Analysis on Manifolds
Lecture notes for the 2009/2010 Master Class

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LECTURE 9
The index of an elliptic operator

9.1. Pseudo-differential operators and Sobolev space

We recall the definition of the Sobolev space $H_s(\mathbb{R}^n)$, for every $s \in \mathbb{R}$, from Definition 4.3.12, The Sobolev space $H_s(\mathbb{R}^n)$ comes equipped with the inner product that makes $\mathcal{F}$ an isometry from $H_s(\mathbb{R}^n)$ to $L^2_s(\mathbb{R}^n)$. The associated norm on $H_s(\mathbb{R}^n)$ is denoted by $\| \cdot \|_s$. Since Fourier transform is an isometry from $L^2(\mathbb{R}^n)$ to itself, we see that

$$H_0(\mathbb{R}^n) = L^2(\mathbb{R}^n) \tag{9.1}$$

as Hilbert spaces. We recall that for $s < t$ we have $H_t(\mathbb{R}^n) \subset H_s(\mathbb{R}^n)$ with continuous inclusion. The intersection of these spaces, denoted $H_\infty(\mathbb{R}^n)$, and equipped with all Sobolev norms $\| \cdot \|_s$, is a Fréchet space.

We note that for all $s \in \mathbb{R} \cup \{\infty\}$ we have $C^\infty_c(\mathbb{R}^n) \subset H_s(R^n) \subset \mathcal{D}'(\mathbb{R}^n)$ with continuous inclusion maps. Thus, the following lemma implies that $H_s(\mathbb{R}^n)$ is a local space in the sense of Lecture 3, Definition 3.1.1.

Lemma 9.1.1. *Let $s \in \mathbb{R} \cup \{\infty\}$. Then the multiplication map $(\varphi, f) \mapsto M_\varphi(f)$, $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, restricts to a continuous bilinear map $\mathcal{S}(\mathbb{R}^n) \times H_s(\mathbb{R}^n) \to H_s(\mathbb{R}^n)$.*

Remark 9.1.2. This should have been the statement of Lemma 4.3.19, at least for $s < \infty$, and a proof should have been inserted.

Proof It suffices to prove this for finite s. It follows from Lemma 7.1.3 that $\mathcal{F}(M_\varphi(f)) = \mathcal{F}(\varphi) * \mathcal{F}(f)$. Since $\mathcal{F}$ is a topological linear isomorphism from $\mathcal{S}(\mathbb{R}^n)$ onto itself, and from $H_s(\mathbb{R}^n)$ onto $L^2_s(\mathbb{R}^n)$, the result of the lemma is equivalent to the statement that convolution defines a continuous bilinear map $\mathcal{S}(\mathbb{R}^n) \times L^2_s(\mathbb{R}^n) \to L^2_s(\mathbb{R}^n)$. This is what we will prove.

Let $\varphi, f, g \in \mathcal{S}(\mathbb{R}^n)$. Then for all $\xi, \eta \in \mathbb{R}^n$ we have

$$(1 + \|\xi - \eta\|)^{-s}(1 + \|\xi\|)^s \leq (1 + \|\eta\|)^{|s|}, \tag{9.2}$$

hence

$$\begin{aligned}
&|\langle g, \varphi * f \rangle| \\
&\quad \leq \iint |\, g(\xi)\varphi(\eta)f(\xi - \eta)\,|\; d\eta d\xi \\
&\quad \leq \int |\varphi(\eta)|(1 + \|\eta\|)^{|s|} \int |g(\xi)|(1 + \|\xi\|)^{-s}|f(\xi - \eta)|(1 + \|\xi - \eta\|)^s \; d\xi d\eta \\
&\quad \leq \int |\varphi(\eta)|\,(1 + \|\eta\|)^{|s|}\; d\eta \; \|f\|_{L^2_s}\|g\|_{L^2_{-s}},
\end{aligned}$$

by the Cauchy-Schwartz inequality for the L^2-inner product. Let $N \in \mathbb{N}$ be such that $|s| - N < -n$; then there exists a continuous seminorm ν_N on $\mathcal{S}(\mathbb{R}^n)$

such that for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$|\varphi(\eta)|\,(1+\|\eta\|)^{|s|} \leq \nu_N(\varphi)(1+\|\eta\|)^{-N+|s|}, \qquad (\eta \in \mathbb{R}^n).$$

We conclude that

$$|\langle g, \varphi * f\rangle\| \leq C\nu_N(\varphi)\|f\|_{L^2_s}\|g\|_{L^2_{-s}},$$

with $C = \int(1+\|\eta\|)^{|s|-N}\,d\eta$ a positive real number. Now this is valid for all g in the dense subspace $\mathcal{S}(\mathbb{R}^n)$ of the space $L^2_{-s}(\mathbb{R}^n)$, whose dual is isometrically isomorphic to $L^2_s(\mathbb{R}^n)$. Therefore,

$$\|\varphi * f\|_{L^2_s} \leq C\nu_N(\varphi)\|f\|_{L^2_s}, \qquad (\varphi, f \in \mathcal{S}(\mathbb{R}^n)).$$

By density of $\mathcal{S}(\mathbb{R}^n)$ in $L^2_s(\mathbb{R}^n)$ it now follows that the convolution product has a continuous bilinear extension to a map $\mathcal{S}(\mathbb{R}^n) \times L^2_s(\mathbb{R}^n) \to L^2_s(\mathbb{R}^n)$. The latter space is included in $\mathcal{S}'(\mathbb{R}^n)$ with continuous inclusion map. Hence the present extension of the convolution product must be the restriction of the convolution product $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. □

As said, it follows from the above lemma that $H_s(\mathbb{R}^n)$ is a functional space. In view of the general discussion in Lecture 3 we may now define the local Sobolev space as follows.

Definition 9.1.3. Let $U \subset \mathbb{R}^n$ be open and let $s \in \mathbb{R} \cup \infty$. The *local Sobolev space* $H_{s,\,\mathrm{loc}}(U)$ is defined to be the space of $f \in \mathcal{D}'(U)$ with the property that $\psi f \in H_s(\mathbb{R}^n)$ for all $\psi \in C^\infty_c(\mathbb{R}^n)$. The space $H_s(\mathbb{R}^n)$ is equipped with the locally convex topology induced by the collection of seminorms $\nu_\psi : f \mapsto \|\psi f\|_s$, for $\psi \in C^\infty_c(U)$.

