Analysis on Manifolds
Lecture notes for the 2009/2010 Master Class

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Received by the editors September, 2009.
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LECTURE 8
Operators between vector bundles

8.1. Operators on manifolds

In this section we shall extend the definition of pseudo-differential operator to sections of vector bundles on a smooth manifold M. We start by recalling the notion of a smooth kernel or smoothing operator between vector bundles on manifolds.

Let M and N be smooth manifolds, and let $\pi_E : E \to M$ and $\pi_F : F \to N$ be smooth complex vector bundles over M and N respectively.

We fix a positive smooth density dy on M, i.e., a smooth section of the density bundle D_M on M that is positive at every point of M. For the definition of the density bundle and the integration of its sections, see Lecture 2.

The exterior tensor product $F \boxtimes E$ is the vector bundle on $N \times M$ defined by

$$F \boxtimes E := \mathrm{pr}_1^*(F) \otimes \mathrm{pr}_2^*(E);$$

here $\mathrm{pr}_1, \mathrm{pr}_2$ denote the projections of $N \times M$ to N and M, respectively. Let k be a smooth section of the bundle $F \boxtimes E^*$ on $N \times M$ and let $f \in \Gamma_c^\infty(M, E)$. For every $x \in N$ we may view $y \mapsto k(x, y)f(y)\ dy$ as a compactly supported density on M with values in the finite dimensional linear space F_x. By using a partition of unity it is readily seen that the integral of the mentioned density depends smoothly on x. Accordingly, we define the complex linear operator $T : \Gamma_c^\infty(M, E) \to \Gamma^\infty(N, F)$ by

$$Tf(x) = \int_M k(x, y)\, f(y)\ dy \qquad (y \in M).$$

Any T of the above form is called a smooth kernel operator or smoothing operator from $\Gamma_c^\infty(M, E)$ to $\Gamma^\infty(N, F)$. Obviously, for such an operator the Schwartz kernel $K_T \in \mathcal{D}'(N \times M, F \boxtimes E^\vee)$ is smooth and given by the formula

$$K_T(x, y) = k(x, y)\mathrm{pr}_2^*(dy).$$

In terms of local trivializations of the bundles, T is given by matrices of scalar smoothing operators. More precisely, this may be described as follows.

We recall that a trivialization of E over an open subset U of M is defined to be a vector bundle isomorphism $\tau : E_U \to U \times \mathbb{C}^k$, with k the rank of E.

A frame of E over an open subset U of M is an ordered set $s_1, \dots, s_k$ of sections in $\Gamma^\infty(U, E)$ such that $s_1(x), \dots, s_k(x)$ is an ordered basis of the fiber E_x, for every $x \in U$. Given a choice of frame $s_1, \dots, s_k$, we have a vector bundle isomorphism $\sigma : U \times \mathbb{C}^k \to E_U$ given by $(x, v) \mapsto a_1 s_1(x) + \cdots + a_k s_k(x)$. The inverse τ of σ is a vector bundle isomorphism $E_U \to U \times \mathbb{C}^k$, i.e., a trivialization of E over U.

Conversely, if $\tau : E_U \to U \times \mathbb{C}^k$ is a trivialization of the bundle, then there exists a unique frame $s_1, \dots, s_k$ of E over U, to which τ is associated in the above manner, i.e.,

$$\tau[\, a_1 s_1(x) + \cdots + a_k s_k(x)\,] = (x, a), \qquad ((x,a) \in U \times \mathbb{C}^k).$$

Assume that E trivializes over the open subset $U \subset M$ and that F trivializes over the open subset $V \subset N$. Then the operator $T_{V,U} : \Gamma_c^\infty(U, E) \to \Gamma^\infty(V, F)$ given by

$$T_{V,U}(f) = (Tf)|_V, \qquad (f \in \Gamma_c^\infty(U, E)),$$

has Schwartz kernel equal to $K\mathrm{pr}_2^*(dy)|_{V\times U}$. Let $s_1, \dots, s_k$ be a local frame for E over U and $t_1, \dots, t_l$ a local frame for F over V, then $T_{V,U}$ is given by

$$T_{U,V}\big(\sum_j f^j s_j\big) = T_j^i(f^j) t_i,$$

with T_j^i uniquely determined smoothing operators $C_c^\infty(U) \to C^\infty(V)$. Conversely, any such collection of smoothing operators defines a smoothing operator from $\Gamma_c^\infty(U, E)$ to $C^\infty(V, F)$. Let $t^1, \dots, t^l$ be the frame of F^* over U dual to $t_1, \dots, t_l$, i.e., $\langle t^i(y), t_j(y)\rangle = \delta_j^i$ for all $y \in V$. Then we note that the Schwartz kernel of T_j^i equals

$$\langle K(x,y), t^i(x) \otimes s_j(y)\rangle\, \mathrm{pr}_2(dy)_{(x,y)}.$$

The space of smooth kernel operators from E to F is denoted by $\Psi^{-\infty}(E, F)$. Since any positive smooth density on M is of the form $c(y)dy$, with c a strictly positive smooth function, the space $\Psi^{-\infty}(E, F)$ is independent of the particular choice of the density dy.

Let $\mathbb{C}_M$ denote the trivial line bundle $M \times \mathbb{C}$. Then the space $\Psi^{-\infty}(\mathbb{C}_M, \mathbb{C}_M)$ may be identified with the space of smoothing operators $C_c^\infty(M) \to C^\infty(M)$, which we previously denoted by $\Psi^{-\infty}(M)$.

We shall now give the definition of a pseudo-differential operator between smooth complex vector bundles $E, F \to M$. We first deal with the case that E and F are trivial bundles on an open subset $U \subset M$. Thus, $E = U \times \mathbb{C}^k$ and $F = U \times \mathbb{C}^l$. Then we have natural identifications $\Gamma_c^\infty(U, E) \simeq C_c^\infty(U, \mathbb{C}^k) \simeq C_c^\infty(U)^k$, and similar identifications for F. Accordingly, we define $\Psi^d(U, E, F) = \Psi^d(E_U, F_U) \subset \mathrm{Hom}(\Gamma_c^\infty(U, E), \Gamma^\infty(U, F))$ by

$$\Psi^d(E_U, F_U) := \mathrm{M}_{l,k}(\Psi^d(U)),$$

the linear space of $l \times k$ matrices with entries in $\Psi^d(U)$. With these identifications, the action of an element $P \in \Psi^d(E_U, F_U)$ on a section $f \in \Gamma_c^\infty(U, E)$ is given by

$$(Pf)_i = \sum_{1 \le j \le k} P_{ij} f_j.$$

Assume that τ_E and τ_F are bundle automorphisms of the trivial bundles $E = U \times \mathbb{C}^k$ and $F = U \times \mathbb{C}^l$, respectively. Thus, τ_E is a map of the form $(x, v) \mapsto (x, \gamma_E(x)v)$ with $\gamma_E : U \to \mathrm{GL}\,(k, \mathbb{C})$ a smooth map, and τ_F is similarly given in terms of smooth map $\gamma_F : U \to \mathrm{GL}\,(l, \mathbb{C})$. Then we have an induced linear automorphism τ_{E*} of $\Gamma_c^\infty(U, E) \simeq C_c^\infty(U)^k$ given by

$$(\tau_{E*} f)(x) = \gamma_E(x) f(x), \qquad (x \in U).$$

Likewise, we have an induced linear automorphism τ_F^* of $\Gamma^\infty(U,F)$, and, accordingly, an induced linear automorphism τ_* of $\mathrm{Hom}(\Gamma_c^\infty(U,E),\Gamma_c^\infty(U,F))$. The latter is given by $\tau_*(Q) = \tau_{F*}\circ Q \circ \tau_{E*}^{-1}$, or

$$\tau_*(Q)(f) = \gamma_F Q(\gamma_E^{-1} f), \qquad (f \in \Gamma_c^\infty(U,E) \simeq C_c^\infty(U)^k).$$

It follows by component wise application of Lemma 7.5.1 that τ_* maps the linear space $\Psi^d(E_U,F_U)$ isomorphically onto itself.

The last observation paves the way for the definition of $\Psi^d(U,E,F) = \Psi^d(E_U,F_U)$ when E and F are smooth complex vector bundles on M that admit trivializations over an open subset $U \subset M$. Let $\tau_E : E \to E' = U \times \mathbb{C}^k$ and $\tau_F : F \to F' = U \times \mathbb{C}^l$ be trivializations; let $\tau_{E*} : \Gamma_c^\infty(U,E) \to \Gamma_c^\infty(U,E')$ be the induced map, and let $\tau_{F*} : \Gamma^\infty(U,F) \to \Gamma^\infty(U,F')$ be defined similarly. Then we define $\Psi^d(U,E,F)$ to be the space of linear maps $Q : \Gamma_c^\infty(U,E) \to \Gamma^\infty(U,F)$ such that

$$\tau_*(Q) := \tau_F^* \circ Q \circ \tau_{E*}^{-1} \in \Psi^d(U,E',F'). \tag{8.1}$$

This definition is independent of the particular choice of the trivializations for E_U and F_U, by the observation made above.

