum $p=\sum_j p_j$, with $p_j\in S^d_c(U)$, and $\mathrm{pr}_1(\mathrm{supp}\,p_j)$ a locally finite collection of subsets of U. For each j we have $\Psi_{p_j}\circ\Psi_q=\Psi_{r_j}$ with r_j given in terms of p_j and q as in (7.3). In particular this implies that $\mathrm{pr}_1(\mathrm{supp}\,r_j)\subset\mathrm{pr}_1(\mathrm{supp}\,p_j)$. It follows that $\{\mathrm{pr}_1(\mathrm{supp}\,r_j)\}$ is a locally finite collection of subsets of U. Therefore, $r=\sum_j r_j$ defines a symbol in $S^{d+e}(U)$. Clearly, r satisfies (7.3), and $\Psi_p\circ\Psi_q=\Psi_r$. □

Lemma 7.1.5. *Let $\mathcal{K}\subset U$ be a compact subset. Then the map ρ defined by (7.4) maps $S^d_{\mathcal{K}}(U)\times S^e_{\mathcal{K}}(U)$ continuous bilinearly to $S^{d+e}_{\mathcal{K}}(U)$.*

Proof By continuity of the map $e^{\langle D_y, \partial_\xi \rangle} : S^d_{\mathcal{K}}(U) \to S^d(\mathbb{R}^n)$ (see Theorem 6.5.2 there exist constants $C > 0, k \in \mathbb{N}$, such that

$$(1 + \|\xi\|)^{-d} |[e^{D_x \partial_\xi} f](x, \xi)| \leq C \nu^d_{\mathcal{K},k}(f),$$

for all $f \in S^d_{\mathcal{K}}(U)$, $x \in \mathcal{K}$ and $\xi \in \mathbb{R}^n$. Let $p \in S^d_{\mathcal{K}}(U)$ and $q \in S^e_{\mathcal{K}}(U)$. Then by application of the above estimate to $f = f_{x,\eta} : (y, \xi) \mapsto p(x, \xi) q(y, \eta)$, and observing that

$$(1 + \|\eta\|)^{-e} \nu^d_{\mathcal{K},k}(f_{x,\eta}) \leq \nu^d_{\mathcal{K},k}(p) \nu^e_{\mathcal{K},k}(q),$$

we find the estimate

$$(1 + \|\xi\|)^{-d-e} |\rho(p, q)(x, \xi)| \leq C \nu^d_{\mathcal{K},k}(p) \nu^e_{\mathcal{K},k}(q), \tag{7.7}$$

for all $(x, \xi) \in U \times \mathbb{R}^n$. (Note that the expression on the left-hand side vanishes for $x \in U \setminus \mathcal{K}$.) We now observe that for $\alpha, \beta \in \mathbb{N}^n$ we have

$$\partial_x^\alpha \partial_\xi^\beta \rho(p, q) = \sum_{\substack{\alpha_1 + \alpha_2 = \alpha \\ \beta_1 + \beta_2 = \beta}} \begin{pmatrix} \alpha \\ \alpha_1 \end{pmatrix} \begin{pmatrix} \beta \\ \beta_1 \end{pmatrix} \rho(\partial_x^{\alpha_1} \partial_\xi^{\beta_1} p, \partial_x^{\alpha_2} \partial_\xi^{\beta_2} q).$$

Combining this with (7.7) we find that for every $l \in \mathbb{N}$ there exists a constant $C_l > 0$, only depending on l, such that

$$\nu^{d+e}_{\mathcal{K},l} \rho(p, q) \leq C\, C_l\, \nu^d_{\mathcal{K},k+l}(p)\, \nu^e_{\mathcal{K},k+l}(q).$$

The asserted continuity follows. □

Exercise 7.1.6. Give a proof of Proposition 6.2.2 based on the idea of continuous extension used in the above proof.

We recall that a pseudo-differential operator $Q \in \Psi^e(U)$ has an operator kernel $K_Q \in \mathcal{D}'(U \times U)$ and is said to be properly supported if and only if the projection maps $\mathrm{pr}_1, \mathrm{pr}_2 : \mathrm{supp}\,(K_Q) \to \mathbb{R}^n$ are proper.

If $B \subset U$ is compact then so is $A := \mathrm{pr}_1(K_Q \cap \mathrm{pr}_2^{-1}(B))$ and it is easily verified that Q maps $C^\infty_B(U)$ into $C^\infty_A(U)$. Hence, Q is a continuous linear endomorphism of $C^\infty_c(U)$. Thus, for any $P \in \Psi^d(U)$ the composition $P \circ Q$ is a well defined continuous linear operator $C^\infty_c(U) \to C^\infty(U)$.

Theorem 7.1.7. *Let $P \in \Psi^d(U)$ and $Q \in \Psi^e(U)$ be properly supported. Then $P \circ Q$ belongs to $\Psi^{d+e}(U)$. Moreover, the full symbol of $P \circ Q$ is given by*

$$\sigma(P \circ Q) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma(P)\, D_x^\alpha \sigma(Q).$$

Proof In view of Lemma 7.1.9 below, there exist symbols $p \in S^d(U)$ and $q \in S^e(U)$ such that $P = \Psi_p$ and $Q = \Psi_q$. Let $K_P \in \mathcal{D}'(U \times U)$ be the distribution kernel of P. Let $\{\psi_j\}$ be a partition of unity on U. Let $p_j = \psi_j p$; then $P = \sum_j P_j$, with $P_j = \Psi_{p_j}$. For each j the set $A_j := \mathrm{pr}_2(\mathrm{pr}_1^{-1}(\mathrm{supp}\,\psi_j) \cap \mathrm{supp}\,K_P)$ is compact and contained in U. Hence, there exists a $\chi_j \in C^\infty_c(U)$ with $\chi_j = 1$ on A_j. It follows that $P_j \circ M_{\chi_j}$ has kernel $(\psi_j \otimes \chi_j) K_P = (\psi_j \otimes 1) K_P$ hence equals P_j. Therefore, $P_j \circ Q = P_j \circ Q_j$, where $Q = M_{\chi_j} \circ Q = \Psi_{q_j}$, with $q_j = \chi_j q$. It follows that $P_j \circ Q = \Psi_{r_j}$, with $r_j \in S^{d+e}(U)$ expressed in terms of p_j and q_j as in formula (7.3). In particular, $\mathrm{pr}_1(\mathrm{supp}\,(r_j)) \subset \mathrm{supp}\,\psi_j$, so that $r = \sum_j r_j$ is a locally finite sum defining an element of $S^{d+e}(U)$. We now have that

$$\Psi_r = \sum_j P_j \circ Q_j = \sum_j P_j \circ Q = P \circ Q.$$

on $C_c^\infty(U)$. From the construction, it follows that $q_j = q$ on an open neighborhood of $\mathrm{supp}\,\psi_j$, so that for all $\alpha \in \mathbb{N}^n$ we have

$$\partial_\xi^\alpha p_j\, D_x^\alpha q_j = \partial_\xi^\alpha p_j D_x^\alpha q.$$

This implies that

$$\begin{aligned} r &\sim \sum_{\alpha\in\mathbb{N}^n} \sum_j \frac{1}{\alpha!} \partial_\xi^\alpha p_j\, D_x^\alpha q_j = \sum_{\alpha\in\mathbb{N}^n} \sum_j \frac{1}{\alpha!} \partial_\xi^\alpha p_j\, D_x^\alpha q \\ &= \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!} \partial_\xi^\alpha p\, D_x^\alpha q. \end{aligned}$$

The result follows. □

Lemma 7.1.8. *Let $T \in \Psi^{-\infty}(U)$ be properly supported. Then there exists a $r \in S^{-\infty}(U)$ such that $T = \Psi_r$.*

Proof Let K be the integral kernel of T. Fix a partition of unity $\{\psi_j\}$ on U. Then $K_j(x,y) = \psi_j(x)K(x,y)$ defines a smooth function with compact support contained in $\mathrm{pr}_1^{-1}(\mathrm{supp}\,\psi_j) \cap \mathrm{supp}\,K$. By Lemma 5.2.5 (see also its proof) there exists a $p \in S^{-\infty}(U)$ with $\mathrm{pr}_1(\mathrm{supp}\,p) \subset \mathrm{pr}_1(\psi_j)$ such that $T_{K_j} = \Psi_{p_j}$. The locally finite sum $\sum_j p_j$ defines an element $p \in S^{-\infty}(U)$, and it is clear that $\Psi_p = \sum_j \Psi_{p_j} = \sum_j T_{K_j} = T_K$ on $C_c^\infty(U)$. □

Lemma 7.1.9. *Let $d \in \mathbb{R} \cup \{\infty\}$ and let $P \in \Psi^d(U)$ be properly supported. Then there exists a $p \in S^d(U)$ such that $P = \Psi_p$.*

Proof In view of the above lemma we may assume that $d > -\infty$. Then by Lemma 6.1.6 we may rewrite P as $P = \Psi_q + T$, with $q \in S^d(U)$ and $T \in \Psi^{-\infty}$ and with Ψ_q and hence also T properly supported. By the previous lemma there exists a $r \in S^{-\infty}(U)$ such that $T = \Psi_r$. The lemma now follows with $p = q + r$. □

We can now deduce the important result that the principal symbol behaves multiplicatively.

