LECTURE 4
Fourier transform

4.1. Schwartz functions

Recall that $L^1(\mathbb{R}^n)$ denotes the Banach space of functions $f : \mathbb{R}^n \to \mathbb{C}$ that are absolutely integrable, i.e., $|f|$ is Lebesgue integrable over $\mathbb{R}^n$. The norm on this space is given by

$$\|f\|_1 = \int_{\mathbb{R}^n} |f(x)|\, dx.$$

Given $\xi \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we put

$$\xi x := \xi_1 x_1 + \cdots + \xi_n x_n.$$

For each $\xi \in \mathbb{R}^n$, the exponential function

$$e^{i\xi} :\ x \mapsto e^{i\xi x},\ \mathbb{R}^n \to \mathbb{C},$$

has absolute value 1 everywhere. Thus, if $f \in L^1(\mathbb{R}^n)$ then $e^{-i\xi} f \in L^1(\mathbb{R}^n)$ for all $\xi \in \mathbb{R}^n$.

Definition 4.1.1. For a function $f \in L^1(\mathbb{R}^n)$ we define its Fourier transform $\hat{f} = \mathcal{F}f : \mathbb{R}^n \to \mathbb{C}$ by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\xi x}\, dx. \tag{4.1}$$

We will use the notation $C_b(\mathbb{R}^n)$ for the Banach space of bounded continuous functions $\mathbb{R}^n \to \mathbb{C}$ equipped with the sup-norm.

Lemma 4.1.2. *The Fourier transform maps $L^1(\mathbb{R})$ continuous linearly to the Banach space $C_b(\mathbb{R}^n)$.*

Proof Let f be any function in $L^1(\mathbb{R}^n)$. The functions $fe^{-i\xi}$ are all dominated by $|f|$ in the sense that $|fe^{-i\xi}| \leq |f|$ (almost) everywhere. Let $\xi_0 \in \mathbb{R}^n$; then it follows by Lebesgue's dominated convergence theorem that $\mathcal{F}f(\xi) \to \mathcal{F}f(\xi_0)$ if $\xi \to \xi_0$. This implies that $\mathcal{F}f$ is continuous. It follows that $\mathcal{F}$ defines a linear map from $L^1(\mathbb{R})$ to $C(\mathbb{R}^n)$. It remains to be shown that $\mathcal{F}$ maps $L^1(\mathbb{R})$ continuously into $C_b(\mathbb{R})$.

For this we note that for $f \in L^1(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$,

$$|\mathcal{F}f(\xi)| = |\int_{\mathbb{R}^n} f(x)\, e^{-i\xi x}\, dx\,| \leq \int_{\mathbb{R}^n} |f(x)\, e^{-i\xi x}|\, dx = \|f\|_1.$$

Thus, $\sup |\mathcal{F}f| \leq \|f\|_1$. It follows that $\mathcal{F}$ is a linear map $L^1(\mathbb{R}) \to C_b(\mathbb{R}^n)$ which is bounded for the Banach topologies, hence continuous. □

Remark 4.1.3. We denote by $C_0(\mathbb{R}^n)$ the subspace of $C_b(\mathbb{R}^n)$ consisting of functions f that vanish at infinity. By this we mean that for any $\epsilon > 0$ there exists a compact set $K \subset \mathbb{R}^n$ such that $|f| < \epsilon$ on the complement $\mathbb{R}^n \setminus K$. It is well known that $C_0(\mathbb{R}^n)$ is a closed subspace of $C_b(\mathbb{R}^n)$, thus a Banach space of its own right.

The well known *Riemann-Lebesgue lemma* asserts that, actually, $\mathcal{F}$ maps $L^1(\mathbb{R}^n)$ into $C_0(\mathbb{R}^n)$.

The above amounts to the traditional way of introducing the Fourier transform. Unfortunately, the source space $L^1(\mathbb{R}^n)$ is very different from the target space $C_b(\mathbb{R}^n)$. We shall now introduce a subspace of $L^1(\mathbb{R}^n)$ which has the advantage that it is preserved under the Fourier transform: the so-called Schwartz space.

Definition 4.1.4. A smooth function $f : \mathbb{R}^n \to \mathbb{C}$ is called rapidly decreasing, or Schwartz, if for all $\alpha, \beta \in \mathbb{N}^n$,

$$\sup_{x \in \mathbb{R}^n} |x^\beta \partial^\alpha f(x)| < \infty. \tag{4.2}$$

The linear space of these functions is denoted by $\mathcal{S}(\mathbb{R}^n)$.

Exercise 4.1.5. Show that the function

$$f(x) = e^{-\|x\|^2}$$

belongs to $\mathcal{S}(x)$.

Condition (4.2) for all α, β is readily seen to be equivalent to the following condition, for all $N \in \mathbb{N}, k \in \mathbb{N}$:

$$\nu_{N,k}(f) := \max_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} (1 + \|x\|)^N |\partial^\alpha f(x)| < \infty.$$

We leave it to the reader to check that $\nu = \nu_{N,k}$ defines a norm, hence in particular a seminorm, on $\mathcal{S}(\mathbb{R}^n)$. We equip $\mathcal{S}(\mathbb{R}^n)$ with the locally convex topology generated by the set of norms $\nu_{N,k}$, for $N, k \in \mathbb{N}$.

The Schwartz space behaves well with respect to the operators (multiplication by) x^α and ∂^β.

Exercise 4.1.6. Let α, β be multi-indices. Show that

$$x^\alpha : f \mapsto x^\alpha f \quad \text{and} \quad \partial^\beta : f \mapsto \partial^\beta f$$

define continuous linear endomorphisms of $\mathcal{S}(\mathbb{R}^n)$.

Exercise 4.1.7.

(a) Show that $\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$, with continuous inclusion map.

(b) Show that
$$C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n),$$
with continuous inclusion maps.

Lemma 4.1.8. *The space $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space.*

Proof As the given collection of seminorms is countable it suffices to show completeness, i.e., every Cauchy sequence in $\mathcal{S}(\mathbb{R}^n)$ should be convergent. Let (f_n) be a Cauchy sequence in $\mathcal{S}(\mathbb{R}^n)$. Then by continuity of the second inclusion in Exercise 4.1.7 (b), the sequence is Cauchy in $C^\infty(\mathbb{R}^n)$. By completeness of the latter space, the sequence f_n converges to f, locally uniformly, in all derivatives. We will show that $f \in \mathcal{S}(\mathbb{R}^n)$ and $f_n \to f$ in $\mathcal{S}(\mathbb{R}^n)$. First, since (f_n) is Cauchy, it is bounded in $\mathcal{S}(\mathbb{R}^n)$. Let $N, k \in \mathbb{N}$; then there exists a constant $C_{N,k} > 0$ such that $\nu_{N,k}(f_n) \le C_{N,k}$, for all $n \in \mathbb{N}$. Let $x \in \mathbb{R}^n$, then from $\partial^\alpha f_n(x) \to \partial^\alpha f(x)$ it follows that
$$(1 + \|x\|)^N \partial^\alpha f_n(x) \to (1 + \|x\|)^N \partial^\alpha f(x), \qquad \text{as} \quad n \to \infty.$$
In view of the estimates $\nu_{N,k}(f_n) \le C_{N,k}$, it follows that $|(1 + \|x\|)^N \partial^\alpha f(x)| \le C_{N,k}$, for all α with $|\alpha| \le k$. This being true for arbitrary x, we conclude that $\nu_{N,k}(f) \le C_{N,k}$. Hence f belongs to the Schwartz space.

Finally, we turn to the convergence of the sequence f_n in $\mathcal{S}(\mathbb{R}^n)$. Let $N, k \in \mathbb{N}$. Let $\epsilon > 0$. Then there exists a constant M such that
$$n, m > M \Rightarrow \nu_{N,k}(f_n - f_m) \le \epsilon/2.$$
Let $|\alpha| \le k$ and fix $x \in \mathbb{R}^n$. Then it follows that
$$(1 + \|x\|)^N |\partial^\alpha f_n(x) - \partial^\alpha f_m(x)| \le \frac{\epsilon}{2}$$
As $\partial^\alpha f_n \to \partial^\alpha f$ locally uniformly, hence in particular pointwise, we may pass to the limit for $m \to \infty$ and obtain the above estimate with f_m replaced by f, for all $x \in \mathbb{R}^n$. It follows that $\nu_{N,k}(f_n - f) < \epsilon$ for all $n \ge M$. □

Another important property of the Schwartz space is the following.

Lemma 4.1.9. *The space $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$.*

Proof Fix a function $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $0 \le \varphi \le 1$ and $\varphi = 1$ on the closed unit ball in $\mathbb{R}^n$. For $k \in \mathbb{N}$ we put
$$\|\varphi\|_{C^k} := \max_{|\alpha| \le k} \sup_{x \in \mathbb{R}^n} |\partial^\alpha \varphi(x)|.$$
For $j \in \mathbb{Z}_+$ define the function $\varphi_j \in C_c^\infty(\mathbb{R}^n)$ by
$$\varphi_j(x) = \varphi(x/j).$$
Let now $f \in \mathcal{S}(\mathbb{R}^n)$. Then $\varphi_j f \in C_c^\infty(\mathbb{R}^n)$ for all $j \in \mathbb{Z}_+$. We will complete the proof by showing that $\varphi_j f \to f$ in $\mathcal{S}(\mathbb{R}^n)$ as $j \to \infty$.

