Thus, by density of $C_c^\infty(\mathbb{R}^n)$, the induced map $\beta_1 : H_s(\mathbb{R}^n) \to \bar{H}_{-s}(\mathbb{R}^n)^*$ is an isometry. Likewise, $\beta_2 : H_{-s}(\mathbb{R}^n) \to \bar{H}_s(\mathbb{R}^n)^*$ is an isometry. From the injectivity of β_1 it follows that β_2 has dense image. Being an isometry, β_2 must then be surjective. Likewise, β_1 is surjective. □

4.5. Rellich's lemma for Sobolev spaces

In this section we will give a proof of the Rellich lemma for Sobolev spaces, which will play a crucial role in the proof of the Fredholm property for elliptic pseudo-differential operators on compact manifolds.

Given $s \in \mathbb{R}$ and a compact subset $K \subset \mathbb{R}^n$, we define

$$H_{s,K}(\mathbb{R}^n) = \{u \in H_s(R^n) \mid \operatorname{supp} u \subset K\}.$$

Lemma 4.5.1. *$H_{s,K}(\mathbb{R}^n)$ is a closed subspace of $H_s(\mathbb{R}^n)$.*

Proof Let $f \in C_c^\infty(\mathbb{R}^n)$. Then the space

$$f^\perp := \{u \in H_s(\mathbb{R}^n) \mid \langle u, f\rangle = 0\}$$

has Fourier transform equal to the space of $\varphi \in L_s^2(\mathbb{R}^n)$ with $\langle \varphi, \mathcal{F}f\rangle = 0$, which is the orthocomplement of $(1+\|\xi\|)^{-2s}\mathcal{F}f$ in $L_s^2(\mathbb{R}^n)$. As this orthocomplement is closed in $L_s^2(\mathbb{R}^n)$, it follows that $f^\perp$ is closed in $H_s(\mathbb{R}^n)$.

We now observe that $H_{s,K}(\mathbb{R}^n)$ is the intersection of the spaces $f^\perp$ for $f \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp} f \cap K = \emptyset$. □

Lemma 4.5.2. (Rellich) *Let $t < s$. Then the inclusion map $H_{s,K}(\mathbb{R}^n) \to H_t(\mathbb{R}^n)$ is compact.*

To prepare for the proof, we first prove the following result, which is based on an application of the Ascoli-Arzéla theorem.

Lemma 4.5.3. *Let B be a bounded subset of the Fréchet space $C^1(\mathbb{R}^n)$. Then B is relatively compact (i.e., has compact closure) as a subset of the Fréchet space $C(\mathbb{R}^n)$.*

Proof Boundedness of B means that every continuous semi-norm of $C^1(\mathbb{R}^n)$ is bounded on B. Let $K \subset \mathbb{R}^n$ be a compact ball. Then there exists a constant $C > 0$ such that $\sup_K \|df\| \leq C$ for all $f \in B$ and each $1 \leq j \leq n$. Since

$$f(x) - f(y) = \int_0^1 df(y + t(x-y))(x-y)\, dt$$

for all $y \in x$, we see that

$$|f(y) - f(x)| \leq C\|x - y\|, \quad \text{for all} \quad (x, y \in K).$$

It follows that the set of functions $B|_K = \{f|_K \mid f \in B\}$ is equicontinuous and bounded in $C(K)$. By application of the Ascoli-Arzèla theorem, the set $B|_K$ is relatively compact in $C(K)$. In particular, if (f_k) is a sequence in B, then there is a subsequence (f_{k_j}) which converges uniformly on K.

Let (f_k) be a sequence in B. We shall now apply the usual diagonal procedure to obtain a subsequence that converges in $C(\mathbb{R}^n)$.

For $r \in \mathbb{N}$ let K_r denote the ball of center 0 and radius r in $\mathbb{R}^n$. Then by repeated application of the above there exists a sequence of subsequences $(f_{k_{1,j}}) \succeq (f_{k_{2,j}}) \succeq \cdots$ ... such that $(f_{k_{r,j}})$ converges uniformly on K_r, for every $r \in \mathbb{N}$.

The sequence $(f_{k_{j,j}})_{j\in\mathbb{N}}$ is a subsequence of all the above sequences. Hence, it converges uniformly on each ball K_r. Therefore, it converges in $C(\mathbb{R}^n)$. $\square$

Remark 4.5.4. By a slight modification of the proof above, one obtains a proof of the compactness of each inclusion map $C^{k+1}(\mathbb{R}^n) \hookrightarrow C^k(\mathbb{R}^n)$. This implies that the identity operator of $C^\infty(\mathbb{R}^n)$ is compact. Equivalently, each bounded subset of $C^\infty(\mathbb{R}^n)$ is relatively compact. A locally convex topological vector space with this property is called Montel.

If B is a subset of $L^2_s(\mathbb{R}^n)$ (see Lecture 4) and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we write

$$\varphi * B := \{\varphi * f \mid f \in B\}.$$

Then $\varphi * B$ is a subset of $L^2_s(\mathbb{R}^n)$.

