Notes: Extra exercises for the course Foliations

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Exercises for Chapter 1

Exercise 1 The purpose of this exercise is to show that the three dimensional sphere decomposes into two solid tori, glued together along their boundaries. Of course the reader is also encouraged to find an immediate visualization of this fact.

Given a point $x \in \mathbb{R}^4$, we agree to write $x = (x', x'')$, with $x', x'' \in \mathbb{R}^2$. The circle group $\mathrm{SO}(2)$ has a smooth action on $\mathbb{R}^4$, given by

$$a \cdot x = (ax', a^{-1}x'').$$

Let Δ denote the closed disc $\|y\|^2 \leq 1/2$ in $\mathbb{R}^2$

(a) Show that the action described induces a free action on the unit sphere $S := S^3$ in $\mathbb{R}^4$.

(b) Put $S_1 = \{x \in S \mid \|x'\| \leq \frac{1}{2}\sqrt{2}\}$ and $S_2 = \{x \in S \mid \|x''\| \leq \frac{1}{2}\sqrt{2}\}$. Show that the map $\varphi_1 : \Delta \times \mathrm{SO}(2) \to S$ given by

$$\varphi_1(y, a) = a \cdot (y, \sqrt{1 - y^2}, 0)$$

is a diffeomorphism onto S_1. Likewise, show that the map $\varphi_2 : \Delta \times \mathrm{SO}(2) \to S$ given by

$$\varphi_2(z, a) = a \cdot (\sqrt{1 - z^2}, 0, z)$$

is a diffeomorphism onto S_2.

(c) Show that $S_1 \cap S_2 \simeq \mathrm{SO}(2) \times \mathrm{SO}(2)$ is the torus.

(d) Show that the orbit space $X := S^3/\mathrm{SO}(2)$ is a Hausdorff topological space, homeomorphic to the 2-sphere. Show that the fibration $S^3 \to X$ is locally trivial with fiber $\mathrm{SO}(2)$. This fibration is known as the Hopf fibration. In the course Lie groups this fibration will naturally occur as the composed fibration $S^3 \simeq \mathrm{SU}(2) \to \mathrm{SO}(3) \to S^2$.

Exercise 2 The result stated in Remark (1) on page 11 can be reformulated as follows. Let $\mathcal{J}$ be an ideal in $\Omega(M)$ which is generated by an everywhere linearly independent collection $\omega_1, \ldots, \omega_q$ of one forms. Then the following two assertions are equivalent, for every $\lambda \in \Omega^1(M)$.

(a) $d\lambda \in \mathcal{J}$;

(b) $d\lambda \wedge \omega_1 \wedge \cdots \omega_q = 0$.

We leave the proof as an exercise to the reader. Hint for the implication '(b) $\Rightarrow$ (a)': first consider an open set U of M on which $\omega_1, \ldots, \omega_q$ extends to a frame for $\wedge^* T^* M|_U$ and show that there exist one forms α_j^U on U such that $d\lambda = \sum_{j=1}^q \alpha_U^j \wedge \omega_j$. Then glue such local results together by using a partition of unity.

Exercise 3 Let ω be an integrable nowhere vanishing one form on M. Show that for every $m \in M$ there exists an open neighborhood $U \ni m$ and a submersion $f : U \to \mathbb{R}$ such that $\omega = gdf$ on U for a nowhere vanishing function $g \in C^\infty(U)$. Hint: define the codimension one subbundle E of TM given by $E_x = \ker \omega_x$. Show that E is locally at m given by a submersion f.

Exercise 4 Show that the 2-sphere has no codimension 1 foliation. Hint: assume that a foliation $\mathcal{F}$ exists. If $\mathcal{F}$ is orientable, consider a suitable vector field. If $\mathcal{F}$ is not orientable, consider the orientation cover.

Exercise 5 Let V be a finite dimensional real linear space, let $A : V \to V$ be linear map and let $f \in F(V)$. Let (A_{ij}) be the matrix of A with respect to the frame f. Thus,

$$Af_j = \sum_i A_{ij} f_i.$$

(a) Show that A is the matrix of $f^{-1} \circ A \circ f \in L(\mathbb{R}^n, \mathbb{R}^n)$ with respect to the standard basis of $\mathbb{R}^n$.

(b) Show that $\det(A_{ij})$ is independent of the choice of f. Therefore, we may as well write $\det A$ for $\det(A_{ij})$.

