Notes on quotients and group actions

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1 Quotients

Let X be a topological space, and R an equivalence relation on X. The set of equivalence classes for this relation is denoted by X/R and the canonical map by $p : X \to X/R$. The quotient X/R is equipped with the quotient topology. This is the finest topology for which the map $p : X \to X/R$ is continuous. Thus, a subset $U \subset X/R$ is open if and only if its preimage $p^{-1}(U)$ is open in X. The quotient topology is uniquely determined by the following universal property. For every topological space Y and every continuous map $f : X \to Y$ that is constant on the equivalence classes, the factored map $\bar{f} : X/R \to Y$ is continuous.

Lemma 1.1 *If X/R is Hausdorff, then R is closed in $X \times X$. If $p : X \to X/R$ is open, the converse is also true.*

Proof: The map $p \times p : X \times X \to X/R \times X/R$ is continuous (with respect to the product topologies). Since X/R is Hausdorff, the diagonal Δ is closed in $X/R \times X/R$. Therefore, its preimage is closed in $X \times X$. We now observe that

$$R = (p \times p)^{-1}(\Delta).$$

Conversely, assume that p is open. Then $p \times p$ is open as well. Since $X \times X \setminus R$ is open, its image under $p \times p$ is open. This image is equal to $(X/R \times X/R) \setminus \Delta$. Hence Δ is closed, and it follows that X/R is Hausdorff. $\square$

Remark 1.2 In general, the quotient topology for $p \times p$ on $X/R \to X/R$ is finer than the product topology (of the quotient topologies on the two factors X/R). If the two topologies coincide, then R closed implies X/R Hausdorff. The topologies coincide if p is open.

Let M be a manifold, and R an equivalence relation on M. In the rest of this section we will address the question whether the quotient space M/R carries a manifold structure such that the natural projection map $p : M \to M/R$ has reasonable properties. The map p should at least be smooth. It seems also reasonable to require that p be a submersion. We will first show that if such a manifold structure exists, it is necessarily unique. The following lemma will be a useful tool.

Lemma 1.3 *Let M, N, Z be manifolds, and let the following be a commutative diagram of maps between sets:*

$$\begin{array}{ccc} & Z & \\ \alpha \swarrow & & \searrow \beta \\ M & \xrightarrow{f} & N \end{array}$$

Assume that α is a surjective submersion, and that β is smooth. Then f is smooth. If in addition β is a submersion, then so is f. If in addition β is surjective, then so is f.

Proof: Let $m \in M$. Then $m = \alpha(z)$ for some $z \in Z$. Using the normal form of α at z, we see that there exists an open neighborhood $\mathcal{O}$ of m in M and a smooth map $s : \mathcal{O} \to Z$ such that $s(m) = z$ and $\alpha \circ s = I$ on $\mathcal{O}$. It follows that $f = f \circ \alpha \circ s = \beta \circ s$ on $\mathcal{O}$. Hence, f is smooth on $\mathcal{O}$. □

Corollary 1.4 *Let $p : M \to N$ be a surjective map between sets. Assume that M has a structure of smooth manifolds. Then N has at most one structure of smooth manifold for which p is a submersion.*

Proof: For $j = 1, 2$, let N_j be N equipped with a structure of smooth manifold such that $p : M \to N_j$ is a submersion. Let $I : N_1 \to N_2$ be the identity map. Then by the above lemma, I is smooth. Similarly, the inverse of I is smooth, hence I is a diffeomorphism. □

We return to the situation of a smooth manifold M equipped with an equivalence relation R. We will briefly say that *M/R is a smooth manifold*, if and only if M/R has a structure of smooth manifold for which $p : M \to M/R$ is a smooth submersion. This structure is then necessarily unique.

In the following lemma we view R as a subset of $M \times M$. Let pr_j denote the projection map $M \times M \to M$ on the j-th component, for $j = 1, 2$. Then the equivalence class of an element $a \in M$ may be described as $[a] = \mathrm{pr}_1((M \times \{a\}) \cap R)$.

Lemma 1.5 *Assume that M/R is a smooth manifold. Then R is a closed submanifold of M. Moreover, the projection maps $\mathrm{pr}_j|_R : R \to M$ are surjective submersions, for $j = 1, 2$.*

Proof: By definition, the projection map p is a submersion, hence $\alpha := I \times p : M \times M \to M \times M/R$ is a submersion as well. The graph $\Gamma = \{(x, p(x)) \mid x \in M\}$ is a smooth submanifold of $M \times M/R$. We note that $R = \alpha^{-1}(\Gamma)$. Hence R is a closed submanifold of $M \times M$ and $\alpha|_R : R \to \Gamma$ is a surjective submersion. Since $\mathrm{pr}_1 \circ (I, p) = I$ on M, it follows that $\mathrm{pr}_1 : \Gamma \to M$ is a surjective submersion. It follows that the composed map $\mathrm{pr}_1 \circ \alpha$ is a surjective submersion $R \to M$. We now observe that the composed map equals $\mathrm{pr}_1|_R$. □

Remark 1.6 The above is a special instance of a more general result, namely that the pull-back of a submersion is a submersion. More precisely, let $f : M \to N$ be a smooth map, and $p : Z \to N$ a submersion. Then set-theoretically the pull-back f^*Z may is defined as the set $\{(m, z) \in M \times Z \mid f(m) = p(z)\}$. This set equals the inverse image of graph(f) under the submersion $I \times p$. Since graph(f) is a smooth submanifold, it follows that f^*Z is a smooth submanifold. Finally, by a similar argument as in the above proof it follows that projection onto the first factor induces a submersion $f^*p : f^*Z \to M$.

The following result, due to R. Godement, expresses that the converse is also true.

Theorem 1.7 *Let R be an equivalence relation on M and assume that R is a closed submanifold of $M \times M$ such that $\mathrm{pr}_1|_R$ is a surjective submersion onto M. Then M/R has a unique structure of smooth manifold such that $p : M \to M/R$ is a submersion.*

We assume the conditions of the theorem to be fulfilled, and will present the proof in a number of lemmas, following the exposition of J.-P. Serre, p. 92. in [Serre]. [1] We begin by observing that, by symmetry of the relation, the projection $\mathrm{pr}_2|_R : R \to M$ is a submersion onto M as well.