The space $L^2_{\mathrm{loc}}(U)$ is defined in a similar fashion. In view of (9.1),

$$L^2_{\mathrm{loc}}(U) = H_{0,\mathrm{loc}}(U),$$

including topologies. Since $H_s(\mathbb{R}^n)$ is a Hilbert space, if follows from the theory developed in Lecture 3 that $H_{s,\,\mathrm{loc}}(U)$ (with the specified topology) is a Fréchet space.

It follows from the Sobolev lemma, Lemma 4.3.16, that $C^\infty_c(\mathbb{R}^n) \subset H_\infty \subset C^\infty(\mathbb{R}^n)$ with continuous inclusion maps. This in turn implies that

$$H_{\infty,\mathrm{loc}}(U) = C^\infty(U),$$

including topologies.

The Sobolev spaces behave very naturally under the action of pseudo-differential operators.

Lemma 9.1.4. *Let $\mathcal{K} \subset \mathbb{R}^n$ be a compact subset, and let $-d, s \in \mathbb{R} \cup \{\infty\}$. Then the map*

$$(p, f) \mapsto \Psi_p(f), \;\; S^d_{\mathcal{K}}(\mathbb{R}^n) \times C^\infty_c(\mathbb{R}^n) \to C^\infty_{\mathcal{K}}(\mathbb{R}^n),$$

has a unique extension to a continuous bilinear map

$$S^d_{\mathcal{K}}(\mathbb{R}^n) \times H_s(\mathbb{R}^n) \to H_{s-d,\mathcal{K}}(\mathbb{R}^n)$$

Proof It suffices to prove this for s, d finite, which we will now assume to be the case. Uniqueness follows from density of $C_c^\infty(\mathbb{R}^n)$ in $H_s(\mathbb{R}^n)$. Thus, it suffices to establish existence. Given $p \in S^d_{\mathcal{K}}(\mathbb{R}^n)$, let $\mathcal{F}_1 p$ denote the Fourier transform of the function $(x, \xi) \mapsto p(x, \xi)$ with respect to the first variable. If $\alpha \in \mathbb{N}^n$, then

$$\begin{aligned}(1 + \|\xi\|)^{-d} |\eta^\alpha \mathcal{F}_1 p(\eta, \xi)| &= |\mathcal{F}_1((1 + \|\xi\|)^{-d} \partial_x^\alpha p)(\eta, \xi)| \\ &\leq \operatorname{vol}(\mathcal{K}) \sup \left[(1 + \|\xi\|)^{-d} |\partial_x^\alpha p(x, \xi)|\right].\end{aligned}$$

It follows that for every $N \in \mathbb{N}$ there exists a continuous seminorm μ_N on $S^d_{\mathcal{K}}(\mathbb{R}^n)$ such that

$$(1 + \|\eta\|)^N (1 + \|\xi\|)^{-d} |\mathcal{F}_1 p(\eta, \xi)| \leq \mu_N(p), \qquad ((\eta, \xi) \in \mathbb{R}^{2n}),$$

for all $p \in S^d_{\mathcal{K}}(\mathbb{R}^n)$.

Let now $p \in S^d_{\mathcal{K}}(\mathbb{R}^n)$ and $f, g \in C_c^\infty(\mathbb{R}^n)$. Then it follows that

$$\begin{aligned}\langle g, \Psi_p f\rangle &= \iint e^{i\xi x} g(x) p(x, \xi) \widehat{f}(\xi)\, d\xi\, dx \\ &= \iint e^{i\xi x} g(x) p(x, \xi) \widehat{f}(\xi)\, dx\, d\xi \\ &= \iint \widehat{g}(\xi - \eta) \mathcal{F}_1 p(\eta, \xi) \widehat{f}(\xi)\, d\eta\, d\xi.\end{aligned}$$

We obtain

$$\begin{aligned}&|\langle g, \Psi_p f\rangle| \\ &\leq \iint F_p(\eta, \xi)\, (1 + \|\xi - \eta\|)^{d-s} |\widehat{g}(\xi - \eta)|\, (1 + \|\xi\|)^s |\widehat{f}(\xi)|\, d\xi\, d\eta \\ &\leq \int \sup_{\xi \in \mathbb{R}^n} F_p(\eta, \xi)\, d\eta\ \ \|g\|_{d-s} \|f\|_s, \qquad (9.3)\end{aligned}$$

where

$$\begin{aligned}F_p(\eta, \xi) &= |\mathcal{F}_1 p(\eta, \xi)| (1 + \|\xi - \eta\|)^{s-d} (1 + \|\xi\|^{-s} \\ &\leq |\mathcal{F}_1 p(\eta, \xi)| (1 + \|\xi\|)^{-d} (1 + \|\eta\|)^{|s-d|} \\ &\leq (1 + \|\eta\|)^{|s-d|-N}\, \mu_N(p).\end{aligned}$$

In the above estimation we have used (9.2) with $s - d$ in place of s. Fix N such that $|s - d| - N < -n$. Then combining the last estimate with (9.3) we see that there exists a constant $C > 0$ such that for all $f, g \in C_c^\infty(\mathbb{R}^n)$ and $p \in S^d_{\mathcal{K}}(\mathbb{R}^n)$,

$$|\langle g, \Psi_p f\rangle| \leq C\, \mu_N(p) \|g\|_{d-s} \|f\|_s.$$

The space $C_c^\infty(\mathbb{R}^n)$ is dense in the Hilbert space $H_{d-s}(\mathbb{R}^n)$ whose dual is isometrically isomorphic with $H_{s-d}(\mathbb{R}^n)$. This implies that

$$\|\Psi_p(f)\|_{s-d} \leq C\, \mu_N(p) \|f\|_s, \qquad (p \in S^d_{\mathcal{K}}(\mathbb{R}^n), f \in C_c^\infty(\mathbb{R}^n)).$$

it follows that the map $(p, f) \mapsto \Psi_p(f)$ has a continuous bilinear extension $\beta : S^d_{\mathcal{K}}(\mathbb{R}^n) \times H_s(\mathbb{R}^n) \to H_{s-d}(\mathbb{R}^n)$. Since β maps the dense subspace $S^d_{\mathcal{K}}(\mathbb{R}^n) \times C_c^\infty(\mathbb{R}^n)$ into the closed subspace $H_{s-d, \mathcal{K}}(\mathbb{R}^n)$ it follows that β maps continuous bilinearly into this closed subspace as well. □

The *local* Sobolev spaces behave very naturally under the action of pseudo-differential operators as well.