Fix $N, k \in \mathbb{N}$. Our goal is to find an estimate for $\nu_{N,k}(\varphi_j f - f)$, independent of f. To this end, we first note that for every multi-index β we have $\partial^\beta \varphi_j(x) = (1/j)^{|\beta|} \partial^\beta \varphi(x/j)$. It follows that
$$\sup_{\mathbb{R}^n} |\partial^\beta \varphi_j| \le \frac{1}{j} \|\varphi\|_{C^k}, \qquad (j \in \mathbb{Z}_+,\ 0 < |\beta| \le k).$$

Let $|\alpha| \leq k$. Then by application of Leibniz' rule we obtain, for all $x \in \mathbb{R}^n$, that

$$|\partial^\alpha(\varphi_j f - f)(x)| \leq |(\varphi_j(x) - 1)\, \partial^\alpha f(x)| + \frac{1}{j}\|\varphi\|_{C^k} \sum_{0\neq\beta\leq\alpha} \binom{\alpha}{\beta} |\partial^{\alpha-\beta} f(x)|.$$

The first term on the right-hand side is zero for $\|x\| \leq j$. For $\|x\| \geq j$ it can be estimated as follows:

$$\begin{aligned} |(\varphi_j(x) - 1)\partial^\alpha f(x)| &\leq (1 + \sup|\varphi|)(1 + j)^{-1}(1 + \|x\|)|\partial^\alpha f(x)| \\ &\leq 2j^{-1}(1 + \|x\|)|\partial^\alpha f(x)|. \end{aligned}$$

We derive that there exists a constant $C_k > 0$, only depending on k, such that for every $N \in \mathbb{N}$,

$$\nu_{N,k}(\varphi_j f - f) \leq \frac{C_k}{j}\nu_{N+1,k}(f).$$

It follows that $\varphi_j f \to f$ in $\mathcal{S}(\mathbb{R}^n)$. □

The following lemma is a first confirmation of our claim that the Schwartz space provides a suitable domain for the Fourier transform.

Lemma 4.1.10. *The Fourier transform is a continuous linear map $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. Moreover, for each $f \in \mathcal{S}(\mathbb{R}^n)$ and all $\alpha \in \mathbb{N}^n$, the following hold.*

(a) $\mathcal{F}(\partial^\alpha f) = (i\xi)^\alpha \mathcal{F}f$;
(b) $\mathcal{F}(x^\alpha f) = (i\partial_\xi)^\alpha \mathcal{F}f$.

Proof Let $f \in \mathcal{S}(\mathbb{R}^n)$ and let $1 \leq j \leq n$. Then it follows by differentiation under the integral sign that

$$\frac{\partial}{\partial\xi_j} \int_{\mathbb{R}^n} f(x)\, e^{-i\xi x}\, dx = \int_{\mathbb{R}^n} f(x)(-ix_j)e^{-i\xi x}\, dx.$$

The interchange of integration and differentiation is justified by the observation that the integrand on right-hand side is continuous and dominated by the integrable function $(1 + \|x\|)^{-n-1}\nu_{n+1,0}(f)$ (check this). It follows that $\mathcal{F}(-x_j f) = \partial_j \mathcal{F}f$. By repeated application of this formula, we see that $\mathcal{F}f$ is a smooth function and that (b) holds. Since the inclusion map $\mathcal{S}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ and the Fourier transform $L^1(\mathbb{R}^n) \to C_b(\mathbb{R}^n)$ are continuous, it follows that $\mathcal{F}$ is continuous from $\mathcal{S}(\mathbb{R}^n)$ to $C_b(\mathbb{R}^n)$. As multiplication by x^α is a continuous endomorphism of the Schwartz space, it follows by application of (b) that $\mathcal{F}$ is a continuous linear map $\mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$.

Let $f \in C_c^\infty(\mathbb{R}^n)$ and $1 \leq j \leq n$. Then by partial integration it follows that

$$\int_{\mathbb{R}^n} \partial_j f(x)e^{-i\xi x}\, dx = (i\xi_j)\int_{\mathbb{R}^n} f(x)e^{-i\xi x}\, dx$$

so that $\mathcal{F}(\partial_j f) = (i\xi_j)\mathcal{F}(f)(\xi)$. By repeated application of this formula, it follows that (a) holds for all $f \in C_c^\infty(\mathbb{R}^n)$. By density of $C_c^\infty(\mathbb{R}^n)$ in $\mathcal{S}(\mathbb{R}^n)$ combined with continuity of the endomorphism $\partial^\alpha \in \mathrm{End}(\mathcal{S})$ and continuity of $\mathcal{F}$ as a map $\mathcal{S}(\mathbb{R}^n) \to C(\mathbb{R}^n)$ it now follows that (a) holds for all $f \in \mathcal{S}(\mathbb{R}^n)$.

It remains to establish the continuity of $\mathcal{F}$ as an endomorphism of $\mathcal{S}(\mathbb{R}^n)$. For this it suffices to show that $\xi^\alpha\partial^\beta\mathcal{F}$ is continuous linear as a map $\mathcal{S}(\mathbb{R}^n) \to C_b(\mathbb{R}^n)$. This follows from $\xi^\alpha\partial^\beta\mathcal{F} = \mathcal{F} \circ (-i\partial)^\alpha(-ix)^\beta$ (by (a), (b)) and the fact that $(-i\partial)^\alpha \circ (-ix)^\beta$ is a continuous linear endomorphism of $\mathcal{S}(\mathbb{R}^n)$. □

Later on, we will see that it is convenient to write

$$D^\alpha = (-i\partial)^\alpha,$$

so that formula (a) of the above lemma becomes

$$\mathcal{F}(D^\alpha f) = \xi^\alpha \mathcal{F} f.$$

Given $a \in \mathbb{R}^n$ we write T_a for the translation $\mathbb{R}^n \to \mathbb{R}^n, x \mapsto x+a$ and T_a^* for the map $C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ induced by pull-back. Thus, $T_a^* f(x) = f(x+a)$.

Lemma 4.1.11. *The map T_a^* restricts to a continuous linear endomorphism of $\mathcal{S}(\mathbb{R}^n)$. Moreover, for all $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$\mathcal{F}(T_a^* f) = e^{i\xi a}\mathcal{F}(f); \quad \mathcal{F}(e^{-iax} f) = T_a^* \mathcal{F} f.$$

Exercise 4.1.12. Prove the lemma.

We write S for the point reflection $\mathbb{R}^n \to \mathbb{R}^n, x \mapsto -x$ and S^* for the induced linear endomorphism of $C^\infty(\mathbb{R}^n)$. It is readily seen that S^* defines a continuous linear endomorphism of $\mathcal{S}(\mathbb{R}^n)$.

Exercise 4.1.13. The map S^* defines a continuous linear endomorphism of $\mathcal{S}(\mathbb{R}^n)$ which commutes with $\mathcal{F}$.

We can now give the full justification for the introduction of the Schwartz space.

Theorem 4.1.14. (Fourier inversion)

(a) *$\mathcal{F}$ is a topological linear isomorphism $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$.*

(b) *The endomorphism $S^*\mathcal{F}\mathcal{F}$ of $\mathcal{S}(\mathbb{R}^n)$ equals $(2\pi)^n$ times the identity operator. Equivalently, for every $f \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F} f(\xi)\, e^{i\xi x}\, d\xi, \qquad (x \in \mathbb{R}^n).$$

Proof We consider the continuous linear operator $\mathcal{T} := S^*\mathcal{F}\mathcal{F}$ from $\mathcal{S}(\mathbb{R}^n)$ to itself. By Lemma 4.1.10 it follows that

$$\mathcal{T} \circ x^\alpha = S^*\mathcal{F} \circ (i\partial)^\alpha \circ \mathcal{F} = S^* \circ (-x)^\alpha \circ \mathcal{F}\mathcal{F} = x^\alpha \circ \mathcal{T}.$$

In other words, $\mathcal{T}$ commutes with multiplication by x^α, for every multi-index α. In a similar fashion it is shown that $\mathcal{T}$ commutes with T_a^*, for every $a \in \mathbb{R}^n$.

We will now show that any continuous linear endomorphism $\mathcal{T}$ of $\mathcal{S}(\mathbb{R}^n)$ with these properties must be equal to a constant times the identity. For this we use the Gaussian function $G(x) = \exp(-\|x\|^2/2)$. Let $f \in C_c^\infty(\mathbb{R}^n)$ and put $\varphi = G^{-1} f$. Then φ is smooth compactly supported as well. Moreover, in view of the formula

$$\begin{aligned} \varphi(x) &= \varphi(0) + \int_0^1 \frac{\partial}{\partial t}\varphi(tx)\, dt \\ &= \varphi(0) + \left[\int_0^1 D\varphi(tx)\, dt\right] x, \end{aligned}$$

we see that there exists a smooth map $L : \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{C})$ such that $\varphi(x) = \varphi(0) + L(x)x$ for all $x \in \mathbb{R}^n$. It is easily seen that each component $L_j(x)$ is smooth with partial derivatives that are all bounded on $\mathbb{R}^n$. Hence, $L_jG \in \mathcal{S}(\mathbb{R}^n)$. It now follows that

$$\begin{aligned} \mathcal{T}(f) &= \mathcal{T}(\varphi G) \\ &= \mathcal{T}(\varphi(0)G) + \mathcal{T}(\sum_j x_j L_j G) \\ &= \varphi(0)\mathcal{T}(G) + \sum_j x_j \mathcal{T}(L_j G). \end{aligned}$$

Evaluating at $x = 0$ we find that $\mathcal{T}(f)(0) = cf(0)$, with c the constant $\mathcal{T}(G)(0)$.