Lemma 1.8 *The projection map $p : M \to M/R$ is open. The quotient topology is Hausdorff.*

Proof: Let $U \subset M$. Then $p^{-1}(p(U)) = \mathrm{pr}_1((M \times U) \cap R)$. Since $\mathrm{pr}_1|_R$ is a submersion, we see that for U open, the set $p^{-1}(p(U))$ is open in M, hence $p(U)$ is open in M/R. Therfore, p is an open map. The last assertion now follows by application of Lemma 1.1. $\square$

Lemma 1.9 *Each equivalence class of R is a closed submanifold of M of dimension* $\dim R - \dim M$.

Proof: The idea is that $[x] = \mathrm{pr}_1(\mathrm{pr}_2^{-1}(\{x\}))$, for every $x \in M$. By symmetry, $\mathrm{pr}_2 : R \to M$ is a submersion as well. Thus, if $x \in M$, then $\mathrm{pr}_2^{-1}(\{x\})$ is a smooth submanifold of R, of dimension $\dim R - \dim M$. Now $\mathrm{pr}_2^{-1}(\{x\}) = (M \times \{x\}) \cap R$ hence $\mathrm{pr}_2^{-1}(\{x\})$ is a smooth submanifold of $M \times \{x\}$. Thus, the projection onto the first coordinate, which equals $[x]$, is a smooth submanifold of M. $\square$

The next lemma expresses that locally near a point $a \in M$, we may parametrize the equivalence classes of R by a smooth submanifold through a that is transversal to $[a]$. Eventually this will give a local coordinatisation of the orbit space near $[a]$.

Lemma 1.10 *Let $a \in M$. Then there exists an open neighborhood $U \ni a$, a closed submanifold S of U and a submersion $q : U \to S$ such that for every $x \in U$, the set $[x] \cap U$ intersects S in the single point $q(x)$.*

Proof: We fix a submanifold S' of M through a such that $T_aS' \oplus T_a[a] = T_aM$. We will eventually show that all assertions hold for a suitable neighborhood U of a in M, and for $S = S' \cap U$.

The manifold S' has codimension $\dim[a] = \dim R - \dim M$. Since $\mathrm{pr}_2|_R$ is a submersion from R onto M it follows that $Z := \mathrm{pr}_2(S') \cap R = (M \times S') \cap R$ is a submanifold of codimension $\dim R - \dim M$ in R. Therefore, Z has the same dimension as M.

We observe that $[a] \times \{a\}$ is a submanifold of Z and that the diagonal $\Delta_{S'}$ of $S' \times S'$ is a submanifold of Z. The dimensions of these submanifolds add up to $\dim M$. Moreover, the tangent spaces at (a, a) are given by $T_a[a] \times \{0\}$ and $\Delta_{T_aS'}$, hence have trivial intersection. It follows that $T_{(a,a)}Z$ is the direct sum of $[a] \times \{0\}$ and $\Delta_{T_aS'}$. We thus see that $\mathrm{pr}_1|_Z : Z \to M$ is a local diffeomorphism at (a, a). It follows that there exists an open neighborhood $\mathcal{O}$ of a in M such that pr_1 is a diffeomorphism from $Z \times (\mathcal{O} \times \mathcal{O})$ onto an open neighborhood U' of a in M. Let s denote the inverse of this isomorphism, then $\mathrm{pr}_1 \circ s = I_{U'}$ and $q = \mathrm{pr}_2 \circ s$ is a submersion from U' onto an open subset of $S' \cap \mathcal{O}$.

Accordingly, $s(x) = (x, q(x))$ for all $x \in U'$, and we see that $U' \subset \mathcal{O}$. If $x \in U' \cap S'$, then $(x, x) \in (\mathcal{O} \times \mathcal{O}) \cap Z$ and $\mathrm{pr}_1(x, x) = \mathrm{pr}_1 s(x)$, hence $s(x) = (x, x)$ and we see that $q(x) = x$. In particular this implies that $q(q(x)) = q(x)$ as soon as $q(x) \in U'$.

[1] J.-P. Serre, Lie algebras and Lie groups; Lecture Notes in Mathematics, 1500, Springer-Verlag, New York, 1992

We now put $U = U' \cap q^{-1}(U' \cap \mathcal{O})$ and $S = S' \cap U$ and claim that U and S satisfy the requirements of the theorem. Indeed, clearly $S \subset U$. If $x \in U$, then $q(x) \in U'$ and therefore $q(q(x)) = q(x) \in U' \cap \mathcal{O}$ from which we see that $q(x) \in U \cap S' = S$.

Finally, let $x \in U$. By what we showed, $q(x) \in S' \cap U = S$. Moreover, if $y \in S$ is such that $y \in [x]$, then it follows that $(x, y) \in (\mathcal{O} \times \mathcal{O}) \cap ((M \times S) \cap R) = (\mathcal{O} \times \mathcal{O}) \cap Z$, hence $(x, y) = \mathrm{pr}_1(x, y)$, from which we deduce that $y = q(x)$. This shows that the intersection of $[x] \cap U$ and S consists of the single point $q(x)$. $\square$

Remark 1.11 A submanifold S with the properties mentioned above is called a slice through the classes of R in U.

If $V \subset M$ is a subset, then by R_V we denote the restriction of R to V. Thus, in terms of graphs, $R_V = (V \times V) \cap R$. The classes of R_V in V are the sets $[x] \cap V$, for $x \in V$. It follows that the inclusion $V \hookrightarrow M$ factors to an inclusion $V/R_V \hookrightarrow M/R$. The projection $V \to V/R_V$ will be denoted by p_V. Then the diagram

$$\begin{array}{ccc} V & \hookrightarrow & M \\ {\scriptstyle p_V} \downarrow & & \downarrow {\scriptstyle p} \\ V/R_V & \hookrightarrow & M/R \end{array}$$

commutes.

Corollary 1.12 *Let $a \in M$. Then there exists an open neighborhood U of a in M such that U/R_U is a manifold.*

Proof: Let U, S, q be as in the above lemma. Then the map $q : U \to S$ factors to a bijection $\bar{q} : U/R_U \to S$. Accordingly, $q = \bar{q} \circ p_U$. We use the bijection $\bar{q}$ to transfer the manifold structure of S to a manifold structure on U/R_U. Then p_U becomes a submersion. $\square$

A subset $V \subset M$ is called saturated if and only if it is a union of equivalence classes for R. Equivalently, this means that $V = p^{-1}(p(V))$. The next step is to extend the manifold structures on open neighborhoods to manifold structures on saturated open neighborhoods.