Proposition 9.1.5. *Let $P \in \Psi^d(U)$ be properly supported, $d \in \mathbb{R} \cup \{-\infty\}$. Then for every $s \in \mathbb{R} \cup \{\infty\}$ the operator $P : \mathcal{D}'(U) \to \mathcal{D}'(U)$ restricts to a continuous linear operator $P_s : H_{s,\mathrm{loc}}(U) \to H_{s-d,\mathrm{loc}}(U)$.*

Proof By Lemma 7.1.9 there exists a $p \in S^d(U)$ such that $P = \Psi_p$. Let $\psi \in C_c^\infty(U)$ and put $B = \mathrm{supp}\,\psi$. Then it suffices to show that the operator $Q := M_\psi \circ P$ is continuous linear from $H_{s,\mathrm{loc}}(U)$ to $H_{s-d}(\mathbb{R}^n)$. We note that $Q = \Psi_q$, where $q \in S_c^d(U) \subset S_c^d(\mathbb{R}^n)$ is given by $q = \psi p$. Since P is properly supported, there exists a compact subset $\mathcal{K}$ of U such that the kernel of Q has support contained in $B \times \mathcal{K}$. Let $\chi \in C_c^\infty(U)$ be such that $\chi = 1$ on an open neighborhood of $\mathcal{K}$. Then $Q = Q \circ M_\chi$ on $C_c^\infty(U)$, hence on $\mathcal{D}'(U)$, hence on $H_{s,\mathrm{loc}}(U)$. Put $A = \mathrm{supp}\,\chi$. Then M_χ is continuous linear $H_{s,\mathrm{loc}}(\mathbb{R}^n) \to H_{s,A}(\mathbb{R}^n)$. Moreover, by Lemma 9.1.4 Ψ_q is continuous linear $H_{s,A}(\mathbb{R}^n) \to H_{s-d,B}(\mathbb{R}^n)$. Therefore, $Q = \Psi_q \circ M_\chi$ is continuous linear $H_{s,\mathrm{loc}}(\mathbb{R}^n) \to H_{s-d}(\mathbb{R}^n)$. $\square$

9.2. Sobolev spaces on manifolds

In the previous section we have seen that the local Sobolev spaces are functional in the sense of Lecture 3. In order to be able to extend these spaces to manifolds, we need to establish their invariance under diffeomorphims. We will do this through characterizing them by elliptic pseudo-differential operators, which are already known to behave well under diffeomorphisms. A first result in this direction is the following.

Proposition 9.2.1. *Let $s \in \mathbb{R}$, and let $P \in \Psi^s(U)$ be a properly supported elliptic pseudo-differential operator. Then*

$$H_{s,\mathrm{loc}}(U) = \{f \in \mathcal{D}'(U) \mid \; Pf \in L^2_{\mathrm{loc}}(U)\}.$$

Proof If $f \in H_{s,\mathrm{loc}}(U)$ then $f \in \mathcal{D}'(U)$ and $Pf \in H_{0,\mathrm{loc}}(U) = L^2_{\mathrm{loc}}(U)$. This proves one inclusion. To prove the converse inclusion, we use that by Theorem 8.4.6 applied with $M = U$ and $E = F = \mathbb{C}_U$, there exists a properly supported $Q \in \Psi^{-s}(U)$ such that $Q \circ P = I + T$, with $T \in \Psi^{-\infty}(U)$ a properly supported smoothing operator. If $f \in \mathcal{D}'(U)$ and $Pf \in L^2_{\mathrm{loc}}(U) = H_{0,\mathrm{loc}}(U)$, then $QPf \in H_{s,\mathrm{loc}}(U)$ and $Tf \in C^\infty(U) \subset H_{s,\mathrm{loc}}(U)$. Hence, $f = QPf - Tf \in H_{s,\mathrm{loc}}(U)$. $\square$

By combination of the above result with the following lemma, it can be shown that the local Sobolev spaces behave well under diffeomorphisms.

Lemma 9.2.2. *Let $U \subset \mathbb{R}^n$ be an open subset and let $d \in \mathbb{R}$. There exists a properly supported elliptic operator in $\Psi^d(U)$.*

Proof Let d be finite. Then the function $p = p_d : \mathbb{R}^{2n} \to \mathbb{C}$ defined by $p(x,\xi) = (1 + \|\xi\|^2)^{d/2}$ belongs to the symbol space $S^d(\mathbb{R}^n)$. Since $p_d p_{-d} = 1$, the operator $R = \Psi_{p_d} \in \Psi^d(\mathbb{R}^n)$ is elliptic. Its restriction $P = R_U$ to U belongs to $\Psi^d(U)$ and has principal symbol $[(p_d)_U]$ hence is elliptic as well. By Lemma 8.3.5 there exists a properly supported $P_0 \in \Psi^d(U)$ such that $P - P_0$ is a smoothing operator. Hence, P_0 has the same principal symbol as P and we see that P_0 is elliptic. $\square$

Let $\varphi : U \to V$ be a diffeomorphism of open subsets of $\mathbb{R}^n$. This diffeomorphism induces a topological linear isomorphism

$$\varphi_* : \mathcal{D}'(U) \to \mathcal{D}'(V). \tag{9.4}$$

Theorem 9.2.3. *Let $\varphi : U \to V$ be a diffeomorphism of open subsets of $\mathbb{R}^n$ and let $s \in \mathbb{R} \cup \{\infty\}$.*

(a) *The map (9.4) restricts to a linear isomorphism*

$$\varphi_{*s} : H_{s,\,\mathrm{loc}}(U) \to H_{s,\,\mathrm{loc}}(V).$$

(b) *The isomorphism φ_{*s} is topological.*

Proof It suffices to prove the result for finite s, the result for $s = \infty$ is then a consequence.