Definition 8.1.1. Let E,F be smooth vector bundles on a manifold M and let $d \in \mathbb{R}$. A pseudo-differential operator of order at most d from E to F is a continuous linear operator $P : \Gamma_c^\infty(E) \to \Gamma^\infty(F)$ with distribution kernel $K_P \in \mathcal{D}'(M\times M, F \boxtimes E^\vee)$ such that the following conditions are fulfilled.

(a) The kernel K_P is smooth outside the diagonal of $M \times M$.

(b) For each $a \in M$ there exists an open neighborhood $U \subset M$ on which E and F admit trivializations and such that the operator $P_U : \Gamma_c^\infty(U,E) \to \Gamma^\infty(U,F)$, $f \mapsto Pf|_U$ belongs to $\Psi^d(E_U,F_U)$.

The space of such pseudo-differential operators is denoted by $\Psi^d(E,F)$.

It follows from the above definition that the space $\Psi^d(E,F)$ transforms naturally under isomorphisms of vector bundles. More precisely, let $\varphi : M_1 \to M_2$ be an isomorphism of smooth manifolds, and let $\varphi_E : E_1 \to E_2$ be a compatible isomorphism of smooth complex vector bundles $\pi_{E_1} : E_1 \to M_1$ and $\pi_{E_2} : E_2 \to M_2$. Here the requirement of compatibility means that the pair (φ_E, φ) is an isomorphism of E_1 and E_2 in the sense of Lecture 2, §3. The isomorphism φ_E induces a linear isomorphism $\varphi_{E*} : \Gamma_c^\infty(M_1,E_1) \to \Gamma_c^\infty(M_2,E_2)$, given by $\varphi_{E*} f = \varphi_E \circ f \circ \varphi^{-1}$.

Likewise, let $\varphi_F : F_1 \to F_2$ be an isomorphism of vector bundles $F_j \to M_j$ which is compatible with φ. Then we have an induced linear isomorphism $\varphi_{F*} : \Gamma^\infty(M_1,F_1) \to \Gamma^\infty(M_2,F_2)$. Moreover, the map

$$\varphi_* : \mathrm{Hom}(\Gamma_c^\infty(M_1,E_1),\Gamma^\infty(M_1,F_1)) \to \mathrm{Hom}(\Gamma_c^\infty(M_2,E_2),\Gamma^\infty(M_2,F_2))$$

given by $\varphi_*(Q) = \varphi_{F*}\circ Q \circ \varphi_{E*}^{-1}$ restricts to a linear isomorphism

$$\varphi_* : \Psi^d(M_1,E_1,F_1) \xrightarrow{\simeq} \Psi^d(M_2,E_2,F_2).$$

We note that it also follows from the above definition that if E and F admit trivializations $\tau_E : E \to E' = U \times \mathbb{C}^k$ and $\tau_F : F \to F' = U \times \mathbb{C}^k$, respectively, then τ_* maps $\Psi^d(U,E,F)$ linearly isomorphically onto $\Psi^d(U,E',F') \simeq \mathrm{M}_{l,k}(\Psi^d(U))$.

Indeed, by the previous remark and the remark below (8.1), it suffices to prove this for $E = M \times \mathbb{C}^k$ and $F = M \times \mathbb{C}^l$. In this case

$$\mathrm{Hom}(\Gamma_c^\infty(E), C^\infty(F)) \simeq \mathrm{M}_{l,k}(\mathrm{Hom}(C_c^\infty(M), C^\infty(M))).$$

If $P \in \mathrm{Hom}(\Gamma_c^\infty(E), C^\infty(F))$, then $P \in \Psi^d(E, F)$ in the sense of Definition 8.1.1 if and only if all its components P_{ij} belong to $\Psi^d(M)$ in the sense of Definition 7.3.1.

Remark 8.1.2. The straightforward analogues of Exercise 7.3.4 and 7.3.5, Lemma 7.3.6, Exercise 7.3.7, Lemma 7.3.8, Exercise 7.3.10 are valid for operators from $\Psi^d(E, F)$, by reduction to trivial bundles and the scalar case, along the lines discussed above. We leave it to the reader to check the details.

It follows from Definition 8.1.1 and the corresponding fact for scalar operators that $\Psi^{-\infty}(E, F)$ equals the intersection of the spaces $\Psi^d(E, F)$, for $d \in \mathbb{R}$.

8.2. The principal symbol, vector bundle case

In this section we shall discuss the definition and basic properties of the principal symbol for a pseudo-differential operator between smooth complex vector bundles $\pi_E : E \to M$ and $\pi_F : F \to M$. We first concentrate on the definition of the appropriate symbol space.

Let $\pi_H : H \to M$ be a vector bundle. We agree to write

$$\Gamma^\infty(T^*M, H) = \{f \in C^\infty(T^*M, H) \mid \forall x \in M : \ f(T_x^*M) \subset H_x\}.$$

There is a natural identification of this space with the space of sections of the pull-back π^*H of the vector bundle H under the map $\pi : T^*M \to M$, so that $\Gamma^\infty(T^*M, H) \simeq \Gamma^\infty(T^*M, \pi^*H)$, but we shall not need this. If H is trivial of the form $H = M \times \mathbb{C}^N$, then the elements of $\Gamma^\infty(T^*M, H)$ are precisely the functions of the form $\xi_x \mapsto (x, f(\xi_x))$ with $f \in C^\infty(T^*M, \mathbb{C}^N)$. It follows that $\Gamma^\infty(T^*M, H) \simeq C^\infty(T^*M, \mathbb{C}^N)$. Accordingly, we define

$$S^d(M, H) := \{f \in \Gamma^\infty(T^*M, V) \mid \forall j : \ f_j \in S^d(M) \ \} \simeq S^d(M)^N.$$

Let $\tau : H_1 \to H_2$ be an isomorphism of two vector bundles on M. Then the map

$$\tau_* : \Gamma^\infty(T^*M, H_1) \to \Gamma^\infty(T^*M, H_2),$$

defined by

$$\tau_*(f)(\xi_x) = \tau_x(f(\xi_x)), \qquad (x \in M, \ \xi_x \in T_x^*M),$$

is a linear isomorphism. Assume now that τ is a bundle automorphism of the trivial bundle $H = M \times \mathbb{C}^N$. Then τ has the form $\tau(x, v) = (x, \tau_x(v))$, with $x \mapsto \tau_x$ a smooth map $M \to \mathrm{GL}\,(N, \mathbb{C})$. It follows that the linear automorphism τ_* of $\Gamma^\infty(T^*M, H) \simeq C^\infty(T^*M)^N$ is given by

$$\tau_*(f)(\xi_x) = \tau_x(f(\xi_x)).$$

It is readily checked that this map restricts to a linear automorphism of the symbol space $S^d(T^*M, H)$.

Now assume the bundle H is trivializable and let $\tau' : H \to H' = M \times \mathbb{C}^N$ be a trivialization. Then we define $S^d(M, H) := \tau_*^{-1} S^d(M, H')$. This definition is independent of the particular choice of the trivialization τ, in view of the preceding discussion.

Definition 8.2.1. Let $H \to M$ be a complex vector bundle. For $d \in \mathbb{R} \cup \{-\infty\}$, we define the symbol space $S^d(M, H)$ to be the space of sections $p \in \Gamma^\infty(T^*M, H)$ such that for every open neighborhood U on which H admits a trivialization, the restriction $p_U := p|_{T^*U}$ belongs to $S^d(U, H_U)$.

Clearly, for $p \in \Gamma^\infty(T^*M, H)$ to belong to $S^d(M, H)$ is suffices that for every $a \in M$ there exists an open trivializing neighborhood U such that $p_U \in S^d(U, H_U)$.

We also note that for $\varphi \in C^\infty(M)$ multiplication by $\pi^*\varphi \in C^\infty(T^*M)$ maps $S^d(T^*M, H)$ linear isomorphically to itself. Accordingly, $S^d(M, H)$ becomes a $C^\infty(M)$-module. It follows that the quotient

$$S^d/S^{d-1}(M, H) := S^d(M, H)/S^{d-1}(M, H)$$

is a $C^\infty(M)$-module as well.