(c) Given two frames f, g let $A = A_g^f$ be the unique linear map $V \to V$ such that $Af_j = g_j$ for all $j = 1, \ldots, n$. Show that f and g have the same orientation if and only if $\det A_g^f > 0$.

Exercise 6 Let E be a vector bundle on the manifold M. Let ϵ be an orientation section on E. Show that the orientation section ϵ is smooth on U if and only if for every frame f defined on an open subset $V \subset U$ the function $m \mapsto \epsilon_m(f(m))$ is smooth on V.

Exercise 7 Let $\pi : E \to M$ be a vector bundle. Assume that M is connected. Show that there exist either no or two smooth orientations on E. Hint: show that given two orientations ϵ_1 and ϵ_2 there exists a unique smooth scalar function $\chi \in C^\infty(M)$ such that $(\epsilon_1)_m = \chi(m)(\epsilon_2)_m$ for all $m \in M$. Investigate χ.

Exercise 8 Show that a vector bundle is orientable if and only if there exists an open covering of M by open sets U_α with frames f_α of E on U_α such that for all α, β such that for all α, β and all $m \in U_\alpha \cap U_\beta$ the frames $f_\alpha(m)$ and $f_\beta(m)$ of E_m have the same orientation.

Exercise 9 Prove the following result. Let $p : Y \to X$ be a continuous covering projection of topological spaces. Assume that X has the structure of a smooth manifold, and that $p^{-1}(\{x\})$ is at most countable for every x. Then Y has a unique manifold structure for which p is a local diffeomorphism.

Exercise 10 Let M be a non-orientable smooth connected manifold, and let $q : \tilde{M} := \mathrm{oc}(TM) \to M$ be the associated two-fold covering by an orientable connected manifold.

For every $m \in M$ we define the map $S_m : \mathrm{oc}(T_mM) \to \mathrm{oc}(T_mM)$ by $S_m(\epsilon) = -\epsilon$. Moreover, we define the map $S : \tilde{M} \to \tilde{M}$ by $S = S_m$ on $\mathrm{oc}(T_mM)$. Show that S is a diffeomorphism of order 2 (i.e., $S^2 = 2$) without fixed points. Show that S reverses every choice of orientation for M.

Conversely, let N be an oriented smooth connected manifold, and let S be a diffeomorphism of N of order 2, which reverses orientation and has no fixed points. Let $\sim$ be the relation on N defined by $x \sim y \iff y \in \{x, Sx\}$. Show that $\sim$ is an equivalence relation and that $M = N/\sim$ has a unique structure of smooth manifold for which the canonical projection $N \to M$ is a submersion. Show that M is not orientable and that $N \simeq \mathrm{oc}(M)$.

Exercise 11 Let M be a smooth manifold, and let G be a finite group acting freely on M by diffeomorphisms. Show that M/G has a unique structure of smooth manifold for which the natural map $p : M \to M/G$ is a submersion. Show that p is a smooth covering projection on which G acts by covering transformations.

Exercise 12 Let $f : X \to Y$ be a smooth map of manifolds. Moreover, let $Z \subset Y$ be a smooth submanifold of codimension q. Then f is said to be transversal to Z if for all $x \in f^{-1}(Z)$ we have

$$\mathrm{image}(df(x)) + T_{f(x)}Z = T_{f(x)}Y.$$

Show that if f is transversal to Z, then $f^{-1}(Z)$ is a submanifold of X of codimension q.

Hint: fix $x \in f^{-1}(Z)$. Then there exists a neighborhood U of $f(x)$ in Y and a submersion $s : U \to \mathbb{R}^q$, such that $Z \cap U = q^{-1}(0)$. Show that $s \circ f$ is a submersion in a suitable neighborhood of x.

Exercise 13 Let $f : N \to M$ be a smooth map of smooth manifolds, and let $p : Y \to M$ be a submersion.

(a) Show that the diagonal $\Delta := \{(m, m) \mid m \in M\}$ is a smooth submanifold of $M \times M$.

(b) Show that the map $f \times p : N \times Y \to M \times M$ is transversal to Δ.

(c) Show that $\Gamma = \{(x, y) \in N \times Y \mid f(n) = p(y)\}$ is a smooth submanifold of $N \times Y$.

(d) Show that the map $\mathrm{pr}_1|_\Gamma : \Gamma \to N$ is a submersion.

We write $f^*Y := \Gamma$ and $f^*p := \mathrm{pr}_1|_\Gamma$. The map $f^*p : f^*Y \to N$ is called the pull-back of the submersion p under f. We write $\tilde{f} := \mathrm{pr}_2|_\Gamma$.