We will first obtain (a) as a consequence of Proposition 9.2.1 and Lemma 9.2.2. By the latter lemma there exists a properly supported elliptic operator $P \in \Psi^d(U)$. Let $P' = \varphi_*(P)$. Then $P' \in \Psi^d(V)$ and $P'\varphi_*(f) = \varphi_*(Pf)$ for all $f \in \mathcal{D}'(U)$. The principal symbol of P' is given by $\sigma^d(P') = \varphi_*(\sigma^d(P))$, hence elliptic. Therefore, P' is elliptic. The kernel of P' has support contained in $(\varphi \times \varphi)(\mathrm{supp}\, K_P)$, hence is properly supported.

By a straightforward application of the substitution of variables formula, it follows that φ_* restricts to a topological linear isomorphism $L^2_{\mathrm{loc}}(U) \to L^2_{\mathrm{loc}}(V)$. Let now $f \in \mathcal{D}'(U)$. Then

$$\begin{aligned} f \in H_{s,\,\mathrm{loc}}(U) \quad &\Longleftrightarrow \quad Pf \in L^2_{\mathrm{loc}}(U) \iff \varphi_*(Pf) \in L^2_{\mathrm{loc}}(V) \\ &\Longleftrightarrow \quad P'\varphi_*(f) \in L^2_{\mathrm{loc}}(V) \iff \varphi_*(f) \in H_{s,\,\mathrm{loc}}(V). \end{aligned}$$

This proves (a). In order to prove (b) we need characterizations of the topology on $H_{s,\,\mathrm{loc}}$ that behave well under diffeomorphisms. These will first be given in two lemmas below. The present proof will be completed right after those lemmas. □

Lemma 9.2.4. *Let $s \geq 0$ and let $P \in \Psi^s(U)$ be properly supported and elliptic. Then the topology on $H_{s,\,\mathrm{loc}}(U)$ is the weakest locally convex topology for which both the inclusion map $j : H_{s,\,\mathrm{loc}}(U) \to L^2_{\mathrm{loc}}(U)$ and the map $P : H_{s,\,\mathrm{loc}}(U) \to L^2_{\mathrm{loc}}(U)$ are continuous.*

Proof For the topology on $H_{s,\,\mathrm{loc}}(U)$ the mentioned maps j and P are continuous with values in $L^2_{\mathrm{loc}}(U)$. Let V be equal to $H_{s,\,\mathrm{loc}}(U)$ equipped with the weakest locally convex topology for which j and P are continuous. Then we must show that the identity map $V \to H_{s,\,\mathrm{loc}}(U)$ is continuous. Let $\varphi \in C_c^\infty(U)$; then it suffices to show that the map $M_\varphi : V \to H_s(\mathbb{R}^n)$ is continuous.

Let $Q \in \Psi^{-s}(U)$ be a properly supported parametrix for P. Then

$$\mathrm{I} = QP + T,$$

with $T \in \Psi^{-\infty}(U)$ a properly supported smoothing operator. Let $A = \mathrm{supp}\,\varphi$. There exists a compact subset $B \subset U$ such that the intersections of both $\mathrm{supp}\, K_Q$ and $\mathrm{supp}\, K_T$ with $A \times U$ are contained in $A \times B$. Let $\psi \in C_c^\infty(U)$ be such that $\psi = 1$ on an open neighborhood of B. Then $M_\varphi \circ Q = M_\varphi \circ Q \circ M_\psi$ and $M_\varphi \circ T = M_\varphi \circ T \circ M_\psi$. We now see that, for all $f \in H_{s,\,\mathrm{loc}}(U)$,

$$\varphi f = M_\varphi(QP + T)(f) = M_\varphi Q M_\psi P f + M_\varphi T M_\psi f.$$

As $M_\varphi \circ Q$ and $M_\varphi \circ T$ define continuous linear maps $H_0(\mathbb{R}^n) = L^2(\mathbb{R}^n) \to H_s(\mathbb{R}^n)$, by Lemma 9.1.4, it follows that there exists a constant $C > 0$ such that

$$\|\varphi f\|_s \leq C(\|\varphi P f\|_0 + \|\psi f\|_0),$$

for all $f \in H_{s,\,\mathrm{loc}}(U)$. The seminorms $f \mapsto \|\varphi P f\|_0$ and $f \mapsto \|\psi f\|_0$ are continuous on V. Hence, $M_\varphi : V \to H_s(\mathbb{R}^n)$ is continuous. □

We will also need a characterization of the topology of $H_{s,\mathrm{loc}}$ by duality, which is invariant under diffeomorphisms for all negative s. Let $D = D_{\mathbb{R}^n}$ denote the density bundle on $\mathbb{R}^n$. Then we have the natural continuous bilinear pairing $\langle\,\cdot\,,\,\cdot\,\rangle : C_c^\infty(\mathbb{R}^n) \times \Gamma_c^\infty(\mathbb{R}^n, D) \to \mathbb{C}$ given by

$$\langle f, \gamma\rangle = \int_{\mathbb{R}^n} f\gamma.$$

This pairing induces a continuous injection of the space $C_c^\infty(\mathbb{R}^n)$ into the topological dual $\mathcal{D}'(\mathbb{R}^n)$ of $\Gamma_c^\infty(\mathbb{R}^n, D)$. This pairing also induces a continuous injection of $\Gamma_c^\infty(\mathbb{R}^n, D)$ into $\mathcal{D}'(\mathbb{R}^n, D) \simeq C_c^\infty(\mathbb{R}^n)'$. We note that the map $g \mapsto g dx$ defines a topological linear isomorphism $C_c^\infty(\mathbb{R}^n) \to \Gamma_c^\infty(\mathbb{R}^n, D)$ and extends to a continuous linear isomorphism $\mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n, D)$. For $s \in \mathbb{R}$, the image of the Sobolev space $H_s(\mathbb{R}^n)$ under this isomorphism, equipped with the transferred Hilbert structure, is denoted by $H_s(\mathbb{R}^n, D)$.

By transposition, the inclusion $\Gamma_c^\infty(\mathbb{R}^n, D) \to H_s(\mathbb{R}^n, D)$ induces a continuous linear map $H_s(\mathbb{R}^n, D)' \to \mathcal{D}'(\mathbb{R}^n)$ which is injective by density of $\Gamma_c^\infty(\mathbb{R}^n, D)$ in $H_s(\mathbb{R}^n, D)$. Of course the induced map is given by $u \mapsto u|_{\Gamma_c^\infty(\mathbb{R}^n, D)}$.