Corollary 7.1.10. *Let $P \in \Psi^d(U)$ and $Q \in \Psi^e(U)$ be properly supported. Then $P \circ Q$ belongs to $\Psi^{d+e}(U)$. Moreover, the principal symbol of $P \circ Q$ of order $d + e$ is given by*

$$\sigma^{d+e}(P \circ Q) = \sigma^d(P)\, \sigma^e(Q).$$

7.2. Invariance of pseudo-differential operators

In order to be able to lift pseudo-differential operators to manifolds, we need to establish invariance under diffeomorphisms. For this it will turn out to be useful to have a different characterization of properly supported pseudo-differential operators.

We recall the definition of the (Fréchet) space $C^\infty(\mathbb{R}^n) \widehat{\otimes} \mathcal{S}(\mathbb{R}^n) = S^{-\infty}(\mathbb{R}^n)$ given in the text before Exercise 5.1.4. Note that $\mathcal{S}(\mathbb{R}^n)$ may be identified with the (closed) subspace of functions in $C^\infty(\mathbb{R}^n) \widehat{\otimes} \mathcal{S}(\mathbb{R}^n)$ that are constant in the x-variable.

For $\varphi \in C^\infty(\mathbb{R}^n) \widehat{\otimes} \mathcal{S}(\mathbb{R}^n)$ the integral

$$W(\varphi)(x) := \int_{\mathbb{R}^n} e^{i\xi x} \varphi(x, \xi)\, dx$$

is absolutely convergent for every $x \in X$ and defines a function $W(\varphi) \in C^\infty(\mathbb{R}^n)$. Moreover, by Lemma 5.2.1 the transform W is continuous linear from $C^\infty(\mathbb{R}^n) \widehat{\otimes} \mathcal{S}(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$.

We now define a more general symbol space as follows. We write (x, ξ, y) for points in $\mathbb{R}^{3n} \simeq \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ and denote by Σ^d the space of smooth functions $\mathbb{R}^{3n} \to \mathbb{C}$ such that for all compact $\mathcal{K} \subset \mathbb{R}^n$ and all $k \in \mathbb{N}$,

$$\nu^d_{\mathcal{K},k}(r) := \max_{|\alpha|,|\beta|,|\gamma| \le k} \sup_{\mathcal{K} \times \mathbb{R}^n \times \mathcal{K}} (1 + |\xi|)^{|\beta| - d} |\partial_x^\alpha \partial_\xi^\beta \partial_y^\beta r(x, \xi, y)| < \infty. \tag{7.8}$$

Note that the symbol space $S^d(\mathbb{R}^n)$ may be viewed as the subspace of Σ^d consisting of functions that are constant in the y-variable.

Let now $r \in \Sigma^d$. Then for each $f \in \mathcal{S}(\mathbb{R}^n)$ the integral

$$r(x, \xi, f) := \int_{\mathbb{R}^n} e^{-i\xi y} r(x, \xi, y) f(y)\, dy$$

converges absolutely and defines a function in $C^\infty(\mathbb{R}^n) \widehat{\otimes} \mathcal{S}(\mathbb{R}^n)$. Moreover, the map $f \mapsto r(\,\cdot\,, f)$ is continuous for the obvious topologies.

The definition of pseudo-differential operator may now be extended to symbols in Σ^d by putting

$$\Psi_r(f)(x) := W(r(\,\cdot\,, \,\cdot\,, f)) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\xi(x-y)} r(x, \xi, y) f(y)\, dy\, d\xi. \tag{7.9}$$

Note that if r is independent of the variable y, then $r \in S^d(\mathbb{R}^n)$ and by carrying out the integration over y we see that (7.9) equals

$$\int_{\mathbb{R}^n} e^{i\xi x} r(x, \xi) \widehat{f}(\xi)\, d\xi,$$

which is compatible with the definition given before. In fact, we have not really extended our class of operators. For $\mathcal{K} \subset \mathbb{R}^n$ a compact subset, let $\Sigma^d_{\mathcal{K}}$ be the closed subspace of Σ^d consisting of all functions $r \in \Sigma^d$ with $\operatorname{supp} r \subset \mathcal{K} \times \mathbb{R}^n \times \mathcal{K}$. Moreover, let Σ^d_c be the union of the spaces $\Sigma^d_{\mathcal{K}}$, for $\mathcal{K} \subset \mathbb{R}^n$ compact.

Proposition 7.2.1. *Let $r \in \Sigma^d_{\mathcal{K}}$. Then*

$$p(x, \xi) = e^{\langle D_y, \partial_\xi \rangle} r(x, \xi, y)|_{y=x}$$

belongs to $S^d(\mathbb{R}^n)$*, and* $\Psi_r = \Psi_p$*. In particular,* Ψ_r *belongs to* $\Psi^d(\mathbb{R}^n)$ *and its full symbol is given by*

$$\sigma^d(\Psi_r) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} D_y^\alpha \partial_\xi^\alpha r(x, \xi, y)|_{y=x}.$$

To prepare for the proof we introduce the space

$$C^\infty_{\mathcal{K},c}(\mathbb{R}^{3n}) := \{\varphi \in C^\infty_c(\mathbb{R}^{3n}) \mid \operatorname{supp} \varphi \subset \mathcal{K} \times \mathbb{R}^n \times \mathcal{K}\}.$$

The following lemma is proved in the same fashion as Lemma 4.1.9.

Lemma 7.2.2. *Let* $d' > d$*. Then* $C^\infty_{\mathcal{K},c}(\mathbb{R}^{3n})$ *is dense in* $\Sigma^d_{\mathcal{K}}$ *for the* $\Sigma^{d'}$*-topology.*

Proof of Proposition 7.2.1 For $r \in \Sigma^d_{\mathcal{K}}$ and fixed $x \in \mathbb{R}^n$ the function $r_x : (y, \xi) \mapsto r(x, \xi, y)$ belongs to $S^d(\mathbb{R}^n)$. Moreover, $x \mapsto r_x$ is a smooth map $\mathbb{R}^n \to S^d_{\mathcal{K}}(\mathbb{R}^n)$, supported by $\mathcal{K}$ and it follows that

$$x \mapsto e^{\langle D_y, \partial_\xi \rangle}(r_x)$$

is a smooth map $\mathbb{R}^n \to S^d_{\mathcal{K}}(\mathbb{R}^n)$. The function $p = p(r)$ is given by the formula

$$p(r)(x, \xi) = e^{\langle D_y, \partial_\xi \rangle}(r_x)(x, \xi).$$

It is readily seen that $r \mapsto p(r)$ is a continuous linear map $\Sigma^d_{\mathcal{K}} \to S^d_{\mathcal{K}}(\mathbb{R}^n)$. Fix $f \in C^\infty_c(\mathbb{R}^n)$. Then both $r \mapsto \Psi_r(f)$ and $r \mapsto \Psi_{r(p)}(f)$ are continuous linear maps $\Sigma^d_{\mathcal{K}} \to C^\infty(\mathbb{R}^n)$, for every $d \in \mathbb{R}$. As $C^\infty_{\mathcal{K},c}(\mathbb{R}^{3n})$ is dense in $\Sigma^d_{\mathcal{K}}$ for the Σ^{d+1}-topology, it suffices to prove the identity $\Psi_r(f) = \Psi_{p(r)}(f)$ for every $p \in C^\infty_{\mathcal{K},c}(\mathbb{R}^{3n})$.

Thus, let $p \in C^\infty_{\mathcal{K},c}(\mathbb{R}^{3n})$ be fixed. Then

$$\begin{aligned} \Psi_r(f) &= \int\!\!\int e^{i\xi(x-y)} r(x, \xi, y) f(y)\, dy\, d\xi \\ &= \int\!\!\int e^{i\xi(x-y)} r_x(y, \xi) \int e^{iy\eta} \widehat{f}(\eta)\, d\eta\, dy\, d\xi \\ &= \int e^{i\eta x}\, p(x, \eta) \widehat{f}(\eta)\, d\eta, \end{aligned}$$

with

$$p(x, \eta) = \int\!\!\int e^{-i(\eta - \xi)(x-y)} r_x(y, \xi) dy d\xi.$$

Write $u(y, \xi) = e^{-i\xi y}$, then it follows that

$$\begin{aligned} p(x, \eta) &= (u * r_x)(x, \eta) \\ &= e^{\langle D_y, \partial_\xi \rangle} r_x(y, \eta)|_{y=x} \\ &= p(r)(x, \eta). \end{aligned}$$

All assertions now follow. □

Corollary 7.2.3. *Let $p \in S_c(\mathbb{R}^n)$. Then Ψ_p is properly supported if and only if there exists a $r \in \Sigma^d_{\mathcal{K}}$ such that*

$$\Psi_p = \Psi_r \qquad \text{on} \quad C_c^\infty(\mathbb{R}^n).$$

For any such r the d-th order principal symbol of p is represented by the symbol $(x, \xi) \mapsto r(x, \xi, x)$.