Lemma 1.13 *Let $U \subset M$ be an open subset such that U/R_U is a manifold. Let $V = p^{-1}(p(U))$. Then the natural map $q : V \to U/R_U$ is a surjective submersion.*

Proof: Let $p_U : U \to U/R_U$ and $p_V : V \to V/R_V$ denote the projection maps. The inclusion map $i : U \to V$ factors to a bijection $j : U/R_U \to V/R_V$. The map q is given by $j \circ q = p_V$. From this we infer that the following diagram commutes

$$\begin{array}{ccc} (M \times U) \cap R & \stackrel{\mathrm{pr}_2}{\to} & U \\ {\scriptstyle \mathrm{pr}_1} \downarrow & & \downarrow {\scriptstyle p_U} \\ V & \stackrel{q}{\to} & U/R_U. \end{array}$$

The top arrow and right vertical arrow represent surjective submersions, and so does the left vertical arrow. By application of Lemma 1.3 it follows that q is a surjective submersion. $\square$

Corollary 1.14 *Let U, V be as above. If U/R_U is a smooth manifold, then so is V/R_V.*

Proof: With notation as in the proof of the above lemma, $j : U/R_U \to V/R_V$ is a bijection and $j \circ q = p_V$. We equip V/R_V with the manifold structure obtained by transport of structure under j. Then p_V becomes a submersion. $\square$

Remark 1.15 It follows from the above that the unique map $r : V \to S$ defined by $[x] \cap S = \{r(x)\}$ is a smooth surjective submersion. Accordingly, the slice S for U is also a slice for V, the smallest saturated open set containing U.

Corollary 1.16 *Every point $a \in M$ is contained in a saturated open neighborhood V such that V/R_V is a manifold.*

Completion of the proof of Theorem 1.7. It follows from the preceding corollary that M has a covering with saturated open neighborhoods U_i such that U_i/R_{U_i} is a smooth manifold for each i.

Let $p : M \to M/R$ be the projection map, then $p|_{U_i} : U_i \to M/R$ factors to a bijection from U_i/R_{U_i} onto the open subset $p(U_i)$ of M/R. We transfer the manifold structure from U_i/R_{U_i} to a manifold structure on $p(U_i)$. Let i, j be indices such that $p(U_i) \cap p(U_j) \neq \emptyset$. Since U_i and U_j are saturated, $p(U_i) \cap p(U_j) = p(U_i \cap U_j)$. The manifold structure from $p(U_i)$ induces a manifold structure $\mathcal{M}_i$ on $p(U_i) \cap p(U_j)$ for which the projection map $p : U_i \cap U_j \to p(U_i) \cap p(U_j)$ is a submersion. Likewise, the manifold structure from $p(U_j)$ induces a manifold structure $\mathcal{M}_j$ on $p(U_i) \cap p(U_j)$ such that $p : U_i \cap U_j \to p(U_i) \cap p(U_j)$ is a surjective submersion. By application of Lemma 1.3 it follows that the two manifold structures coincide.

We conclude that M/R has a unique manifold structure for which all inclusion maps $U_i/R_{U_i} \to M/R$ are diffeomorphisms onto open subsets. For this manifold structure, the projection map $p : M \to M/R$ is a submersion. □

2 Quotients for group actions

In this section we apply Theorem 1.7 to suitable group actions. Let G be a group acting (from the right) on a locally compact Hausdorff topological space M by homeomorphisms. Associated to this group action is the equivalence relation R on M defined by $(x, y) \in R \iff xG = yG$. Thus, the classes of R are the right G-cosets. We write $M/G = M/R$ for the associated coset space.

Lemma 2.1 *The quotient map $p : M \to M/G$ is open.*

Proof: Let U be open. Then $p^{-1}p(U) = \cup_{g \in G} Ug$, which is a union of open sets hence open. It follows that $p(U)$ is open. □

Corollary 2.2 *Let G be a group acting on M by homeomorphisms. Then the quotient topology on M/G is Hausdorff if and only if the map*

$$\alpha : M \times G \to M \times M, \quad (x, g) \mapsto (x, xg). \tag{2.1}$$

has closed image.

Proof: Since $p : M \to M/G$ is open, the quotient topology is Hausdorff if and only if R is closed in $M \times M$, see Lemma 1.1. We now observe that R equals the image of α. □

The action of G on M is called free if for each $x \in M$ the stabilizer subgroup $G_x = \{g \in G \mid xg = x\}$ is trivial. It is called free and properly discontinuous if for each $x \in M$ there exists a neighborhood U such that $U \cap Ug = \emptyset$ for every $g \in G$, $g \neq e$. Equivalently, this means that the sets Ug, for $g \in G$ are mutually disjoint.

Lemma 2.3 *Let G have a free action on M by homeomorphisms. Then the following conditions are equivalent:*

(a) *The action is properly discontinuous and the relation R is closed in $M \times M$.*

(b) *The map α is proper (G equipped with the discrete topology).*

If any of these conditions is fulfilled, each G-orbit is closed in M.

Proof: First assume (a). Since R is closed, and p is open, the quotient space M/G is Hausdorff. It follows that for every $x, y \in M$ with $xG \neq yG$ there exist open neighborhoods $U \ni x$ and $V \ni y$ such that $UG \cap VG = \emptyset$. On the other hand, if $xG = yG$, then we may choose an open neighborhood U of x such that the Ug, for $g \in G$, are mutually disjoint. Let $y = xh$ and put $V = Uh$. Then $Ug \cap V \neq \emptyset$ for only a single g. Thus in any case, given two elements $x, y \in M$ there exist open neighborhoods U of x and V of y such that $Ug \cap V \neq \emptyset$ for at most finitely many $g \in G$. Let now $C \subset M$ be compact and let $x \in C$. Then by using compactness, it follows that x has an open neighborhood U such that $Ug \cap C \neq \emptyset$ for only finitely many $g \in G$. By using compactness once more, it follows that the collection of $g \in G$ such that $C \cap Cg \neq \emptyset$ is finite.

Finally, let $K \subset M \times M$ be compact. Put $C = \mathrm{pr}_1(K) \cup \mathrm{pr}_2(K)$. Then C is compact and $K \subset C \times C$. Let F be the finite set of $g \in G$ such that $Cg \cap C \neq \emptyset$. Then $\alpha(x, g) \in K$ implies that $x \in C$ and $g \in F$. Hence $\alpha^{-1}(K)$ is a closed subset of the compact set $C \times F$. It follows that $\alpha^{-1}(K)$ is compact. Hence (b).