Let $U \subset M$ be open and let $\mathcal{K} \subset U$ be compact. Then we write $S^d_{\mathcal{K}}(U, H_U)$ for the subspace of $S^d(U, H_U)$ consisting of p with support in $\mathcal{K}$ in the sense that $p_{U\setminus\mathcal{K}} = 0$. Equivalently, this means that the function $p : T^*U \to H_U$ vanishes on $\pi^*(U \setminus \mathcal{K})$. The extension of such a function to T^*M by the requirement $u(\xi_x) = 0_x \in H_x$ for every $\xi_x \in T^*M \setminus T^*U$ belongs to $S^d_{\mathcal{K}}(M, H)$. Accordingly, we have a linear injection

$$S^d_{\mathcal{K}}(U, H_U) \hookrightarrow S^d_{\mathcal{K}}(M, H).$$

Let $S^d_c(U, H_U)$ denote the union of the spaces $S^d_{\mathcal{K}}(U, H_U)$, for $\mathcal{K} \subset U$ compact. Then $S^d_c(U, H_U) \hookrightarrow S^d_c(M, H)$. Accordingly, we have an induced linear injection

$$S^d_c(U, H_U)/S^{d-1}_c(U, H_U) \hookrightarrow S^d_c(M, H)/S^{d-1}_c(M, H).$$

We will now see that the definition of the principal symbol map can be generalized to the context of bundles. Let E, F be two complex vector bundles on M. The principal symbol map associated with $\Psi^d(E, F)$ will be a map

$$\sigma^d : \Psi^d(M, E, F) \to S^d(M, \underline{\mathrm{Hom}}\,(E, F))/S^{d-1}(M, \underline{\mathrm{Hom}}\,(E, F)).$$

Here $\underline{\mathrm{Hom}}\,(E, F)$ is the vector bundle on M whose fiber at $x \in M$ is given by

$$\underline{\mathrm{Hom}}\,(E, F)_x = \mathrm{Hom}_{\mathbb{C}}(E_x, F_x).$$

If $U \subset M$ is an open subset on which both E and F admit trivializations $\tau_U : E_U \to U \times \mathbb{C}^k$ and $\tau_F : F_U \to U \times \mathbb{C}^l$, then the bundle $\underline{\mathrm{Hom}}\,(E, F)$ admits the trivialization

$$\tau : \underline{\mathrm{Hom}}\,(E, F)_U \to U \times \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^l)$$

given by

$$\tau_x(T) = (\tau_F)_x \circ T \circ (\tau_E)_x^{-1}.$$

Let $\varphi : E \to F$ be a vector bundle homomorphism. Then the map $\underline{\varphi} : x \mapsto \varphi_x \in \mathrm{Hom}(E_x, F_x)$ defines a smooth section of the bundle $\underline{\mathrm{Hom}}\,(E, F)$. Using trivializations we readily see that the map $\varphi \mapsto \underline{\varphi}$ defines a linear isomorphism

$$\mathrm{Hom}(E, F) \xrightarrow{\simeq} \Gamma^\infty(\underline{\mathrm{Hom}}\,(E, F)).$$

Initially we will give the definition of principal symbol for trivial bundles. Assume that $E = M \times \mathbb{C}^k$ and $F = M \times \mathbb{C}^l$ so that $\underline{\mathrm{Hom}}\,(E, F) = M \times \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^l) \simeq \mathrm{M}_{l,k}(\mathbb{C})$. Then

$$S^d(M, \underline{\mathrm{Hom}}\,(E, F)) \simeq \mathrm{M}_{l,k}(S^d(M))$$

and, accordingly,

$$S^d(M, \underline{\mathrm{Hom}}\,(E, F))/S^{d-1}(M, \underline{\mathrm{Hom}}\,(E, F)) \simeq \mathrm{M}_{l,k}(S^d(M)/S^{d-1}(M))$$

In this setting of trivial bundles, we define the principal symbol map $\sigma^d = \sigma^d_{E,F}$ component wise by

$$\sigma^d(P)_{ij} := \sigma^d(P_{ij}), \qquad (1 \le i \le l,\ 1 \le j \le k).$$

Assume now that τ_E and τ_F are automorphisms of the trivial bundles E and F, respectively and let τ be the induced automorphism of $\underline{\mathrm{Hom}}\,(E, F)$. We denote by τ_* the induced automorphisms of $\Psi^d(E, F)$ and of the quotient space $S^d(M, \underline{\mathrm{Hom}}\,(E, F))/S^{d-1}(M, \underline{\mathrm{Hom}}\,(E, F))$.

Lemma 8.2.2. *For every $P \in \Psi^d(E, F)$,*

$$\sigma^d(\tau_*(P)) = \tau_*(\sigma^d(P)).$$

Proof We observe that for $f \in \Gamma^\infty(E) \simeq C^\infty(M)^k$ we have

$$(\tau_*(P)f)_i = \sum_{r,s,j} (\tau_{Fx})_{ir} P_{rs} (\tau_{Ex}^{-1})_{sj} f_j$$

so that by Lemma 7.5.1

$$\begin{aligned} \sigma^d(\tau_*(P)_{ij}) &= \sum_{r,s} (\tau_F)_{ir} (\tau_E^{-1})_{sj} \sigma^d(P_{rs}) \\ &= \sum_{r,s} (\tau_F)_{ir} (\tau_E^{-1})_{sj} \sigma^d(P)_{rs} \\ &= (\tau_* \sigma^d(P))_{ij}. \end{aligned}$$

This implies that $\sigma^d(\tau_*(P))_{ij} = \sigma^d(\tau_*(P)_{ij}) = (\tau_* \sigma^d(P))_{ij}$. □

If E and F admit trivializations $\tau_E : E \to E' = M \times \mathbb{C}^k$ and $\tau_F : F \to F' = M \times \mathbb{C}^l$ we define the principal symbol map $\sigma^d_{E,F}$ on $\Psi^d(E, F)$ by requiring the following diagram to be commutative

$$\begin{array}{ccc} \Psi^d(E, F) & \xrightarrow{\tau_*} & \Psi^d(E', F') \\ \sigma^d_{E,F} \downarrow & & \downarrow \sigma^d_{E',F'} \\ S^d/S^{d-1}(M, \underline{\mathrm{Hom}}\,(E, F)) & \xrightarrow{\tau_*} & S^d/S^{d-1}(M, \underline{\mathrm{Hom}}\,(E', F')). \end{array} \tag{8.2}$$

Finally, we come to the case that $E \to M$ and $F \to M$ are arbitrary complex vector bundles of rank k and l respectively.

Lemma 8.2.3. *Let $P \in \Psi^d(E, F)$. Then there exists a unique*

$$\sigma^d(P) = \sigma^d_{E,F} \in S^d(M, \underline{\mathrm{Hom}}\,(E, F))/S^{d-1}(\underline{\mathrm{Hom}}\,(E, F))$$

such that for every open subset $U \subset M$ on which both E and F admit trivializations,

$$\sigma^d(P)_U = \sigma^d_{E_U,F_U}(P_U). \tag{8.3}$$

Proof Uniqueness is obvious. We will establish existence. Let $\{U_j\}$ be an open cover of M consisting of open subsets on which both E and F admit trivializations. We may assume that $\{U_j\}$ is locally finite and that $\{\psi_j\}$ is a partition of unity on M with $\operatorname{supp}\psi_j \subset U_j$ for all j. Given $P \in \Psi^d(E,F)$ we define $\sigma^d(P)$ by

$$\sigma^d(P) = \sum_j \psi_j \sigma^d_{E_{U_j},F_{U_j}}(P_{U_j}).$$

As this is a locally finite sum, it defines an element of $S^d(M,H)/S^{d-1}(M,H)$. It remains to verify (8.3) for an open subset U on which E and F admit trivializations τ_U. With $\sigma^d(P)$ as just defined we have

$$\begin{aligned}
\sigma^d(P)_U &= \sum_j \psi_j|_U\, \sigma^d_{E_{U_j},F_{U_j}}(P_{U_j})_{U\cap U_j} \\
&= \sum_j \psi_j|_U\, \sigma^d_{E_U,F_U}(P_U)_{U\cap U_j} \\
&= \sum_j \psi_j|_U\, \sigma^d_{E_U,F_U}(P_U) \\
&= \sigma^d_{E_U,F_U}\big(\sum_j M_{\psi_j|U} \circ P_U\big) \\
&= \sigma^d_{E_U,F_U}(P_U).
\end{aligned}$$

□

Definition 8.2.4. Let $P \in \Psi^d(E,F)$. The d-th order principal symbol of P is defined to be the unique element $\sigma^d(P) \in S^d/S^{d-1}(M, \underline{\mathrm{Hom}}\,(E,F))$ satisfying the properties of Lemma 8.2.3.

Obviously, $P \mapsto \sigma^d(P)$ is a linear map. As should be expected, it follows from the above definition that the principal symbol map behaves well under bundle isomorphisms. Consider isomorphisms $\tau_E : E_1 \to E_2$ and $\tau_F : F_1 \to F_2$ of vector bundles on M. Then the definitions have been given in such a way that the following diagram commutes

$$\begin{array}{ccc}
\Psi^d(M,E_1,F_1) & \xrightarrow{\sigma^d} & S^d(M,\underline{\mathrm{Hom}}\,(E_1,F_1))/S^{d-1}(M,\underline{\mathrm{Hom}}\,(E_1,F_1)) \\
\tau_* \downarrow & & \downarrow \tau_* \\
\Psi^d(M,E_2,F_2) & \xrightarrow{\sigma^d} & S^d(M,\underline{\mathrm{Hom}}\,(E_2,F_2))/S^{d-1}(M,\underline{\mathrm{Hom}}\,(E_2,F_2))
\end{array} \tag{8.4}$$

The local version of this result is true because of the local requirement (8.2). The global validity follows by the uniqueness part of the characterization of the symbol map in Lemma 8.2.3.

