Notes: Orientations on vector bundles

Erik van den Ban

Fall 2006

1 Orientation on a linear space

Let V be a finite dimensional real linear space, of dimension n. By a frame in V we mean an ordered basis $f_1, \ldots, f_n$ of V. Let $F(V)$ denote the set of frames in V. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. Then given a frame $(f_i) \in F$, there is a unique linear map $f : \mathbb{R}^n \to V$ such that $f(e_i) = f_i$ for all $1 \leq i \leq n$. Clearly, the linear map f is an isomorphism of linear spaces. Conversely, every linear isomorphism $f : \mathbb{R}^n \to V$ gives rise to the frame $(f_i = f(e_i))$. Let $L(\mathbb{R}^n, V)$ denote the (finite dimensional linear) space of linear maps $\mathbb{R}^n \to V$ and let $L_{\rm iso}(\mathbb{R}^n, V)$ denote the subset of invertible ones. Then the map $f \mapsto (f(e_i))$ gives a bijection from $L_{\rm iso}(\mathbb{R}^n, V)$ onto $F(M)$. We will use this bijection to identify the two sets. In particular, since $L_{\rm iso}(\mathbb{R}^n, V)$ is an open subset of the finite dimensional linear space $L(\mathbb{R}^n, V)$ we obtain a structure of smooth manifold on $F(M)$.

Given two frames $f, g \in F(M)$, the linear map $A = g^{-1}f : \mathbb{R}^n \to \mathbb{R}^n$ is invertible, hence has a non-zero determinant. We will say that f and g have the same orientation, notation $f \sim g$ if and only if this determinant is positive. Clearly, $\sim$ is an equivalence relation on $F(M)$, and $F(M)/\sim$ consists of two elements. The elements of $F(M)/\sim$ are called *orientations* of V. An oriented finite dimensional real linear space is a finite dimensional real linear space V together with a fixed choice of an orientation.

Exercise 1.1 Let $A : V \to V$ be linear map and let $f \in F(V)$. Let (A_{ij}) be the matrix of A with respect to the frame f. Thus,

$$Af_j = \sum_i A_{ij} f_i.$$

(a) Show that A is the matrix of $f^{-1} \circ A \circ f \in L(\mathbb{R}^n, \mathbb{R}^n)$ with respect to the standard basis of $\mathbb{R}^n$.

(b) Show that $\det(A_{ij})$ is independent of the choice of f. Therefore, we may as well write $\det A$ for $\det(A_{ij})$.

(c) Given two frames f, g let $A = A_g^f$ be the unique linear map $V \to V$ such that $Af_j = g_j$ for all $j = 1, \ldots, n$. Show that f and g have the same orientation if and only if $\det A_g^f > 0$.

In the following it will be convenient to view the set of orientations on V somewhat differently. Given an orientation $o \in F(M)/\sim$ we define a map $\epsilon = \epsilon_o : F(M) \to \{\pm 1\}$ by $\epsilon(f) = +1$ if $f \in o$ and by $\epsilon(f) = -1$ if $f \notin o$. Then $\epsilon : F(V) \to \{-1, 1\}$ is a surjective map which is constant on the classes for $\sim$. The set of such maps is denoted by ${\rm or}(V)$. It is easily checked that the map $o \mapsto \epsilon_o$ is a bijection from $F(M)/\sim$ onto $\mathrm{or}(V)$. Indeed, the inverse map is given by $\epsilon \mapsto o_\epsilon$, where $o_\epsilon = \{f \in F(M) \mid \epsilon(f) = 1\}$. In particular, it follows that $\mathrm{or}(V)$ consists of two elements. If $\epsilon \in \mathrm{or}(V)$ then $\mathrm{or}(V) = \{-\epsilon, \epsilon\}$. In the following we shall identify the elements of $\mathrm{or}(V)$ with the classes for $\sim$ in $F(V)$ in the above fashion. Given a choice $\epsilon \in \mathrm{or}(V)$ we say that a frame f is positively oriented if $\epsilon(f) = 1$ and that it is negatively orientend if $\epsilon(f) = -1$.

Yet another view on orientation is obtained by using alternating n-forms. Let $\bigwedge^n V^*$ be the (1-dimensional) space of alternating n-forms on V. Given a non-zero element $\omega \in \bigwedge^n V^* \setminus \{0\}$ we obtain a non-zero function $F(M) \to \mathbb{R}$ given by $f \mapsto \omega(f) = \omega(f_1, \ldots, f_n)$. We note that $f \sim g$ if and only if $\omega(f)$ and $\omega(g)$ have the same sign. Indeed, $\omega(f) = f^*\omega(e_1, \ldots, e_n)$ and a similar formula for $\omega(g)$, so that

$$\omega(g) = \omega(gf^{-1}f) = \det(gf^{-1})\omega(f).$$

Given a non-zero n-form ω we define the orientation $\epsilon_\omega \in \mathrm{or}(V)$ by $\epsilon_\omega(f) = \mathrm{sign}\,\omega(f)$. The map $\omega \mapsto \epsilon_\omega$ is surjective to $\mathrm{or}(V)$. Two forms belong to the same fiber for this map if and only if they differ by a positive scalar factor. Accordingly, the map induces a bijection

$$(\bigwedge^n V^* \setminus \{0\}) \,/\, \mathbb{R}^+ \ \stackrel{\simeq}{\longrightarrow}\ \mathrm{or}(V).$$

We will use this bijection to identify the elements of the spaces on both sides.