For the converse implication, assume (b). Then $R = \mathrm{image}(\alpha)$ is closed. Moreover, let $x \in M$. Then x has a compact neighborhood C. Then $\alpha^{-1}(C \times C) = \{(x, g) \in C \times G \mid xg \in C\}$ is compact. Therefore the projection of this set onto the second coordinate is compact, hence a finite subset $F \subset G$. We note that F equals $\{(g \in G \mid C \cap Cg \neq \emptyset\}$. Since the action is free, we may chose, for every $f \in F \setminus \{e\}$ open neighborhoods V_f of x and W_f of xf such that $V_f \cap W_f = \emptyset$. Put $U_f = V_f \cap W_f f^{-1}$. Then $U_f \cap U_f f = \emptyset$. Let U be an open neighborhood of x contained in C and in $\cap_{f \in F \setminus \{e\}} U_f$. Then $Ug \cap U = \emptyset \Rightarrow g = e$. □

Lemma 2.4 *Let M be a smooth manifold, equipped with a free and properly discontinuous action of a group G of diffeomorphims. Assume moreover that M/G is Hausdorff. Then M/G is a manifold. Moreover, $p : M \to M/G$ is a smooth covering, for which G acts by covering maps on M.*

Proof: Let $x \in M$. Then there exists an open set $U \ni x$ such that the set Ug, for $g \in G$, are mutually disjoint. Since M is second countable, it follows that G is at most countable. Hence G may be viewed as a zero dimensional manifold, and accordingly, $M \times G$ may be viewed as a manifold of the same dimension as M. Since G acts by diffeomorphisms, the map (2.1) is an injective immersion. Since α is proper, it is an embedding onto the closed submanifold R. It follows that $M/G = M/R$ is a smooth manifold. The assertion that G acts by covering transformations is immediate. □

We will now consider the situation of a right action by a Lie group G on M. We first recall some basic facts from the theory of Lie groups. A **Lie group** is a group $(G, \cdot, e)$ equipped with the structure of a smooth manifold such that the multiplication map $m : (x, y) \mapsto xy,\ G \times G \to G$ and the inversion map $\iota : x \mapsto x^{-1},\ G \to G$ are smooth. Given $a \in G$ the maps $r_a : G \to G, x \mapsto xa$ and $l_a : G \to G, x \mapsto ax$ are diffeomorphisms from G onto itself. A vector field v on G is said to be right-invariant if and only if $r_a^* v = v$ for all $a \in G$. The linear space of such vector fields is denoted by $\mathfrak{X}_R(G)$. Let $\mathfrak{g} = T_e G$. Then for every $X \in \mathfrak{g}$ there is a unique right invariant vector field $X_R \in \mathfrak{X}_R(G)$ with $X_R(e) = X$. It is given by $X_R(a) = dr_a(e)X$. Accordingly, the map $\mathfrak{X}_R(G) \to \mathfrak{g}$ defined by $v \mapsto v(e)$ is a linear isomorphism. The space $\mathfrak{X}_R(G)$ is closed under Lie brackets of vector fields; in other words, $\mathfrak{X}_R(G)$ is a sub Lie algebra of the algebra $\mathfrak{X}(G)$ of all vector fields on G. By transport of structure, there exists a unique structure of Lie algebra on $\mathfrak{g}$ for which the isomorphism $\mathfrak{X}_R(G) \to \mathfrak{g}$ becomes an isomorphism of Lie algebras. Equippee with this structure, $\mathfrak{g}$ is called the Lie algebra of the Lie group G.

A one-parameter subgroup of G is a smooth group homomorphism from $(\mathbb{R}, +, 0)$ to G. For every $X \in \mathfrak{g}$ there exists a unique one-parameter subgroup $\alpha_X : \mathbb{R} \to G$ with $\alpha_X'(0) = X$. In fact, it coincides with the integral curve of the vector field X_R with initial point e. The exponential map $\exp : \mathfrak{g} \to G$ is defined by

$$\exp X := \alpha_X(1).$$

The exponential map is smooth; moreover, the one-parameter subgroup α_X may be expressed in terms of the exponential map by

$$\alpha_X(t) = \exp tX.$$

In particular, this implies that

$$\exp(s+t)X = \exp sX \exp tX, \qquad \left.\frac{d}{dt}\right|_{t=0} \exp tX = X.$$

The latter formula is equivalent to $d\exp(0) = I_{\mathfrak{g}}$. In particular, we see that the exponential map is a local diffeomorphism at 0.

The right action of G on M is said to be smooth if the action map $M \times G \to M$, $(m, g) \mapsto mg$ is smooth. In this setting, every $X \in \mathfrak{g}$ defines a smooth vector field X_M on M, given by the formula

$$X_M(m) = \left.\frac{d}{dt}\right|_{t=0} m \exp tX. \tag{2.2}$$

The action of G on M is called proper if the map (2.1) is proper. The situation of Lemma 2.4 may be viewed as a special case of the next result, with G a zero-dimensional (hence discrete) group.

Lemma 2.5 *Let $(m, g) \mapsto mg$ be a smooth action of a Lie group G on a smooth manifold M. If the action is free and proper, then M/G has a unique structure of smooth manifold such that the natural projection map $p : M \to M/G$ is a submersion.*

Proof: We first observe that freeness of the action is equivalent to injectivity of α.

For $Y \in \mathfrak{g}$, the Lie algebra of G, let Y_M denote the vector field on M given by (2.2). Let now $x \in M, g \in G$; then the derivative of α at (x, g) is given by

$$\begin{aligned} d\alpha(x,g)(X, dl_g(e)Y) &= d_1\alpha(x,g)X + \left.\frac{d}{dt}\right|_{t=0} \alpha(x, g\exp tY) \\ &= (X, dr_g(x)X) + (0, Y_M(xg)). \end{aligned}$$

From this it follows that α is an immersion. Since α is proper, it follows that α is an embedding onto a closed submanifold. Now R is the image of α, hence it remains to be verified that $\mathrm{pr}_1 : R \to M$ is a submersion. This is immediate from the fact that $\mathrm{pr}_1 \circ \alpha$ equals the projection map $M \times G \to M$, hence is a submersion. The result now follows by application of Theorem 1.7. □

3 Fiber bundles associated with group actions

Let B, F be smooth manifolds. A **fiber bundle** over a manifold B (the base manifold) with fiber modelled on F is a smooth manifold M together with a submersion $p : M \to B$ such that for every $a \in B$ there exists an open neighborhood $U \ni a$ and a diffeomorphism $\tau : p^{-1}(U) \to U \times F$ (trivialization) such that the following diagram commutes:

$$\begin{array}{ccc} U \times F & \xrightarrow{\tau} & \pi^{-1}(U) \\ & {}_{\mathrm{pr}_1}\searrow \quad \swarrow_{\pi} & \\ & U & \end{array}$$

In particular, this means that every fiber $p^{-1}(x)$, for $x \in B$ is diffeomorphic to F.