Lemma 8.2.5. *Let $\psi, \chi \in C^\infty(M)$. Then, for all $P \in \Psi^d(E,F)$,*

$$\sigma^d(M_\psi \circ P \circ M_\chi) = \psi\chi\sigma^d(P).$$

Proof For trivializable bundles E and F the result is a straightforward consequence of the analogous result in the scalar case. Let U be any open subset of M on which both E and F admit trivializations. Then

$$\begin{aligned}\sigma^d(M_\psi \circ P \circ M_\chi)_U &= \sigma^d_U(M_{\psi|U} \circ P_U \circ M_{\chi|U}) \\ &= \big(\psi|_U \chi|_U \sigma^d_U(P_U) \\ &= \big(\psi\chi\sigma^d(P)\big)_U.\end{aligned}$$

The result follows. □

Theorem 8.2.6. *The principal symbol map σ^d induces a linear isomorphism*

$$\Psi^d(E,F)/\Psi^{d-1}(E,F) \xrightarrow{\simeq} S^d(M, \underline{\mathrm{Hom}}\,(E,F))/S^{d-1}(M, \underline{\mathrm{Hom}}\,(E,F)).$$

Proof If E, F are trivial, then the result is an immediate consequence of the analogous result in the scalar case. If E, F are trivializable, the result is still true in view of the commutativity of the diagram (8.4). Let now E, F be arbitrary complex vector bundles on M and put $H = \underline{\mathrm{Hom}}\,(E,F)$. We must show that the principal symbol map $\sigma^d : \Psi^d(E,F) \to S^d(M,H)/S^{d-1}(M,H)$ has kernel $\Psi^{d-1}(E,F)$ and is surjective. Let $P \in \Psi^d(E,F)$, then $\sigma^d(P) = 0$ if and only if for every open subset $U \subset M$ on which both E and F admit a trivialization, $\sigma^d(P)_U = 0$. The latter condition is equivalent to $\sigma^d_U(P_U) = 0$, hence by the first part of the proof to $P_U \in \Psi^{d-1}(E_U, F_U)$. It follows that $\ker \sigma^d = \Psi^{d-1}(E,F)$.

To establish the surjectivity, let $p \in S^d(M,H)$ and let $[p]$ denote its class in the quotient $S^d(M,H)/S^{d-1}(M,H)$. Let $\{U_j\}$ be an open cover of M such that both E and F admit trivializations over U_j, for all j. We may choose the covering such that there exists a partition of one, $\{\psi_j\}$, with $\mathrm{supp}\,\psi_j \subset U_j$ for all j. By the first part of the proof, there exists for each j a pseudo-differential operator $P_j \in \Psi^d(E_{U_j}, F_{U_j})$, such that

$$\sigma^d(P_j) = [p]_{U_j} \in S^d(U_j, H_{U_j})/S^{d-1}(U_j, H_{U_j}).$$

For each j we fix $\chi_j \in C^\infty_c(U_j)$ such that $\chi_j = 1$ on $\mathrm{supp}\,\psi_j$. Then $M_{\psi_j} \circ P_j \circ M_{\chi_j}$ is a pseudo-differential operator in $\Psi^d(E,F)$ with distribution kernel supported by $\mathrm{supp}\,\psi_j \times \mathrm{supp}\,\chi_j$. It follows that the distribution kernels are locally finitely supported. Hence

$$P := \sum_j M_{\psi_j} \circ P_j \circ M_{\chi_j}$$

is a well-defined pseudo-differential operator in $\psi^d(E,F)$. Let U be any relatively compact open subset of M on which both E and F admit trivializations, then

$$\sigma^d(P)_U = \sum_j \big(\psi_j \sigma^d(P_j)\big)_U = \sum_j \psi_j|_U [p]_U = [p]_U,$$

with only finitely many terms of the sums different from zero. It follows that $\sigma^d(P) = [p]$. We have established the surjectivity of the principal symbol map. □

8.3. Symbol of adjoint and composition

We now turn to the behavior of the principal symbol when passing to adjoints. Let $E \to M$ and $F \to M$ be complex vector bundles on M of rank k and l, respectively. Let E^* and F^* be the dual bundles of E and F respectively. We recall that $E^\vee := E^* \otimes D_M$ and $F^\vee = F^* \otimes D_M$, with D_M the density bundle on M.

Lemma 8.3.1. *Let V, W be finite dimensional complex linear spaces, and let L be a one-dimensional complex linear space. Then the map $T \mapsto T \otimes \mathrm{I}_L$ defines a natural isomorphism*

$$\mathrm{Hom}_{\mathbb{C}}(V, W) \simeq \mathrm{Hom}_{\mathbb{C}}(V \otimes L, W \otimes L).$$

Proof Straightforward. □

Corollary 8.3.2. *The map*

$$\underline{\mathrm{Hom}}\,(F^*, E^*)_x \ni T_x \mapsto T_x \otimes \mathrm{I}_{D_{Mx}} \in \underline{\mathrm{Hom}}\,(F^\vee, E^\vee)_x$$

defines a natural isomorphism of vector bundles.

Given $p \in S^d(M, \underline{\mathrm{Hom}}\,(E, F))$, we define $p^\vee : T^*M \to \underline{\mathrm{Hom}}\,(F^\vee, E^\vee)$ by

$$p^\vee(\xi_x) = p(-\xi_x)^* \otimes \mathrm{I}_{D_{Mx}},$$

for $x \in M$ and $\xi_x \in T_x^*M$. Then clearly, $p^\vee \in S^d(M, \underline{\mathrm{Hom}}\,(F^\vee, E^\vee))$. The map $p \to p^\vee$ is readily seen to define a linear isomorphism

$$S^d(M, \underline{\mathrm{Hom}}\,(E, F)) \to S^d(M, \underline{\mathrm{Hom}}\,(F^\vee, E^\vee),$$

for every $d \in \mathbb{R} \cup \{-\infty\}$. Moreover, $p^{\vee\vee} = p$ for all p. Accordingly, we have an induced linear isomorphism

$$S^d/S^{d-1}(M, \underline{\mathrm{Hom}}\,(E, F)) \xrightarrow{\simeq} S^d/S^{d-1}(M, \underline{\mathrm{Hom}}\,(F^\vee, E^\vee)),$$

denoted by $\sigma \mapsto \sigma^\vee$.

Let $P \in \Psi^d(E, F)$. As P is a continuous linear operator $\Gamma_c^\infty(E) \to \Gamma^\infty(F)$, its adjoint P^t is a continuous linear operator from the topological linear dual $\Gamma^\infty(F)' = \mathcal{E}'(F^\vee)$ to the topological linear dual $\Gamma_c^\infty(E)' = \mathcal{D}'(E^\vee)$. We recall that the natural continuous bilinear pairing $\Gamma^\infty(E^\vee) \times \Gamma_c^\infty(E) \to \mathbb{C}$ defined by

$$\langle f, g\rangle = \int_M (f, g)$$

induces a natural continuous linear embedding $\Gamma^\infty(E^\vee) \hookrightarrow \mathcal{D}'(E^\vee)$. Likewise, we have a natural continuous linear embedding $\Gamma_c^\infty(F^\vee) \hookrightarrow \mathcal{E}'(F^\vee)$.

Lemma 8.3.3. *Let $P \in \Psi^d(E, F)$. The adjoint P^t restricts to a continuous linear map $\Gamma_c^\infty(F^\vee) \to \Gamma^\infty(E^\vee)$. The restricted map is a pseudo-differential operator in $\Psi^d(F^\vee, E^\vee)$ with principal symbol given by*

$$\sigma^d(P^t) = \sigma^d(P)^\vee.$$

Proof Let $U \subset M$ be an open subset. Then it is readily seen from the definitions that $(P^t)_U = (P_U)^t$ and that $(\sigma^d(P)^\vee)_U = (\sigma^d(P)_U)^\vee = \sigma^d(P_U)^\vee$. Therefore the result is of a local nature, and we may as well assume that E and F admit trivializations on M. Let $e_1, \ldots, e_k$ be a frame for E and let $f_1, \ldots, f_l$ be a frame for F. Let $e^1, \ldots, e^k$ be the dual frame for the dual bundle E^*; i.e., $(e^i, e_j) = \delta_{ij}$, for all $1 \leq i, j \leq k$. Similarly, let $f^1, \ldots, f^l$ be the dual frame for the dual bundle F^*. Let dm be choice of smooth positive density on M, then $\{dm\}$ constitutes a frame for the density bundle D_M. It follows that $e^1 dm, \ldots, e^k dm$ is a frame for $E^\vee$ and $f^1 dm, \ldots, f^l dm$ a frame for $F^\vee$.