We now use that $\mathcal{T}$ commutes with translation:

$$\mathcal{T}(f)(x) = [T_x^*\mathcal{T}(f)](0) = \mathcal{T}(T_x^* f)(0) = c\,T_x^* f(0) = cf(x).$$

This proves the claim that $\mathcal{T} = cI$. To complete the proof of (b) we must show that $c = (2\pi)^n$. This is the subject of the exercise below.

It follows from (b) and the fact that S^* commutes with $\mathcal{F}$ that $\mathcal{F}$ has $(2\pi)^{-n}S^*\mathcal{F}$ as a continuous linear two-sided inverse. Hence, $\mathcal{F}$ is a topological linear automorphism of $\mathcal{S}(\mathbb{R}^n)$.

Exercise 4.1.15. We consider the Gaussian function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = e^{-\frac{1}{2}x^2}$.

(a) Show that $\mathcal{F}g$ satisfies the differential equation $\frac{d}{dx}\mathcal{F}g = -x\mathcal{F}g$.
(b) Determine the Fourier transform $\mathcal{F}g$.
(c) Prove that for the Gaussian function $G : \mathbb{R}^n \to \mathbb{R}$ we have $\mathcal{T}(G) = (2\pi)^n G$.

In order to get rid of the constant $(2\pi)^n$ in formulas involving Fourier inversion, we change the normalization of the measures dx and $d\xi$ on $\mathbb{R}^n$, by requiring both of these measures to be equal to $(2\pi)^{-n/2}$ times Lebesgue measure. The definition of $\mathcal{F}$ is now changed by using formula (4.1) but with the new normalization of measures. Accordingly, the Fourier inversion formula becomes, for $f \in \mathcal{S}(\mathbb{R}^n)$,

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)\, e^{i\xi x}\, d\xi. \tag{4.3}$$

4.2. Convolution

The Schwartz space is also very natural with respect to convolution. In the following we shall make frequent use of the following easy estimates, for $x, y \in \mathbb{R}^n$

$$(1 + \|x\|)(1 + \|y\|)^{-1} \leq (1 + \|x + y\|) \leq (1 + \|x\|)(1 + \|y\|). \tag{4.4}$$

The inequality on the right is an easy consequence of the triangle inequality. The inequality on the left follows from the one on the right if we first substitute $-y$ for y and then, in the resulting inequality, $x + y$ for x.

Assume that $f_1, f_2 : \mathbb{R}^n \to \mathbb{C}$ are continuous functions with

$$\nu_N(f_j) := \sup(1 + \|x\|)^N |f_j(x)| < \infty$$

for all $N \in \mathbb{N}$ (Schwartz functions are of this type). Then it follows that

$$\begin{aligned} |f_j(x)| &= (1+\|x\|)^{-N}(1+\|x\|)^N|f_j(x)| \\ &\leq (1+\|x\|)^N \nu_N(f_j) \end{aligned}$$

for all $x \in \mathbb{R}^n$. Therefore,

$$\begin{aligned} f_1(y)f_2(x-y) &\leq (1+\|y\|)^{-M}(1+\|x-y\|)^{-N}\nu_M(f_1)\nu_N(f_2) \\ &\leq (1+\|y\|)^{N-M}(1+\|x\|)^{-N}\nu_M(f_1)\nu_N(f_2). \end{aligned}$$

Choosing $N = 0$ and $M > n$ we see that the function $y \mapsto f_1(y)f_2(x-y)$ is integrable for every $x \in \mathbb{R}^n$.

Definition 4.2.1. For $f, g \in \mathcal{S}(\mathbb{R}^n)$ we define the convolution product $f * g : \mathbb{R}^n \to \mathbb{C}$ by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy.$$

Lemma 4.2.2.

(a) *The convolution product defines a continuous bilinear map*

$$(f,g) \mapsto f * g, \quad \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n).$$

(b) *For all* $f, g \in \mathcal{S}(\mathbb{R}^n)$,

$$\mathcal{F}(f * g) = \mathcal{F}f\mathcal{F}g \qquad \textit{and} \qquad \mathcal{F}(fg) = \mathcal{F}f * \mathcal{F}g.$$

Proof Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and let α be a multi-index of order at most k. Let $K \in \mathbb{N}$. Then it follows from the above estimates with $f_1 = f$ and $f_2 = \partial^\alpha g$ that

$$(1+\|x\|)^K|f(y)\partial^\alpha g(x-y)| \leq (1+\|y\|)^{N-M}(1+\|x\|)^{K-N}\nu_{M,0}(f)\nu_{N,k}(g).$$

We now choose $N = K$ and $M > N + n$. Then the function on the right-hand side is integrable with respect to y. It now follows by differentiation under the integral sign that the function $f * g$ is smooth and that for all α we have $\partial^\alpha(f * g) = f * \partial^\alpha g$. Moreover, it follows from the estimate that

$$\nu_{K,k}(f * g) \leq \nu_{M,0}(f)\nu_{N,k}(g)\int_{\mathbb{R}^n}(1+\|y\|)^{N-M}\, dy.$$

We thus see that the map $(f,g) \mapsto f * g$ is continuous bilinear from $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.

Moreover, the above estimates justify the following application of Fubini's theorem:

$$\begin{aligned} \mathcal{F}(f * g)(\xi) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)g(x-y)e^{-i\xi x}\, dy\, dx \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)g(x-y)e^{-i\xi x}\, dx\, dy \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)g(z)e^{-i\xi(z+y)}\, dz\, dy \\ &= \mathcal{F}f(\xi)\mathcal{F}g(\xi). \end{aligned}$$

To obtain the second equality of (b), we use that $S^*\mathcal{F} = \mathcal{F}S^*$ is the inverse to $\mathcal{F}$ (by our new normalization of measures). Put $\varphi = \mathcal{F}S^*f$ and $\psi = \mathcal{F}S^*g$. Then $fg = \mathcal{F}\varphi\mathcal{F}\psi = \mathcal{F}(\varphi * \psi)$. By application of $\mathcal{F}$ we now readily verify that

$$\mathcal{F}(fg) = S^*(\varphi * \psi) = S^*(\varphi) * S^*(\psi) = \mathcal{F}f * \mathcal{F}g.$$

□

Corollary 4.2.3. *The convolution product $*$ on $\mathcal{S}(\mathbb{R}^n)$ is continuous bilinear, associative and commutative, turning $\mathcal{S}(\mathbb{R}^n)$ into a commutative continuous algebra.*

Proof This follows from the above lemma combined with the fact that $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a topological linear isomorphism. □

Exercise 4.2.4. By using Fourier transform, show that the algebra $(\mathcal{S}(\mathbb{R}^n), +, *)$ has no unit element.

On $\mathcal{S}(\mathbb{R}^n)$ we define the L^2-inner product $\langle \cdot\,, \cdot \rangle_{L^2}$ by

$$\langle f, g\rangle_{L^2} = \int_{\mathbb{R}^n} f(x)\,\overline{g(x)}\,dx.$$

Accordingly, the space $L^2(\mathbb{R}^n)$ may be identified with the Hilbert completion of $\mathcal{S}(\mathbb{R}^n)$.

Proposition 4.2.5. *Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then $\langle \mathcal{F}f, \mathcal{F}g\rangle_{L^2} = \langle f, g\rangle_{L^2}$. The Fourier transform has a unique extension to a surjective isometry $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.*

Proof We define the function $\check{g} : \mathbb{R}^n \to \mathbb{C}$ by

$$\check{g}(x) = \overline{g(-x)}.$$

Then g belongs to the Schwartz space, and $\mathcal{F}(\check{g}) = \overline{\mathcal{F}g}$. Moreover,

$$\langle f, g\rangle_{L^2} = f * \check{g}(0).$$

By the Fourier inversion formula it follows that the latter expression equals

$$\int_{\mathbb{R}^n} \mathcal{F}(f * \check{g})(\xi)\,d\xi = \int_{\mathbb{R}^n} \mathcal{F}f(\xi)\overline{\mathcal{F}g(\xi)}\,d\xi = \langle \mathcal{F}f, \mathcal{F}g\rangle_{L^2}.$$

Thus, $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is an isometry for $\langle \cdot\,, \cdot \rangle_{L^2}$. Since $C_c^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, so is $\mathcal{S}(\mathbb{R}^n)$ and it follows that $\mathcal{F}$ has a unique continuous linear extension to an endomorphism of the Hilbert space $L^2(\mathbb{R}^n)$; moreover, the extension is an isometry. Likewise, S^* is isometric hence extends to an isometric endomorphism of $L^2(\mathbb{R}^n)$. By density of $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ the composition of extended maps $S^*\mathcal{F}$ is a two-sided inverse to the extended map $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Therefore, $\mathcal{F}$ is surjective. □

4.3. Tempered distributions and Sobolev spaces

By means of the Fourier transform we shall give a different characterization of Sobolev spaces, which will turn out to be very useful in the context of pseudo-differential operators. We start by introducing the notion of tempered distribution.

Definition 4.3.1. The elements of $\mathcal{S}'(\mathbb{R}^n)$, the continuous linear dual of the Fréchet space $\mathcal{S}(\mathbb{R}^n)$, are called *tempered distributions.*

Here we note that a linear functional $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is continuous if and only if there exist constants $N, k \in \mathbb{N}$ and $C > 0$ such that

$$|u(f)| \leq C\,\nu_{N,k}(f) \quad \text{for all} \quad f \in \mathcal{S}(\mathbb{R}^n).$$

The name distributions is justified by the following observation. By transposition the continuous inclusions

$$C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$$

give rise to continuous linear transposed maps between the continuous linear duals of these spaces. Here we assume to have the duals equipped with the strong dual topologies (of uniform convergence on bounded sets). Moreover, as $C_c^\infty(\mathbb{R}^n)$ is dense in both $\mathcal{S}(\mathbb{R}^n)$ and $C^\infty(\mathbb{R}^n)$, it follows that the transposed maps are injective:

$$\mathcal{E}'(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\mathbb{R}^n).$$

We note that the transposed maps are given by restriction. Thus, $\mathcal{E}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is given by $u \mapsto u|_{\mathcal{S}(\mathbb{R}^n)}$. Moreover, the map $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ is given by $v \mapsto v|_{C_c^\infty(\mathbb{R}^n)}$. In this sense tempered distributions may be viewed as distributions.