Let G be a Lie group. A **principal fiber bundle** with **structure group** G over B is a fiber bundle $p : M \to B$ (with fiber modelled on G) together with a smooth right action of G on M such that for every $a \in B$ there exists an open neighborhood $U \ni a$ and a trivialization $\tau : p^{-1}(U) \to U \times G$ in which the action becomes trivialized in the sense that the following diagram commutes, for all $g \in G$:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\tau} & U \times G \\ {\scriptstyle \alpha_g}\downarrow & & \downarrow{\scriptstyle 1\times r_g} \\ p^{-1}(U) & \xrightarrow{\tau} & U \times G \end{array}$$

Here α_g denotes the right action $M \to M$, $x \mapsto xg$ and r_g denotes the right multiplication $G \to G$, $x \mapsto xg$.

Principal fiber bundles appear naturally in connection with proper free actions of G.

Theorem 3.1

(a) *Let M be a smooth manifold, equipped with a proper and free action of G. Then M/G has a unique structure of smooth manifold such that the canonical projection $p : M \to M/G$ is a submersion. Furthermore, $p : M \to M/G$ is a principal fiber bundle with structure group G.*

(b) *Let $p : M \to B$ be a principal fiber bundle with structure group G. Then the action of G on M is proper and free, and the map p factors to a diffeomorpism $M/G \simeq B$.*

Proof: Let M be as in (a). Then the first assertion follows from Lemma 2.5. For the second assertion, we refer to the standard literature. One may also consult my lecture notes on Lie groups. Assertion (b) is a rather straightforward consequence of the definition of a principal fiber bundle.

Remark 3.2 A covering projection $M \to B$ may be viewed as a fiber bundle with discrete fiber. In this spirit, if G is discrete, we retrieve the old equivalence of free and proper discrete actions to coverings.

We shall now discuss the **associated bundle** construction. Let M be a smooth manifold equipped with a proper and free smooth right action of a Lie group G. Then the natural map $p : M \to M/G$ defines a principal fiber bundle with structure group G. Let F be a smooth manifold equipped with a smooth left action of G. Then we may define a right action of G on $M \times F$ by $(y, z)g = (yg, g^{-1}z)$. This action is proper and free, since the action of G on M is already so. It follows that $M \times_G F$ has a unique structure of smooth manifold such that the projection map $q : M \times F \to M \times_G F$ is a submersion. We note that $p \circ \mathrm{pr}_1 : M \times F \to M/G$ induces a map $\pi : M \times_G F \to M/G$ which is smooth by Lemma 1.3.

Lemma 3.3 *The map $\pi : M \times_G F \to M/G$ defines a fiber bundle with fiber modelled on F.*

Proof: We must show that $\pi : M \times_G F \to M/G$ admits suitable local trivializations. For this, let $a \in M/G$. Then the principal fiber bundle $p : M \to M/G$ admits a local trivialization τ over some open neighborhood U of a. The map $s : x \mapsto \tau^{-1}(x, e)$ defines a section of p over U. We define the map $\psi : U \times F \to p^{-1}(U)$ by $\psi(x, z) = q(s(x), z))$. Then clearly, $p \circ \psi = \mathrm{pr}_1$. Thus, if ψ is a diffeomorphism onto, it follows that ψ^{-1} is a local trivialization of π over U. We will finish the proof by showing that ψ is indeed a diffeomorphism onto. We leave it to the reader to verify that ψ is a bijection. Thus, it remains to be shown that ψ is a local diffeomorphism. For dimensional reasons it suffices to show that ψ is a submersion.

Fix $x \in U$. From the fact that $M \to M/G$ is a principal bundle, it follows that the map $j : X \mapsto X_M(s(x))$ is an injective linear map $\mathfrak{g} \to T_{s(x)}M$ such that

$$\mathrm{im}\,(ds(x)) \oplus \mathrm{im}\,(j) = T_{s(x)}M.$$

Let $y \in F$ and consider the map $k : \mathfrak{g} \to T_{(s(x),y)}(M \times F)$ given by $k(X) = X_{M \times F}(s(x), y) = (j(X), -X_F(y))$. Then it follows that

$$T_{(s(x),y)}(M \times F) = T_{s(x)}M \times T_yF = \mathrm{im}\,(ds(x) \times I)\ \oplus \mathrm{im k}.$$

We now observe that $dq(s(x), y)$ vanishes on $\mathrm{im}\, k$. Hence, $d\psi(x, y) = dq(s(x), y) \circ (ds(x) \times I)$ is surjective from $T_x(M/G) \times T_yF$ onto $T_{\psi(x,y)}(M \times_G F)$. It follows that ψ is submersive at every point of $U \times F$. $\square$

4 Background on Riemannian manifolds

Let M be a smooth manifold. A Riemannian metric on M is a family of positive definite inner products g_x on T_xM, for $x \in M$, such that the map $g : x \mapsto g_x$ is a smooth covariant two-tensor. Equivalently, smoothness means that for every pair of smooth vector fields $X, Y \in \mathfrak{X}(M)$ the function $g(X, Y)$ is smooth.

A Riemannian manifold is a smooth manifold M equipped with a Riemannian metric g.

Let N be a second smooth manifold, and $\varphi : N \to M$ a smooth map. Then the pull-back $\varphi^*(g)$, defined by

$$\varphi^*(g)_x(\xi, \eta) = g_{\varphi(x)}(d\varphi(x)\xi, d\varphi(x)\eta),$$

defines a covariant two-tensor on N. If φ is a local diffeomorphism, then $\varphi^*(g)$ is a Riemannian metric on N.

If N is equipped with a Riemannian metric h, then the map φ is called an isometry if and only if the derivative $d\varphi(x) : T_xN \to T_{\varphi(x)}M$ is an isometric linear map, for every $x \in N$. If φ is a local diffeomorphism this means precisely that $\varphi^*(g) = h$.

If $\gamma : [0, 1] \to M$ is a C^1-curve , its length $L(\gamma)$ is defined by the formula

$$L(\gamma) := \int_0^1 \|\gamma'(t)\|_{\gamma(t)}\, dt,$$

where $\|\cdot\|_a$ denotes the norm associated with g_a, for $a \in M$. Given two points $a, b \in M$, let $\mathcal{C}_{a,b}$ denote the set of C^1-curves $\gamma : [0, 1] \to M$ with initial point $\gamma(0) = a$ and end point $\gamma(1) = b$. We put

$$d(a, b) = \inf_{\gamma \in \mathcal{C}_{a,b}} L(\gamma).$$

It can be shown that d defines a metric on M whose topology coincides with the original topology of M.