Let $P \in \Psi^d(E, F)$. The operator P has components P_{ij} relative to the frames $\{e_j\}$ and $\{f_i\}$. Given $\varphi, \psi \in C_c^\infty(M)$, we have

$$\psi P_{ij} \varphi = (\psi f^i, P(\varphi e_j)).$$

The components of the adjoint operator P^t are given by

$$\varphi (P^t)_{ji}(\psi) dm = (\varphi e^j, P^t(\psi f^i dm)).$$

Integrating the densities on both sides of the equality over M we find that

$$\langle \varphi dm, (P^t)_{ji} \psi \rangle = \langle P(\varphi e^j), \psi f^i dm \rangle = \langle P_{ij}(\varphi), \psi dm \rangle.$$

This implies that $(P^t)_{ji}$ equals the adjoint of P_{ij} in the sense of Lemma 7.5.2. Hence, $P^t \in \Psi^d(F^\vee, E^\vee)$ and

$$\sigma^d(P^t)_{ji}(\xi_x) = \sigma^d(P)_{ij}(-\xi_x) = \big(\sigma^d(P)^\vee(\xi_x)\big)_{ji}.$$

The result follows. □

Let $P \in \Psi^d(E, F)$. Then it follows by application of the lemma above that P extends to a continuous linear map $\mathcal{E}'(E) \to \mathcal{D}'(F)$. Indeed, the extension equals the adjoint of the map $P^t : \Gamma_c^\infty(F^\vee) \to \Gamma^\infty(E^\vee)$. The extension is unique by density of $\Gamma_c^\infty(E)$ in $\mathcal{E}'(E)$.

It follows from the definitions that if the distributional kernel K_P has support $S \subset M \times M$, then the distributional kernel of the adjoint operator P^t has support $S^t = \{(y, x) \in M \times M \mid (x, y) \in S\}$. In analogy with the scalar case, the operator P is said to be *properly supported* if the restricted projection maps $\mathrm{pr}_j|_S : S \to M$ are proper, for $j = 1, 2$. Thus, if P is properly supported, then so is P^t. In this case P^t maps $\Gamma_c^\infty(F^\vee)$ continuous linearly into $\Gamma_c^\infty(E^\vee)$, and we see that P extends to a continuous linear operator $\mathcal{D}'(E) \to \mathcal{D}'(F)$.

Lemma 8.3.4. *Let $P \in \Psi^d(E, F)$ be properly supported. Then the continuous linear operator $P : \mathcal{D}'(E) \to \mathcal{D}'(F)$ is pseudo-local, i.e., for all $u \in \mathcal{D}'(E)$ we have*

$$\mathrm{singsupp}\,(Pu) \subset \mathrm{singsupp}\, u.$$

Proof Let $a \in M \setminus \mathrm{supp}\, u$. Then there exists an open neighborhood $\mathcal{O} \ni a$ with $\mathcal{O} \cap \mathrm{supp}\, u = \emptyset$. Let $\psi \in C_c^\infty(\mathcal{O})$ be equal to 1 on a neighborhood of a. By paracompactness of $\mathrm{supp}\, u$ there exists a smooth functions $\chi \in C^\infty(M)$ such that $\chi = 1$ on a neighborhood of $\mathrm{supp}\, u$ and such that $\mathrm{supp}\, \psi \cap \mathrm{supp}\, \chi = \emptyset$. It follows that $T := M_\psi \circ P \circ M_\chi$ is a properly supported smoothing operator. Hence $\psi P u = \psi P(\chi u) = T u$ is smooth. It follows that Pu is smooth in a neighborhood of a. □

As in the scalar case, modulo a smoothing operator each pseudo-differential operator can be represented by a properly supported one.

Lemma 8.3.5. *Let $\Omega \subset M \times M$ be an open neighborhood of the diagonal. Then for each $P \in \Psi^d(E,F)$ there exists a properly supported $P_0 \in \Psi^d(E,F)$ with $\operatorname{supp} K_{P_0} \subset \Omega$ such that $P - P_0 \in \Psi^{-\infty}(E,F)$.*

Proof The proof is an obvious adaptation of the proof of Lemma 6.1.6. By Lemma 6.1.7 there exists a locally finite open covering $\{U_j\}_{j\in J}$ of M such that for all $i,j \in J$, $U_i \cap U_j \neq \emptyset \Rightarrow U_i \times U_j \subset \Omega$. There exists a partition of unity $\{\psi_j\}$ with $\psi_j \in C_c^\infty(U_j)$. For each j we choose a $\chi_j \in C_c^\infty(U_j)$ which equals 1 on an open neighborhood of $\operatorname{supp} \psi_j$. We now define

$$P_0 = \sum_{j\in J} M_{\psi_j} \circ P \circ M_{\chi_j}.$$

The j-th term in the above sum is a pseudo-differential operator of order d with distribution kernel supported in $\operatorname{supp} \psi_j \times \operatorname{supp} \chi_j$. As this is a locally finite collection of sets, it follows that $P_0 \in \Psi^d(E,F)$. Moreover, the distribution kernel of P_0 has support contained in the union of the sets $U_j \times U_j$ which is contained in Ω.

Since $\operatorname{supp} \psi_j \cap \operatorname{supp}(1-\chi_j) = \emptyset$, the operator

$$T_j := M_{\psi_j} \circ P \circ M_{1-\chi_j}$$

is a smooth kernel operator, with smooth kernel supported inside the set $U_j \times M$. Since these sets form a locally finite collection in $M \times M$, the sum $T = \sum_j T_j$ is a well defined smoothing operator in $\Psi^{-\infty}(E,F)$. It is now readily checked that $P - P_0 = T$. □

We end this section with a discussion of the composition of two properly supported pseudo-differential operators. To prepare for this, we will first study the product of bundle-valued symbols. Let E_1, E_2, E_3 be complex vector bundles on M. Given $p \in S^d(M, \underline{\mathrm{Hom}}(E_1,E_2)$ and $q \in S^e(M, \underline{\mathrm{Hom}}(E_2,E_3)$ we define $qp : T^*M \to \mathrm{Hom}(E_1,E_3)$ by

$$qp : T_x^*M \ni \xi_x \mapsto q(\xi_x) \circ_x p(\xi_x) \in \underline{\mathrm{Hom}}(E_1,E_3)_x,$$

where $\circ_x$ denotes the composition map from $\underline{\mathrm{Hom}}(E_1,E_2)_x \times \underline{\mathrm{Hom}}(E_2,E_3)_x$ to $\underline{\mathrm{Hom}}(E_1,E_3)_x$.

Lemma 8.3.6. *The assignment $(p,q) \mapsto qp$ defines a bilinear map*

$$S^d(M, \underline{\mathrm{Hom}}(E_1,E_2)) \times S^e(M, \underline{\mathrm{Hom}}(E_2,E_3)) \to S^{d+e}(M, \underline{\mathrm{Hom}}(E_1,E_3)).$$

Proof The bilinearity of the assignment as a map into $\Gamma^\infty(T^*M, \underline{\mathrm{Hom}}(E_1,E_3))$ is obvious. We will prove the remaining assertion that the assignment has image contained in S^{d+e}.

If U is an open subset of M, then for $p \in S^d(M, \underline{\mathrm{Hom}}(E_1,E_2))$ and $q \in S^e(M, \underline{\mathrm{Hom}}(E_2,E_3))$ we have $(qp)_U = q_U p_U$. Therefore, the result is of a local nature, and we may as well assume that each E_j admits a trivialization, for

$j = 1, 2, 3$. For each such j, let $e_{j1}, \dots, e_{jk_j}$ be a frame for E_j. Then the symbol p has components $p_{\beta\alpha} \in S^d(M)$ given by

$$p(\xi_x)(e_{1\alpha}(x)) = \sum_{\beta=1}^{k_2} p_{\beta\alpha}(\xi_x) e_{2\beta}(x)$$

for all $x \in M$ and $\xi_x \in T_x^*M$. Likewise, the symbol q has components $q_{\gamma\beta} \in S^e(M)$ given by

$$q(\xi_x)(e_{2\beta}(x)) = \sum_{\gamma=1}^{k_3} q_{\gamma\beta}(\xi_x) e_{3\gamma}(x).$$

It follows that the γ-component of $q(\xi_x)p(\xi_x)(e_{1\alpha}(x))$ relative to the basis $\{e_{3\gamma}(x)\}$ of E_{3x} is given by

$$(qp)_{\gamma\alpha} = \sum_{\beta=1}^{k_2} q_{\gamma\beta} p_{\beta\alpha}.$$

This shows that $qp \in S^{d+e}(M, \underline{\mathrm{Hom}}\,(E_1, E_3))$. □

It follow from this lemma that the product map induces a bilinear map

$$S^d/S^{d-1}(M, \underline{\mathrm{Hom}}\,(E_1, E_2)) \times S^e/S^{e-1}(M, \underline{\mathrm{Hom}}\,(E_2, E_3)) \longrightarrow S^{d+e}/S^{d+e-1}(M, \underline{\mathrm{Hom}}\,(E_1, E_3))$$

denoted $(\sigma_1, \sigma_2) \mapsto \sigma_2\sigma_1$.