(e) Show that f^*p, f^*Y is characterized by the following universal property. For every smooth manifold Z and every pair of smooth maps $q : Z \to N$ and $g : Z \to Y$ such that $p \circ g = f \circ q$ there exists a unique smooth map $\varphi : Z \to f^*Y$ such that $\tilde{f} \circ \varphi = g$ and $f^*p \circ \varphi = q$. Draw a diagram to illustrate this property.

(f) Assume now that $p : Y \to M$ is a fiber bundle, i.e., p allows local trivializations. Show that $f^*p : p^*Y \to N$ is a fiber bundle modeled on the same fiber. Formulate the universal property in terms of fiber bundles $Z \to N$. The fiber bundle $p^*Y \to N$ is called the pull-back of $Y \to M$ under f.

(g) Same question, but now for $p : Y \to M$ a vector bundle. The associated vector bundle $f^*p : f^*Y \to N$ is called the pull-back of the vector bundle $Y \to M$.

Exercises for Chapter 2

Exercise 14 Let $(M, \mathcal{F})$ be a foliated manifold and let $p : TM \to N(\mathcal{F})$ be the natural vector bundle homomorphism from the tangent bundle of M onto the quotient bundle $N(\mathcal{F}) := TM/T\mathcal{F}$. Then $p_* : Y \mapsto p \circ Y$ defines a mapping $\mathfrak{X}(M) \to \Gamma(N(\mathcal{F}))$.

(a) Show that the map p_* is a surjective linear map, with kernel $\mathfrak{X}(\mathcal{F})$.

Let $X \in \mathfrak{X}(\mathcal{F})$. The flow of X is denoted by $\varphi : (t, x) \mapsto \varphi^t(x) = \varphi(t, x)$. Its domain $\mathcal{D}$ is an open subset of $\mathbb{R} \times M$.

(b) Show that for every $(t, x) \in \mathcal{D}$ the derivative $d\varphi^t(x)$ maps $T_x\mathcal{F}$ onto $T_{\varphi^t(x)}(\mathcal{F})$, hence induces a bijective linear isomorphism $d\varphi^t(x)_*$ from $T_xM/T_x\mathcal{F}$ onto $T_{\varphi(t,x)}M/T_{\varphi(t,x)}\mathcal{F}$.

For σ a smooth section of $N(\mathcal{F})$ we define the Lie derivative $\mathcal{L}_X\sigma \in \Gamma(N(\mathcal{F}))$ by

$$\mathcal{L}_X\sigma(x) = \frac{d}{dt}\bigg|_{t=0} d\varphi^{-t}(x)_*\, \sigma(\varphi^t(x)).$$

(c) Show that for any vector field $Y \in \mathfrak{X}(M)$ we have

$$\mathcal{L}_X(p_*Y) = p_*[X, Y].$$

(d) Show that a vector field Y is projectable if and only if $\mathcal{L}_X(p_*Y) = 0$ for all $X \in \mathfrak{X}(\mathcal{F})$.

Let now g be a positive semi-definite inner product on TM such that $\ker g_x = T_x\mathcal{F}$ for every $x \in M$.

(e) Show that there exists a unique positive definite inner product $\bar{g}$ on the quotient bundle $N(\mathcal{F})$ such that $p^*\bar{g} = g$.

(f) Give a definition of Lie derivative $\mathcal{L}_X$ operating on sections of the tensor bundle $N(\mathcal{F})^* \otimes N(\mathcal{F})^*$ in such a fashion that

$$p^*\mathcal{L}_X\bar{g} = \mathcal{L}_Xg,$$

for every $X \in \mathfrak{X}(M)$. Show that the metric g is transverse on $(M, \mathcal{F})$ if and only if

$$\mathcal{L}_X\bar{g} = 0 \qquad \text{for all} \qquad X \in \mathfrak{X}(\mathcal{F}).$$

Exercise 15 Let $(M, \mathcal{F})$ be a foliated manifold.

(a) Show that for every $x_0 \in M$ there exists an open neighborhood U of x and a collection of projectable vector fields $X_1, \ldots, X_n$ such that $X_1(x), \ldots, X_n(x)$ is a basis for T_xM, for all $x \in U$.