We recall that the operators $x^\alpha\cdot$ and ∂^α on $\mathcal{D}'(\mathbb{R}^n)$ are defined through transposition:

$$x^\alpha u = u \circ (x^\alpha \cdot), \quad \text{and} \quad \partial^\alpha u = u \circ (-\partial)^\alpha,$$

for $u \in \mathcal{D}'(\mathbb{R}^n)$.

Exercise 4.3.2. Show that $\mathcal{S}'(\mathbb{R}^n)$ is stable under the operators ∂^α and x^α for all multi-indices α.

We recall that there is a natural continuous linear injection $L^2_{\text{loc}}(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\mathbb{R}^n)$. If $\varphi \in L^2_{\text{loc}}(\mathbb{R}^n)$ then the associated distribution is given by

$$f \mapsto \langle \varphi, f \rangle := \int_{\mathbb{R}^n} \varphi(x) f(x)\, dx, \quad C_c^\infty(\mathbb{R}^n) \to \mathbb{C}.$$

Lemma 4.3.3. *The continuous linear injection $L^2(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\mathbb{R}^n)$ maps $L^2(\mathbb{R}^n)$ continuously into $\mathcal{S}'(\mathbb{R}^n)$.*

Proof Denote the injection by j. Let $\varphi \in L^2(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$. Fix $N > n/2$. Then

$$\begin{aligned}\langle \varphi, f\rangle &= \int_{\mathbb{R}^n} \varphi(x) f(x)\, dx \\ &\leq \int_{\mathbb{R}^n} \varphi(x)(1+\|x\|)^{-N} \nu_{N,0}(f)\, dx \\ &\leq C\, \|\varphi\|_2\, \nu_{N,0}(f)\end{aligned}$$

where C is the L^2-norm of $(1+\|\xi\|)^{-N}$. It follows that the pairing $(\varphi, f) \mapsto \langle \varphi, f\rangle$ is continuous bilinear $L^2(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$. This implies that j maps $L^2(\mathbb{R}^n)$ continuously into $\mathcal{S}'(\mathbb{R}^n)$. □

The inclusion $\mathcal{S}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ is continuous. Accordingly, the natural injection $\mathcal{S}(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ maps $\mathcal{S}(\mathbb{R}^n)$ continuous linearly into $\mathcal{S}'(\mathbb{R}^n)$.

Exercise 4.3.4. Let $s \in \mathbb{R}$. We denote by $L^2_s(\mathbb{R}^n)$ the space of $f \in L^2_{\text{loc}}(\mathbb{R}^n)$ with $(1+\|x\|)^s f \in L^2(\mathbb{R}^n)$. Equipped with the inner product

$$\langle f, g\rangle_{L^2,s} := \int_{\mathbb{R}^n} f(x)\overline{g(x)}(1+\|x\|)^{2s}\, dx$$

this space is a Hilbert space.

Show that the continuous linear injection $L^2_s(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ maps $L^2_s(\mathbb{R}^n)$ continuously into $\mathcal{S}'(\mathbb{R}^n)$.

The following result will be very useful for our understanding of Sobolev spaces.

Proposition 4.3.5. *The Fourier transform has a continuous linear extension to a continuous linear map* $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. *For all* $u \in \mathcal{S}'(\mathbb{R}^n)$ *and* $f \in \mathcal{S}(\mathbb{R}^n)$ *we have*

$$\langle \mathcal{F}u, f\rangle = \langle u, \mathcal{F}f\rangle.$$

The extension to $\mathcal{S}'(\mathbb{R}^n)$ *is compatible with the previously defined extension to* $L^2(\mathbb{R}^n)$.

Remark 4.3.6. It can be shown that $C_0^\infty(\mathbb{R}^n)$, hence also $\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{S}'(\mathbb{R}^n)$. Therefore, the continuous linear extension is uniquely determined. However, we shall not need this.

Proof The Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous linear. Therefore its tranposed $\mathcal{F}^t : u \mapsto u \circ \mathcal{F}$ is a continuous linear map $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$.

We claim that $\mathcal{F}^t$ restricts to $\mathcal{F}$ on $\mathcal{S}(\mathbb{R}^n)$. Indeed, let us view $\varphi \in \mathcal{S}(\mathbb{R}^n)$ as a tempered distribution. Then by a straightforward application of Fubini's theorem, it follows that, for all $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}\langle \mathcal{F}^t\varphi, f\rangle &= \langle \varphi, \mathcal{F}f\rangle \\ &= \int_{\mathbb{R}^n} \varphi(\xi) \int_{\mathbb{R}^n} f(x) e^{-i\xi x}\, dx\, d\xi \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \varphi(\xi) e^{-i\xi x}\, d\xi\, f(x)\, dx \\ &= \langle \mathcal{F}\varphi, f\rangle.\end{aligned}$$

This establishes the claim. We have thus shown that $\mathcal{F}$ has $\mathcal{F}^t$ as a continuous linear extension to $\mathcal{S}'(\mathbb{R}^n)$.

It remains to prove the asserted compatibility. Let $u \in L^2(\mathbb{R}^n)$. There exists a sequence of Schwartz functions $u_n \in \mathcal{S}(\mathbb{R}^n)$ such that $u_n \to u$ in $L^2(\mathbb{R}^n)$ for $n \to \infty$. It follows that $\mathcal{F}u_n \to \mathcal{F}u$ in $L^2(\mathbb{R}^n)$, hence also in $\mathcal{S}'(\mathbb{R}^n)$, by Lemma 4.3.3. On the other hand, we also have $u_n \to u$ in $\mathcal{S}'(\mathbb{R}^n)$ by the same lemma. Hence $\mathcal{F}^t u_n \to \mathcal{F}^t u$ by what we proved above. Since $\mathcal{F}^t = \mathcal{F}$ on $\mathcal{S}(\mathbb{R}^n)$ it follows that $\mathcal{F}u_n = \mathcal{F}^t u_n$ for all n. Thus, $\mathcal{F}u = \mathcal{F}^t u$. $\square$

From now on, we shall denote the extension of $\mathcal{F}$ to $\mathcal{S}'(\mathbb{R}^n)$ by the same symbol $\mathcal{F}$. The following lemma is proved in the same spirit as the lemma above. We leave the easy proof to the reader.

Lemma 4.3.7. *The operators ∂^α, $x^\alpha\cdot$, T_a^* and $e^{ia}\cdot$ have (unique) continuous linear extensions to endomorphisms of $\mathcal{S}'(\mathbb{R}^n)$. For $u \in \mathcal{S}'(\mathbb{R}^n)$ we have*

$$\partial^\alpha u = u \circ (-\partial)^\alpha, \quad x^\alpha u = u \circ x^\alpha, \quad T_a^* u = u \circ T_{-a}^*, \quad e^{ia} u = u \circ e^{ia}.$$

The formulas (a),(b) of Lemma 1.1.10 and the formulas of Lemma 1.1.11 are valid for $f \in \mathcal{S}'(\mathbb{R}^n)$.

Lemma 4.3.8. *Let $u \in \mathcal{E}'(\mathbb{R}^n)$. Then $\mathcal{F}u$ is a smooth function. Moreover, for every $\xi \in \mathbb{R}^n$,*

$$\mathcal{F}u(\xi) = \langle u, e^{-i\xi} \rangle.$$

Proof We sketch the proof. Not all details can be worked out because of time constraints. Let $f \in C_c^\infty(\mathbb{R}^n)$. Then the function $\varphi : \xi \mapsto f(\xi)e^{-i\xi}$ with values in the Fréchet space $C^\infty(\mathbb{R}^n)$ is smooth and compactly supported. This implies that $\xi \mapsto u(\varphi(\xi))$ is smooth and compactly supported. Now

$$u(\varphi(\xi)) = f(\xi)u(e^{-i\xi})$$

and since f was arbitrary, we see that $\hat{u} : \xi \mapsto u(e^{-i\xi})$ is a smooth function.

Furthermore, the integral for $\mathcal{F}f$ may be viewed as an integral of the $C^\infty(\mathbb{R}^n)$-valued function φ. This means that in $C^\infty(\mathbb{R}^n)$ it can be approximated by $C^\infty(\mathbb{R}^n)$-valued Riemann sums. This in turn implies that

$$\begin{aligned} \langle \mathcal{F}u, f \rangle &= \langle u, \mathcal{F}f \rangle \\ &= u\Big(\int_{\mathbb{R}^n} \varphi(\xi)\, d\xi\Big) \\ &= \int_{\mathbb{R}^n} u(\varphi(\xi))\, d\xi \\ &= \int_{\mathbb{R}^n} f(\xi) u(e^{-i\xi})\, d\xi \\ &= \langle \hat{u}, f \rangle. \end{aligned}$$

Since this is true for any $f \in C_c^\infty(\mathbb{R}^n)$, it follows that $\hat{u} = \mathcal{F}u$. $\square$

We recall from Definition 2.2.10 that for $r \in \mathbb{N}$ the Sobolev space $H_r(\mathbb{R}^n)$ is defined as the space of distributions $u \in \mathcal{D}'(\mathbb{R}^n)$ such that $\partial^\alpha f \in L^2(\mathbb{R}^n)$ for each $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq r$. In particular, taking $\alpha = 0$ we see that $H_r(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$. Hence also $H_r(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$.