The perfectness of the pairing of Lemma 4.3.18 can now be expressed as follows.

Lemma 9.2.5. *Let $s \in \mathbb{R}$. The image of the injection $H_s(\mathbb{R}^n, D)' \to \mathcal{D}'(\mathbb{R}^n)$ equals $H_{-s}(\mathbb{R}^n)$. The associated bijection $H_s(\mathbb{R}^n, D)' \to H_{-s}(\mathbb{R}^n)$ is a topological linear isomorphism.*

More generally, if $U \subset \mathbb{R}^n$ is an open subset, we define $H_{s,\mathrm{comp}}(U, D)$ to be the image of $H_{s,\mathrm{comp}}(U)$ in $\mathcal{D}'(U, D)$ under the map $f \mapsto f dm$, equipped with the topology that makes this map a topological isomorphism. The natural inclusion $\Gamma_c^\infty(U, D) \to H_{s,\mathrm{comp}}(U, D)$ induces a continuous linear injection

$$H_{s,\mathrm{comp}}(U, D)' \hookrightarrow \mathcal{D}'(U).$$

Here, the space on the left is equipped with the strong dual topology.

Corollary 9.2.6. *Let $s \in \mathbb{R}$. Then the image of $H_{s,\mathrm{comp}}(U, D)'$ in $\mathcal{D}'(U)$ equals $H_{-s,\mathrm{loc}}(U)$. The associated bijection $H_{s,\mathrm{comp}}(U, D)' \to H_{-s,\mathrm{loc}}(U)$ is a topological linear isomorphism.*

Proof Let $j : H_{s,\mathrm{comp}}(U)' \to \mathcal{D}'(U)$ denote the natural linear injection. Let $\varphi \in C_c^\infty(M)$. Then for all $u \in H_{s,\mathrm{comp}}(U)'$ we have

$$M_\varphi \circ j(u) = j(u \circ M_\varphi).$$

The map $u \mapsto u \circ M_\varphi$ is continuous linear $H_{s,\mathrm{comp}}(U)' \to H_s(\mathbb{R}^n)'$. It follows that $u \mapsto M_\varphi \circ j(u) = j(M_\varphi u)$ is continuous linear $H_{s,\mathrm{comp}}(U)' \to H_{-s}(\mathbb{R}^n)$. Since this holds for all φ, it follows that j is continuous linear $H_{s,\mathrm{comp}}(U)' \to H_{-s,\mathrm{loc}}(U)$ as stated.

Conversely, let $\mathcal{K} \subset U$ be compact. Let $\mathcal{K}'$ be a compact neighborhood of $\mathcal{K}$ in U. Then $H_{s,\mathcal{K}}(\mathbb{R}^n)$ is contained in the closure of $C^\infty_{\mathcal{K}'}(\mathbb{R}^n)$ in $H_s(\mathbb{R}^n)$. Let $v \in H_{-s,\mathrm{loc}}(U)$ and let $\varphi \in C^\infty_c(U)$ be such that $\varphi = 1$ on a neighborhood of $\mathcal{K}'$. Then $M_\varphi v \in H_{-s}(\mathbb{R}^n)$, hence, $M_\varphi v = j(k_\varphi(v))$ for a unique $k_\varphi(v) \in H_s(\mathbb{R}^n)'$. Moreover, the map $k_\varphi : H_{-s,\mathrm{loc}}(\mathbb{R}^n) \to H_s(\mathbb{R}^n)'$ is continuous linear by the above lemma. The restriction of $k_\varphi(v)$ to $C^\infty_{\mathcal{K}'}(\mathbb{R}^n)$ is independent of the choice of φ, and therefore so is the map $k_{\mathcal{K}} : v \mapsto k_\varphi(v)|_{H_{s,\mathcal{K}}}$. The map $k_{\mathcal{K}}$ is continuous linear. Moreover, if $\mathcal{K}_1 \subset \mathcal{K}_2$ are compact subsets of U then $k_{\mathcal{K}_1}(v) = k_{\mathcal{K}_2}(v)|\mathcal{K}_1$. It follows that there exists a unique linear map $k : H_{-s,\mathrm{loc}}(U) \to H_{s,\mathrm{comp}}(U)'$ such that $k_A(v) = k(v)|_A$ for all $v \in H_{-s,\mathrm{loc}}(U)$ and all $A \subset U$ compact. As all the k_A are continuous, it follows that k is continuous. Now $j \circ k = \mathrm{I}$ and it follows that j defines a continuous linear isomorphism from $H_{s,\mathrm{comp}}(U)'$ onto $H_{-s,\mathrm{loc}}(U)$. $\square$

Completion of the proof of Theorem 9.2.3. First assume that $s \geq 0$. Let $P \in \Psi^s(U)$ be a properly supported elliptic operator, and let $P' = \varphi_*(P)$ be as in part (a) of the proof. Then by Lemma 9.2.4 the topology of $H_s(U)$ is the weakest locally convex topology for which both the inclusion $j_U : H_s(U) \to L^2_{\mathrm{loc}}(U)$ and the map $P : H_s(U) \to L^2_{\mathrm{loc}}(U)$ are continuous. Likewise, the topology on $H_s(V)$ is the weakest for which both the inclusion map $j_V : H_s(V) \to L^2_{\mathrm{loc}}(V)$ and $P' : H_s(V) \to L^2_{\mathrm{loc}}(V)$ are continuous. The map $\varphi_* : L^2_{\mathrm{loc}}(U) \to L^2_{\mathrm{loc}}(V)$ is a topological linear isomorphism and the following diagrams commute:

$$\begin{array}{ccc} H_s(U) & \xrightarrow{j_U} & L^2_{\mathrm{loc}}(U) \\ \varphi_{*s}\downarrow & & \downarrow \varphi_* \\ H_s(V) & \xrightarrow{j_V} & L^2_{\mathrm{loc}}(V) \end{array} \qquad \begin{array}{ccc} H_s(U) & \xrightarrow{P} & L^2_{\mathrm{loc}}(U) \\ \varphi_{*s}\downarrow & & \downarrow \varphi_* \\ H_s(V) & \xrightarrow{P'} & L^2_{\mathrm{loc}}(V) \end{array}$$

It follows that φ_{*s} is a topological linear isomorphism.

