Notex on Fredholm (and compact) operators

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Abstract

In these separate notes, we give an exposition on Fredholm operators between Banach spaces. In particular, we prove the theorems stated in the last section of the first lecture [1].

Contents

[1] emphasize that some of the extra-material is just for your curiosity and is not needed for the promised proofs. It is a good exercise for you to cross out the parts which are not needed

1 Fredholm operators: basic properties

Let $\mathbb{E}$ and $\mathbb{F}$ be two Banach spaces. We denote by $\mathcal{L}(\mathbb{E},\mathbb{F})$ the space of bounded linear operators from $\mathbb{E}$ to $\mathbb{F}$.

Definition 1.1 *A bounded operator $T : \mathbb{E} \longrightarrow \mathbb{F}$ is called Fredholm if* $\mathrm{Ker}(A)$ *and* $\mathrm{Coker}(A)$ *are finite dimensional. We denote by $\mathcal{F}(\mathbb{E},\mathbb{F})$ the space of all Fredholm operators from $\mathbb{E}$ to $\mathbb{F}$.*
The index of a Fredholm operator A is defined by

$$\mathrm{Index}(A) := \dim(\mathrm{Ker}(A)) - \dim(\mathrm{Coker}(A)).$$

Note that a consequence of the Fredholmness is the fact that $R(A) = \mathrm{Im}(A)$ is closed. Here are the first properties of Fredholm operators.

Theorem 1.2 *Let $\mathbb{E}$, $\mathbb{F}$, $\mathbb{G}$ be Banach spaces.*

(i) If $B : \mathbb{E} \longrightarrow \mathbb{F}$ and $A : \mathbb{F} \longrightarrow \mathbb{G}$ are bounded, and two out of the three operators A, B and AB are Fredholm, then so is the third, and

$$\mathrm{Index}(A \circ B) = \mathrm{Index}(A) + \mathrm{Index}(B).$$

(ii) If A is Fredholm, then so is A^*, and

$$\mathrm{Index}(A^*) = -\mathrm{Index}(A).$$

(iii) $\mathcal{F}(\mathbb{E},\mathbb{F})$ is an open subset of $\mathcal{L}(\mathbb{E},\mathbb{F})$, and

$$\mathrm{Index} : \mathcal{F}(\mathbb{E},\mathbb{F}) \longrightarrow \mathbb{Z}$$

is locally constant.

Proof: Part (i) is a purely algebraic result. We prove that if A and B are Fredholm, then so is AB (the other cases following from the arguments bellow). First of all we have a short exact sequence

$$0 \longrightarrow \mathrm{Ker}(B) \longrightarrow \mathrm{Ker}(AB) \xrightarrow{B} \mathrm{Im}(B) \cap \mathrm{Ker}(A) \longrightarrow 0,$$

and this proves that AB has finite dimensional kernel with

$$\dim(\mathrm{Ker}(AB)) = \dim(\mathrm{Ker}(B)) + \dim(\mathrm{Ker}(A) \cap \mathrm{Im}(B)).$$

Next, we have the exact sequence

$$0 \longrightarrow \frac{\mathrm{Im}(B) + \mathrm{Ker}(A)}{\mathrm{Im}(B)} \longrightarrow \frac{\mathbb{F}}{\mathrm{Im}(B)} \xrightarrow{A} \frac{\mathbb{G}}{\mathrm{Im}(AB)} \longrightarrow \frac{G}{\mathrm{Im}(A)} \longrightarrow 0,$$

where the first map is the obvious inclusion, and the last one is the obvious projection. All the spaces in this sequence, except maybe $\mathrm{Coker}(AB)$, are finite dimensional (the first one is isomorphic to $\mathrm{Ker}(A)/\mathrm{Ker}(A) \cap \mathrm{Im}(B)$, so we deduce that also $\mathrm{Coker}(AB)$ is finite dimensional and

$$\dim(\mathrm{Coker}(AB)) = \dim(\mathrm{Coker}(A)) + \dim(\mathrm{Coker}(B)) - \dim(\mathrm{Ker}(A)) + \dim(\mathrm{Ker}(A) \cap \mathrm{Im}(B)).$$

Combining the last two identities, we get the desired equation for the index.