Let $\langle\,\cdot\,,\,\cdot\,\rangle$ be a Riemannian metric on M. Let $T\mathcal{F}^\perp$ be the associated normal bundle, i.e.,

$$(T\mathcal{F}^\perp)_x := (T_x\mathcal{F})^\perp, \qquad (x \in M).$$

Given a vector field $Y \in \mathfrak{X}(M)$ we write $Y = Y^{(t)} + Y^{(n)}$ with tangential component $Y^{(t)} \in \mathfrak{X}(\mathcal{F})$ and normal component $Y^{(n)} \in \Gamma(T\mathcal{F}^\perp)$.

(b) Show that Y is projectable if and only if its normal component $Y^{(n)}$ is projectable.

We define the positive semi-definite metric g on TM by $g(X,Y) = \langle X^{(n)}, Y^{(n)} \rangle$.

(c) Show that for all $X \in \mathfrak{X}(\mathcal{F})$ and all projectable vector fields $Y, Z \in \mathfrak{X}(M)$,

$$Xg(Y,Z) = \mathcal{L}_X g(Y,Z).$$

(d) Show that the metric g is transverse on $(M, \mathcal{F})$ if and only if

$$Xg(Y,Z) = 0$$

for all $X \in \mathfrak{X}(\mathcal{F})$ and all normal projectable vector fields $Y, Z \in \mathfrak{X}(M)$.

Exercise 16 (Foliated Lie group action) Let M be a manifold, G a compact Lie group, and assume that a smooth left action $G \times M \to M$, $(g,x) \mapsto gx$ is given. We assume that the action is foliated, i.e., $\dim G_x$ is a constant function of $x \in M$.

Fix $a \in M$. It is known that there exists a connected smooth submanifold S of M, containing a, invariant under G_a and such that the map

$$\varphi : G \times S \to M,$$

given by $\varphi(g,x) = gx$, factors to a diffeomorphism of $G \times_{G_a} S$ onto an open subset U of M. (This is the slice theorem; a submanifold S with these properties is said to be a slice for the action in the point a. For more details, we refer to the notes 'Quotients and actions').

(a) Show that for $x \in S$ the isotropy subgroup G_x is an open subgroup of G_a. In particular, conclude that G_a/G_x is a finite group.

(b) Let K be the kernel of the natural map $G_a \to \mathrm{Diff}(S)$. Show that K has finite index in G_a and conclude that $G/K \to G/G_a$ is a finite covering with covering group G_a/K.

The action by G determines a unique foliation $\mathcal{F}$ whose leaves are the G-orbits. We fix the leaf $L := Ga$ and will study its holonomy group.

(c) Show that the natural map $g \mapsto ga$ factors to a covering $G/K \simeq L$ with covering group G_a/K.

(d) Let $\alpha : [0,1] \to L$ be a continuous curve with $\alpha(0) = a$. and let $\tilde{\alpha}$ be its lifting to G/K, with initial point $\tilde{\alpha}(0) = \bar{e} := eK$. Show that $\mathrm{hol}^{\tilde{\alpha}(1)S,S}(\alpha)$ is the germ at a of the map $x \mapsto \tilde{\alpha}(1)x$.

(e) Now assume that α is a loop in L, based at a, so that $\tilde{\alpha}(1) = \bar{e} \cdot [\alpha]$, for the natural right action of the fundamental group $\Pi_1(L,a)$ on the covering space $(G/K, \bar{e})$. Show that

$$\mathrm{hol}^{S,S}(\alpha) : x \mapsto \tilde{\alpha}(1)x.$$

(f) Show that the holonomy group of L at a equals the finite group G_a/K.

Exercise 17 (The tubular neighborhood theorem) For background to this exercise, read Chapter 3 of the notes 'Quotients and group actions' and Section 2 of the course notes 'Foliation theory'.

We assume that L is a compact submanifold of the smooth manifold M. Let M be equipped with a Riemannian metric $\langle \cdot\, ,\, \cdot \rangle$ and let N be the associated normal bundle of L in M, defined by

$$N = \{(x, \xi) \in TM \mid x \in L,\, \xi \perp T_x L\}.$$

We denote the natural inclusion map $L \to N$, $x \mapsto (x, 0)$, by j_L. For each $x \in L$, we denote the natural inclusion map $N_x \to TM$ by $i_{N,x}$. We define the map φ from an open neighborhood of L (in N) to M by

$$\varphi(x, \xi) = \mathrm{Exp}_x(\xi).$$

Here Exp denotes the exponential map associated with the Riemannian metric on M.

(a) Calculate $d\varphi(x, 0)(dj_L(x)X + di_{N,x}(0)Y)$ for $x \in L$, $X \in T_x L$ and $Y \in N_x$.

(b) Show that $d\varphi(x, 0)$ is a bijective linear map from $T_{(x,0)}N$ onto $T_x M$, for each $x \in L$.