Proof The only if part as well as the statement about the principal symbol follows from Proposition 7.2.1 above. For the if part, assume that Ψ_p is properly supported. Let $\mathcal{K}$ be a compact subset of $\mathbb{R}^n$ such that $\operatorname{supp} p \subset \mathcal{K} \times \mathbb{R}^n$. Let K_p be the distribution kernel of Ψ_p. Then $\mathcal{K} \times \mathbb{R}^n \cap \operatorname{supp} K_p$ is compact, hence contained in a product of the form $K \times \mathcal{K}'$, with $\mathcal{K}' \subset \mathbb{R}^n$ compact. Let V be any open neighborhood of $\mathcal{K}'$ in $\mathbb{R}^n$. Then there exists a $\chi \in C_c^\infty(U)$ which is identically one on a neighborhood of $\mathcal{K}'$. It follows that the $\Psi_p = \Psi_p \circ M_\chi$ on $C_c^\infty(\mathbb{R}^n)$. This implies that for all $f \in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned} \Psi_p(f)(x) &= \Psi_p(\chi f(x)) \\ &= \int e^{i\xi x} p(x, \xi) \mathcal{F}(\chi f)(\xi)\, d\xi \\ &= \int e^{i\xi x} p(x, \xi) \int e^{-i\xi y} \chi(y) f(y)\, dy\, d\xi \\ &= \Psi_r(f)(x), \end{aligned}$$

with $r(x, \xi, y) = p(x, \xi)\chi(y)$. □

We now turn to the actual proof of the invariance. Given two points $x, y \in \mathbb{R}^n$, we denote the line segment from x to y by $[x, y]$. Thus,

$$[x, y] = \{x + t(y - x) \mid t \in [0, 1]\}.$$

We agree to write $\mathrm{M}_n(\mathbb{R})$ for the space of $n \times n$-matrices with real entries, and $\mathrm{GL}\,(n, \mathbb{R})$ for the subset of invertible matrices.

Lemma 7.2.4. *Let U be an open subset of $\mathbb{R}^n$.*

(a) *There exists an open neighborhood Ω of the diagonal $\operatorname{diag}(U)$ in $U \times U$ such that $(x, y) \in \Omega \Rightarrow [x, y] \subset U$.*
(b) *If Ω is any such neighborhood, and if $f : U \to \mathbb{R}^n$ is a smooth map, then there exists a smooth map $T : \Omega \to \mathrm{M}_n(\mathbb{R})$ such that*

$$f(y) - f(x) = T(x, y)(y - x), \qquad (x, y \in \Omega).$$

(c) *Any continuous map $T : \Omega \to \mathrm{M}_n(\mathbb{R})$ with property (b) satisfies $T(x, x) = df(x)$ for all $x \in U$.*

Proof The map $a : [0, 1] \times (U \times U) \to \mathbb{R}^n$ given by $a(t, x, y) = x + t(y - x)$ is continuous and maps $[0, 1] \times \operatorname{diag}(U)$ to U. The preimage $a^{-1}(U)$ of U under a is open in $[0, 1] \times U \times U$ and contains $[0, 1] \times \operatorname{diag}(U)$. By compactness of $[0, 1]$ it contains a subset of the form $[0, 1] \times \Omega$ with Ω an open neighborhood of $\operatorname{diag}(U)$ in $U \times U$. For $(x, y) \in \Omega$ we have $[x, y] = a([0, 1] \times \{(x, y)\}) \subset U$.

Let f be as in (b). Then

$$f(y) - f(x) = \int_0^1 \partial_t f(x + t(y - x))\, dt = T(x, y)(y - x),$$

with

$$T(x,y) = \int_0^1 df(x + t(y-x))\, dt.$$

Clearly, T is a smooth map $\Omega \to \mathrm{M}_n(\mathbb{R})$.

Let now $T : \Omega \to \mathrm{M}_n(\mathbb{R})$ be any continuous map satisfying property (b). Then for each $v \in \mathbb{R}^n$ we have

$$df(x)v = \lim_{t\to 0} t^{-1}[f(x+tv) - f(x)] = \lim_{t\to 0} T(x, x+tv)v = T(x,x)v.$$

□

If $\varphi : M \to N$ is a diffeomorphism of smooth manifolds, and $P : C_c^\infty(M) \to C^\infty(M)$ a continuous linear operator, then the push-forward of P by φ, denoted $\varphi_*(P)$, is defined to be the continuous linear operator $\varphi_*(P) : C_c^\infty(N) \to C^\infty(N)$ given by

$$\varphi_*(P)(f) = P(f \circ \varphi) \circ \varphi^{-1}.$$

Proposition 7.2.5. *Let $\varphi : U \to V$ be a smooth diffeomorphism between open subsets of $\mathbb{R}^n$, with inverse ψ. Let Ω be an open neighborhood of $\mathrm{diag}\,(V)$ in $V \times V$ and assume that $T : \Omega \to \mathrm{GL}\,(n, \mathbb{R})$ is a smooth map such that $\psi(y) - \psi(x) = T(x,y)(y-x)$ for all $(x,y) \in \Omega$ (such a pair Ω, T exists). Let $\mathcal{K}$ be a compact subset of U such that $\varphi(\mathcal{K}) \times \varphi(\mathcal{K}) \subset \Omega$.*

(a) *For every $d \in \mathbb{N}$ and $r \in \Sigma^d_{\mathcal{K}}$, the function $\varphi_*(r) : \mathbb{R}^{3n} \to \mathbb{C}$ defined by*

$$(x,\eta,y) \mapsto r(\psi(x), T(x,y)^{-1t}\eta, \psi(y))\, |\det d\psi(y)|\, |\det T(x,y)|^{-1}$$

on $\mathcal{K} \times \mathbb{R}^n \times \mathcal{K}$, and by $\varphi_*(r) = 0$ elsewhere, belongs to $\Sigma^d_{\varphi(\mathcal{K})}$.

(b) *For every $d \in \mathbb{N}$, the map $r \mapsto \varphi_*(r)$ is continuous linear $\Sigma^d_{\mathcal{K}} \to \Sigma^d_{\varphi(\mathcal{K})}$.*

(c) *For every $d \in \mathbb{N}$ and all $r \in \Sigma^d_{\mathcal{K}}$,*

$$\varphi_*(\Psi_r) = \Psi_{\varphi_*(r)}.$$

(d) *The principal symbol of $\varphi_*(\Psi_r)$ is represented by the symbol*

$$(x,\xi) \mapsto r(\psi(x), d\varphi(\psi(x))^t\xi, \psi(x)).$$

Proof The proof of (a) and (b) is straightforward, be it somewhat tedious. Fix $f \in C_c^\infty(V)$. Then the equality $\varphi_*(\Psi_r)(f) = \Psi_{\varphi_*(r)}(f)$ is equivalent to

$$\Psi_r(f \circ \varphi) \circ \psi = \Psi_{\varphi_*(r)}(f). \tag{7.10}$$

The map $r \mapsto \Psi_r(f\circ\varphi)\circ\psi$ is continuous linear $\Sigma^d_{\mathcal{K}} \to C^\infty_{\mathcal{K}}(U)$ and so is the map $r \mapsto \varphi^*(\Psi_{\varphi^*(r)})$, for every $d \in \mathbb{N}$. Now $C^\infty_{\mathcal{K},c}(\mathbb{R}^{3n})$ is dense in $\Sigma^d_{\mathcal{K}}$ for the Σ^{d+1}-topology. Therefore, it suffices to prove the equality (7.10) for all $r \in C^\infty_{\mathcal{K},c}(\mathbb{R}^{3n})$.

Fix $r \in C^\infty_{\mathcal{K},c}(\mathbb{R}^{3n})$ and let $x \in U$. Then

$$\begin{aligned}
\Psi(f\circ\varphi)(\psi(x)) &= \int\int e^{i\xi\psi(x)-i\xi z}\, r(\psi(x),\xi,z)\, f(\varphi(z))\, dz\, d\xi \\
&= \int\int e^{i\xi(\psi(x)-\psi(y))}\, r(\psi(x),\xi,\psi(y))\, f(y)\, |\det d\psi(y)|\, dy\, d\xi \\
&= \int\int e^{i[T(x,y)^t\xi](x-y)}\, r(\psi(x),\xi,\psi(y))\, f(y)\, |\det d\psi(y)|\, d\xi\, dy \\
&= \int\int e^{i\eta(x-y)}\varphi_*(r)(x,\eta,y)\, f(y)\, d\eta\, dy \\
&= \Psi_{\varphi_*(r)}(f)(x).
\end{aligned}$$

This establishes (c).