Lemma 4.3.9. *Let $r \in \mathbb{N}$. Then*

$$H_r(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid (1 + \|\xi\|)^r \mathcal{F}(u) \in L^2(\mathbb{R}^n)\}.$$

Proof Let $u \in H_r(\mathbb{R}^n)$ and let α be a multi-index of order at most r. Then $\partial^\alpha u \in L^2(\mathbb{R}^n)$. It follows that

$$(i\xi)^\alpha \mathcal{F}u = \mathcal{F}(\partial^\alpha u) \in L^2(\mathbb{R}^n).$$

In view of the lemma below this implies that $(1 + \|\xi\|)^r \mathcal{F}u \in L^2(\mathbb{R}^n)$.

Conversely, let $u \in \mathcal{S}'(\mathbb{R}^n)$ and assume that $(1 + \|\xi\|)^r \mathcal{F}u \in L^2(\mathbb{R}^n)$. Then $\mathcal{F}u$ is locally square integrable, and in view of the obvious estimate

$$|\xi^\alpha| \leq (1 + \|\xi\|)^{|\alpha|}, \qquad (\xi \in \mathbb{R}^n)$$

it follows that $(i\xi)^\alpha \mathcal{F}u \in L^2(\mathbb{R}^n)$. We conclude that

$$\partial^\alpha u = S^* \mathcal{F}((i\xi)^\alpha \mathcal{F}u) \in L^2(\mathbb{R}^n).$$

□

Lemma 4.3.10. *Let $r \in \mathbb{N}$. There exists a constant $C > 0$ such that for all $\xi \in \mathbb{R}^n$,*

$$(1 + \|\xi\|)^r \leq C \sum_{|\alpha| \leq r} |\xi^\alpha|;$$

here ξ^0 should be read as 1.

Proof It is readily seen that there exists a constant $C > 0$ such that

$$(1 + \sqrt{n}|t|)^r \leq C(1 + |t|^r), \qquad (t \in \mathbb{R}),$$

where $|t|^0 \equiv 1$. Let $\xi \in \mathbb{R}^n$ and assume that k is an index such that $|\xi_k|$ is maximal. Then $\|\xi\| \leq \sqrt{n}|\xi_k|$. Hence,

$$(1 + \|\xi\|)^r \leq (1 + \sqrt{n}|\xi_k|)^r \leq C(1 + |\xi_k|^r) \leq C \sum_{|\alpha| \leq r} |\xi^\alpha|.$$

□

Exercise 4.3.11. Show that the Fourier transform maps $H_r(\mathbb{R}^n)$ bijectively onto $L^2_r(\mathbb{R}^n)$. Thus, by transfer of structure, $H_r(\mathbb{R}^n)$ may be given the structure of a Hilbert space. Show that this Hilbert structure is not the same as the one introduced in Definition 2.2.10, but that the associated norms are equivalent.

The characterization of $H_r(\mathbb{R}^n)$ given above allows generalization to arbitrary real r.

Definition 4.3.12. Let $s \in \mathbb{R}$. We define the Sobolev space $H_s(\mathbb{R}^n)$ of order s to be the space of $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $(1 + \|\xi\|)^s \mathcal{F}f \in L^2(\mathbb{R}^n)$, equipped with the inner product

$$\langle f, g \rangle_s = \int_{\mathbb{R}^n} \mathcal{F}f(\xi)\overline{\mathcal{F}g(\xi)}\, (1 + \|\xi\|)^{2s}\, d\xi.$$

Equipped with this inner product, the Sobolev space $H_s(\mathbb{R}^n)$ is a Hilbert space. The associated norm is denoted by $\| \cdot \|_s$.

Exercise 4.3.13. The Heaviside function $H : \mathbb{R} \to \mathbb{R}$ is defined as the characteristic function of the interval $[0, \infty)$. For $R > 0$ we define u_R to be the characteristic function of $[0, R]$.

(a) Show that $u_R, H \in \mathcal{S}'(\mathbb{R})$ and that $u_R \to H$ in $\mathcal{S}'(\mathbb{R})$ (pointwise) as $R \to \infty$.
(b) Determine $\mathcal{F}u_R$ for every $R > 0$.
(c) Show that $u_R \in H_s(\mathbb{R})$ for every $s < \frac{1}{2}$, but not for $s = \frac{1}{2}$.
(d) Determine $\mathcal{F}H$ and show that $H \notin H_s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$.

Lemma 4.3.14. *Let $s \in \mathbb{R}$. Then $\mathcal{S}(\mathbb{R}^n) \subset H_s(\mathbb{R}^n)$, with continuous inclusion map. Furthermore, $C_c^\infty(\mathbb{R}^n)$ is dense in $H_s(\mathbb{R}^n)$.*

Proof If $f \in \mathcal{S}(\mathbb{R}^n)$ then $\mathcal{F}f \in \mathcal{S}(\mathbb{R}^n)$. Moreover, let $N \in \mathbb{N}$ be such that $N > s + n/2$. Then $N = s + n/2 + \epsilon$, with $\epsilon > 0$, hence

$$|\mathcal{F}f(\xi)|^2\,(1 + \|\xi\|)^{2s} \;\leq\; \nu_{N,0}(\mathcal{F}f)^2\,(1 + \|x\|)^{-n-2\epsilon}.$$

This implies that $f \in H_s(\mathbb{R}^n)$ and that

$$\|f\|_s \;\leq\; \nu_{N,0}(\mathcal{F}f)\;\|(1 + \|x\|)^{-n-2\epsilon}\|_{L^1}^{1/2}.$$

Since $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is continuous, it follows from this estimate that the inclusion map $\mathcal{S} \to H_s$ is continuous.

For the assertion about density it suffices to show that the orthocomplement of $C_c^\infty(\mathbb{R}^n)$ in the Hilbert space $H_s(\mathbb{R}^n)$ is trivial. Let $u \in H_s(\mathbb{R}^n)$, and assume that $\langle u, f\rangle_s = 0$ for all $f \in \mathcal{S}(\mathbb{R}^n)$. This means that

$$\int_{\mathbb{R}^n} \mathcal{F}u(\xi)\,\mathcal{F}f(\xi)\;(1 + \|\xi\|)^{2s}\,d\xi = 0, \qquad (f \in \mathcal{S}(\mathbb{R}^n)).$$

Therefore, the tempered distribution $\mathcal{F}u(\xi)\,(1 + \|\xi\|)^{2s}$ vanishes on the space $\mathcal{F}(C_c^\infty(\mathbb{R}^n))$. The latter space is dense in $\mathcal{F}(\mathbb{R}^n)$, since $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{F}$ is a topological linear automorphism of $\mathcal{S}(\mathbb{R}^n)$. We conclude that $\mathcal{F}u = 0$, hence $u = 0$. □

We conclude this section with two results that will allow us to define the local versions of the Sobolev spaces.

Lemma 4.3.15. *Let $s \in \mathbb{R}^n$. Then convolution $(f, g) \mapsto f * g$, $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ has a unique extension to a continuous bilinear map $\mathcal{S}(\mathbb{R}^n) * L^2_s(\mathbb{R}^n) \to L^2_s(\mathbb{R}^n)$.*

In the proof we will need a particular type of estimate that will be useful at a later stage as well. Specifically, for every $s \in \mathbb{R}$ the following estimate is valid for all $x, y \in \mathbb{R}^n$:

$$(1 + \|x + y\|)^s \leq (1 + \|x\|)^{|s|}(1 + \|y\|)^s.$$

It suffices to check the estimate for $s = r\epsilon$, with $r > 0$ and $\epsilon = \pm 1$. By monotonicity of the function $z \mapsto z^r$ on $]\,0, \infty\,[$ it suffices to check the estimate for $s = \pm 1$. In these cases, the estimate follows from (4.4).

Proof Let $f, g \in C_c^\infty(\mathbb{R}^n)$. Then for all $x, y \in \mathbb{R}^n$ we have

$$(1+\|x\|)^s|f(y)g(x-y)| \leq (1+\|y\|)^{|s|}|f(y)|(1+\|x-y\|)^s|g(x-y)|.$$

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$. Then multiplying the above expression by $|\varphi(y)|$, followed by integration against $dxdy$, application of Fubini's theorem and of the Cauchy-Schwartz inequality for the L^2-inner product, we find

$$|\langle(1+\|x\|)^s f * g, \varphi\rangle| \leq \int_{\mathbb{R}^n}(1+\|y\|)^{|s|}|f(y)|\,dy\,\|g\|_{L^2,s}\,\|\varphi\|_{L^2}.$$

Since this holds for arbitrary $\varphi \in C_c^\infty(\mathbb{R}^n)$, we obtain

$$\|(1+\|x\|)^s(f * g)\|_{L^2} \leq \|(1+\|y\|)^{|s|}f\|_{L^1}\,\|g\|_{L^2,s}.$$

The expression on the left-hand side equals $\|f * g\|_{L^2,s}$. Fix $N \in \mathbb{N}$ such that $|s| - N < -n$. Then the L^1-norm on the right-hand side is dominated by $C\nu_{N,0}(f)$, with C equal to the L^1-norm of the function $(1+\|y\|)^{|s|-N}$. It follows that

$$\|f * g\|_{L^2,s} \leq C\nu_{N,0}(f)\,\|g\|_{L^2,s}.$$

As $C_c^\infty(\mathbb{R}^n)$ is dense in both $\mathcal{S}(\mathbb{R}^n)$ and $L^2_s(\mathbb{R}^n)$, the result follows. □