We now turn to the composition of pseudo-differential operators. If $P \in \Psi^d(E_1, E_2)$ is a properly supported pseudo-differential operator then P maps $\Gamma_c^\infty(E_1)$ continuous linearly to $\Gamma_c^\infty(E_2)$. Thus, if $Q \in \Psi^e(E_2, E_3)$ then the composition $Q \circ P$ is a well-defined continuous linear operator $\Gamma_c^\infty(E_1) \to \Gamma^\infty(E_3)$.

Theorem 8.3.7. *Let $P \in \Psi^d(E_1, E_2)$ and $Q \in \Psi^e(E_2, E_3)$ be properly supported. Then the composition $Q \circ P$ is a properly supported pseudo-differential operator in $\Psi^{d+e}(E_1, E_3)$ with principal symbol given by*

$$\sigma^{d+e}(Q \circ P) = \sigma^e(Q)\sigma^d(P). \tag{8.5}$$

Proof We first assume that $d = e = -\infty$ so that both P and Q are smoothing operators and will show that $Q \circ P$ is a smoothing operator. For this it suffices to be shown that the kernel of $Q \circ P$ is smooth at each point $(a, b) \in M \times M$. Let U and W be relatively compact open neighborhoods of a and b on which both E and F admit trivializations. Let $\chi \in C_c^\infty(U)$ be equal to 1 on an open neighborhood of a and let $\chi' \in C_c^\infty(W)$ be equal to 1 on an open neighborhood of b. Then the kernel of $M_{\chi'} \circ Q \circ P \circ M_\chi$ equals the kernel of $Q \circ P$ on an open neighborhood of (b, a), so that it suffices to show that $M_{\chi'} \circ Q \circ P \circ M_\chi$ is a smoothing operator.

Let A be a compact subset of M such that $\mathrm{supp}\, K_P \subset A \times U$, and $\mathrm{supp}\, K_Q \subset W \times A$. Let $\{V_j\}$ be a finite cover of A by open subsets of M on which each of the bundles E, F admits a trivialization. Let $\psi_j \in C_c^\infty(V_j)$ be functions such that $\sum_j \psi_j = 1$ on an open neighborhood of A. For each j, let $\psi_j' \in C_c^\infty(V_j)$ be such that $\psi_j' = 1$ on an open neighborhood of $\operatorname{supp}\psi_j$. Then

$$M_{\chi'} \circ Q \circ P \circ M_\chi = \sum_j Q_j \circ P_j,$$

where $Q_j = M_{\chi'} \circ Q \circ M_{\psi_j'}$ and $P_j = M_{\psi_j} \circ P \circ M_\chi$. It suffices to show that each of the operators $Q_j \circ P_j$ is smoothing. Fix j. Let $e_{11}, \dots, e_{1k_1}$ be a frame of E_1 on U, $e_{21}, \dots, e_{2k_2}$ a frame of E_2 on V_j and $e_{31}, \dots, e_{3k_3}$ a frame of E_3 on W. Let $P_{j\beta\alpha}$ be the components of $P_j : \Gamma_c^\infty(U, E_1) \to \Gamma_c^\infty(V_j, E_2)$ relative to the first two frames, and let $Q_{j\gamma\beta}$ be the components of $Q_j : \Gamma_c^\infty(V_j, E_2) \to \Gamma_c^\infty(W, E_3)$ relative to the second pair of frames. These components are scalar smoothing operators. The components of $Q_j \circ P_j : \Gamma_c^\infty(U) \to \Gamma_c^\infty(W)$ are given by

$$(Q_j \circ P_j)_{\gamma\alpha} = \sum_\beta Q_{j\gamma\beta} \circ P_{j\beta\alpha}.$$

As all operators in this sum are smoothing, it follows that $Q_j \circ P_j : \Gamma_c^\infty(U, E_1) \to \Gamma_c^\infty(W, E_3)$ is smoothing. As $Q_j \circ P_j$ vanishes on the complement of $\operatorname{supp}\chi$ and has image contained in $C_c^\infty(W)$, it follows that $Q_j \circ P_j$ is a smoothing operator.

We now assume that $d \in \mathbb{R}$ and $e = -\infty$ and will show that $Q \circ P$ is smoothing. Let $U \subset M$ be a relatively compact open subset on which each of the bundles E_j, for $j = 1, 2, 3$ admits a trivialization. Let $\chi \in C_c^\infty(U)$ then it suffices to show that $M_\chi \circ Q \circ P$ is smoothing. Let $\psi \in C_c^\infty(U)$ be such that $\psi = 1$ on an open neighborhood of $\operatorname{supp}\chi$. Then $M_\chi \circ Q$ differs from $M_\chi \circ Q \circ M_\psi$ by a smoothing operator, hence, by the first part of the proof it suffices to show that $M_\chi \circ Q \circ M_\psi \circ P$ is smoothing. Let $\chi' \in C_c^\infty(U)$ be such that $\chi' = 1$ on an open neighborhood of $\operatorname{supp}\chi$. Then $M_\chi \circ Q \circ M_\psi \circ P = Q_0 \circ P_0$, where $Q_0 = M_\chi \circ Q \circ M_\psi$ and $P_0 = M_{\chi'} \circ P$. As P is properly supported, there exists a compact subset $B \subset M$ such that $\operatorname{supp} K_P \cap U \subset M \subset U \times B$. Let $\{V_j\}$ be a finite open cover of B such that the bundle E_1 admits a trivialization on each of the sets V_j. Let $\psi_j \in C_c^\infty(V_j)$ be such that $\sum_j \psi_j = 1$ on an open neighborhood of B. Then $P_0 = \sum_j P_j$, where $P_j = P_0 \circ M_{\psi_j}$. It suffices to show that each operator $Q_0 \circ P_j$ is smoothing. Fix j, let $e_{11}, \dots, e_{1,k_1}$ be a frame of E_1 on V_j, let $e_{21}, \dots e_{2k_2}$ be a frame of E_2 on U and $e_{31}, \dots, e_{3k_3}$ a frame of E_3 on U. Then in terms of components of the operators $Q_0 : \Gamma_c^\infty(U, E_2) \to \Gamma_c^\infty(U, E_3)$ and $P_j : \Gamma_c^\infty(V_j, E_1) \to \Gamma_c^\infty(U, E_2)$ the operator $Q_0 \circ P_j : \Gamma_c^\infty(V_j, E_1) \to \Gamma_c^\infty(U, E_3)$ has components given by

$$(Q_0 \circ P_j)_{\gamma\alpha} = \sum_{\beta=1}^{k_2} (Q_0)_{\gamma\beta} \circ (P_j)_{\beta\alpha} \tag{8.6}$$

The operators $(P_j)_{\beta\alpha}$ are smoothing with kernels whose compact supports are contained in $U \times V_j$. Extending these kernels with value zero outside their supports, we obtain kernels on $M \times M$ such that the identities (8.6) still hold for the associated scalar operators. As $(Q_0)_{\gamma\beta} \in \Psi^e(M)$ for all γ, β and $(P_j)_{\beta\alpha} \in \Psi^{-\infty}(M)$ for all α, β, it follows from Theorem 7.5.3 that all components of $Q_0 \circ P_j$ are smoothing operators. Hence, $Q_0 \circ P$ is smoothing. We conclude that $Q \circ P$ is smoothing.

Likewise, if $e = -\infty$ and $d \in \mathbb{R}$, then $Q \circ P$ is a smoothing operator. To see this we may either imitate the argument in the previous part of the proof, or combine the result of that part with Lemma 8.3.3.

Finally, we discuss the case that e, d are arbitrary. We will show that $Q \circ P \in \Psi^{d+e}(E_1, E_3)$. Let U be an open subset of M on which each of the bundles E_j, for $j = 1, 2, 3$ trivializes. Let $\chi \in C_c^\infty(U)$; then it suffices to show that $M_\chi \circ Q \circ P \in \Psi^{d+e}$. Let $\chi' \in C_c^\infty(U)$ be such that $\chi' = 1$ on $\operatorname{supp} \chi$. Then $M_\chi \circ Q$ equals $Q_0 := M_\chi \circ Q \circ M_{\chi'}$ modulo a smoothing operator, hence by the first part of the proof it suffices to show that $Q_0 \circ P$ belongs to Ψ^{d+e}. The latter operator equals $Q_0 \circ M_\psi \circ P$ for $\psi \in C_c^\infty(U)$ such that $\psi = 1$ on an open neighborhood of $\operatorname{supp} \chi'$. Let $\psi' \in C_c^\infty(U)$ be equal to 1 on an open neighborhood of $\operatorname{supp} \psi$, then $M_\psi \circ P$ equals $P_0 := M_\psi \circ P \circ M_{\psi'}$ modulo a smoothing operator, hence it suffices to show that $Q_0 \circ P_0 \in \Psi^{d+e}(E_1, E_3)$. As the supports of the kernels of Q_0 and P_0 are contained in $U \times U$, it suffices to show that $(Q_0)_U \circ (P_0)_U \in \Psi^{d+e}((E_1)_U, (E_3)_U)$.