For (d) we note that $T(x,x) = d\psi(x)$ so that the principal symbol of $\Psi_{\varphi_*(r)}$ is given by

$$(x,\xi) \mapsto \varphi_*(r)(x,\xi,x) = \varphi(\psi(x), d\psi(x)^{-1t}\xi, \psi(x)).$$

Now use that $d\psi(x)^{-1} = dh(\psi(x))$. □

Theorem 7.2.6. *Let $\varphi : U \to V$ be a diffeomorphism of open subsets of $\mathbb{R}^n$. Then the following assertions are valid.*

(a) *For each $d \in \mathbb{R} \cup \{-\infty\}$ and all $P \in \Psi^d(U)$ the operator $\varphi_*(P)$ belongs to $\Psi^d(V)$.*

(b) *Let the principal symbol of P be represented by $p \in S^d(U)$. Then the principal symbol of $\varphi_*(P)$ is represented by*

$$\varphi_*(p) : (x,\xi) \mapsto p(\varphi^{-1}(x), d\varphi(\varphi^{-1}(x))^t\xi).$$

Proof Let $\psi : V \to U$ be the inverse to φ and let Ω, T be as in the statement of Proposition 7.2.5. First, assume that $d = -\infty$ and let $P \in \Psi^{-\infty}(U)$. Then P is an integral operator T_K with integral kernel $K \in C^\infty(U\times U)$. Let $f \in C^\infty_c(V)$ and $x \in V$, then by substitution of variables

$$\begin{aligned}
\varphi_*(P)(f)(x) &= \int_U K(\psi(x),z) f(\varphi(z))\, dz \\
&= \int_V K(\psi(x),\psi(y))|\det d\psi(y)|\, f(y)dy
\end{aligned}$$

from which we see that $\varphi_*(P)$ is the integral transformation with smooth integral kernel $\tilde{K} \in C^\infty(V\times V)$ given by

$$\widetilde{K}(x,y) = K(\psi(x),\psi(y))\, |\det d\psi(y)|.$$

We now assume that $d \in \mathbb{R}$ and that $P \in \Psi^d(U)$. Then by Lemma 6.1.6 we may write $P = \Psi_p + T$, with $T \in \Psi^{-\infty}(U)$ and $p \in S^d(U)$ such that Ψ_p is properly supported and has distribution kernel K_p supported inside $\Omega_U := (\varphi\times\varphi)^{-1}(\Omega)$. Since $\varphi_*(T)$ is a smoothing operator by the first part of the proof, it suffices to show that $\varphi_*(\Psi_p) \in \Psi^d(V)$. For this we proceed as follows.

Let $\{\chi_j\}$ be a partition of unity on U such that $\operatorname{supp}\chi_j \times \operatorname{supp}\chi_j \subset \Omega_U$ for every j. For each j we select an open neighborhood U_j of $\operatorname{supp}\chi_j$ with $U_j \times U_j \subset \Omega_U$ and a function $\chi'_j \in C^\infty_c(U_j)$ which is identically 1 on $\operatorname{supp}\chi_j$.

Then $P_j = P_j \circ M_{\chi_j'} + T_j$, with T_j a smoothing operator supported in $\operatorname{supp} \chi_j \times U_j$ and with $P_j \circ M_{\xi_j} = \Psi_{r_j}$, where $r_j(x,\xi,y) = \chi_j(x)p(x,\xi)\chi_j'(y)$.

We now observe that $\mathcal{K}_j = \operatorname{supp} \chi_j'$ is compact and that $\varphi(\mathcal{K}_j) \times \varphi(\mathcal{K}_j) \subset \Omega$. Morover, $r_j \in \Sigma^d_{\mathcal{K}_j}$. It follows that $\varphi_*(\Psi_{r_j}) = \Psi_{\varphi_*(r_j)}$. The supports of the kernels of the operators T_j form a locally finite set, so that $T = \sum_j T_j$ is a smoothing operator. Hence, so is $\varphi_*(T)$. The sum $\sum_j P_{r_j}$ is locally finite, and therefore so is $Q = \sum_j \varphi_*(P_{r_j})$. Hence $Q \in \Psi^d(V)$. We conclude that $\varphi_*(P) = Q + \varphi_*(T)$ is a pseudo-differential operator on V of order d. Its principal symbol equals the principal symbol of Q, which is represented by the symbol $q \in S^d(V)$ given by

$$\begin{aligned} q(x,\xi) &= \sum_j \varphi_*(r_j)(x,\xi,x) \\ &= \sum_j r_j(\psi(x), d\varphi(\psi(x))^t\xi, \psi(x)) \\ &= \sum_j p_j(\psi(x), d\varphi(\psi(x))^t\xi) \\ &= p(\psi(x), d\varphi(\psi(x))^t\xi) = \varphi_*(p)(x,\xi). \end{aligned}$$

□

7.3. Pseudo-differential operators on a manifold, scalar case

In view of the results of the previous section we can now extend the notion of a pseudo-differential operator to a smooth manifold M of dimension n.

Definition 7.3.1. Let $d \in \mathbb{R} \cup \{-\infty\}$. A pseudo-differential operator P of order d on M is a continuous linear operator $C_c^\infty(M) \to C^\infty(M)$ given by a distribution kernel $K_P \in \mathcal{D}'(M \times M, \mathbb{C}_M \boxtimes D_M)$ such that the following conditions are fulfilled.

(a) The kernel K_P is smooth outside the diagonal of $M \times M$.
(b) For each $a \in M$ there exists a chart (U_κ, κ) containing a such that the operator $P_\kappa : C_c^\infty(\kappa(U)) \to C^\infty(\kappa(U))$ given by

$$P_\kappa(f)(\kappa(x)) = P(f \circ \kappa)(x), \qquad (x \in U)$$

belongs to $\Psi^d(\kappa(U))$.

Remark 7.3.2. Of course, by the Schwartz kernel theorem, each continuous linear operator $P : C_c^\infty(M) \to C^\infty(M)$ is in particular continuous linear $C_c^\infty(M) \to \mathcal{D}'(M)$, hence given by a distribution kernel $K_P \in \mathcal{D}'(M \times M, D_M \otimes \mathbb{C}_M)$. In the above formulation, the existence of the kernel is demanded in order not to rely on the kernel theorem.

Condition (a) asserts that K_P has singular support contained in the diagonal of M, whereas condition (b) stipulates that the singularity along the diagonal is of the same type as that of the kernel of a pseudo-differential operator on $\mathbb{R}^n$.

If $\varphi : M \to N$ is a diffeomorphism then by the nature of the definition the map $\varphi_* : \Psi^d(M) \to \Psi^d(N)$ is readily seen to be a linear isomorphism. Before proceeding we will show that the definition of pseudo-differential operator coincides with the old one in case M is an open subset of $\mathbb{R}^n$.

Lemma 7.3.3. *Let M be an open subset of $\mathbb{R}^n$ and let $P : C_c^\infty(M) \to C^\infty(M)$ be a continuous linear operator with distribution kernel K_P. Then the following statements are equivalent.*

(1) *Conditions (a) and (b) of the above definition are fulfilled.*
(2) *P is a pseudo-differential operator in the sense of Definition 5.2.4.*

Proof Clearly (2) implies (1), since M can be taken as the coordinate patch. We assume (1) and will prove (2).

We first assume that $d = -\infty$. Then requirements (a) and (b) of Definition 7.3.1 guarantee that K_P is smooth on all of M, hence that P is an integral operator with smooth kernel $K_P \in \Gamma^\infty(M \times M, \mathbb{C}_M \boxtimes D_M)$. Thus, P is an operator in $\Psi^{-\infty}$ in the sense of Definition 7.3.1.

We now assume that $d \in \mathbb{R}$. By Lemma 6.1.6 each $a \in M$ has an open neighborhood U_a in M such that the operator $P_a : C_c^\infty(U_a) \to C^\infty(U_a)$ given by $f \mapsto (Pf)|_{U_a}$ may be written as $P_a = \Psi_{p_a} + T_a$, with p_a a symbol in $\Psi^d(U_a)$ and with $T_a \in \Psi^{-\infty}(U_a)$ a smoothing operator.