Lemma 4.1 *Let M be a smooth manifold. Then there exists a Riemannian metric on M.*

Proof: Let $n = \dim M$. Let $\{(U_i, \chi_i)\}_{i\in I}$ be a locally finite covering of M by means of coordinate charts with compact closure. Let h denote the standard Euclidean metric on $\mathbb{R}^n$, then $g_i = \chi_i^*(h)$ is a Riemannian metric on U_i. Let $\{\psi_i\}$ be a partition of unity subordinate to the covering $\{U_i\}$ then $g = \sum_i \psi_i g_i$ is a smooth covariant 2-tensor on M. It is readily verified that g is symmetric and positive definite at every point, hence defines a Riemannian metric. □

It follows from the above that every smooth manifold is metrizable.

Let (M, g) be a Riemannian manifold. Then for every $a \in M$ there exists an open neighborhood $U \ni a$ such that for every $p, q \in M$ there exists a unique curve γ with initial point p and end point q which minimizes the length functional, i.e., $d(p, q) = L(\gamma)$. This unique curve is called a local geodesic. In local coordinates $x^1, \dots, x^n$, it can be described by means of a second order ordinary differential equation of the form

$$\frac{d^2\gamma^k}{dt^2} + \sum_{ij} \Gamma^k_{ij} \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} = 0.$$

Here the **Christoffel symbols** Γ^k_{ij} are smooth functions on the coordinate patch associated with the given local coordinates. These symbols are explicitly expressible in terms of the components g_{ij} of the metric in the same local coordinates.

The reader should keep in mind the example of $\mathbb{R}^n$ equipped with the standard Euclidean metric. In that situation the Christoffel symbols vanish, and the local geodesics are the curves of the form $t \mapsto a + tv$, for $a, v \in \mathbb{R}^n$.

A geodesic on M is defined to be any C^2 (hence C^∞-) curve $\gamma : J \to M$, with J an open interval, satisfying the above equation in local coordinates.

domain of definition cannot be increased. It follows from the existence and uniqueness theorem for the initial value problem for ordinary differential equations of the second order, that for every $a \in M$ and every $\xi \in T_aM$ there exists a unique maximal geodesic $\gamma_{a,\xi} : J_{a,\xi} \to M$, with $J_{a,\xi} \subset \mathbb{R}$ an open interval containing 0, such that

$$\gamma(0) = a, \qquad \text{and} \qquad \gamma'(0) = \xi.$$

As in the theory of vector fields, it can be shown that the set D of $(a, \xi, t) \in TM \times \mathbb{R}$ with $t \in J_{a,\xi}$ is open. For $a \in M$, we define the open subset D_a of T_aM to consist of all ξ with $(a, \xi, 1) \in D$. Moreover, we define the exponential map $\exp_a : D_a \to M$ by

$$\exp_a \xi = \gamma_{a,\xi}(1).$$

From this definition it can be deduced that

$$\gamma_{a,\xi}(t) = \exp_a t\xi,$$

for all $t \in J_{a,\xi}$. In particular, it follows by differentiation with respect to t at $t = 0$, that

$$d \exp_a(0) = I_{T_aM}.$$

Hence, $\exp_a$ is a local diffeomorphism at a. Here we mention that M is said to be geodesically complete if $J_{a,xi} = \mathbb{R}$ for all $(a, \xi) \in TM$. This implies that $\exp_a$ is defined on all of T_aM.

Using the material described in the above summary, we can now prove the following result.

Lemma 4.2 *Let M, N be two Riemannian manifolds, and let $\varphi : M \to N$ be an isometric local diffeomorphim. Let $a \in M$ and put $b = \varphi(a)$. Then there exists an open neighborhood U of 0 in T_aM such that*

$$\varphi \circ \exp_a = \exp_b \circ\, d\varphi(a) \qquad \text{on} \quad U.$$

Proof: Let U, V be open balls centered at 0 in T_aM and T_bN respectively, such that $\exp_a$ is a diffeomorphism from U into M, $\exp_b$ is a diffeomorphism from V into N, and $\varphi(\exp_a U) \subset \exp_b(V)$. Then for every $\xi \in U$, the curve $t \mapsto \exp_a(t\xi)$, $[0, 1] \to M$ is a geodesic. Since φ is an isometry, it follows that $c : t \mapsto \varphi(\exp_a \xi))$ is a geodesic in N, with $c(0) = b$. Put $\eta = c'(0)$. Then it follows that $c(t) = \exp_b t\eta$. By the chain rule it follows that $\eta = d\varphi(a)\xi$. Hence, $\varphi(\exp_a t\xi) = \exp_b t d\varphi(a)\xi$. Now substitute $t = 1$ to finish the proof. □

Corollary 4.3 *Let M, N be two connected Riemannian manifolds, and let $\varphi, \psi : M \to N$ be two isometric local diffeomorphims. Then the following assertions are equivalent.*

(a) $\varphi = \psi$;

(b) *There exists a point $a \in M$ such that $\varphi(a) = \psi(a)$, and $d\varphi(a) = d\psi(a)$.*

Proof: Clearly, (a) implies (b). Conversely, assume (b). We consider the set A of points $x \in M$ with $\varphi(x) = \psi(x)$ and $d\varphi(x) = d\psi(x)$. Then A is non-empty and closed. We will finish the proof by showing that A is open as well. Fix $p \in A$ and put $q = \varphi(p) = \psi(p)$. By the previous result, we may fix an open ball U around 0 in T_pM such that $\exp_p$ is a diffeomorphism from U into M and such that $\varphi \circ \exp_p = \exp_p \circ\, d\varphi(p)$ on U, and a similar equation with ψ in place of φ. Since $d\varphi(p) = d\psi(p)$, it follows that $\varphi = \psi$ on $\exp_p(U)$, which is an open neighborhood of p. We now observe that $\exp_p U \subset A$. □

Corollary 4.4 *Let M be a connected Riemannian manifold, $a \in M$, and let $I(M,a)$ be the group of isometric diffeomorphisms of M onto itself, fixing the point a. Then $\nu : \varphi \mapsto d\varphi(a)$ defines an injective group homomorphism from $I(M,a)$ into $\mathrm{O}\,(T_aM)$.*

Proof: From the chain rule, it follows that ν is a group homomorphism. Injectivity follows by application of the previous corollary. □

Remark 4.5 The group $I(M)$ of all isometries of M is a topological group for the compact open topology. It can be shown, see, e.g., [Hel][2], Ch. IV., Thm. 2.5, that $I(M)$ is a locally compact group. Moreover, the subgroup $I(M,a)$ is compact. It follows that ν maps $I(M,a)$ homeomorphically onto a closed subgroup of $\mathrm{O}\,(T_aM)$. Hence, $I(M,a)$ carries the structure of a compact Lie group.