Part (ii) is easy.

For (iii), let $A \in \mathcal{F}(\mathbb{E},\mathbb{F})$. We choose complements $\mathbb{E}_1$ of $\mathrm{Ker}(A)$ in $\mathbb{E}$, and $\mathbb{F}_2$ of $\mathrm{Im}(A)$ in $\mathbb{F}$. This is possible because $\mathrm{Ker}(A)$ is finite dimensional, and because $\mathrm{Im}(A)$ is closed of finite codimension, respectively. Denote by $i_1 : \mathbb{E}_1 \longrightarrow \mathbb{E}$ the canonical inclusion and by $p : \mathbb{F} \longrightarrow \mathrm{Im}(A)$ the projection. To any operator $S \in \mathcal{L}(\mathbb{E},\mathbb{F})$ we associate the operator $S_0 = pSi : \mathbb{E}_1 \longrightarrow \mathrm{Im}(A)$. Since A_0 is clearly an isomorphism, there exists $\epsilon > 0$ so that, for all S such that $||S - A|| < \epsilon$, S_0 is an isomorphism. For such an S we can also say that $S_0 = pSi$ is Fredholm of index zero. But p is Fredholm of index $-\dim(\mathrm{Ker}(A))$ while i is Fredholm of index $\dim(\mathrm{Coker}(A))$. Using (i), S must be Fredholm and

$$0 = \mathrm{Index}(S_0) = -\dim(\mathrm{Ker}(A)) + \mathrm{Index}(S) + \dim(\mathrm{Coker}(A)).$$

In conclusion, for $||S - A|| < \epsilon$, S is Fredholm of index equal to $\mathrm{Index}(A)$. $\square$

2 Compact operators: basic properties

Definition 2.1 *A linear map $T : \mathbb{E} \longrightarrow \mathbb{F}$ is said to be compact if for any bounded sequence $\{x_n\}$ in $\mathbb{E}$, $\{T(x_n)\}$ has a convergent subsequence.*

Equivalently, compact operators are those linear maps $T : \mathbb{E} \longrightarrow \mathbb{F}$ with the property that $T(B_{\mathbb{E}}) \subset \mathbb{F}$ is relatively compact, where $B_{\mathbb{E}}$ is the unit ball of $\mathbb{E}$. Here are the first properties of compact operators.

Proposition 2.2 *Let $\mathbb{E}$, $\mathbb{F}$ and $\mathbb{G}$ be Banach spaces.*

(i) $\mathcal{K}(\mathbb{E}, \mathbb{F})$ is a closed subspace of $\mathcal{L}(\mathbb{E}, \mathbb{F})$.

(ii) given $T \in \mathcal{L}(\mathbb{E}, \mathbb{F})$, $S \in \mathcal{L}(\mathbb{F}, \mathbb{G})$, if T or S is compact, then so is $T \circ S$.

(iii) $T \in \mathcal{L}(\mathbb{E}, \mathbb{F})$ is compact if and only if $T^* \in \mathcal{L}(\mathbb{F}^*, \mathbb{E}^*)$ is.

In particular, $\mathcal{K}(\mathbb{E})$ is a closed two-sided $*$-ideal in $\mathcal{L}(\mathbb{E})$.

Note that, if $\mathbb{E} = \mathcal{H}$ is a Hilbert space, then $\mathcal{K}(\mathcal{H})$ is the unique non-trivial (norm-)closed ideal in $\mathcal{L}(\mathcal{H})$.

Proof: We prove that, if $T_n \longrightarrow T$ and T_n are all compact, then T is compact. Since $T(B_{\mathbb{E}})$ is bounded and $\mathbb{F}$ is Banach, it suffices to show that $T(B_{\mathbb{E}})$ is precompact, i.e. that it can be covered by a finite number of balls of arbitrarily small radius ϵ. So, let $\epsilon > 0$. Choose n such that $||T_n - T|| < \epsilon/2$ and cover $T_n(B_{\mathbb{E}})$ by a finite number of balls $B(f_i, \epsilon/2)$. Then the balls $B(f_i, \epsilon)$ cover $T(B_{\mathbb{E}})$.