Thus, to show that $Q \circ P$ belongs to $\Psi^{d+e}(E_1, E_3)$ we may as well assume that E_1, E_2, E_3 are trivial on M from the start. In this situation, $\Psi^d(E_1, E_2) \simeq M_{k_2,k_1}(\Psi^d(M))$ and $\Psi^e(E_2, E_3) \simeq M_{k_3 k_2}(\Psi^e(M))$. Moreover, the composition $Q \circ P$ has components

$$(Q \circ P)_{\gamma\alpha} = \sum_\beta Q_{\gamma\beta} \circ P_{\beta\alpha}.$$

It follows by application of Theorem 7.5.3 that these components belong to $\Psi^{d+e}(M)$ and have principal symbols given by

$$\sigma^{d+e}((Q \circ P)_{\gamma\alpha}) = \sum_\beta \sigma^e(Q_{\gamma\beta}) \circ \sigma^d(P_{\beta\alpha}).$$

It follows that

$$\sigma^{d+e}(Q \circ P)_{\gamma\alpha} = \big(\sigma^e(Q)\sigma^d(P)\big)_{\gamma\alpha}$$

whence (8.5).

We now assume to be in the general situation again. It remains to be shown that $R := Q \circ P$ is properly supported. Let $A \subset M$ be compact. Then there exists a compact subset $B \subset M$ such that $\operatorname{supp} K_P \cap (M \times A) \subset B \times A$. There exists a compact subset $C \subset M$ such that $\operatorname{supp} K_Q \cap (M \times B) \subset C \times B$. It now easily follows that $\operatorname{supp} K_R \cap (M \times A) \subset C \times A$. Thus, $\mathrm{pr}_2|_{\operatorname{supp} K_R} : \operatorname{supp} K_R \to M$ is proper. The properness of $\mathrm{pr}_1|_{\operatorname{supp} K_R}$ is established in a similar way. □

Exercise 8.3.8. Let $P \in \Psi^d(E, F)$. Show that the following two assertions are equivalent.

(a) The operator P is properly supported.
(b) The operator P maps $\Gamma_c^\infty(E)$ continuous linearly to $\Gamma_c^\infty(F)$ and its adjoint P^t maps $\Gamma_c^\infty(F^\vee)$ continuous linearly to $\Gamma_c^\infty(E^\vee)$.

8.4. Elliptic operators, parametrices

We assume that $E \to M$ and $F \to M$ are complex vector bundles on M of rank k and l, respectively.

The symbol space $S^0(M, \underline{\mathrm{End}}\,(E))$ has a distinguished element 1_E given by

$$1_E : T_x^*M \ni \xi_x \mapsto \mathrm{I}_{E_x} \in \underline{\mathrm{End}}\,(E)_x, \qquad (x \in M).$$

Likewise, $S^0(M, \underline{\mathrm{End}}\,(F))$ has a distinguished element 1_F.

A symbol $p \in S^d(M, \underline{\mathrm{Hom}}\,(E, F))$ is said to be *elliptic* (of order d) if there exists a symbol $q \in S^{-d}(M, \underline{\mathrm{Hom}}\,(F, E)$ such that

$$pq - 1_F \in S^{-1}(M, \underline{\mathrm{End}}\,(F)), \quad \text{and} \quad qp - 1_E \in S^{-1}(M, \underline{\mathrm{End}}\,(E)).$$

For the classes $[p] \in S^d/S^{d-1}$ and $[q] \in S^{-d}/S^{-d-1}$ this means precisely that

$$[p][q] = [1_F], \quad \text{and} \quad [q][p] = [1_E].$$

Thus, the notion of ellipticity factors to the quotient space S^d/S^{d-1}.

Definition 8.4.1. A pseudo-differential operator $P \in \Psi^d(E, F)$ is said to be elliptic if its principal symbol $\sigma^d(P) \in S^d/S^{d-1}(M, \underline{\mathrm{Hom}}\,(E, F))$ is elliptic.

The notion of ellipticity of a pseudo-differential operator generalizes the similar notion for a differential operator.

Lemma 8.4.2. *Let $d \in \mathbb{N}$ and let P be a differential operator of order d from E to F. Then the following assertions are equivalent.*

(a) *P is elliptic as differential operator;*
(b) *P is elliptic as a pseudo-differential operator in $\Psi^d(E, F)$.*

Proof Let p be the principal symbol of P as a differential operator. Then $p \in \Gamma^\infty(T^*M, \underline{\mathrm{Hom}}\,(E, F))$ and for each $x \in M$, the map $\xi \mapsto p(x, \xi)$ is a $\mathrm{Hom}(E_x, F_x)$-valued polynomial function on T_xM^* that is homogeneous of degree d. By using local trivializations of E and F one sees that p is a symbol in $S^d(M, \underline{\mathrm{Hom}}\,(E, F))$, and that its class in S^d/S^{d-1} is the principal symbol of P in the sense of pseudo-differential operators.

Now assume (a). This means that $p(x, \xi_x)$ is an invertible homomorphism $E_x \to F_x$ for every $x \in M, \xi_x \in T_x^*M \setminus \{0\}$. Let $\chi : T^*M \to \mathbb{R}$ be a smooth function such that $\pi : \operatorname{supp} \chi \to M$ is proper and such that $\chi = 1$ on a neighborhood of M; as usual we identify M with the image of the zero section in T^*M. The existence of such a function can be established locally (relative to M) by using an atlas of M, and globally by using a partition of unity subordinate to this atlas.

We define the smooth function $q \in \Gamma^\infty(T^*M, \underline{\mathrm{Hom}}\,(F, E))$ by

$$q(\xi_x) := (1 - \chi(\xi_x))p(\xi_x)^{-1}, \qquad (x \in M,\ \xi_x \in T_x^*M).$$

By a local analysis we readily see that $q \in S^{-d}(M, \underline{\mathrm{Hom}}\,(F, E))$. Moreover, from the definition of q it follows that

$$q(\xi_x)p(\xi_x) - I_{E_x} = \chi(\xi_x)I_{E_x}, \quad \text{and} \quad p(\xi_x)q(\xi_x) - I_{F_x} = \chi(\xi_x)I_{F_x}.$$

This implies that $qp - 1_E \in S^{-\infty}(E, E)$ and $pq - 1_F \in S^{-\infty}(F, F)$. Hence (b) follows.

Conversely, assume that (b) holds. Let p be the principal symbol of the differential operator P introduced above. Then $[p] = \sigma^d(P)$ in the sense of pseudo-differential operators, hence there exists a $q \in S^{-d}(M, \underline{\mathrm{Hom}}\,(F, E))$ such that

$[q][p] = [1_E]$ and $[p][q] = [1_F]$. It follows that there exists a $r \in S^{-1}(M, \underline{\mathrm{End}}\,(E))$ such that

$$q(\xi_x)p(\xi_x) = \mathrm{I}_{E_x} + r(\xi_x), \qquad (x \in M,\ \xi_x \in T_x^*M).$$

Fix $x \in M$, and choose norms on the finite dimensional spaces T_x^*M and $\mathrm{End}(E_x)$. Then it follows that

$$q(\xi_x)p(\xi_x) - \mathrm{I}_{E_x} = \mathcal{O}(1 + \|\xi_x\|)^{-1} \qquad (\|\xi_x\| \to \infty).$$

This implies that $\det(q(\xi_x)p(\xi_x)) \to 1$ for $\|\xi_x\| \to \infty$. Hence, $p(\xi_x)$ is an invertible element of $\mathrm{Hom}(E_x, F_x)$ for $\|\xi_x\|$ sufficiently large. By homogeneity of $p|_{T_x^*M}$ this implies that $p(\xi_x)$ is invertible for all $\xi \in T_x^*M \setminus \{0\}$. As this holds for any $x \in M$, the operator P is elliptic as a differential operator. □

Corollary 8.4.3. *Let $P \in \Psi^d(E, F)$ be properly supported and elliptic. Then there exists a properly supported $Q \in \Psi^{-d}(F, E)$ such that $QP - I \in \Psi^{-1}(E, E)$ and $PQ - I \in \Psi^{-1}(F, F)$.*

Proof By Definition 8.4.1 there exists a $q \in S^{-d}/S^{-d-1}(M, \underline{\mathrm{Hom}}\,(F, E))$ such that $\sigma^d(P)q = [1_E]$ and $\sigma^d(p)q = [1_F]$. By Theorem 8.2.6 there exists a $Q \in \Psi^{-d}(F, E)$ with $\sigma^{-d}(Q) = q$. By Lemma 8.3.5 there exists such a Q such that in addition Q is properly supported. It now follows from Theorem 8.3.7 that $\sigma^0(Q \circ P) = [1_E]$ and $\sigma^0(P \circ Q) = [1_F]$. As $[1_E]$ is the principal symbol of the identity operator $\mathrm{I}_E : \Gamma_c^\infty(E) \to \Gamma_c^\infty(E)$, it follows that $Q \circ P - \mathrm{I}_E \in \Psi^{-1}(E, E)$, by Theorem 8.2.6. Likewise, $P \circ Q - \mathrm{I}_F \in \Psi^{-\infty}(F, F)$. □

The above corollary has the remarkable improvement that Q may be adapted in such a way that $QP - I$ and $PQ - I$ become smooth kernel operators. The proof of this fact is based on the following principle involving series of pseudo-differential operators. That principle in turn is the appropriate generalization of the similar principle for symbols, as formulated in Lemma 5.5.1.