This proves (b) for $s \geq 0$. We will complete the proof by proving (b) for $s \leq 0$, using the duality expressed in Lemma 9.2.5. We put $t = -s$, so that $t \geq 0$. By the validity of (b) for $s \geq 0$ it follows that the map $\varphi_* : \mathcal{D}'(U) \to \mathcal{D}'(V)$, restricts to a topological linear isomorphism $\varphi_{*t} : H_{t,\mathrm{comp}}(U) \to H_{t,\mathrm{comp}}(V)$. On the other hand, the map φ_* restricts to the topological linear isomorphism $\varphi_* : C^\infty_c(U) \to C^\infty_c(V)$ given by $f \mapsto f \circ \varphi^{-1}$. The map $f \mapsto f dx$ defines a topological linear isomorphism from $\mathcal{D}'(U)$ to $\mathcal{D}'(U, D)$. Likewise, $g \mapsto g dy$ defines a topological linear isomorphism from $\mathcal{D}'(V)$ to $\mathcal{D}'(V, D)$. Now

$$\varphi_*(f dx) = \varphi_*(f)\varphi_*(dx) = M_J \varphi_*(f) dy,$$

where $J : V \to (0, \infty)$ is the positive smooth function given by

$$J(y) = |\det D\varphi(\varphi^{-1})|^{-1}.$$

The map M_J defines a topological linear automorphism of $\mathcal{D}'(V)$ and restricts to a topological linear isomorphism of $H_t(V)$. We conclude that φ_* defines a topological linear isomorphism $\mathcal{D}'(U, D) \to \mathcal{D}'(V, D)$ and restricts to a topological linear isomorphism $\varphi_{*t} : H_{t,\text{comp}}(U, D) \to H_{t,\text{comp}}(V, D)$. Its restriction to $\Gamma_c^\infty(U, D)$ is given by $f dx \mapsto M_J(\varphi^{-1})^*(f)dy$, hence defines a topological linear isomorphism $\Gamma_c^\infty(U, D) \to \Gamma_c^\infty(V, D)$. By taking transposed maps in the commutative diagram

$$\begin{array}{ccc} H_{t,\text{comp}}(U, D) & \longrightarrow & H_{t,\text{comp}}(V, D) \\ \uparrow & & \uparrow \\ \Gamma_c^\infty(U, D) & \longrightarrow & \Gamma_c^\infty(V, D) \end{array}$$

we obtain the commutative diagram

$$\begin{array}{ccc} \mathcal{D}'(U) & \overset{\varphi_*^{-1}}{\longleftarrow} & \mathcal{D}'(V) \\ \uparrow & & \uparrow \\ H_{t,\text{comp}}(U, D)' & \overset{\simeq}{\longleftarrow} & H_{t,\text{comp}}(V, D)'. \end{array}$$

Here the bottom arrow is the transpose of a topological linear isomorphism, hence a topological linear isomorphism of its own right.

In view of Lemma 9.2.5, and using $s = -t$, we see that $\varphi_* : \mathcal{D}'(U) \to \mathcal{D}'(V)$ restricts to a topological linear isomorphism $\varphi_{*s} : H_{s,\text{loc}}(U) \to H_{s,\text{loc}}(V)$. □

In particular, it follows from Theorem 9.2.3 that $H_{s,\text{loc}}$ is an invariant local functional space in the terminology of Lecture 3. It follows that for E a complex vector bundle on a smooth manifold the spaces of sections $H_{s,\text{comp}}(M, E)$ and $H_{s,\text{loc}}(M, E)$ are well defined locally convex topological vector spaces. Moreover, the first of these spaces is contained in the second with continuous inclusion map, and the second space is a Fréchet space.

Lemma 9.2.7. *Let $s, t \in \mathbb{R}$ and let $s < t$. Then $H_{t,\text{loc}}(M, E) \subset H_{s,\text{loc}}(M, E)$ with continuous inclusion map. If M is compact, this inclusion map is compact.*

Proof Let $\{U_j\}_{j \in J}$ be a cover of M by relatively compact open coordinate patches on which the bundle E admits a trivialization. Passing to a locally finite refinement, we may assume that the index set J is countable. Let φ_j be a partition of unity subordinate to the cover. For each $j \in J$ we write $K_j = \text{supp}\, \varphi_j$. Then the map $f \mapsto (\varphi_j f)_{j \in J}$ defines a continuous linear embedding

$$H_{s,\text{loc}}(M, E) \longrightarrow \prod_{j \in J} H_{s,K_j}(U_j, E),$$

for every $s \in \mathbb{R}$. Via a trivialization of E over U_j we may identify $H_{s,K_j}(U_j, E) \simeq H_{s,K_j}(U_j)^k$. As $H_{t,K_j}(U_j)^k \subset H_{s,K_j}(U_j)^k$, for $s < t$, with continuous inclusion map, the first assertion of the lemma follows.

If M is compact, we may take the covering such that the index set J is finite. Then by the above reasoning and application of Rellich's lemma, Lemma 4.5.2, it follows that the following diagram commutes, and that the inclusion map represented by the vertical arrow on the right is the finite direct product of compact maps:

$$\begin{array}{ccc} H_s(M, E) & \longrightarrow & \prod_{j \in J} H_{s,K_j}(U_j, E) \\ \uparrow & & \uparrow \\ H_t(M, E) & \longrightarrow & \prod_{j \in J} H_{t,K_j}(U_j, E). \end{array}$$

As the maps represented by the horizontal arrows are embeddings, it follows that the inclusion $H_t(M,E) \to H_s(M,E)$, represented by the vertical arrow on the left, is compact as well. □

We define

$$H_{\infty,\mathrm{loc}}(M,E) = \cup_{s\in\mathbb{R}}\, H_{s,\mathrm{loc}}(M,E)$$

equipped with the weakest topology for which all inclusion maps $H_{\infty,\mathrm{loc}}(M,E) \to H_{s,\mathrm{loc}}(M,E)$ are continuous. Then by an argument similar to the one used in the proof of the above lemma, it follows from the corresponding local statement (see 9.1.5), that $H_{\infty,\mathrm{loc}}(M,E) = \Gamma^\infty(M,E)$, as topological linear (Fréchet) spaces.