5 Actions of Lie groups

In this section, we assume that a smooth left action of a Lie group G on a smooth manifold M is given. As before we consider the associated linear map $\mathfrak{g} \to \mathfrak{X}(M)$ defined by

$$X_M(m) = \frac{d}{dt}\bigg|_{t=0} \exp tXm, \qquad (m \in M),$$

for $X \in \mathfrak{g}$. Here we warn the reader that $X \mapsto X_M$ is an anti-homomorphism of Lie algebras (the analogous map for a right action is a homomorphism). We note that the flow φ of the vector field X_M is given by

$$\varphi^t(m) = \exp tX\, m.$$

Let now $a \in M$, then the natural map $G \to M$, given by $g \mapsto ga$ factors to a bijection from G/G_a onto the orbit Ga. The following basic result of Lie group theory is very important in this context.

Lemma 5.1 *Every subgroup H of the Lie group G which is closed in the sense of topology has a unique structure of Lie group for which the inclusion map $i_H : H \to G$ is a smooth embedding onto a closed submanifold.*

Proof: For a proof, see, e.g. [BtD][3] or my lecture notes on Lie groups. □

In the setting of the above lemma, let $\mathfrak{h} := T_eH$ be equipped with the Lie algebra structure associated with H. Then the inclusion map $i_H : H \to G$ is a group homomorphism and an embedding of manifolds. From this it is not difficult to obtain that $di_H(e) : \mathfrak{h} \to \mathfrak{g}$ is an injective homomorphism of Lie algebras. Accordingly, we identify $\mathfrak{h}$ with a subalgebra of $\mathfrak{g}$. Furthermore, the following diagram commutes

$$\begin{array}{ccc} H & \xrightarrow{i_H} & G \\ \downarrow & & \downarrow \\ \mathfrak{h} & \xrightarrow{di_H(e)} & \mathfrak{g}. \end{array} \tag{5.3}$$

[2][Hel]: S. Helgason, Differential Geometry and Symmetric Spaces, Academic Press, 1962

[3][BtD]: T. Bröcker, T. tom Dieck, Representations of compact Lie groups. Graduate Texts in Mathematics, 98. Springer-Verlag, New York, 1985

In the setting of the above lemma, with H a closed subgroup of the Lie group G, the action of H on G by right multiplication is readily seen to be proper and free, turning the natural map $\pi : G \to G/H$ into a principal fiber bundle. From a local trivialization we see that $d\pi(e) : \mathfrak{g} \to T_{eH}(G/H)$ is a submersion with kernel $\ker d\pi(e) = T_e(\pi^{-1}(eH)) = T_eH = \mathfrak{h}$. It follows that $d\pi(e)$ factors to a linear isomorphism from $\mathfrak{g}/\mathfrak{h}$ onto $T_{eH}(G/H)$. From now on we shall use this identification without further comment.

The isotropy group G_a is closed, hence a closed Lie subgroup in the above sense. This implies that $G \to G/G_a$ is a principal fiber bundle with structure group G_a. The induced map $G/G_a \to M$ is readily seen to be smooth. Moreover, the following result is valid.

Lemma 5.2 *The natural map $j : G/G_a \to M$ is an injective immersion with image Ga.*

Proof: We consider the map $J : G \to M$, $g \mapsto ga$. Its tangent map at e is given by $dJ(e)X = X_M(a)$. If $X \in \mathfrak{g}$ and $dJ(e)X = 0$ then it follows that X_M has a zero at a, hence its integral curve $t \mapsto \exp tXa$ is constant. It follows that $\exp tX \in G_a$ for all t. Differentiating at $t = 0$ we find that $X \in \mathfrak{g}_a$ (the Lie algebra of G_a).

Conversely, if $X \in \mathfrak{g}_a$, then $\exp_{G_a} tX \in G_a$, hence $\exp tX \in G_a$, by commutativity of the diagram (5.3). This implies that $\exp tXa = a$ for all t, hence $dJ(e)(X) = X_M(a) = 0$. It follows that $\ker dJ(e) = \mathfrak{g}_a$. Since $j \circ \pi = J$, it follows that $dj(eG_a) \circ d\pi(eG_a) = dJ(e)$. Since $d\pi(eG_a)$ factors to a linear isomorphism from $\mathfrak{g}/fg_a = \mathfrak{g}/\ker dJ(e)$ onto $T_{eG_a}(G/G_a)$, it follows that $dj(eG_a)$ is injective.

For $g \in G$ let $l_g : G/G_a \to G/G_a$ denote the left multiplication by g on G/G_a, and let $\alpha_g : M \to M$ denote the left multiplication by g on M. Both maps are diffeomorphisms, and $\alpha_g \circ j = j \circ l_g$. Using this homogeneity, and the chain rule for differentiation we infer that $dj(gG_a)$ is injective for all $g \in G$. □

Corollary 5.3 *In the setting of the above lemma, the map $X \mapsto X_M(a)$ factors to a linear isomorphism from the quotient space $\mathfrak{g}/\mathfrak{g}_a$ onto the tangent space $T_a(Ga)$ of the orbit Ga.*

Proof: See the proof of the previous result. □

In the following we assume that the Lie group G is connected, so that its orbits Ga are connected injectively immersed submanifolds of M. The action of G on the manifold M is said to be foliated if $\dim G_a$ is constant as a function of $a \in M$. In this case all G-orbits have the same dimension. We will show that they are actually the leaves of a foliation on M.