By paracompactness of M there exists a partition of unity $\{\chi_j\}$ on M such that for each j the support of χ_j is contained in some set U_{a_j} as above. We put $P_j = P_{a_j}$, $p_j = p_{a_j}$ and $T_j = T_{a_j}$. Then $P_j = \Psi_{p_j} + T_j$. For each j we choose a $\chi_j' \in C_c^\infty(U_j)$ such that $\chi_j' = 1$ on an open neighborhood of $\operatorname{supp} \chi_j$. Then $\chi_j(1 - \chi_j') = 0$ so $T_j' := M_{\chi_j} P M_{(1-\chi_j')}$ has kernel $[\chi_j \otimes (1 - \chi_j')]K_P$ which is smooth. The supports of these kernels form a locally finite collection, so that $T' = \sum_j T_j'$ is a smoothing operator.

Moreover, we may write

$$M_{\chi_j} \circ P \circ M_{\chi_j'} = \Psi_{q_j} + M_{\chi_j} \circ T_j \circ M_{\chi_j'},$$

with $q_j \in S^d(M)$ supported by $\operatorname{supp} \psi_j$. It follows that $q = \sum_j q_j$ is a locally finite sum and defines an element of $S^d(M)$. Moreover, the smooth kernels of the operators $M_{\chi_j} \circ T_j \circ M_{\chi_j'}$ are locally finitely supported in $M \times M$ so that the operators sum up to a smoothing operator T. For $f \in C_c^\infty(M)$ we now have that

$$Pf = \sum_j \chi_j P(\chi_j' f) + T'(f) = \Psi_q(f) + T(f) + T'(f)$$

and (2) follows. □

If M is a smooth manifold, $P : C_c^\infty(M) \to C^\infty(M)$ a continuous linear operator, and $U \subset M$ an open subset, we agree to write P_U for the operator $C_c^\infty(U) \to C^\infty(U)$ given by

$$P_U f = (Pf)|_U, \qquad (f \in C_c^\infty(U)).$$

The following results are now easy consequences of the definitions.

Exercise 7.3.4. If $K \in \mathcal{D}'(M \times M)$ is the distribution kernel of P, then the restriction $K|_{U\times U}$ is the distribution kernel of P_U.

Exercise 7.3.5. Let M be a smooth manifold, and $U \subset M$ an open subset. Then for each $P \in \Psi^d(M)$, the operator P_U belongs to $\Psi^d(U)$.

In the sequel we shall make frequent use of the following results.

Lemma 7.3.6. *Let $P \in \Psi^d(M)$ and let $\chi \in C^\infty(M)$.*

(a) *Let $\psi \in C^\infty(M)$ be such that* $\operatorname{supp}\psi \cap \operatorname{supp}\chi = \emptyset$. *Then*
$$M_\chi \circ P \circ M_\psi \in \Psi^{-\infty}(M).$$

(b) *Let $\chi' \in C_c^\infty(M)$ be such that $\chi' = 1$ on an open neighborhood of $\operatorname{supp}\chi$. Then*
$$M_\chi \circ P - M_\chi \circ P \circ M_{\chi'} \in \Psi^{-\infty}(M).$$
Likewise, $P \circ M_\chi - M_{\chi'} \circ P \circ M_\chi \in \Psi^{-\infty}(M)$.

(c) *Let $\{P_j\}$ is a collection of operators from $\Psi^d(M)$ such that the supports* $\operatorname{supp} K_{P_j}$ *of the distribution kernels form a locally finite collection of subsets of $M \times M$. Then*
$$\sum_j P_j \in \Psi^d(M).$$

Proof Let K_P denote the distribution kernel of P. The distribution kernel K' of the operator $M_\chi \circ P \circ M_\psi$ equals $K' = (\chi \otimes \psi)K_P$. Since $\chi \otimes \psi = 0$ on an open neighborhood of the diagonal in $M \times M$, the kernel K' is smooth. Hence (a).

We turn to (b). There exists a function $\varphi \in C^\infty(M)$ such that $\varphi = 1$ on an open neighborhood of $\operatorname{supp}\chi$ and such that $\chi' = 1$ on an open neighborhood of $\operatorname{supp}\varphi$. It follows that $(1-\chi)$ and φ have disjoint supports. Hence
$$M_\chi \circ P - M_\psi \circ M_\chi = M_\psi \circ (M_\varphi \circ P \circ M_{1-\chi}) \in \Psi^{-\infty}(M).$$
The second statement of (b) is proved in a similar way.

It remains to prove (c). Let $Q = \sum_j P_j$. Then Q is a well defined continuous linear operator $C_c^\infty(M) \to C^\infty(M)$ with distribution kernel $K_Q = \sum K_{P_j}$. On the complement of the diagonal in $M \times M$ the kernel K_Q is a locally finite sum of smooth functions, hence a smooth function. Let $a \in M$. There exists a coordinate patch $U \ni a$ whose closure in M is compact. The collection J of indices j for which $\operatorname{supp} K_{P_j} \cap (U \times U) \neq \emptyset$ is finite. It follows that the kernel of Q_U equals
$$K_Q|_{U\times U} = \sum_{j\in J} K_{P_j}|_{U\times U} = \sum_{j\in J} K_{P_{jU}}.$$
Hence, Q_U equals the finite sum $\sum_{j\in J} P_{jU}$ and belongs to $\Psi^d(U)$. It follows that $Q \in \Psi^d(M)$. □

Exercise 7.3.7. Let M be a smooth manifold, and $P : C_c^\infty(M) \to C^\infty(M)$ a continuous linear operator with a distribution kernel that it smooth outside the diagonal in $M \times M$. Let $\{U_j\}$ be an open covering of M. If $P_{U_j} \in \Psi^d(U_j)$ for each j, then $P \in \Psi^d(M)$.

The following result indicates that pseudo-differential operators modulo smoothing operators behave like sections of a sheaf.

Lemma 7.3.8. *Let $\{U_j\}$ be an open covering of the manifold M.*

(a) *Let $P, Q \in \Psi^d(M)$ be such that $P_{U_j} = Q_{U_j}$ for all j. Then $P - Q \in \Psi^{-\infty}(M)$.*

(b) *Assume that for each j a pseudo-differential operator $P_j \in \Psi^d(U_j)$ is given. Assume furthermore that $P_i = P_j$ on $C_c^\infty(U_i \cap U_j)$ for all indices i, j with $U_i \cap U_j \neq \emptyset$. Then there exist a $P \in \Psi^d(M)$ such that $P_{U_j} - P_j \in \Psi^{-\infty}(U_j)$ for all j. The operator P is uniquely determined modulo $\Psi^{-\infty}(M)$.*

Proof Let K_P and K_Q denote the distribution kernels of P and Q, respectively. Then $K_P|_{U_j \times U_j}$ and $K_Q|_{U_j \times U_j}$ are the distribution kernels of P_{U_j} and Q_{U_j}, respectively. It follows that $K_{P-Q} = K_P - K_Q$ is smooth on each of the sets $U_j \times U_j$, hence on the diagonal of $M \times M$. As K_{P-Q} is already smooth outside the diagonal, it follows that K_{P-Q} is smooth on $M \times M$. Hence, $P - Q \in \Psi^{-\infty}(M)$.

We turn to (b). Let $\Omega = \cup_j U_j \times U_j$. Then Ω is an open neighborhood of the diagonal in $M \times M$. Let $K_j \in \mathcal{D}'(U_j \times U_j)$ denote the distribution kernel of P_j. Let $U_{ij} := U_i \cap U_j \neq \emptyset$. Then from the assumption it follows that $K_i = K_j$ on $(U_i \times U_i) \cap (U_j \times U_j) = U_{ij} \times U_{ij}$. From the gluing property of the sheaf $\mathcal{D}'$ on Ω it follows that there exists a $K \in \mathcal{D}'(\Omega)$ such that $K = K_j$ on $U_j \times U_j$ for all j.

We will now use a cut off function to extend K to all of $M \times M$, leaving it unchanged on an open neighborhood of the diagonal. Let $\{\chi_\nu\}$ be a partition of unity of M, subordinate to the covering $\{U_j\}$. For each ν we select $j(\nu)$ such that $\operatorname{supp} \chi_\nu \subset U_{j(\nu)}$ and we fix a function $\chi'_\nu \in C_c^\infty(U_{j(\nu)})$ which is identically 1 on an open neighborhood of $\operatorname{supp} \chi_\nu$. The functions $\chi_\nu \otimes \chi'_\nu$ form a locally finitely supported family of functions in $C_c^\infty(\Omega)$. Put

$$\psi := \sum_\nu \chi_\nu \otimes \chi'_\nu.$$

Let $x \in M$ and let N_x be the finite collection of indices ν with $x \in \operatorname{supp} \psi_\nu$. Then the functions χ'_ν, for $\nu \in N_x$ are all 1 on a common open neighborhood V_x of x in M. Moreover, $\sum_{\nu \in N_x} \chi_\nu$ equals 1 on an open neighborhood U_x of x in M. It follows that $\psi = 1$ on $U_x \times V_x$. Hence, $\psi = 1$ on an open neighborhood of the diagonal in $M \times M$. Put $P = \sum_\nu M_{\chi_\nu} \circ P_{j(\nu)} \circ M_{\chi'_\nu}$. Then P is a pseudo-differential operator on M with kernel equal to

$$K_P = \sum_\nu (\chi_\nu \otimes \chi'_\nu) K_{P_j} = \psi K.$$

For each $j \in U$ we have that P_{U_j} has kernel

$$K_P|_{U_j \times U_j} = \psi K|_{U_j \times U_j} = \psi K_j.$$

It follows that $K_P - K_{P_j}$ is smooth on $U_j \times U_j$, hence $P_{U_j} - P_j \in \Psi^{-\infty}(U_j)$. The uniqueness statement follows from (a). □

In fact, with a bit more effort it can be shown that $\Psi^d/\Psi^{-\infty}$ defines a sheaf of vector spaces on M. More precisely, for two open subsets $U \subset V$ of M the map $P \mapsto P_U$, $\Psi^d(V) \to \Psi^d(U)$ induces a restriction map $\Psi^d(V)/\Psi^{-\infty}(V) \to \Psi^d(U)/\Psi^{-\infty}(U)$ which we claim to define a sheaf. The following exercise prepares for the proof of this fact.