Theorem 9.2.8. *Let E, F be vector bundles on the smooth manifold M. Let $d \in \mathbb{R}\cup\{-\infty\}$ and $s \in \mathbb{R}\cup\{\infty\}$. Finally, let $P \in \Psi^d(E,F)$ be properly supported. Then $P : \mathcal{D}'(M,E) \to \mathcal{D}'(M,F)$ restricts to a continuous linear operator $P_s : H_{s,\mathrm{loc}}(M,E) \to H_{s-d,\mathrm{loc}}(M,F)$.*

Proof First assume that $d = -\infty$, so that P is a properly supported smoothing operator. Then P is continuous linear $\mathcal{D}'(M,E) \to \Gamma^\infty(M,F)$. Since the inclusion $H_{s,\mathrm{loc}}(M,E) \to \mathcal{D}'(M,E)$ is continuous linear, it follows that the restriction $P_s : H_{s,\mathrm{loc}}(M,E) \to \Gamma^\infty(M,F) = H_{\infty,\mathrm{loc}}(M,F)$ is continuous linear.

Now assume that $\{U_j\}_{j\in J}$ is a cover of M with relatively compact open coordinate patches. By paracompactness, we may assume that J is countable and that the cover is locally finite. Let $\{\psi_j\}$ be a partition of unity subordinate to this cover. For each $j \in J$ we select a function $\chi_j \in C_c^\infty(U_j)$ such that $\chi_j = 1$ on an open neighborhood of $\operatorname{supp}\psi_j$. Put $P_j := M_\psi \circ P \circ M_{\chi_j}$. Then it follows that

$$T_j = M_{\psi_j} \circ P - P_j$$

is a properly supported smoothing operator. The supports of the kernels of T_j form a locally finite set, hence $T = \sum_j T_j$ is a well-defined smoothing operator. Moreover, $P_* = \sum_j P_j$ is a well-defined operator in $\Psi^d(E,F)$, which is properly supported by local finiteness of the cover $\{U_j\}$. Moreover,

$$P = P_* + T,$$

so T is properly supported. By the first part of the proof, T maps $H_{s,\mathrm{loc}}(M,E)$ continuously into $C^\infty(M,F)$, hence also continuously into $H_{s-d}(M,F)$. Thus, it suffices to show that P_* is continuous linear $H_{s,\,\mathrm{loc}}(M,E) \to H_{s-d,\mathrm{loc}}(M,F)$. Let $\varphi \in C_c^\infty(M)$. Then it suffices to show that $M_\varphi \circ P_*$ is continuous linear $H_{s,\,\mathrm{loc}}(M,E) \to H_{s-d,B}(M,F)$, where $B = \operatorname{supp}\varphi$. Now $M_\varphi \circ P_* = \sum_j M_\varphi \circ P_j$, the sum extending over the finite set of j for which $\operatorname{supp}\psi_j \cap B \neq \emptyset$. Thus, it suffices to establish the continuity of $(P_j)_s : H_{s,\mathrm{loc}}(M,E) \to H_{s-d,\mathrm{loc}}(M,F)$. This is equivalent to the continuity of $(P_j)_{U_j} : H_{s,\mathrm{loc}}(U_j,E) \to H_{s-d,\mathrm{loc}}(U_j,F)$ which by triviality of the bundles E_{U_j} and F_{U_j} follows from the local scalar result, Proposition 9.1.5. □

Lemma 9.2.9. *Let M be a smooth manifold, and $E \to M$ be a vector bundle. For every $s \in \mathbb{R}$ the natural pairing $\Gamma_c^\infty(M,E) \times \Gamma^\infty(M,E^\vee) \to \mathbb{C}$ has a unique extension to a continuous bilinear pairing*

$$H_{s,\mathrm{comp}}(M,E) \times H_{-s,\mathrm{loc}}(M,E^\vee) \to \mathbb{C}.$$

Moreover, the pairing is perfect, i.e., the induced maps

$$H_{s,\mathrm{comp}}(M,E) \to H_{-s,\mathrm{loc}}(M,E^\vee)', \quad H_{-s,\mathrm{loc}}(M,E^\vee) \to H_{s,\mathrm{comp}}(M,E)'$$

are topological linear isomorphisms.

Proof The proof will be given in an appendix. □

9.3. The index of an elliptic operator

We are now finally prepared to show that every elliptic operator between vector bundles E, F over a compact manifold M has a well defined index.

Theorem 9.3.1. *Let E, F be vector bundles over a compact manifold M. Let $d \in \mathbb{R}$ and let $P \in \Psi^d(E,F)$ be elliptic. Then the transpose $P^t \in \Psi^d(F^\vee, E^\vee)$ is elliptic as well.*

The operator $P_\infty : \Gamma^\infty(M,E) \to \Gamma^\infty(M,F)$ has a finite dimensional kernel, and closed image of finite codimension.

For all $s \in \mathbb{R}$ the operator $P_s : H_s(m,E) \to H_{s-d}(M,F)$ is Fredholm and has index

$$\mathrm{index}\, P_s = \mathrm{index}\,(P_\infty).$$

In particular, the index is independent of s.

Proof Let $p \in S^d(M, \underline{\mathrm{Hom}}\,(E,F))$ be a representative of the principal symbol $\sigma^d(P)$. Then the principal symbol of P^t is represented by

$$p^\vee : (x,\xi) \mapsto p(x,-\xi)^* \otimes \mathrm{I}_{D_x}.$$

Since p is elliptic, $p^\vee$ is elliptic as well, and we conclude that P^t is elliptic.