We now prove (iii) (the remaining statements are immediate). Assume first that T is relatively compact, and let $K \subset \mathbb{F}$ be the closure of $T(B_{\mathbb{E}})$ (compact). Let v_n be a sequence in the unit ball of $\mathbb{F}^*$. We want to prove that $T^*(v_n) = v_n \circ T$ has a convergent subsequence. We consider the space $\mathcal{C}(K)$ of continuous functions on K, and the subspace $\mathcal{H}$ consisting of the restrictions $\phi_n = v_n|_K$. We claim we can apply Ascoli to $\mathcal{H}$. Equicontinuity: since $||v_n|| \leq 1$, we have

$$|\phi_n(x) - \phi_n(y)| \leq ||x - y||$$

for all x and y. Equiboundedness: since $||v_n|| \leq 1$ and any $y \in K$ has norm less then $||T||$, we have

$$|\phi_n(y)| \leq ||T||$$

for all $y \in K$ and all n. By Ascoli, we find a subsequence of ϕ_n, which we may assume is ϕ_n itself, which is convergent in norm. We use that it is Cauchy:

$$sup_{y \in K} |\phi_n(y) - \phi_m(y)| \longrightarrow 0.$$

Since $T(B_{\mathbb{E}}) \subset K$, this clearly implies that $T^*(v_n)$ is Cauchy in $\mathbb{E}^*$, hence convergent. For the converse of (iii), we apply the first half to conclude that $T^{**} : \mathbb{E}^{**} \longrightarrow \mathbb{F}^{**}$ is compact. Viewing $\mathbb{E} \subset \mathbb{E}^{*}*$ as a closed subspace, and similarly for $\mathbb{F}$, we have $T(B_{\mathbb{E}}) = T^{**}(B_{\mathbb{E}})$-relatively compact. □

Next, we discuss the relationship with finite rank operators.

Definition 2.3 *A linear map $T : \mathbb{E} \longrightarrow \mathbb{F}$ is said to be of finite rank if it is continuous and its image is a finite dimensional space. We denote by $\mathcal{K}_{fin}(\mathbb{E}, \mathbb{F})$ the space of compact operators from $\mathbb{E}$ to $\mathbb{F}$.*

Equivalently, $\mathcal{K}_{\text{fin}}(\mathbb{E}, \mathbb{F})$ is the image of the canonical inclusion

$$\mathbb{E}^* \otimes \mathbb{F} \longrightarrow \mathcal{L}(\mathbb{E}, \mathbb{F}), \quad \sum e_i^* \otimes f_i \mapsto \sum e_i^*(-) f_i$$

i.e. the finite rank operators are those of type $T(x) = \sum e_i^*(x) f_i$ (finite sum) with $e_i^* \in \mathbb{E}^*$, $f_i \in \mathbb{F}$. It is clear that

$$\mathcal{K}_{\text{fin}}(\mathbb{E}, \mathbb{F}) \subset \mathcal{K}(\mathbb{E}, \mathbb{F}) \subset \mathcal{L}(\mathbb{E}, \mathbb{F})$$

and $\mathcal{K}_{\text{fin}}(\mathbb{E},\mathbb{F})$ has all the properties of $\mathcal{K}(\mathbb{E},\mathbb{F})$ from the previous proposition, except from being closed. All we can say in general is that

$$\overline{\mathcal{K}_{\text{fin}}(\mathbb{E},\mathbb{F})} \subset \mathcal{K}(\mathbb{E},\mathbb{F}),$$

and the next proposition[2] gives conditions on $\mathbb{F}$ so that this inclusion becomes equality. For this, we recall that a Schauder basis for $\mathbb{F}$ is a countable family $\{f_k : k \geq 1\}$ of elements of $\mathbb{F}$ with the property that each $y \in \mathbb{F}$ can be uniquely written as

$$y = \sum_{k=1}^{\infty} t_k f_k$$

with t_k-scalars. Clearly, any separable Hilbert space admits a Schauder basis, but also spaces such as L^p with $p \geq 1$ do.