Exercise 7.3.9. Let Ω be smooth manifold, and let $\{\Omega_j\}_{j\in J}$ be an open cover of Ω. Assume that for each pair of indices (i,j) with $\Omega_{ij} := \Omega_i \cap \Omega_j \neq \emptyset$ a smooth function $g_{ij} \in C^\infty(\Omega_{ij})$ is given such that

$$g_{ij} + g_{jk} + g_{kj} = 0 \qquad \text{on} \quad \Omega_{ijk} := \Omega_i \cap \Omega_j \cap \Omega_k$$

for all i, j, k with $\Omega_{ijk} \neq \emptyset$. Show that there exist functions $g_j \in C^\infty(U_j)$ such that $g_i - g_j = g_{ij}$ for all i, j. Hint: select a partition of unity $\{\psi_\alpha\}_{\alpha\in\mathcal{A}}$ on Ω which is subordinate to the covering $\{\Omega_j\}$. Thus, a map $j : \mathcal{A} \to J$ is given such that $\operatorname{supp}\psi_\alpha \subset U_{j(\alpha)}$. Now consider $g_j := \sum_\alpha \psi_\alpha g_{jj(\alpha)}$.

Exercise 7.3.10.

(a) Show that with the restriction maps defined above the assignment $U \mapsto \Psi^d(U)/\Psi^{-\infty}(U)$ defines a presheaf on M.
(b) Show that $U \mapsto \Psi^d(U)/\Psi^{-\infty}(U)$ satisfies the restriction properety of a sheaf.
(c) Use the previous exercise combined with the arguments of the proof of Lemma 7.3.8 to show that $U \mapsto \Psi^d(U)/\Psi^{-\infty}(U)$ has the gluing property.

7.4. The principal symbol on a manifold

We will now extend the definition of principal symbol of a pseudo-differential operator to the setting of a manifold M. Let $\pi : T^*M \to M$ denote the cotangent bundle of M.

First we consider a coordinate patch U of M. Let $\kappa : U \to U'$ be a diffeomorphism onto an open subset U' of $\mathbb{R}^n$. We consider the induced diffeomorphism $T^*\kappa : T^*U \to U \times (\mathbb{R}^n)^* \simeq U' \times \mathbb{R}^n$ given by

$$T^*\kappa(\xi_x) = (\kappa(x)\,,\, \xi_x \circ T_x\kappa^{-1}), \qquad (x \in U,\ \xi_x \in T_x^*M).$$

Pull-back by the inverse of $T^*\kappa$ induces a linear isomorphism

$$\kappa_* : C^\infty(T^*U) \to C^\infty(U' \times \mathbb{R}^n).$$

For $d \in \mathbb{R} \cup \{-\infty\}$ we define

$$S^d(U) = \{p \in C^\infty(T^*U) \mid \kappa_*(p) \in S^d(U')\}.$$

It follows from Lemma 5.1.6 that this space is independent of the choice of κ.

Definition 7.4.1. We define $S^d(M)$ to be the space of smooth functions $p : T^*M \to \mathbb{C}$ with the property that for each $a \in M$ there exists a coordinate patch U containing a such that $p|_{T^*U} \in S^d(U)$.

If $\varphi : M \to N$ is a diffeomorphism of manifolds, then it follows from the above definition that the induced linear isomorphism $\varphi_* : C^\infty(T^*M) \to C^\infty(T^*N)$ maps $S^d(M)$ onto $S^d(N)$. Moreover, if $\Omega \subset M$ is an open subset then the restriction map $C^\infty(T^*M) \to C^\infty(T^*\Omega)$, $p \mapsto p|_{T^*\Omega}$ maps $S^d(M)$ to $S^d(\Omega)$.

If Ω is an open subset of M, and $\mathcal{K} \subset \Omega$ a compact subset, we define $S^d_{\mathcal{K}}(\Omega)$ to be the space of $p \in S^d(\Omega)$ with $\operatorname{supp} p \subset \pi^{-1}(\mathcal{K})$. Finally, we define $S^d_{\mathcal{K}}(\Omega)$ to be the union of the spaces $S^d_{\mathcal{K}}(\Omega)$, for $\mathcal{K} \subset \Omega$ compact. We note that elements of $S^d_c(\Omega)$ may be viewed as elements of $S^d_c(M)$ by defining them to be zero on $T^*M \setminus T^*\Omega$.

If (U, κ) is a chart of M, then the map $\kappa_* : C^\infty(T^*U) \to C^\infty(\kappa(U) \times \mathbb{R}^n)$ induces a linear bijection

$$\kappa_* : \; S^d(U)/S^{d-1}(U) \xrightarrow{\simeq} S^d(\kappa(U))/S^{d-1}(\kappa(U)).$$

The following definition is justified by Theorem 7.2.6 (b).

Definition 7.4.2. Let U be a coordinate patch of M. We define the map $\sigma^d_U : \Psi(U) \to S^d(U)/S^{d-1}(U)$ by

$$\kappa_* \sigma^d_U(P) = \sigma^d_{\kappa(U)}(\kappa_* P),$$

for κ a coordinate system on U. (Here $\sigma^d_{\kappa(U)}$ denotes the principal symbol map of $S^d(\kappa(U))$, defined in Definition 5.4.7.)

If $U \subset M$ is open, then the restriction map $p \mapsto p|_{T^*\Omega}$ induces a map $S^d(M)/S^{d-1}(M) \to S^d(\Omega)/S^{d-1}(\Omega)$, which we shall denote by $\sigma \mapsto \sigma_\Omega$. The support of an element $\sigma \in S^d(M)/S^{d-1}(M)$ is defined to be the complement of the largest open subset $\Omega \subset M$ such that $\sigma_\Omega = 0$.

Lemma 7.4.3. *Let $d \in \mathbb{R}$. There exists a unique linear map $\sigma^d : \Psi^d(M) \to S^d(M)/S^{d-1}(M)$ such that for every coordinate patch $U \subset M$ we have*

$$\sigma^d(P)_U = \sigma^d_U(P_U).$$

Proof Uniqueness of the map follows from the fact that an element $\sigma \in S^d(M)/S^{d-1}(M)$ is completely determined by its restrictions to the sets of an open covering of M. We will establish existence by using a partition of unity.

Let χ_j be a partition of unity on M such that for each j, the support $\mathcal{K}_j$ of χ_j is contained in a coordinate patch U_j. We define

$$\tilde{\sigma}^d : P \mapsto \sum_j \sigma^d_{U_j}((M_{\psi_j} \circ P)_{U_j})$$

Here we note that the term corresponding to j may be viewed as an element of $S^d(M)/S^{d-1}(M)$ with support contained in $\mathcal{K}_j$. In particular, the sum is locally finite and defines an element of $S^d(M)/S^{d-1}(M)$. Let $P \in \Psi^d(M)$ and let (U, κ) be a chart of M. We will show that $\tilde{\sigma}^d(P)_U = \sigma^d_U(P_U)$.