To see this, we note that every G-orbit is an injectively immersed submanifold of dimension $\dim G - d$, where $d = \dim G_a$. Thus, $a \mapsto E_a := T_a(Ga)$ defines a subbundle E of the tangent bundle. It suffices to show that E is locally integrable. To see this, let $a \in M$ and let $\mathfrak{v} \subset \mathfrak{g}$ be a linear subspace such that $fv \oplus \mathfrak{g}_a = \mathfrak{g}$. Moreover, let S be a submanifold of M such that $a \in S$ and $T_aS \oplus T_a(Ga) = T_aM$. We consider the map $\varphi : \mathfrak{v} \times S \to M$, $(X, z) \mapsto \exp X\, z$. Its derivative at $(0, a)$ equals $(X, \zeta) \mapsto X + \zeta$ hence is a linear isomorphism. It follows that there exist open neighborhoods $\mathcal{O}$ of 0 in $\mathfrak{v}$ and V of a in S such that the restricted map $\varphi : \mathcal{O} \times V \to M$ is a diffeomorphism onto an open subset $U \subset M$. Fix $X_0 \in \mathcal{O}$ and $y \in V$ and put $b = \exp X_0y$. The map $X \mapsto \varphi(X_0 + X, y)$ maps into the G-orbit $Gb = Gy$. Hence, its derivative $D_1\varphi(X_0, y)$ maps into E_b. For dimensional reasons, it follows that $D_1(\varphi(X_0, y)$ has image E_b. We conclude that $D\varphi$ maps the subbundle $\mathrm{pr}_1^*T\mathcal{O}$ isomorphically onto $E|_U$. Therefore, E is locally integrable. □

We now come to the important slice theorem for proper actions.

Theorem 5.4 *Let the smooth left action of the Lie group G on M be foliated and proper. Then for every $a \in M$ there exists a smooth submanifold S of M such that:*

(a) *$a \in S$ and S is G_a-invariant;*

(b) *the natural map $\beta : G \times S \to M$, $(g, y) \mapsto gy$, factors to a diffeomorphism $\bar{\beta}$ from $G \times_{G_a} S$ onto an open G-invariant neighborhood of Ga in M.*

Remark 5.5 Since $\bar{\beta}$ intertwines the G-actions, it follows that β brings the action of G on a neighborhood of the orbit Ga in the standard form $G \times_{G_a} S$.

Proof: First we note that G_a is compact by properness of the action. By the method of averaging we obtain a G_a-invariant Riemannian metric on M. We now select an open ball U centered at 0 in T_aM such that $\exp_a : U \to M$ is a diffeomorphism onto an open subset. Let $\mathfrak{v}$ be the orthocomplement of $T_a(Ga)$ in T_aM and put $V = U \cap \mathfrak{v}$. Then $S = \exp_a(V)$ is a smooth submanifold of M. Moreover, G_a acts by isometries, so if $g \in G_a$, then the tangent map $d\alpha_g(a)$ of $\alpha_g : m \mapsto gm$ leaves V invariant. From $\exp_a \circ d\alpha_g(a) = \alpha_g \circ \exp_a$ we see that the action by G_a leaves S invariant.

Since the action of G_a on G by right multiplication is proper and free, it follows that $G \times_{G_a} S$ has a unique structure of smooth manifold for which the natural map $q : G \times S \to G \times_{G_a} S$ is a submersion. It follows that the natural map $\beta : G \times S \to M$ factors to a smooth map $\bar{\beta} : G \times_{G_a} S \to M$.

Select a subspace $\mathfrak{c} \subset \mathfrak{g}$ complementary to $\mathfrak{g}_a$, i.e., $\mathfrak{c} \oplus \mathfrak{g}_a = \mathfrak{g}$. Then the map $\mathfrak{c} \to G/G_a$, given by $X \mapsto \exp X G_a$ is a local diffeomorphism at 0. It follows that the map $\mathfrak{c} \times S \to G \times_F S$, $(X, y) \mapsto q(\exp X, y)$ is a local diffeomorphism at $(0, a)$. It follows from the proof of the previous lemma that the restriction of β to $\exp \mathfrak{c} \times S$ is a local diffeomorphism at (e, a). We conclude that $\bar{\beta}$ is a local diffeomorphism at $eG_a \times a$. Choosing the radius of U sufficiently small, we may arrange that $\bar{\beta}$ maps an open neighborhood of $eG_a \times S$ diffeomorphically onto an open neighborhood of a in M. By homogeneity it follows that the map $\bar{\beta}$ is a local diffeomorphism of $G \times_{G_a} S$ onto an open G-invariant neighborhood of Ga in M.

By properness of the action, there exists a compact subset $C \subset G$ such that $gS \cap S \neq \emptyset \Rightarrow g \in C$. Since G_a is compact, we may replace C by CG_a to arrange that C is right G_a-invariant. The map $\bar{\beta}$ is injective on the image $C_a := q(C \times \{a\})$ in $G \times_{G_a} S$. In view of Lemma 5.6 below, we may select an open neighborhood $\mathcal{O}$ of C_a on which $\bar{\beta}$ is a diffeomorphism, and by replacing U by a smaller neighborhood, we may arrange that $\bar{\beta}$ is injective on $q(C \times S)$.

We will now complete the proof by showing that $\bar{\beta}$ is injective on $G \times_{G_a} S$. Indeed, assume that $g_1, g_2 \in G$ and $y_1, y_2 \in S$ and that $\bar{\beta}(q(g_1, y_1)) = \bar{\beta}(q(g_2, y_2))$. Then $g_2^{-1} g_1 y_1 = y_2$, and therefore $g_2^{-1} g_1 \in C$. Since $\bar{\beta}$ is injective on $q(C \times S)$ it follows that $q(g_2^{-1} g_1, y_1) = q(e, y_2)$. This implies that $q(g_1, y_1) = q(g_2, y_2)$, and we see that $\bar{\beta}$ is injective. $\square$

Lemma 5.6 *Let X, Y be manifolds, and $f : X \to Y$ a local diffeomorphism. Let $C \subset X$ be compact, and assume that $f|_C$ is injective. Then there exists an open neighborhood U of C such that $f|_U$ is a diffeomorphism from U onto an open subset of Y.*

Proof: There exists a decreasing sequence $\{U_j\}$ of relatively compact open neighborhoods of C whose intersection is C. Assume that f is not injective on U_j, for each j. Then there exist sequences of points $x_j, y_j \in U_j$ with $x_j \neq y_j$ such that $f(x_j) = f(y_j)$. The sequence (x_j, y_j) in $M \times M$ has a limit point (x, y). Clearly $x, y \in C$ and $f(x) = f(y)$, hence $x = y$. Fix an open neighborhood $\mathcal{O}$ of x on which f is injective. Then $x_j, y_j \in \mathcal{O}$ for some j, and $f(x_j) = f(y_j)$, hence $x_j = y_j$, contradiction. It follows that f is injective on a suitable open neighborhood of C. The result follows. $\square$