Proposition 2.4 *If $\mathbb{F}$ admits a Schauder basis then an operator $T \in \mathcal{L}(\mathbb{E},\mathbb{F})$ is compact if and only if it is the limit of a sequence of finite rank operators; in other words,*

$$\mathcal{K}(\mathbb{E},\mathbb{F}) = \overline{\mathcal{K}_{fin}(\mathbb{E},\mathbb{F})}.$$

PROOF: We still have to show that any compact T is a limit of finite rank ones. Let $\{f_k : k \geq 1\}$ be a Schauder basis, and let $f^k : \mathbb{F} \longrightarrow \mathbb{C}$ be the coordinate functions. It is known that the Schauder basis can be chosen such that f^k are continuous. We put $T_k \in \mathcal{L}(\mathbb{E},\mathbb{F})$,

$$T_k(x) = \sum_{i}^{k} f^i(T(x)) f_i.$$

To prove $T_k \to T$, let $\epsilon > 0$. For any x of norm less then 1, we find N such that

$$\sum_{i=k}^{\infty} f^i(T(x)) f_i|| < \epsilon$$

for all $k \geq N$. But since $T(B_{\mathbb{E}})$ is relatively compact, we can choose N uniform with respect to $x \in B_{\mathbb{E}}$. Hence $||T - T_k|| < \epsilon$ for all $k \geq N$. □

Finally, to give an alternative description of compact operators, we recall that a linear map $T : \mathbb{E} \longrightarrow \mathbb{F}$ is said to be completely continuous if it carries weakly convergent sequences into norm convergent sequences.

Proposition 2.5 *Any compact operator $T : \mathbb{E} \longrightarrow \mathbb{F}$ is completely continuous. The converse is true if $\mathbb{E}$ is reflexive.*

3 Compact operators: the Fredholm alternative

In this section, $\mathbb{E} = \mathbb{F}$ (a Banach space). One of the versions of the Fredholm alternative says that, if K is a compact operator on $\mathbb{E}$, then the associated equation $x = Kx + y$ behaves like in the finite dimensional case: if the homogeneous equation $x = Kx$ has only the trivial solution $x = 0$, then the inhomogeneous equations

$$x = Kx + y$$

has a unique solution $x \in \mathbb{E}$, for every $y \in \mathbb{E}$. More precisely, we have the following:

Proposition 3.1 *For $K \in \mathcal{K}(\mathbb{E})$, the following are equivalent:*

[2] this proposition is just for your curiosity.

(i) $1-K$ *is injective.*

(ii) $1-K$ *is surjective.*

(iii) $1-K$ *is bijective.*

The general version of the Fredholm alternative is best expressed in terms of Fredholm operators.

Theorem 3.2 *For any compact operator K on $\mathbb{E}$, $1-K$ is a Fredholm operator of index zero.*

Before turning to the proofs, let us point out that these results are naturally cast as properties of the spectrum of compact operators [3]. Recall that, for an operator $T : \mathbb{E} \longrightarrow \mathbb{E}$, the spectrum $\sigma(T)$ consists of those complex numbers λ with the property that $\lambda - T$ is not invertible. A particular case of this is when the equation $Tx = \lambda x$ has a non-trivial solution $x \in \mathbb{E}$. In this case λ is called an eigenvalue of T, the space $N_\lambda = \{x \in \mathbb{E} : Tx = \lambda x\}$ is called the λ-eigenspace of T, and the set of all eigenvalues of T is denoted by $\sigma_p(T)$ (called the point-spectrum of T). With these, we have:

Theorem 3.3 *Assume that $\mathbb{E}$ is infinite dimensional. For any compact operator $K \in \mathcal{K}(\mathbb{E})$,*

(i) $\sigma(K) = \sigma_p(K) \cup \{0\}$*, and this is either finite or it is a countable sequence of complex number converging to zero.*

(ii) for any non-zero eigenvalue λ, the corresponding eigenspace $N_\lambda(K)$ is finite dimensional.