Lemma 4.3.16. *Let $s \in \mathbb{R}$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $u \in H_s(\mathbb{R}^n)$. Then $\varphi u \in H_s(\mathbb{R}^n)$. Moreover, the associated multiplication map $\mathcal{S}(\mathbb{R}^n) \times H_s(\mathbb{R}^n) \to H_s(\mathbb{R}^n)$ is continuous bilinear.*

Proof We recall that by definition the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is an isometry for the norms $\|\cdot\|_s$ (from $H_s(\mathbb{R}^n)$) and $\|\cdot\|_{L^2,s}$. From the above lemma it now follows that the multiplication map $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ has a unique extension to a continuous bilinear map $\mathcal{S}(\mathbb{R}^n) \times H_s(\mathbb{R}^n) \to H_s(\mathbb{R}^n)$. We need to check that this extension coincides with the restriction of the multiplication map $\mathcal{S}(\mathbb{R}^n) \times \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$. Fix $f \in \mathcal{S}(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$. Then we must show that $\langle fg, \varphi\rangle = \langle g, f\varphi\rangle$ for all $g \in H_s(\mathbb{R}^n)$. By continuity of the expressions on both sides in g (verify this!), it suffices to check this on the dense subspace $C_c^\infty(\mathbb{R}^n)$, where it is obvious. □

In particular, it follows that $C_c^\infty(\mathbb{R}^n)H_s(\mathbb{R}^n) \subset H_s(\mathbb{R}^n)$. Therefore, we may define local Sobolev spaces.

Let $U \subset \mathbb{R}^n$ be open, and let $s \in \mathbb{R}$. We define the local Sobolev space $H_{s,\mathrm{loc}}$ in the usual way, as the space of distributions $u \in \mathcal{D}'(U)$ such that $\chi u \in H_s(\mathbb{R}^n)$ for every $\chi \in C_c^\infty(\mathbb{R}^n)$. At a later stage we will prove invariance of the local Sobolev spaces under diffeomorphisms, so that the notion of $H_{s,\mathrm{loc}}$ can be lifted to sections of a vector bundle on a smooth manifold.

Exercise 4.3.17. This exercise is a continuation of Exercise 4.3.13. Show that the Heaviside function $H = 1_{[0,\infty)}$ belongs to $H_{s,\mathrm{loc}}(\mathbb{R}^n)$ for every $s < \frac{1}{2}$ but not for $s = \frac{1}{2}$.

4.4. Some useful results for Sobolev spaces

We note that for $s < t$ the estimate $\|f\|_s \le \|f\|_t$ holds for all $f \in H_t(\mathbb{R}^n)$. Accordingly, we see that

$$H_t(\mathbb{R}^n) \subset H_s(\mathbb{R}^n), \quad \text{for} \quad s < t,$$

with continuous inclusion map. We also note that, by the Plancherel theorem for the Fourier transform, $H_0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$. Accordingly,

$$H_s(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset H_{-s}(\mathbb{R}^n) \qquad (s \ge 0). \tag{4.5}$$

Lemma 4.4.1. *Let $\alpha \in \mathbb{N}^n$. Then $\partial^\alpha : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ restricts to a continuous linear map $H_s(\mathbb{R}^n) \to H_{s-|\alpha|}(\mathbb{R}^n)$, for every $s \in \mathbb{R}$.*

Proof This is an immediate consequence of the definitions. □

Given $k \in \mathbb{N}$ we define $C_b^k(\mathbb{R}^n)$ to be the space of C^k-functions $f : \mathbb{R}^n \to \mathbb{C}$ with

$$s_k(f) := \max_{|\alpha|\le k} \sup_{x\in\mathbb{R}^n} |D_x^\alpha f(x)| < \infty.$$

Equipped with the norm s_k, this space is a Banach space.

Lemma 4.4.2. (Sobolev lemma) *Let $k \in \mathbb{N}$ and let $s > k + n/2$. Then*

$$H_s(\mathbb{R}^n) \subset C_b^k(\mathbb{R}^n)$$

with continuous inclusion map.

Proof In view of the previous lemma, it suffices to prove this for $k = 0$. We then have $s = n/2 + \epsilon$, with $\epsilon > 0$. Let $u \in C_c^\infty(\mathbb{R}^n)$, then

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^n} \mathcal{F}u(\xi) e^{i\xi x}\, dx \\ &= \int_{\mathbb{R}^n} e^{i\xi x} \mathcal{F}u(\xi)(1 + \|\xi\|)^s (1 + \|\xi\|)^{-n/2-\epsilon}\, d\xi \end{aligned}$$

From this we read off that u is bounded continuous, and

$$\sup |u| \le \|u\|_s \, \|(1 + \|\xi\|)^{-n/2-\epsilon}\|_{L^2}.$$

It follows that the inclusion $C_c^\infty \subset C_b$ is continuous with respect to the H_s topology on the first space. By density the inclusion has a unique extension to a continuous linear map $H_s \to C_b$. By testing with functions from $\mathcal{S}$ we see that the latter map coincides with the inclusion of these spaces viewed as subspaces of $\mathcal{S}'$. □

In accordance with the above embedding, we shall view $H_s(\mathbb{R}^n)$, for $s > k + n/2$, as a subspace of $C_b^k(\mathbb{R}^n)$. We observe that as an important consequence we have the following result. Put

$$H_\infty(\mathbb{R}^n) = \bigcap_{s\in\mathbb{R}} H_s(\mathbb{R}^n).$$

Corollary 4.4.3.

(a) $H_\infty(\mathbb{R}^n) \subset C_b^\infty(\mathbb{R}^n)$.

(b) $H_\infty(\mathbb{R}^n)$ *equals the space of smooth functions $f \in C^\infty(\mathbb{R}^n)$ with $\partial^\alpha f \in L^2(\mathbb{R}^n)$, for all $\alpha \in \mathbb{R}^n$.*

Proof Assertion (a) is an immediate consequence of the previous lemma. For (b) we note that $H_r \subset H_s$ for $s < r$. We see that $H_\infty(\mathbb{R}^n)$ is the intersection of the spaces $H_r(\mathbb{R}^n)$, for $r \in \mathbb{N}$. Now use the original definition of $H_r(\mathbb{R}^n)$, Definition 2.2.10. □

Let V, W be topological linear spaces. Then a pairing of V and W is a continuous bilinear map $\beta : V \times W \to \mathbb{C}$. The pairing induces a continuous map $\beta_1 : V \to W^*$ by $\beta_1(v) : w \mapsto \beta(v, w)$ and similarly a map $\beta_2 : W \to V^*$; the stars indicate the continuous linear duals of the spaces involved. The pairing is called non-degenerate if both the maps β_1 and β_2 are injective. It is called perfect if it is non-degenerate, and if β_1 is an isomorphism $V \to W^*$, and β_2 an isomorphism $W \to V^*$.

If V is a complex linear space, we denote by $\bar{V}$ the conjugate space. This is the complex space which equals V as a real linear space, whereas the complex scalar multiplication is given by $(z, v) \mapsto \bar{z}v$.

If V is a Banach space, the continuous linear dual V^* is equipped with the dual norm $\|\cdot\|^*$, given by

$$\|u\|^* = \sup\{|u(x)| \mid x \in V, \|x\| \le 1\}.$$

This dual norm also defines a norm on the conjugate space $\bar{V}^*$.

If H is a Hilbert space with inner product $\langle \cdot , \cdot \rangle$, then the associated norm $\|\cdot\|$ may be characterized by

$$\|v\| = \sup_{\|w\| \le 1} |\langle v, w\rangle|$$

It follows that $v \mapsto \langle v, \cdot \rangle$ induces a linear isomorphism $\varphi : H \to \bar{H}^*$ which is an isometry for the norm on H and the associated dual norm on H^*. The isometry φ may be used to transfer the Hilbert structure on H to a Hilbert structure on $\bar{H}^*$, called the dual Hilbert structure. It is readily seen that the norm associated with this dual Hilbert structure equals the dual norm $\|\cdot\|^*$ defined above.

Lemma 4.4.4. *Let $s \in \mathbb{R}$. Then the L^2-inner product $\langle \cdot , \cdot \rangle$ on $C_c^\infty(\mathbb{R}^n)$ extends uniquely to a continuous bilinear pairing $H_s(\mathbb{R}^n) \times \bar{H}_{-s}(\mathbb{R}^n) \to \mathbb{C}$. The pairing is perfect and induces isometric isomorphisms $H_s(\mathbb{R}^n) \simeq \bar{H}_{-s}(\mathbb{R}^n)^*$ and $\bar{H}_{-s}(\mathbb{R}^n) \simeq H_s(\mathbb{R}^n)^*$.*

Proof Let $f, g \in C_c^\infty(\mathbb{R}^n)$. Then

$$\begin{aligned} \langle f, g\rangle_{L^2} &= \int_{R^n} \mathcal{F}f(\xi)\mathcal{F}g(\xi)\, d\xi \\ &= \int_{R^n} \mathcal{F}f(\xi)(1 + \|\xi\|)^s \mathcal{F}g(\xi)(1 + \|\xi\|)^{-s}\, d\xi. \end{aligned}$$

By the Cauchy-Schwartz inequality, it follows that the absolute value of the latter expression is at most $\|f\|_s\|g\|_{-s}$. By density of $C_c^\infty(\mathbb{R}^n)$ in $H_s(\mathbb{R}^n)$, this implies the assertion about the extension of the pairing. The above formulas also imply that

$$\sup_{g \in C_c^\infty(\mathbb{R}^n), \|g\|_{-s}=1} \langle f, g\rangle = \|f\|_s.$$

Thus, by density of $C_c^\infty(\mathbb{R}^n)$, the induced map $\beta_1 : H_s(\mathbb{R}^n) \to \bar{H}_{-s}(\mathbb{R}^n)^*$ is an isometry. Likewise, $\beta_2 : H_{-s}(\mathbb{R}^n) \to \bar{H}_s(\mathbb{R}^n)^*$ is an isometry. From the injectivity of β_1 it follows that β_2 has dense image. Being an isometry, β_2 must then be surjective. Likewise, β_1 is surjective. □

4.5. Rellich's lemma for Sobolev spaces

In this section we will give a proof of the Rellich lemma for Sobolev spaces, which will play a crucial role in the proof of the Fredholm property for elliptic pseudo-differential operators on compact manifolds.