Lemma 4.5.5. *Let $s \in \mathbb{R}$ and let $B \subset L^2_s(\mathbb{R}^n)$ be bounded. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then the set $\varphi * B$ is a relatively compact subset of $C(\mathbb{R}^n)$.*

Proof In view of the previous lemma, it suffices to prove that $f * B$ is bounded in $C^1(\mathbb{R}^n)$. For this we note that for each $1 \leq j \leq n$,

$$\begin{aligned} &|\frac{\partial}{\partial x_j}[\varphi(x-y)f(y)]| \\ &= \ |\partial_j\varphi(x-y)|(1+\|y\|)^{-s}(1+\|y\|)^s|f(y)| \\ &\leq \ (1+\|x\|)^{-s}|(1+\|x-y\|)^{-s}\|\partial_j\varphi(x-y)\|(1+\|y\|)^s|f(y)|. \end{aligned}$$

The right-hand side can be dominated by an integrable function of y, locally uniformly in x. It now follows by differentiation under the integral sign that $\varphi * f \in C^1(\mathbb{R}^n)$, that $\partial_j(\varphi * f) = \partial_j\varphi * f$ and that

$$\|\partial_j(\varphi * f)(x)\| \leq (1+\|x\|)^{-s}\,\|\varphi\|_{L^2,-s}\,\|f\|_{L^2,s}.$$

This implies that the set $\varphi * B$ is bounded in $C^1(B)$, hence relatively compact in $C(\mathbb{R}^n)$. $\square$

Proposition 4.5.6. *Let $s > t$ and let B be a bounded subset of $L^2_s(\mathbb{R}^n)$ which at the same time is a relatively compact subset of $C(\mathbb{R}^n)$. Then B is relatively compact in $L^2_t(\mathbb{R}^n)$.*

Proof For $R > 0$ we denote by 1_R the characteristic function of the closed ball $B(R) := \bar{B}(0;R)$. Then for each $r \in \mathbb{R}$, the map $f \mapsto 1_R f$ gives the orthogonal projection from $L^2_s(\mathbb{R}^n)$ onto the closed subspace $L^2_{s,B(R)}$ of functions with support in $B(R)$. We now observe that the following estimate holds for every $f \in L^2_s(\mathbb{R}^n)$:

$$\begin{aligned} \|(1-1_R)f\|^2_{L^2,t} \ &= \ \int_{\|x\|\geq R}(1+\|x\|)^{2t-2s}(1+\|x\|)^{2s}\|f(t)\|^2\,dt \\ &\leq \ (1+R)^{2(t-s)}\|f\|^2_{L^2,s}. \end{aligned}$$

Fix $M > 0$ such that $\|f\|_{L^2,s} \leq M$ for all $f \in B$. Then we see that

$$\|(1 - 1_R)f\|_{L^2,t} \leq M\,(1 + R)^{t-s}, \qquad (f \in B).$$

Let now (f_k) be a sequence in B. Then (f_k) has a subsequence (f_{k_j}) which converges in $C(\mathbb{R}^n)$, i.e., there exists a function $f \in C(\mathbb{R}^n)$ such that $f_{k_j} \to f$ uniformly on each compact set $K \subset \mathbb{R}^n$. It easily follows from this that $1_R f_{k_j}$ is a Cauchy-sequence in $L^2_t(\mathbb{R})$, for each $R > 0$. We will show that f_{k_j} is actually a Cauchy sequence in $L^2_t(\mathbb{R}^n)$. By completeness of the latter space, this will complete the proof.

Let $\epsilon > 0$. We fix $R > 0$ such that

$$M(1 + R)^{t-s} < \frac{1}{3}\epsilon.$$

There exists a constant $N > 0$ such that

$$i, j \geq N \Rightarrow \|1_R f_{k_i} - 1_R f_{k_j}\|_{L^2,t} < \frac{1}{3}.$$

It follows that for all $i, j > N$,

$$\begin{aligned} &\|f_{k_i} - f_{k_j}\|_{L^2,t} \\ &\leq \|1_R(f_{k_i} - f_{k_j})\|_{L^2,t} + \|(1 - 1_R)f_{k_i}\|_{L^2,t} + \|(1 - 1_R)f_{k_j}\|_{L^2,t} \\ &< \epsilon. \end{aligned}$$

□

Proof of Lemma 4.5.2 Let $K \subset \mathbb{R}^n$ be compact and let B be a bounded subset of $H_{s,K}(\mathbb{R}^n)$. Fix a smooth compactly supported function $\chi \in C^\infty_c(\mathbb{R}^n)$ that is 1 on a neighborhood of K. Then $\chi f = f$ for all $f \in B$. It follows that

$$\mathcal{F}(B) = \varphi * \mathcal{F}(B),$$

with $\varphi = \mathcal{F}(\chi) \in \mathcal{S}(\mathbb{R}^n)$. By Lemma 4.5.5 it now follows that $\mathcal{F}(B)$ is both bounded in $L^2_s(\mathbb{R}^n)$ and a relatively compact subset of $C(\mathbb{R}^n)$. By the previous proposition, this implies that $\mathcal{F}(B)$ is relatively compact in $L^2_t(\mathbb{R}^n)$. As $\mathcal{F}$ is an isometry from $H_t(\mathbb{R}^n)$ to $L^2_t(\mathbb{R}^n)$, it follows that B is relatively compact in $H_t(\mathbb{R}^n)$. □