Since M is compact, the spaces $H_s(M,E)$ and $H_s(M,F)$ carry a Banach topology. Let $Q \in \Psi^{-d}(F,E)$ be a parametrix. Then $QP = \mathrm{I}_E + T$, with $T \in \Psi^{-\infty}(E,E)$ a smoothing operator. The operator T is continuous $H_s(M,E) \to C^\infty(M,E)$, hence continuous $H_s(M,E) \to H_{s+1}(M,E)$. As $H_{s+1}(M,E) \to H_s(M,E)$ with compact inclusion map, it follows that $T_s : H_s(E) \to H_s(E)$ is a compact operator. It follows that $Q_{s-d} \circ P_s = (\mathrm{I}_E)_s + T_s$, hence P_s has left inverse Q_{s-d} modulo a compact operator. Likewise, from $PQ - I_F = T' \in \Psi^{-\infty}(F,F)$ we see that the operator P_s has right inverse Q_{s-d} modulo a compact operator. This implies that P_s is Fredholm. In particular, $\ker P_s$ is finite dimensional.

Since $\ker P_\infty \subset \ker P_s$ and $\ker P_s \subset \Gamma^\infty(M,E)$ by the elliptic regularity theorem, Corollary 8.4.8, it follows that $\ker P_s = \ker P_\infty$, for all $s \in \mathbb{R}$. In particular, $\ker P_\infty$ is finite dimensional.

Likewise, $(P^t)_s : H_s(M, F\vee) \to H_{s-d}(M, E^\vee)$ is Fredholm, $(\ker P^t)_\infty$ is finite dimensional and $\ker(P^t)_s = \ker(P^t)_\infty$ for all s.

We consider the natural continuous bilinear pairing

$$(f,g) \mapsto \langle f,g \rangle, \quad \Gamma^\infty(M,E) \times \Gamma^\infty(M,E^\vee) \to \mathbb{C}. \tag{9.5}$$

The operator $P^t : \Gamma^\infty(M,F^\vee)) \to \Gamma^\infty(M,E^\vee)$ satisfies

$$\langle Pf, h \rangle = \langle f, P^t h \rangle \tag{9.6}$$

for all $f \in \Gamma^\infty(M,E)$ and $h \in \Gamma^\infty(M,F^\vee)$. The natural bilinear pairing (9.5) extends uniquely to a continuous bilinear pairing $H_s(M,E)\times H_{-s}(M,E^\vee) \to \mathbb{C}$, which is perfect by Lemma 9.2.9. The map P_s is the continuous linear extension of $P : \Gamma^\infty(M,E) \to \Gamma^\infty(M,F)$ to a map $H_s(M,E) \to H_{s-d}(M,F)$. Similarly, P^t extends to a continuous linear map $(P^t)_{d-s} : H_{d-s}(M,F^\vee) \to H_{-s}(M,E^\vee)$. By density and continuity, the identity (9.6) implies that more generally,

$$\langle (P^t)_{d-s}f, g\rangle = \langle f, P_s g\rangle$$

for all $f \in H_{d-s}(M,F^\vee)$ and $g \in H_s(M,E)$. In other words,

$$(P_s)^t = (P^t)_{d-s}.$$

This implies that $\ker(P_s)^t = \ker(P^t)_{d-s} = \ker(P^t)_\infty$.

The annihilator of $\operatorname{im} P_s$ in $H_{d-s}(M,F^\vee)$ relative to the natural pairing

$$H_{s-d}(M,F) \times H_{d-s}(M,F^\vee) \to \mathbb{C}$$

equals $\ker(P_s)^t$ hence $\ker(P^t)_\infty$. By perfectness of the pairing, it follows that

$$\begin{aligned}(\ker P^t)_\infty &\simeq \{u \in H_{d-s}(M,F)' \mid u = 0 \text{ on } \operatorname{im} P_s\}\\ &\simeq [H_{d-s}(M,F)/\overline{\operatorname{im}(P_s)}]'.\end{aligned}$$

Since P_s is Fredholm, its image is closed and of finite codimension. Hence $\ker(P^t)_\infty \simeq \operatorname{coker}(P_s)^*$ and it follows that

$$\operatorname{index}(P_s) = \dim\ker P_\infty - \dim\ker(P^t)_\infty.$$

We will complete the proof by showing that $\ker(P^t)_\infty \simeq (\operatorname{coker} P_\infty)^*$, naturally. For this we note that by the elliptic regularity theorem, Theorem 8.4.8,

$$\operatorname{im}(P_s) \cap \Gamma^\infty(M,F) = \operatorname{im} P_\infty.$$

Since $\Gamma^\infty(M,F) \subset H_{s-d}(M,F)$ with continuous inclusion map, and since $\operatorname{im}(P_s)$ is closed in $H_{s-d}(M,F)$, it follows that $\operatorname{im}(P_\infty)$ is closed in $\Gamma^\infty(M,F)$. The annihilator of $\operatorname{im} P$ in $\Gamma^\infty(M,F)' = \mathcal{D}'(M,F^\vee)$ equals the kernel of $(P^t)_{-\infty} : \mathcal{D}'(M,F^\vee) \to \mathcal{D}'(M,E^\vee)$, which in turn equals $\ker(P^t)_\infty$ by the elliptic regularity theorem. This implies that

$$\ker(P^t)_\infty \simeq [\Gamma^\infty(M,F)/\operatorname{im}(P_\infty)]'.$$

As the first of these spaces is finite dimensional, P_∞ has finite dimensional cokernel, and $\ker(P^t)_\infty \simeq [\Gamma^\infty(M,F)/\operatorname{im}(P_\infty)]^*$. □

For obvious reasons, the integer $\operatorname{index}(P_\infty)$ is called the index of P and will more briefly be denoted by $\operatorname{index}(P)$. The following result asserts that the index depends on P through its principal symbol.

Lemma 9.3.2. *Let M be compact, and E, F complex vector bundles on M. Let $P, P' \in \Psi^d(E,F)$ be elliptic operators. Then*

$$\sigma^d(P) = \sigma^d(P') \Rightarrow \operatorname{index}(P) = \operatorname{index}(P').$$

Proof From the equality of the principal symbols, it follows that $P - P' = Q \in \Psi^{d-1}(E,F)$. Let $s \in \mathbb{R}$. The operator Q maps $H_s(M,E)$ to $H_{s-d+1}(M,F)$. The latter space is contained in $H_{s-d}(M,F)$, with compact inclusion map. It follows that Q is compact as an operator $H_s(M,E) \to H_{s-d}(M,F)$. We conclude that $P_s - P'_s$ is compact, hence $\operatorname{index}(P_s) = \operatorname{index}(P'_s)$. □