Given $s \in \mathbb{R}$ and a compact subset $K \subset \mathbb{R}^n$, we define

$$H_{s,K}(\mathbb{R}^n) = \{u \in H_s(R^n) \mid \operatorname{supp} u \subset K\}.$$

Lemma 4.5.1. *$H_{s,K}(\mathbb{R}^n)$ is a closed subspace of $H_s(\mathbb{R}^n)$.*

Proof Let $f \in C_c^\infty(\mathbb{R}^n)$. Then the space

$$f^\perp := \{u \in H_s(\mathbb{R}^n) \mid \langle u, f\rangle = 0\}$$

has Fourier transform equal to the space of $\varphi \in L_s^2(\mathbb{R}^n)$ with $\langle \varphi, \mathcal{F}f\rangle = 0$, which is the orthocomplement of $(1 + \|\xi\|)^{-2s}\mathcal{F}f$ in $L_s^2(\mathbb{R}^n)$. As this orthocomplement is closed in $L_s^2(\mathbb{R}^n)$, it follows that $f^\perp$ is closed in $H_s(\mathbb{R}^n)$.

We now observe that $H_{s,K}(\mathbb{R}^n)$ is the intersection of the spaces $f^\perp$ for $f \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp} f \cap K = \emptyset$. □

Lemma 4.5.2. (Rellich) *Let $t < s$. Then the inclusion map $H_{s,K}(\mathbb{R}^n) \to H_t(\mathbb{R}^n)$ is compact.*

To prepare for the proof, we first prove the following result, which is based on an application of the Ascoli-Arzéla theorem.

Lemma 4.5.3. *Let B be a bounded subset of the Fréchet space $C^1(\mathbb{R}^n)$. Then B is relatively compact (i.e., has compact closure) as a subset of the Fréchet space $C(\mathbb{R}^n)$.*

Proof Boundedness of B means that every continuous semi-norm of $C^1(\mathbb{R}^n)$ is bounded on B. Let $K \subset \mathbb{R}^n$ be a compact ball. Then there exists a constant $C > 0$ such that $\sup_K \|df\| \leq C$ for all $f \in B$ and each $1 \leq j \leq n$. Since

$$f(x) - f(y) = \int_0^1 df(y + t(x-y))(x-y)\, dt$$

for all $y \in x$, we see that

$$|f(y) - f(x)| \leq C\|x - y\|, \quad \text{for all} \quad (x, y \in K).$$

It follows that the set of functions $B|_K = \{f|_K \mid f \in B\}$ is equicontinuous and bounded in $C(K)$. By application of the Arzèla–Ascoli theorem (see next section), the set $B|_K$ is relatively compact in $C(K)$. In particular, if (f_k) is a sequence in B, then there is a subsequence (f_{k_j}) which converges uniformly on K.

Let (f_k) be a sequence in B. We shall now apply the usual diagonal procedure to obtain a subsequence that converges in $C(\mathbb{R}^n)$.

For $r \in \mathbb{N}$ let K_r denote the ball of center 0 and radius r in $\mathbb{R}^n$. Then by repeated application of the above there exists a sequence of subsequences $(f_{k_{1,j}}) \succeq (f_{k_{2,j}}) \succeq \cdots$ ... such that $(f_{k_{r,j}})$ converges uniformly on K_r, for every $r \in \mathbb{N}$.

The sequence $(f_{k_{j,j}})_{j\in\mathbb{N}}$ is a subsequence of all the above sequences. Hence, it converges uniformly on each ball K_r. Therefore, it converges in $C(\mathbb{R}^n)$. □

Remark 4.5.4. By a slight modification of the proof above, one obtains a proof of the compactness of each inclusion map $C^{k+1}(\mathbb{R}^n) \hookrightarrow C^k(\mathbb{R}^n)$. This implies that the identity operator of $C^\infty(\mathbb{R}^n)$ is compact. Equivalently, each bounded subset of $C^\infty(\mathbb{R}^n)$ is relatively compact. A locally convex topological vector space with this property is called Montel.

If B is a subset of $L^2_s(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we write

$$\varphi * B := \{\varphi * f \mid f \in B\}.$$

Then $\varphi * B$ is a subset of $L^2_s(\mathbb{R}^n)$.

Lemma 4.5.5. *Let $s \in \mathbb{R}$ and let $B \subset L^2_s(\mathbb{R}^n)$ be bounded. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then the set $\varphi * B$ is a relatively compact subset of $C(\mathbb{R}^n)$.*

Proof In view of the previous lemma, it suffices to prove that $f * B$ is bounded in $C^1(\mathbb{R}^n)$. For this we note that for each $1 \leq j \leq n$,

$$\begin{aligned}
&|\frac{\partial}{\partial x_j}[\varphi(x-y)f(y)]| \\
&= |\partial_j\varphi(x-y)|(1+\|y\|)^{-s}(1+\|y\|)^s|f(y)| \\
&\leq (1+\|x\|)^{|s|}|(1+\|x-y\|)^{-s}\|\partial_j\varphi(x-y)\|(1+\|y\|)^s|f(y)|.
\end{aligned}$$

The right-hand side can be dominated by an integrable function of y, locally uniformly in x. It now follows by differentiation under the integral sign that $\varphi * f \in C^1(\mathbb{R}^n)$, that $\partial_j(\varphi * f) = \partial_j\varphi * f$ and that

$$\|\partial_j(\varphi * f)(x)\| \leq (1+\|x\|)^{|s|}\,\|\varphi\|_{L^2,-s}\,\|f\|_{L^2,s}.$$

This implies that the set $\varphi * B$ is bounded in $C^1(B)$, hence relatively compact in $C(\mathbb{R}^n)$. □

Proposition 4.5.6. *Let $s > t$ and let B be a bounded subset of $L^2_s(\mathbb{R}^n)$ which at the same time is a relatively compact subset of $C(\mathbb{R}^n)$. Then B is relatively compact in $L^2_t(\mathbb{R}^n)$.*

Proof For $R > 0$ we denote by 1_R the characteristic function of the closed ball $B(R) := \bar{B}(0;R)$. Then for each $r \in \mathbb{R}$, the map $f \mapsto 1_R f$ gives the orthogonal projection from $L^2_s(\mathbb{R}^n)$ onto the closed subspace $L^2_{s,B(R)}$ of functions with support in $B(R)$. We now observe that the following estimate holds for every $f \in L^2_s(\mathbb{R}^n)$:

$$\begin{aligned}
\|(1-1_R)f\|^2_{L^2,t} &= \int_{\|x\|\geq R}(1+\|x\|)^{2t-2s}(1+\|x\|)^{2s}\|f(t)\|^2\,dt \\
&\leq (1+R)^{2(t-s)}\|f\|^2_{L^2,s}.
\end{aligned}$$

Fix $M > 0$ such that $\|f\|_{L^2,s} \leq M$ for all $f \in B$. Then we see that

$$\|(1 - 1_R)f\|_{L^2,t} \leq M\,(1+R)^{t-s}, \qquad (f \in B).$$

Let now (f_k) be a sequence in B. Then (f_k) has a subsequence (f_{k_j}) which converges in $C(\mathbb{R}^n)$, i.e., there exists a function $f \in C(\mathbb{R}^n)$ such that $f_{k_j} \to f$ uniformly on each compact set $K \subset \mathbb{R}^n$. It easily follows from this that $1_R f_{k_j}$ is a Cauchy-sequence in $L^2_t(\mathbb{R})$, for each $R > 0$. We will show that f_{k_j} is actually a Cauchy sequence in $L^2_t(\mathbb{R}^n)$. By completeness of the latter space, this will complete the proof.