(c) For $\epsilon > 0$ we define the open neighborhood $N(\epsilon)$ of L in N by

$$N(\epsilon) = \{(x, \xi) \in N \mid \|\xi\|_x < \epsilon\}.$$

Show that for $\epsilon > 0$ sufficiently close to zero the map φ is a diffeomorphism from $N(\epsilon)$ onto an open neighborhood of L in M.

Let q be the codimension of L in M. In the following we assume that there exists an open ball $B = B(0; R)$ in $\mathbb{R}^q$ and a diffeomorphism ψ of $L \times B$ onto an open neighborhood of L in M such that $\psi(x, 0) = x$ for all $x \in L$.

(d) Show that for $\epsilon > 0$ sufficiently small there exists an embedding $\alpha : N(\epsilon) \to L \times \mathbb{R}^q$ such that $\psi \circ \alpha = \varphi$.

(e) Show that the map $\tau : N \to L \times \mathbb{R}^q$ defined by

$$\tau(x, \xi) = (x, d(\mathrm{pr}_2 \circ \alpha \circ i_{N,x})(0)\xi)$$

is smooth, and that $\mathrm{pr}_2 \circ \tau(x, \cdot\,)$ is a linear bijection from N_x onto $\mathbb{R}^q$.

(f) Show that it follows from the assumption made before (d) that the normal bundle N is trivial.

Exercise 18 We consider the foliation $\mathcal{F}$ in $\mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q$ whose leaves are of the form $\mathbb{R}^{n-q} \times \{b\}$, with $b \in \mathbb{R}^q$. Let V_1 be an open subset of $\mathbb{R}^{n-q}$ containing 0. Let $B(\epsilon)$ denote the open ball in $\mathbb{R}^q$ of center 0 and radius ϵ. Assume there exists an embedding $\nu : V_1 \times B(\epsilon) \to \mathbb{R}^n$ such that $\nu(x, 0) = (x, 0)$ for all $x \in V_1$. Let V denote the image of ν and let $r : V \to V$ be the map defined by $r(\nu(x, y)) = \nu(x, 0) = (x, 0)$, for $(x, y) \in V_1 \times B(\epsilon)$.

(a) Show that $\mathbb{R}^{n-q} \times \{0\} \oplus \ker dr(0) = \mathbb{R}^n$.

(b) Show that the map $\psi : V \to \mathbb{R}^n$ defined by

$$\psi(x, y) = (\mathrm{pr}_1 r(x, y), y)$$

is a local diffeomorphism at 0.

(c) Show that by replacing V_1 by a smaller neighborhood and $\epsilon > 0$ by a smaller positive constant, we may arrange that ψ is a diffeomorphism from V onto an open neighborhood of 0 in $\mathbb{R}^n$. Show that $\varphi(x, 0) = (x, 0)$ for all $x \in V_1$.

(d) Show that (φ, V) is a foliation chart for $\mathcal{F}$ and that

$$\varphi(r(x, y)) = (\mathrm{pr}_1\varphi(x, y), 0)$$

for all $(x, y) \in V$.

We now assume that M is a manifold of dimension n, that $\mathcal{F}$ is a codimension q foliation and that L is a compact leaf. Let $r : \mathcal{N} \to L$ be a tubular neighborhood of L in M.

(e) Show that for every $a \in L$ there exists an open neighborhood $V_1 = V_{a,1} \subset L$ such that for every ($(n - q)$-dimensional) chart (U_1, χ) of L with $U_1 \subset V_1$ there exists a foliation chart (U, φ) with the following properties:

(i) $U \cap L = U_1$, and $\mathrm{pr}_1 \circ \varphi = \chi$ on U_1;

(ii) $\varphi(U) = \chi(U_1) \times \mathbb{R}^q$;

(iii) $r(U) \subset U_1$;

(iv) $\varphi \circ r = (\mathrm{pr}_1 \circ \varphi, 0)$ on U.

This explains the choice of foliation charts covering the leaf L suggested in the proof of the Local Reeb Stability Theorem in Section 2.3 of the book.

Exercise 19 Let M be smooth manifold of dimension n and G a finite group of diffeomorphisms of M. Let $a \in M$.