Let $\psi \in C^\infty_c(U)$. Then it suffices to show that $\psi\tilde{\sigma}^d(P)_U = \psi\sigma^d_U(P_U)$. From the definition given above it follows that $\psi\tilde{\sigma}^d(P)_U = \tilde{\sigma}^d(\psi P)_U$. Let J be the finite collection of indices for which $\operatorname{supp}\psi_j \cap \operatorname{supp}\psi \neq \emptyset$. For each $j \in J$ we fix a function $\chi_j \in C^\infty_c(U_j \cap U)$ which equals one on $\operatorname{supp}\psi\psi_j$. Then it follows that

$$\begin{aligned}\psi\tilde{\sigma}^d(P)_U &= \sum_{j\in J}\sigma^d_{U_j}(M_{\psi\psi_j}\circ P_{U_j})\\ &= \sum_{j\in J}\sigma^d_{U_j}(M_{\psi\psi_j}\circ P_{U_j}\circ M_{\chi_j})\\ &= \sigma^d_U(\sum_{j\in J}M_{\psi\psi_j}\circ P_{U_j}\circ M_{\chi_j}).\end{aligned}$$

Now modulo smoothing operators from $\Psi^{-\infty}(U)$ we have

$$\sum_{j\in J}M_{\psi\psi_j}\circ P_{U_j}\circ M_{\chi_j} = \sum_{j\in J}M_\psi M_{\psi_j}\circ P_U = M_\psi\circ P_U.$$

Hence,

$$\psi\tilde{\sigma}^d(P)_U = \sigma^d_U(M_\psi\circ P_U) = \psi\sigma^d_U(P_U).$$

The result follows. □

Theorem 7.4.4. *The principal symbol map $\sigma^d : \Psi^d(M)\to S^d(M)/S^{d-1}(M)$ induces a linear isomorphism*

$$\sigma^d:\ \Psi^d(M)/\Psi^{d-1}(M)\xrightarrow{\simeq} S^d(M)/S^{d-1}(M).$$

Proof We will first establish surjectivity. Let $p\in S^d(M)$. Let χ_j be a partition of unity on M such that for each j the support of χ_j is contained in a coordinate patch U_j. For each j there exists an operator $P_j\in\Psi^d(U_j)$ such that $\sigma^d_{U_j}(P_j) = p_{U_j} + S^{d-1}(U_j)$. For each j we select a function $\chi'_j\in C^\infty_c(U_j)$ such that $\chi'_j = 1$ on an open neighborhood of $\operatorname{supp}\chi_j$. Then

$$P = \sum_j M_{\chi_j}\circ P_j\circ M_{\chi'_j}$$

defines an element of $\Psi^d(M)$. We will show that $\sigma^d(P) = p + S^{d-1}(M)$. Let $\psi\in C^\infty_c(M)$ have support inside a coordinate patch U. Then it suffices to show that $\psi\sigma^d(P) = \psi p + S^{d-1}(M)$. Let $\psi'\in C^\infty_c(U)$ be identically one on a neighborhood of $\operatorname{supp}\psi$. Then

$$\begin{aligned}\psi\sigma^d(P) &= \psi\sigma^d_U(P_U) = \sigma^d_U(M_\psi\circ P_U\circ M_{\psi'})\\ &= \sum_j\sigma^d_U(M_{\psi\chi_j}\circ P_U\circ M_{\psi'})\\ &= \sum_j\sigma^d_U(M_{\psi\chi_j}\circ P_U\circ M_{\psi'\chi'_j})\\ &= \sum_j\sigma^d_{U_j}(M_{\psi\chi_j}\circ P_{U_j}\circ M_{\psi'\chi'_j})\\ &= \sum_j\psi\chi_j p + S^{d-1}(M) = \psi p + S^{d-1}(M).\end{aligned}$$

It remains to be established that $\sigma^d : \Psi^d(M)\to S^d(M)/S^{d-1}(M)$ has kernel equal to $\Psi^{d-1}(M)$. Clearly, the latter space is contained in the kernel. Conversely, let $P\in\Psi^d(M)$ and assume that $\sigma^d(P) = 0$. Then it follows that

$\sigma_U^d(P_U) = 0$ for each coordinate patch U. In view of Corollary 5.4.8 it follows that $P_U \in \Psi^{d-1}(U)$ for each coordinate patch U. This in turn implies that $P \in \Psi^{d-1}(M)$ by Definition 7.3.1. □

7.5. Symbol calculus on a manifold

In this section we will discuss results concerning the principal symbols of adjoints and products of pseudo-differential operators on the manifold M. The proofs of these results will consist of reduction to the analogous local results.

Our first goal is to understand the behavior of the principal symbol under left and right composition with multiplication by smooth functions. Let $\chi \in C^\infty(M)$ and $p \in S^d(M)$, then the function $\pi^*(\chi)p : T_x^*M \ni \xi_x \mapsto \chi(x)p(\xi_x)$ belongs to $S^d(M)$ again. Indeed, this is an easy consequence of the analogous property in the local case. Accordingly, the space $S^d(M)$ becomes a $C^\infty(M)$-module and we write χp for $\pi^*(\chi)p$. As $S^{d-1}(M)$ is a submodule, it follows that the quotient $S^d(M)/S^{d-1}(M)$ is a $C^\infty(M)$-module in a natural way.

Lemma 7.5.1. *Let $P \in \Psi^d(M)$ and $\chi, \psi \in C^\infty(M)$. Then $M_\chi \circ P \circ M_\psi \in \Psi^d(M)$ and*

$$\sigma^d(M_\chi \circ P \circ M_\psi) = \chi\psi\sigma^d(P).$$

Proof The first assertion is a straightforward consequence of the definition and the fact that

$$(M_\chi \circ P \circ M_\psi)_U = M_{\chi|U} \circ P_U \circ M_{\psi|U}$$

for every open subset $U \subset M$.

Let now $U \subset M$ be a coordinate patch. Then

$$\begin{aligned} \sigma^d(M_\chi \circ P \circ M_\psi)_U &= \sigma_U^d((M_\chi \circ P \circ M_\psi)_U) \\ &= \sigma_U^d(M_{\chi|U} \circ P_U \circ M_{\psi|U}) \\ &= (\chi\psi)|_U \sigma_U^d(P_U) = [\chi\psi\sigma^d(P)]_U. \end{aligned}$$

The result follows. □

Our next goal is to understand the symbol of the adjoint of a pseudo-differential operator relative to a smooth positive density dm on M. We assume such a density to be fixed for the rest of this section.

Given two smooth functions $f \in C_c^\infty(M)$ and $g \in C^\infty(M)$, we agree to write

$$(\, f \,,\, g \,) = \int_M f(x)g(x)\, dm(x).$$

The above pairing has a unique extension to a continuous bilinear pairing $C_c^\infty(M) \times \mathcal{D}'(M) \to \mathbb{C}$.

Lemma 7.5.2. *Let $P \in \Psi^d(M)$. Then there exists a unique continuous linear operator $R : \mathcal{E}'(M) \to \mathcal{D}'(M)$ such that*

$$(\, Pf \,,\, g \,) = (\, f \,,\, Rg \,), \qquad \text{for all} \quad f, g \in C_c^\infty(M).$$

The operator R belongs to $\Psi^d(M)$ and has principal symbol given by

$$\sigma^d(R)(\xi_x) = \sigma^d(P)(-\xi_x), \qquad (x \in M,\ \xi_x \in T_x^*M).$$

Proof The operator $\tilde{P} : f \mapsto (Pf)\, dm$ is continuous linear from $C_c^\infty(M)$ to $\Gamma^\infty(D_M)$, where D_M denotes the density bundle on M. It follows that the transposed $\tilde{P}^t : \mathcal{E}'(M) \to \mathcal{D}'(M, D_M)$ is continuous linear, hence of the form $u \mapsto R(u)\, dm$ with R a uniquely determined continuous linear operator $\mathcal{E}'(M) \to \mathcal{D}'(M)$. Clearly the operator R satisfies

$$(Pf \,,\, g) = \langle (Pf)\, dm, g\rangle = \langle f, \tilde{P}^t g\rangle = \langle f, (Rg) dm\rangle = (f \,,\, Rg).$$

Moreover, the operator R is uniquely determined by this property. It remains to be shown that $R \in \Psi^d(M)$. For this we first note that R has the distribution kernel $K_R \in \mathcal{D}'(M \times M, \mathbb{C}_M \boxtimes D_M)$ given by

$$K_R(x, y)(dm(x) \otimes 1) = K_P(y, x)(dm(y) \otimes 1),$$

with $K_P \in \mathcal{D}'(M \times M, \mathbb{C}_M \boxtimes D_M)$ the distribution kernel of P. Of course this equality should be interpreted in distribution sense. From the smoothness of K_P outside the diagonal, it follows that K_R is smooth outside the diagonal. Let now $U \subset M$ be a coordinate patch, with associated coordinate system $\kappa : U \to U' \subset \mathbb{R}^n$. Then there exists a unique strictly positive function $J \in C^\infty(U')$ such that $\kappa_*(dm) = J\, dx$. Let $f, g \in C_c^\infty(U)$, then it follows that

$$\begin{aligned}
\langle f dm, Rg\rangle &= (Pf \,,\, g) \\
&= \langle \kappa_*(Pf), \kappa_* g\, \kappa_*(dm)\rangle \\
&= \langle \kappa_*(P_U)\kappa_* f, J\kappa_* g\, dx\rangle \\
&= \langle \kappa_* f, \kappa_*(P_U)^t[J\kappa_* g]\, dx\rangle \\
&= \langle \kappa_*(f\, dm), J^{-1}\kappa_*(P_U)^t[J\kappa_*(g)]\rangle.
\end{aligned}$$