We now turn to the proofs of these results. We will use the Riesz lemma:

Lemma 3.4 *If $M \subset \mathbb{E}$ is a closed subspace, $M \neq \mathbb{E}$, then for every $\epsilon > 0$, there exists $x_\epsilon \in \mathbb{E}$ such that*

$$||x_\epsilon|| = 1, d(x_\epsilon, M) > 1 - \epsilon.$$

PROOF: Choose $x \in \mathbb{E} - M$ and put $d = d(x, M) > 0$. Since $d(x, M) < d/(1-\epsilon)$, we find $m_\epsilon \in M$ such that $||x - m_\epsilon|| < d/(1-\epsilon)$. Put

$$x_\epsilon = \frac{x - m_\epsilon}{||x - m_\epsilon||}.$$

□

Let us also point out the following simple consequence, known as the Theorem of Riesz, which is interesting on its own, and which immediately implies (ii) of Theorem 3.3.

Corollary 3.5 *If the unit ball of a Banach space $\mathbb{E}$ is compact, then $\mathbb{E}$ is finite dimensional.*

PROOF: Cover $B_{\mathbb{E}}$ by a finite number of balls of radius 1/2. Denote by M the subspace spanned by the centers of these balls; if $M \neq \mathbb{E}$, we can apply the previous lemma with $\epsilon = 1/2$ and we obtain a contradiction. In conclusion, $\mathbb{E} = M$ is finite dimensional. □

PROOF:[(of Proposition 3.1 and of Theorem 3.2)] We first claim that, for any compact operator K, the image of $1-K$ is closed in $\mathbb{E}$. Denote $S = 1 - K$, $N = \mathrm{Ker}(S)$. Consider $y \in \overline{S(\mathbb{E})}$, and write

$$y = \lim_{n \to \infty} S(x_n)$$

for some sequence $\{x_n\}$ in $\mathbb{E}$. We will show that $\{x_n\}$ may be chosen to be bounded. From the compactness of K, this implies that $\{x_n\}$ may be assumed to converge to an element $x \in \mathbb{E}$, hence $y = S(x) \in S(\mathbb{E})$. To achieve the boundedness of $\{x_n\}$, it suffices to show that $d(x_n, N)$ is bounded. Indeed, in this case we find $a_n \in N$ such that $\{||x_n - a_n||\}$ is bounded and we may replace x_n by $x_n - a_n$.

[3] again, this (i.e. the next theorem) is just for your curiosity.

So, let us assume that $\{d(x_n, N)\}$ is unbounded and we will obtain a contradiction. First of all, we may assume that this unbounded sequence converges to ∞. Put

$$z_n = \frac{1}{d(x_n, N)} x_n.$$

This has the properties:

$$d(z_n, N) = 1, \lim_{n \to \infty} S(z_n) = 0.$$

We may assume that z_n is bounded (otherwise, by the first property above we find $z_n^{'} \in z_n + N$ such that $||z_n^{'}|| \leq 2$ and $\{z_n^{'}\}$ has the same properties). Since K is compact, we may also assume that $K(z_n)$ converges to an element $a \in \mathbb{E}$. From the properties of z_n, we find that $d(a, N) = 1$ and that $z_n = S(z_n) + K(z_n)$ converges to a. The last statement and the definition of a imply that $K(a) = a$, i.e. $a \in N$, which contradicts $d(a, N) = 1$. This finishes the proof of the fact that $\mathrm{Im}(1 - K)$ is closed.

With this property proven, to finnish the proof of Proposition 3.1, one can go on with a "direct" argument that does not use any of the properties of Fredholm operators. Alternatively, one can now prove Theorem 3.2, which clearly implies the proposition.

PROOF: [(end of proof of Proposition 3.1)] We now prove that (i) implies (ii). Hence, let us assume that S is injective and $S(\mathbb{E}) \neq \mathbb{E}$. We consider the decreasing sequence of subspaces of $\mathbb{E}$:

$$\ldots \subset \mathbb{E}_3 \subset \mathbb{E}_2 \subset \mathbb{E}_1 = \mathbb{E}$$

where $\mathbb{E}_n = S^n(\mathbb{E})$. Note that $K(\mathbb{E}_n) \subset \mathbb{E}_n$. Since the restriction of K to each $\mathbb{E}_n$ is compact, the first part of the proof implies that each $\mathbb{E}_n$ is a closed subspace of $\mathbb{E}_{n-1}$, while the injectivity of S implies that these inclusions are proper. From the Riesz Lemma we find $x_n \in \mathbb{E}_n$ with