(a) Show that there exists an open neighborhood $\mathcal{O}$ of a such that for all $g \in G$ we have

$$\mathcal{O} \cap g\mathcal{O} \neq \emptyset \Rightarrow g \in G_a.$$

(b) Show that for every open neighborhood V of a there exists an open neighborhood U of a in V, a diffeomorphism φ from U onto the open unit ball $B = B(0; 1) \subset \mathbb{R}^n$ and an embedding $\mu : G \to \mathrm{O}(n)$ such that

$$\varphi(gx) = \mu(g)\varphi(x), \qquad (x \in G).$$

Hint: use a suitable Riemannian metric on M.

(c) Show that for every open neighborhood V of a there exists an open neighborhood U as above which is G-stable.

(d) Show that there exists a diffeomorphism $\psi : B(0; 1) \to \mathbb{R}^n$ such that for all $A \in \mathrm{O}(n)$ we have $\psi \circ A = A \circ \psi$

(e) Prove assertions (b) and (c), but this time with $B = \mathbb{R}^n$.

Exercise 20 Prove (iv) of Proposition 2.12. Hint: Apply Lemma 2.11 with domain $\mu(V)$ instead of V and with map $\lambda \circ \mu$ instead of f.

Exercise 21 We assume that Q is an orbifold of dimension n, that (U, G, φ) is an orbifold chart for Q and that $b \in \varphi(U)$.

(a) Let $a \in U$ be such that $\varphi(a) = b$ and let $\mathcal{O}$ be an open neighborhood of a in U. Show that there exist an orbifold chart $(\mathbb{R}^n, H, \psi)$, with H a finite subgroup of $\mathrm{O}(n)$ isomorphic to G_a, and an embedding $\lambda : (\mathbb{R}^n, H, \psi) \to (U, G, \varphi)$ with $\lambda(0) = a$ and $\lambda(\mathbb{R}^n) \subset \mathcal{O}$.

(b) Show that Q possesses an orbifold atlas consisting of orbifold charts of the form $(\mathbb{R}^n, K_i, \chi_i)$, with K_i a finite subgroup of $\mathrm{O}(n)$.

Exercise 22 Show that FQ has the structure of an orbifold for which $\pi : FQ \to Q$ is an orbifold map.

Exercise 23 If M is a smooth manifold, and FM its frame bundle, then we have a natural smooth map $FM \times \mathbb{R}^n \to TM$, defined by

$$((x, f), v) \mapsto (x, fv).$$

Show that this map factors to a vector bundle isomorphism

$$FM \times_{\mathrm{GL}(n,\mathbb{R})} \mathbb{R}^n \to TM.$$

Use this idea to define the tangent bundle TQ of an orbifold Q. Show that the tangent bundle need not be a manifold, but that it admits an orbifold structure for which the natural projection $TQ \to Q$ becomes an orbifold map.

Exercise 24 The purpose of this exercise is to provide hints for Exercise 2.18 of the book. That exercise can be reduced to the following slightly simpler situation (of course here there is still some work to do). We assume that Q is an orbifold and that H is a finite subgroup of the group of orbifold automorphisms of Q. The simplification is that we assume that H has a simultaneous fixed point $a \in Q$.

(a) Show that there exists an orbifold chart (U, G, φ) of Q with $\varphi^{-1}(a)$ consisting of one point, x say. Thus, x is a fixed point for G.

(b) Show that there exists an open G-invariant neighborhood U_0 of x such that every $h \in H$ has a lift $\tilde{h} : U_0 \to U$. Show that $\tilde{h}x = x$ and that $H\varphi(U_0) \subset \varphi(U)$.

Let $\mathcal{O}$ be the intersection of the sets $h\varphi(U_0)$, for $h \in H$. Let V be the connected component of the set $U_0 \cap \varphi^{-1}(\mathcal{O})$ containing x.

(c) Show that V is G-invariant. Show also that for each $h \in H$ and $\tilde{h} : U_0 \to U$ a lift we have $\tilde{h}(V) = V$.

(d) Show that $\varphi(V)$ is H-invariant.

Let H_V be the image of the map $h \mapsto h|_{\varphi(V)}$. Let A be the group of diffeomorphisms of V mapping G-orbits to G-orbits. Let $\bar{A}$ be its natural image in the group of orbifold automorphisms of $\varphi(V)$ and let $p : A \to \bar{A}$ be the natural epimorphism. Finally, let $\tilde{H} = p^{-1}(H_V)$.

(e) Show that the natural sequence

$$1 \to G \to \tilde{H} \to H_V \to 1$$

is short exact.

(f) Let $\pi : Q \to Q/H$ be the canonical projection. Show that $(V, \tilde{H}, \pi \circ \varphi|_V)$ is an orbifold chart for Q/H