From this we conclude that

$$\kappa_*(R_U) = M_{J^{-1}} \circ \kappa_*(P_U)^t \circ M_J.$$

In view of Lemma 5.4.2 and Proposition 6.2.2 it now follows that $\kappa_*(R_U) \in \Psi^d(U')$ with principal symbol given by

$$\begin{aligned}
\sigma^d(\kappa_*(R_U))(x, \xi) &= \sigma^d(M_{J^{-1}} \circ \kappa_*(P_U)^t \circ M_J)(x, \xi) \\
&= J(x)^{-1} J(x) \sigma^d(\kappa_*(P_U))(x, -\xi) = \sigma^d(\kappa_*(P_U))(x, -\xi).
\end{aligned}$$

We conclude that $R_U \in \Psi^d(U)$ with principal symbol given by

$$\begin{aligned}
\sigma_U^d(R_U)([T_m\kappa]^t\xi) &= \sigma^d(\kappa_*(R_U))(\kappa(m), \xi) \\
&= \sigma^d(\kappa_*(P_U))(\kappa(m), -\xi) = \sigma_U^d(P_U)(-[T_m\kappa]^t\xi),
\end{aligned}$$

for all $m \in U$, $\xi \in \mathbb{R}^n$. □

We will end this section by discussing the principal symbol of the composition of two properly supported pseudo-differential operators.

From the similar local property of principal symbols, it follows that multiplication induces a bilinear map

$$S^d(M)/S^{d-1}(M) \times S^e(M)/S^{e-1}(M) \to S^{d+e}(M)/S^{d+e-1}(M)$$

for all $d, e \in \mathbb{R} \cup \{-\infty\}$.

Theorem 7.5.3. *Let $P \in \Psi^d(M)$ and $Q \in \Psi^e(M)$ be properly supported. Then $P \circ Q$ belongs to $\Psi^{d+e}(M)$ and has principal symbol given by*

$$\sigma^{d+e}(P \circ Q) = \sigma^d(P)\, \sigma^e(Q). \tag{7.11}$$

Proof We first consider the case that $e = -\infty$, i.e., Q is a smoothing operator. We will show that in this case $P \circ Q$ is smoothing. Let $K \in \mathcal{D}'(M \times M, \mathbb{C}_M \boxtimes D_M)$ be the distribution kernel of $P \circ Q$. Let a be a point of M, U a coordinate patch containing a and $\psi \in C_c^\infty(U)$ a function with $\psi(a) \neq 0$. Then it suffices to show that $(1 \otimes \psi)K$ is smooth. The latter requirement is equivalent to the requirement that $P \circ Q \circ M_\psi$ be smoothing. Now this can be seen as follows. Since Q is proper, there exists a compact subset $\mathcal{K} \subset M$ such that the $\mathrm{supp}\,(1 \otimes \psi)K \subset \mathcal{K} \times \mathrm{supp}\,\psi$. Fix a non-vanishing smooth density dm on M. We may write $K_Q = \widetilde{K}_Q(1 \otimes dm)$, with $\widetilde{K}_Q \in \mathcal{D}'(M \times M)$. Then the map $k_Q : z \mapsto \widetilde{K}_Q(\,\cdot\,, z)\psi(z)$ is smooth from M to $C^\infty_{\mathcal{K}}(M)$. It follows that the map $k : z \mapsto P(k_Q(z))$ is smooth from M to $C^\infty(M)$. For each $f \in C_c^\infty(M)$ we have

$$\begin{aligned} P \circ Q(\psi f) &= P[\int_M \widetilde{K}_Q(\,\cdot\,, z)\psi(z) f(z) dz] \\ &= \int_M P[\widetilde{K}_Q(\,\cdot\,, z)\psi(z)] f(z)\, dz \\ &= \int_M k(z) f(z)\, dz \end{aligned}$$

which implies that $P \circ Q \circ M_\psi$ has the smooth integral kernel $(x, y) \mapsto k(z)(x)(1 \otimes dm)$. We conclude that $P \circ Q$ is smoothing whenever Q is.

Combining the above with Lemma 7.5.2 we see that $P \circ Q$ is also a smoothing operator whenever P is.

It remains to consider the case of arbitrary $d, e \in \mathbb{R}$. We will first show that $P \circ Q$ is a pseudo-differential operator of order $d + e$. Let $\mathcal{K} \subset M$ be compact. Then there exists a compact subset $\mathcal{K}' \subset M$ such that $\mathrm{supp}\, K_Q \cap (M \times \mathcal{K}) \subset \mathcal{K}' \times \mathcal{K}$ and a compact subset $\mathcal{K}'' \subset M$ such that $\mathrm{supp}\, K_P \cap (M \times \mathcal{K}') \subset \mathcal{K}'' \times \mathcal{K}'$. Then Q maps $C^\infty_{\mathcal{K}}(M)$ continuous linearly into $C^\infty_{\mathcal{K}'}(M)$ and P maps the latter space continuous linearly into $C^\infty_{\mathcal{K}''}(M)$. It follows that the composition $P \circ Q$ is continuous linear $C^\infty_{\mathcal{K}}(M) \to C^\infty_{\mathcal{K}''}(M)$. This implies that $R = P \circ Q$ is a continuous linear map $C_c^\infty(M) \to C_c^\infty(M)$.

Let ψ_j be a partition of unity on M such that each ψ_j is supported in a coordinate patch. Then by Lemma 7.3.6 (c) it suffices to show that each operator $R_j = R \circ M_{\psi_j}$ belongs to $\Psi^{d+e}(M)$. We fix j for the moment, put $\psi = \psi_j$ and let $U = U_j$ be a coordinate patch containing $\mathrm{supp}\,\psi$. There exists a compactly supported function $\chi \in C_c^\infty(U)$ such that $\chi = 1$ on an open neighborhood of $\mathrm{supp}\,\psi$. Now $M_{(1-\chi)} \circ Q \circ M_\psi$ is smoothing, so by the first part of the proof its left composition with P is smoothing as well. Put

$$Q_j = M_\chi \circ Q \circ M_\psi.$$

Then it suffices to show that $P \circ Q_j$ belongs to $\Psi^{d+e}(M)$. We select $\psi' \in C_c^\infty(U)$ such that $\psi' = 1$ on an open neighborhood of $\mathrm{supp}\,\chi$ and $\chi' \in C_c^\infty(U)$ such that $\chi' = 1$ on an open neighborhood of $\mathrm{supp}\,\psi'$. Then $M_{1-\chi'} \circ P \circ M_{\psi'}$ is smoothing, and by the first part of the proof, so is $[M_{1-\chi'} \circ P \circ M_{\psi'}]Q_j$. Put

$$P_j = M_{\chi'} \circ P \circ M_{\psi'}.$$

Then $P = P_j + M_{1-\chi'} \circ P \circ M_{\psi'}$ hence it suffices to show that $P_j \circ Q_j$ belongs to $\Psi^{d+e}(M)$. As the distribution kernels of the operators P_j, Q_j are contained in $\operatorname{supp} \chi' \times \operatorname{supp} \psi' \subset U \times U$, this result follows from the local result, Theorem 7.1.7. We conclude that $P \circ Q \in \Psi^{d+e}(M)$. By application of Corollary 7.1.10 we obtain that

$$\sigma^{d+e}(P_j \circ Q_j) = \sigma^d(P_j)\sigma^d(Q_j).$$

From the above it follows furthermore that $P \circ Q \circ M_{\psi_j} = P_j \circ Q_j$ modulo a smoothing operator, hence

$$\begin{aligned} \psi_j \sigma^{d+e}(P \circ Q) &= \sigma^{d+e}(P_j \circ Q_j) = \sigma^d(P_j)\sigma^e(Q_j) \\ &= \chi_j' \psi_j' \chi_j \psi_j \sigma^d(P)\sigma^d(Q) = \psi_j \sigma^d(P)\sigma^e(Q). \end{aligned}$$

This holds for every j. The identity (7.11) follows. □

Exercise 7.5.4. Let $P \in \Psi^d(M)$.

(a) Show that P has a unique extension to a continuous linear map $\mathcal{E}'(M) \to \mathcal{D}'(M)$.
(b) Show that the extension is pseudo-local.

Exercise 7.5.5. Let dm be positive smooth density on M and let $P \in \Psi^d(M)$.

(a) Show that there exists a unique $P^* \in \Psi^d(M)$ such that

$$(\, Pf \,,\, \overline{g}\,) = (\, f \,,\, \overline{P^* g}\,) \qquad \text{for all} \quad f, g \in C_c^\infty(M).$$

(b) Show that the principal symbol of P^* is given by

$$\sigma^d(P^*) = \overline{\sigma^d(P)}.$$