$$||x_n|| = 1, d(x_n, \mathbb{E}_{n+1}) \geq \frac{1}{2}.$$

However, for each $n > m$ one has

$$Kx_n - Kx_m = Kx_n - x_m + Sx_m \in E_n - x_m + E_{m1} \subset E_{m1} - x_m,$$

hence

$$||Kx_n - Kx_m|| > \frac{1}{2},$$

and then $\{Kx_n\}$ cannot have a convergent subsequence, which contradicts the compactness of K.

Finally, the inverse implication (ii) $\Longrightarrow$ (i) is a consequence of the direct implication and the general fact that $\mathrm{Ker}(T^*) = \mathrm{Im}(T)^{\perp}$: if $S = 1 - K$ is surjective, it follows that $Ker(S^*) = \mathrm{Im}(S)^{\perp} = \{0\}$, i.e. S^* must be injective. Applying (i) $\Longrightarrow$ (ii) to K^* (we do know that K^* is compact!), S^* is surjective, hence $\mathrm{Ker}(S) = \mathrm{Im}(S^*)^{\perp} = \{0\}$, i.e. S is injective. □

PROOF: [(end of the proof of Theorem 3.2), hence also of proof 2 of Proposition 3.1)] The Riesz Lemma immediately implies that $\mathrm{Ker}(1 - K)$ is finite dimensional. Applying this to K^*, we deduce that also $\mathrm{Im}(1 - K)^{\perp} = \mathrm{Ker}(1 - K^*)$ is finite dimensional. Since $\mathrm{Im}(1 - K)$ is closed (see the first part of the previous proof), we deduce $\mathrm{Im}(1 - K)$ is of finite codimension. Hence $1 - K$ is Fredholm. We then have a continuous family $\{1 - tK : t \in \mathbb{R}\}$ of Fredholm operators. By the properties of the index, the index at $t = 1$ coincides with the index at $t = 0$, which is zero. □

□

PROOF:[(of Theorem 3.3)] The only thing still to be proven is that $\sigma_p(K)$ is either finite, or a countable sequence converging to zero. It suffices to show that for any sequence $\{\lambda_n\}$ of distinct eigenvalues of K which converge to λ (finite or infinite), $\lambda = 0$. Assume that $\{\lambda_n\}$ is such a sequence. Choose eigenvectors x_n corresponding to λ_n, $x_n \neq 0$ and put

$$\mathbb{E}_n = \mathrm{span}\{x_1, \ldots, x_n\}.$$

Since the λ_i are distinct, it follows that

$$\mathbb{E}_1 \subset \mathbb{E}_2 \subset \ldots$$

is a strictly increasing sequence of subspaces of $\mathbb{E}$. From the Riesz Lemma with $\epsilon = 1/2$ we find

$$u_n \in \mathbb{E}_n, \quad ||u_n|| = 1, \quad d(u_n, \mathbb{E}_{n-1}) > \frac{1}{2}.$$

Note also that

$$T(\mathbb{E}_n) \subset \mathbb{E}_n, \quad (T - \lambda_m \mathrm{Id})(\mathbb{E}_m) \subset \mathbb{E}_{m-1}.$$

We deduce that for $m > n$,

$$\frac{Tu_n}{\lambda_n} - \frac{Tu_m}{\lambda_m} \in E_n + E_{m-1} - u_m = E_{m-1} - u_m,$$

hence

$$||\frac{Tu_n}{\lambda_n} - \frac{Tu_m}{\lambda_m}|| \geq \frac{1}{2},$$

and $\{Tu_n/\lambda_n\}$ cannot have a convergent subsequence. But, since T is compact, $\{Tu_n\}$ does posses a convergent subsequence, so λ must equal 0. □

4 The relation between Fredholm and compact operators

We have already seen from the Fredholm alternative that, for any compact operator $K \in \mathcal{K}(\mathbb{E})$, $1 - K$ is a Fredholm operator of index zero. Much more precisely, we have the following:

Theorem 4.1 *Compact perturbations do not change Fredholmness and do not change the index, and zero index is achieved only by compact perturbations of invertible operators.*
More precisely:

(i) If $K \in \mathcal{K}(\mathbb{E}, \mathbb{F})$ and $A \in \mathcal{F}(\mathbb{E}, \mathbb{F})$, then $A + K \in \mathcal{F}(\mathbb{E}, \mathbb{F})$ and $\mathrm{Index}(A + K) = \mathrm{Index}(A)$.