We put

$$\Psi(E, F) = \bigcup_{d \in \mathbb{R}} \Psi^d(E, F).$$

Definition 8.4.4. Let $\{d_j\}$ be a sequence of real numbers with $\lim_{j\to\infty} d_j = -\infty$. Let $Q_j \in \Psi^{d_j}(E, F)$, for $j \in \mathbb{N}$. Let $Q \in \Psi(E, F)$. Then

$$Q \sim \sum_{j=0}^{\infty} Q_j$$

means that for each $d \in \mathbb{R}$ there exists an $N \in \mathbb{N}$ such that for all $k \geq N$

$$Q - \sum_{j=0}^{k} Q_j \in \Psi^d(E, F).$$

Theorem 8.4.5. *Let $\{d_\nu\}_{\nu\in\mathbb{N}}$ be a sequence of real numbers with $\lim_{\nu\to\infty} d_\nu = -\infty$ and let for each $\nu \in \mathbb{N}$ a pseudo-differential operator $Q_\nu \in \Psi^{d_\nu}(E, F)$ be given. Then there exists a properly supported $Q \in \Psi(E, F)$ such that*

$$Q \sim \sum_{\nu=0}^{\infty} Q_\nu. \tag{8.1}$$

The operator Q is uniquely determined modulo $\Psi^{-\infty}(E,F)$.

Proof If Q' is a second pseudo-differential operator with this property, then it follows from the definition of $\sim$ that $Q - Q'$ belongs to $\Psi^d(E,F)$ for every d. Hence, $Q - Q' \in \Psi^{-\infty}(E,F)$ and uniqueness follows.

Let $d = \max_\nu d_\nu$. In view of Lemma 8.3.5 it suffices to establish the existence of an operator $Q \in \Psi^d(E,F)$ such that (8.1).

First we consider the case that M is an open subset of $\mathbb{R}^n$ and that E and F are trivial of the form $E = M \times \mathbb{C}^k$ and $F = M \times \mathbb{C}^l$. Then $\Psi(E,F) = M_{l,k}(\Psi(M))$. Let $1 \leq i \leq l$ and $1 \leq j \leq k$. For every $\nu \in \mathbb{N}$ there exists a symbol $(q_\nu)_{ij} \in S^{d_\nu}(M)$ such that

$$(Q_\nu)_{ij} = \Psi_{(q_\nu)_{ij}}.$$

By Lemma 5.5.1 there exists a symbol $q_{ij} \in S^d(M)$ such that

$$q_{ij} \sim \sum_{\nu \in \mathbb{N}} (q_\nu)_{ij}.$$

Let $Q \in \Psi^d(E,F)$ be the pseudo-differential operator with $Q_{ij} = \Psi_{q_{ij}}$ for all $1 \leq i \leq l$, $1 \leq j \leq k$. Then Q satisfies (8.1).

We now turn to the case that both E and F admit trivializations $\tau_E : E \to E' = M \times \mathbb{C}^k$ and $\tau_F : F \to F' = M \times \mathbb{C}^l$. Then by the first part of the proof there exists a $P \in \Psi(E',F')$ such that

$$P \sim \sum_{\nu \in \mathbb{N}} \tau_*(Q_\nu).$$

Put $Q = \tau_*^{-1}(P)$, then Q satisfies (8.1). It remains to establish the general case.

Let $\{U_j\}_{j\in J}$ be an open cover of M such that both bundles E and F admit trivializations on U_j, for every $j \in J$. We may select such a cover with the additional property that it is locally finite and that there exists a partition of unity $\{\psi_j\}_{j\in J}$ such that $\operatorname{supp}\psi_j \subset U_j$ for all $j \in J$. By the first part of the proof there exists for each j an operator $Q_j \in \Psi^d(E_{U_j}, F_{U_j})$ such that

$$Q_j \sim \sum_{\nu \in \mathbb{N}} (Q_\nu)_{U_j}.$$

Clearly, for all i,j such that $U_{ij} := U_i \cap U_j \neq \emptyset$, both operators $(Q_i)_{U_{ij}}$ and $(Q_j)_{U_{ij}}$ have the expansion $\sum_\nu (Q_\nu)_{U_{ij}}$. This implies that the difference of these operators belongs to $\Psi^{-\infty}(U_{ij})$. By the gluing property for $\Psi^d/\Psi^{-\infty}$, see Exercise 7.3.10 and Remark 8.1.2, it follows that there exists a $Q \in \Psi^d(E,F)$ such that $Q_{U_j} - Q_j \in \Psi^{-\infty}(E_{U_j}, F_{U_j})$ for all j. It follows that for all j we have

$$Q_{U_j} \sim \sum_{\nu \in \mathbb{N}} (Q_\nu)_{U_j}.$$

This implies (8.1). □

Theorem 8.4.6. *Let E, F be two complex vector bundles on a manifold M. Let $P \in \Psi^d(E,F)$ be a properly supported elliptic pseudo-differential operator. Then* *there exists a properly supported pseudo-differential operator $Q \in \Psi^{-d}(F, E)$ such that*

$$QP - I \in \Psi^{-\infty}(E, E). \tag{8.2}$$

The operator Q is uniquely determined modulo $\Psi^{-\infty}(F, E)$ and satisfies

$$PQ - I \in \Psi^{-\infty}(F, F). \tag{8.3}$$

Remark 8.4.7. An operator Q with the above properties is called a *parametrix* for P.

Proof It follows from Corollary 8.4.3 that there exists a properly supported operator $Q_0 \in \Psi^{-d}(F, E)$ such that $Q_0P - I \in \Psi^{-1}(E, E)$ and $PQ_0 - I \in \Psi^{-1}(E, E)$. Put $R = I - Q_0P$. Then R is properly supported. It follows that $R^k \in \Psi^{-k}(E, E)$. Hence, there exists a pseudo-differential operator $A \in \Psi^0(E, E)$ such that

$$A \sim \sum_{k=0}^{\infty} R^k.$$

It is now a straightforward matter to verify that $A(I - R) - I \in \Psi^{-n}$ for all $n \in \mathbb{N}$. It follows that $A(I-R)-I \in \Psi^{-\infty}$. Put $Q = AQ_0$. Then $Q \in \Psi^{-d}(F, E)$ is properly supported and

$$QP - I = AQ_0P - I = A(I - R) - I \in \Psi^{-\infty}(E, E).$$

This shows the existence of Q such that (8.2). We will show that Q also satisfies (8.3). Put $B = QP - I$. Then B is a properly supported smoothing operator in $\Psi^{-\infty}(E, E)$. By what we proved so far there exists a properly supported operator $P_1 \in \Psi^d(E, F)$ such that $P_1Q - I \in \Psi^{-\infty}(F, F)$. The operator $C := P_1Q - I$ is properly supported. We now observe that $P_1QP = P(I + B) = P + PB$, and that $P_1QP = (I + C)P_1 = P_1 + CP_1$. Hence $P - P_1 \in \Psi^{-\infty}(E, F)$ and we conclude that $D := PQ-I \in \Psi^{-\infty}(F, F)$. This establishes the existence of Q.

To establish uniqueness, let $Q' \in \Psi^{-d}(F, E)$ be a properly supported operator with the same property as Q. Then $E := Q'P - I$ is smoothing. It follows that

$$Q' - Q = Q'(PQ) - Q'D - Q = EQ - Q'D$$

is a smoothing operator. □

Corollary 8.4.8. *Let $P \in \Psi^d(M, E, F)$ be a properly supported elliptic pseudo-differential operator. Then for all $u \in \mathcal{D}'(E)$ we have*

$$\operatorname{singsupp} u = \operatorname{singsupp} Pu.$$

In particular, if Pu is smooth, then u is smooth.

Proof Since P is pseudo-local, $\operatorname{singsupp} Pu \subset \operatorname{singsupp} u$. Let $Q \in \Psi^{-d}$ be a properly supported parametrix for P. Then $QP - I$ is a properly supported smoothing operator, hence $QPu - u$ is smooth. Since Q is pseudo-local, it follows that $\operatorname{singsupp} u \subset \operatorname{singsupp}(QPu) \subset \operatorname{singsupp} Pu$. □

Remark 8.4.9. Let $P : \Gamma^\infty(E) \to \Gamma^\infty(F)$ be a differential operator of order d. Then by locality of P is follows that $\langle Pf, g\rangle = 0$ for all $f \in \Gamma_c^\infty(E)$ and $g \in \Gamma_c^\infty(F^\vee)$ such that $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$. This implies that the distribution kernel of P is supported by the diagonal of $M \times M$. In particular, P is properly supported. It follows that the above corollary applies to elliptic differential operators.