Let $\epsilon > 0$. We fix $R > 0$ such that

$$M(1+R)^{t-s} < \frac{\epsilon}{3}.$$

There exists a constant $N > 0$ such that

$$i, j \geq N \Rightarrow \|1_R f_{k_i} - 1_R f_{k_j}\|_{L^2,t} < \frac{\epsilon}{3}.$$

It follows that for all $i, j > N$,

$$\begin{aligned} &\|f_{k_i} - f_{k_j}\|_{L^2,t} \\ &\leq \ \|1_R(f_{k_i} - f_{k_j})\|_{L^2,t} + \|(1-1_R)f_{k_i}\|_{L^2,t} + \|(1-1_R)f_{k_j}\|_{L^2,t} \\ &< \ \epsilon. \end{aligned}$$

□

Proof of Lemma 4.5.2 Let $K \subset \mathbb{R}^n$ be compact and let B be a bounded subset of $H_{s,K}(\mathbb{R}^n)$. Fix a smooth compactly supported function $\chi \in C_c^\infty(\mathbb{R}^n)$ that is 1 on a neighborhood of K. Then $\chi f = f$ for all $f \in B$. It follows that

$$\mathcal{F}(B) = \varphi * \mathcal{F}(B),$$

with $\varphi = \mathcal{F}(\chi) \in \mathcal{S}(\mathbb{R}^n)$. By Lemma 4.5.5 it now follows that $\mathcal{F}(B)$ is both bounded in $L^2_s(\mathbb{R}^n)$ and a relatively compact subset of $C(\mathbb{R}^n)$. By the previous proposition, this implies that $\mathcal{F}(B)$ is relatively compact in $L^2_t(\mathbb{R}^n)$. As $\mathcal{F}$ is an isometry from $H_t(\mathbb{R}^n)$ to $L^2_t(\mathbb{R}^n)$, it follows that B is relatively compact in $H_t(\mathbb{R}^n)$. □

4.6. The Arzèla–Ascoli theorem

The Arzèla–Ascoli theorem gives a useful characterization for relative compactness of a set of continuous functions on a locally compact metric space X, which in addition is σ-compact, i.e., X is the union of countably many compact subsets. In the following we assume X to be a such a metric space.

The space $C(X)$ of continuous functions $X \to \mathbb{C}$, is equipped with the locally convex topology of uniform convergence on compact subsets. More precisely, for $K \subset X$ a compact subset we define the seminorm $\|\cdot\|_K$ by

$$\|f\|_K := \sup_{x \in K} |f(x)|, \quad (f \in C(X).$$

The space $C(X)$ is equipped with the topology induced by the seminorms $\|\cdot\|_K$ for $K \subset X$ compact. Note that $C(X)$ is a Fréchet space with this topology. In the special case X compact, the topology is induced by the sup-norm $\|\cdot\|_X$, and $C(X)$ is a Banach space.

Definition 4.6.1. Let $\mathcal{F} \subset C(X)$ be a set of continuous functions on the compact metric space X. The set $\mathcal{F}$ is said to be equicontinuous at a point $a \in X$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in X$ with $d(x, a) < \delta$ we have

$$|f(x) - f(a)| < \epsilon, \qquad \text{for all } f \in \mathcal{F}.$$

The set $\mathcal{F}$ is said to be equicontinuous if it is equicontinuous at every point of X.

Exercise 4.6.2. Assume that X is compact metric space and let $\mathcal{F} \subset C(X)$. Show that $\mathcal{F}$ is equicontinuous if and only if $\mathcal{F}$ is uniformly equicontinuous, i.e., for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $f \in \mathcal{F}$ and all $x, y \in X$,

$$d(x, y) < \delta \;\Rightarrow\; |f(x) - f(y)| < \epsilon.$$

Hint: use the open covering property.

A set $\mathcal{F} \subset C(X)$ is said to be relatively compact if its closure $\bar{\mathcal{F}}$ in $C(X)$ is compact. Since every Fréchet space is metrizable, the latter is equivalent to $\bar{\mathcal{F}}$ being sequentially compact.

Theorem 4.6.3. (Arzèla-Ascoli) *Let X be a locally compact and σ-compact metric space, and $\mathcal{F} \subset C(X)$. Then the following assertions are equivalent.*

(a) *The set $\mathcal{F}$ is relatively compact in $C(X)$.*
(b) *The set $\mathcal{F}$ is equicontinuous and pointwise bounded.*

Proof We first assume that X is compact. Assume (a). Fix $a \in X$. Then the map $\mathrm{ev}_a : C(X) \to \mathbb{C}$ is continuous. Therefore, $\mathrm{ev}_a(\mathcal{F}) = \{f(a) \mid f \in \mathcal{F}\}$ is relatively compact, hence bounded in $\mathbb{C}$. This implies that $\mathcal{F}$ is pointwise bounded.

Let $\bar{\mathcal{F}}$ denote the closure of $\mathcal{F}$ in $C(X)$. Let $\epsilon > 0$. Then by compactness of $\bar{\mathcal{F}}$ there exists a finite collection of functions $f_j \in \bar{\mathcal{F}}$, $1 \leq j \leq k$, such that the balls $B(f_j; \epsilon/2)$ in $C(X)$ cover $\bar{\mathcal{F}}$. By compactness of X, each f_j is uniformly continuous. Hence there exists a $\delta > 0$ such that for all $x, y \in X$ with $d(x, y) < \delta$ and all j we have $|f_j(x) - f_j(y)| < \epsilon/3$. Let now $f \in \bar{\mathcal{F}}$. Then there exists a j such that $\|f - f_j\|_X < \epsilon/3$. It follows that for all $x, y \in X$ with $d(x, y) < \delta$ we have

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)| \\ &\leq 2\|f - f_j\|_X + |f_j(x) - f_j(y)| \\ &< \epsilon. \end{aligned}$$

This shows that $\bar{\mathcal{F}}$, and hence $\mathcal{F}$, is (uniformly) equicontinuous.

For the converse, assume (b). Then it is easily seen that the closure $\bar{\mathcal{F}}$ is equicontinuous and pointwise bounded as well.

Since $C(X)$ is metric, it suffices to show that $\bar{\mathcal{F}}$ is sequentially compact, or, equivalently, that every sequence in $\bar{\mathcal{F}}$ has a converging subsequence.

We will first show the validity of the following claim. Let (f_j) be any sequence in $\bar{\mathcal{F}}$ and $\epsilon > 0$. Then by passing to a subsequence we may arrange that for all k, l we have $\|f_k - f_l\|_X < \epsilon$.

To establish the claim, let $\epsilon > 0$. Then for every $a \in X$ there exists a $\delta_a > 0$ such that for for all $x \in X$ with $d(x, a) < \delta_a$ and all $f \in \bar{\mathcal{F}}$ we have $|f(x) - f(a)| < \epsilon/4$.

By compactness of X, there exist finitely many points $a_1, \ldots, a_r$ such that the open balls $B_X(a_i, \delta_{a_i})$ cover X. Fix i. Then the sequence $(f_j(a_i))$ is bounded, hence has a converging subsequence. We see that we may replace (f_j) by a subsequence to arrange that $(f_j(a_i))$ converges, for every $1 \leq i \leq r$. Thus we may pass to yet another subsequence of (f_j) to arrange that for all j, k, i we have

$$|f_j(a_i) - f_k(a_i)| < \epsilon/4.$$

Let $x \in X$. Select i such that $d(x, a_i) < \delta_{a_i}$. Then we find that for all j, k,

$$\begin{aligned} |f_j(x) - f_k(x)| &\leq |f_j(x) - f_j(a_i)| + |f_j(a_i) - f_k(a_i)| + |f_k(a_i) - f_k(x)| \\ &< 3\epsilon/4. \end{aligned}$$

It follows that the obtained subsequence satisfies $\|f_j - f_k\|_X \leq 3\epsilon/4 < \epsilon$, for all j, k. This establishes the claim.

Let now (f_j) be a sequence in $\bar{\mathcal{F}}$. Applying the above claim repeatedly we obtain a sequence of subsequences

$$(f_j) \succeq (f_{1,j}) \succeq (f_{2,j}) \succeq \cdots$$

such that for all k, i, j we have

$$\|f_{k,i} - f_{k,j}\| < 2^{-k}.$$

The sequence $(f_{k,k})_{k \in \mathbb{N}}$ is a subsequence of (f_j) and satisfies $\|f_{k,k} - f_{l,l}\|_X < 2^{-k}$ for all $k < l$. One now readily verifies that the sequence $(f_{k,k})$ is Cauchy for the sup-norm on $C(X)$ hence converges to a function $f \in C(X)$. Thus $\bar{\mathcal{F}}$ is sequentially compact, and (a) follows.

The general situation can be reduced to the present one by application of a diagonal argument, see the exercise below. □

Exercise 4.6.4. Let the metric space X be locally compact and σ-compact. Let (f_j) be a sequence in X such that the set $\mathcal{F} := \{f_j \mid j \in \mathbb{N}\}$ is equicontinuous.

(a) Show that there exists a countable sequence (K_j) of compact subsets of X such that $K_j \subset \text{int}\,(K_{j+1})$ and $\cup_j K_j = X$.
(b) Use a diagonal argument to show that (f_j) has a subsequence (f_{j_ν}) which converges uniformly on every set K_l, for $l \in \mathbb{N}$.
(c) Show that the sequence (f_{j_ν}) converges in $C(X)$.
(d) Complete the proof of Theorem 4.6.3.

Exercise 4.6.5. Let X be a locally compact metric space. Let $C_b(X)$ denote the set of bounded continuous functions $X \to \mathbb{C}$. Equipped with the supnorm $\|\cdot\|_X$ this space is a Banach space.

Let $F : X \to [0, \infty[$ be a continuous function which vanishes at infinity, i.e., for every $\epsilon > 0$ there exists a compact set $K \subset X$ such that $F(x) < \epsilon$ for all $x \in X \setminus K$.

Let $\mathcal{F}$ be an equicontinuous subset of $C_b(X)$ which is dominated by F, i.e. $|f(x)| \leq F(x)$ for all $f \in \mathcal{F}$ and $x \in X$.

(a) Let (f_j) be a sequence in $\mathcal{F}$. Show that for every $\epsilon > 0$ there exists a subsequence (f_{j_k}) of (f_j) such that
$$\|f_{j_k} - f_{j_l}\|_X < \epsilon$$
for all k, l.

(b) Show that $\mathcal{F}$ is relatively compact in $C_b(X)$.