(ii) If $A \in \mathcal{F}(\mathbb{E}, \mathbb{F})$, then $\mathrm{Index}(A) = 0$ *if and only if $A = A_0 + K$ for some invertible operator A_0 and some compact operator K.*

There is yet another relation between Fredholm and compact operators, know as the Atkinson characterization of Fredholm operators:

Theorem 4.2 *Fredholmness= invertible modulo compact operators.*
More precisely, given a bounded operator $A : \mathbb{E} \longrightarrow \mathbb{F}$, the following are equivalent:

(i) A is Fredholm.

(ii) A is invertible modulo compact operators, i.e. there exist and operator $B \in \mathcal{L}(\mathbb{F}, \mathbb{E})$ and compact operators K_1 and K_2[4] such that

$$BA = 1 + K_1, AB = 1 + K_2.$$

We now turn to the proofs of these results.

PROOF:[(of Theorem 4.2)] Assume first the existence of B, K_1 and K_2. Since identity plus compact is Fredholm, we deduce that the kernel of A is finite dimensional (since is included in the kernel of $1 + K_1$) and, similarly, the cokernel of A is finite dimensional. Hence A is Fredholm.

[4]from the proof we will see that one can actually choose K_1 and K_2 to be finite rank operators, and B so that $ABA = A$, $BAB = B$.

Assume now that A is Fredholm. Choose a complement $\mathbb{E}_1$ of $\mathrm{Ker}(A)$ in $\mathbb{E}$ and a complement $\mathbb{F}_1$ of $\mathrm{Im}(A)$ in $\mathbb{F}$. Then $A_1 = A|_{\mathbb{E}_1}$ is an isomorphism from $\mathbb{E}_1$ into $\mathrm{Im}(A)$ and we define B such that $B = (A_1)^{-1}$ on $\mathrm{Im}(A)$ and $B = 0$ on $\mathbb{F}_2$. Then the resulting K_1 will be a projection onto $\mathrm{Ker}(A)$ and $1 + K_2$ will be a projection onto $\mathrm{Im}(A)$; hence K_1 and K_2 will have the desired properties. □

PROOF:[(of Theorem 4.1)] Part (i) follows easily from Atkinson's characterization and the Fredholm alternative: choose B, K_1 and K_2 as above. We deduce that B is itself Fredholm of index $-\mathrm{index}(A)$ (here we used the additivity of the index and the Fredholm alternative). We remark that $(A + K)B = 1 + (K_1 + KB)$ and $BA = 1 + (K_2 + BK)$, where $K_1 + KB$ and $K_2 + BK$ are compact. We then deduce that $A + K$ is Fredholm of index equal to $-\mathrm{index}(B) = \mathrm{index}(A)$.

We still have to prove that $\mathrm{Index}(A) = 0$ can only happen for compact perturbations of invertible operators. As above, we choose a complement $\mathbb{E}_1$ of the kernel of A and a complement $\mathbb{F}_1$ of the image of A. With respect to these decompositions, A is just $(x, y) \mapsto (A_1(x), 0)$, where $A_1 : \mathbb{E}_1 \longrightarrow \mathrm{Im}(A)$ is an isomorphism (the restriction of A to $\mathbb{E}_1$). That A has zero index means that the dimension of $\mathrm{Ker}(A)$ equals to the dimension of $\mathbb{F}_1$ (bot finite!). Choosing an isomorphism $\phi : \mathrm{Ker}(A) \longrightarrow \mathbb{F}_1$, the map $K : (x, y) \mapsto (0, \phi(y))$ is compact and $A + K = (A_1, \phi)$ is an isomorphism. □