2 Orientation on vector bundles

Let M be a smooth manifold and let $\pi : E \to M$ be a vector bundle of rank n on M. If U is an open subset of M, then by a (smooth) frame f of E on U we mean an n-tuple $f = (f_1, \ldots, f_n$ of smooth sections $f_j : U \to E$ such that for every $x \in M$ the tuple $(f_1(x), \ldots, f_n(x))$ is a frame for E_x. Given such a frame f, we define the map $\hat{f} : U \times \mathbb{R}^n \to E|_U$ by $\hat{f}(x, \xi) = f(x)(\xi) = \sum_{j=1}^n \xi_j f_j$. Then it is readily seen that $\tau_f = \hat{f}^{-1}$ is a trivialization of the bundle $E|_U$. Conversely, if $\tau : E|_U \to U \times \mathbb{R}^n$ is a trivialization, then the functions $f_j(x) = \tau^{-1}(x, e_j)$ define a smooth frame for E on U. It follows that giving a local frame is equivalent to giving a local trivialization of the bundle.

An orientation ϵ on E is the choice of an element $\epsilon_m \in \mathrm{or}(E_m)$ for every $m \in M$. An orientation is said to be smooth at a point a if there exists an open neighborhood U of a and a smooth frame f of $E|_U$ such that $m \mapsto \epsilon_m(f(m))$ is a smooth function $U \to \{-1, 1\}$. Note that a function $U \to \{\pm 1\}$ is smooth if and only if it is locally constant. The orientation is said to be smooth on an open set U if it is smooth at every point of U.

Exercise 2.1 Show that the orientation ϵ is smooth on U if and only if for every frame f defined on an open subset $V \subset U$ the function $m \mapsto \epsilon_m(f(m))$ is smooth on V.

Definition 2.2 The vector bundle $\pi : E \to M$ is said to be orientable if and only if there exists a smooth orientation for E

Definition 2.3 The manifold M is said to be orientable if and only if the tangent bundle TM is orientable.

Exercise 2.4 Let $\pi : E \to M$ be a vector bundle. Assume that M is connected. Show that there exist either no or two smooth orientations on E. Hint: show that given two orientations ϵ_1 and ϵ_2 there exists a unique smooth scalar function $\chi \in C^\infty(M)$ such that $(\epsilon_1)_m = \chi(m)(\epsilon_2)_m$ for all $m \in M$. Investigate χ.

Lemma 2.5 *Let $\pi : E \to M$ be a vector bundle on the manifold M. Then the following assertions are equivalent.*

(a) *The vector bundle E is orientable.*

(b) *There exists a smooth non-vanishing section ω of the line bundle $\bigwedge^n E^* = \coprod_{m\in M} \bigwedge^n E^*_m$.*

Proof: Assume (b). Let ω be a non-vanishing section. Then we define $\epsilon_m \in \mathrm{or}(E_m)$ by $\epsilon_m(f) = \mathrm{sign}\, \omega_m(f)$ for $f \in F(E_m)$. By smoothness of ω, it follows that ϵ is smooth.

Assume (a). Let $a \in M$ and let $f_1, \dots, f_n$ be a local frame for $E|_U$ on some open neighborhood $U = U_a$ of a. Then $\epsilon(f)$ is a locally constant function with values in $\{\pm 1\}$. Replacing f_1 by $-f_1$ if necessary, we may assume that $\epsilon_a(f(a)) = 1$. Replacing U by a smaller neighborhood if necessary, we may assume that $\epsilon(f) = 1$ on U. In other words, f is a positively oriented local frame with respect to the smooth orientation ϵ. Let $f^1 \dots f^n$ be the dual frame for E^*. Then $\omega = f^1 \wedge \cdots \wedge f^n$ is a non-vanishing smooth section of $\bigwedge^n E^*$ over U. Moreover, for every $m \in U$, the form ω_m is positively oriented with respect to ϵ_m.

In view of the above, there exists an open cover U_i, $i \in I$, of M together with non-vanishing sections ω_i of $\bigwedge^n E^*|_{U_i}$ such that $\omega_i(m)$ is ϵ_m-positively oriented for every $i \in I$ and $m \in U_i$. There exists a partition of unity $\{\psi_k\}$ subordinate to U_i. This means that $\psi_k \in C_c^\infty(M)$, $0 \le \psi_k \le 1$, for every k there exists a $i_k \in I$ such that $\mathrm{supp}\, \psi_k \subset U_{i_k}$, and finally, $\sum_k \psi_k = 1$, with locally finite sum.

We now claim that $\omega = \sum_k \psi_k \psi_k \omega_{i_k}$ is a smooth non-vanishing section of $\bigwedge^n E^*$ which is everywhere positively oriented with respect to ϵ. The proof of this claim is left as an execise to the reader. $\square$

Remark 2.6 Note that in the above proof we have actually shown that the form ω may be chosen such that it is everywhere positively oriented with respect to the orientation ϵ.

Exercise 2.7 Show that a vector bundle is orientable if and only if there exists an open covering of M by open sets U_α with frames f_α of E on U_α such that for all α, β such that for all α, β and all $m \in U_\alpha \cap U_\beta$ the frames $f_\alpha(m)$ and $f_\beta(m)$ of E_m have the same orientation.

3 The orientation cover of a vector bundle

Let $\pi : E \to M$ be a rank n vector bundle on the smooth manifold M. We consider the disjoint union

$$\mathrm{oc}(E) = \coprod_{m\in M} \mathrm{or}(E_m).$$

In the previous section we defined an orientation ϵ on E to be section $\epsilon : M \to \mathrm{oc}(E)$. The purpose of this section is to put a manifold structure on $\mathrm{oc}(E)$ such that the natural map $\mathrm{oc}(E) \to M$ becomes a fiber bundle (with fiber diffeomorphic to $\{\pm 1\}$) and such that smoothness of an orientation corresponds to smoothness of the orientation viewed as a section.

Consider the line bundle $L = \bigwedge^n E^*$, viewed as a manifold. Let 0_L denote the image of the zero section in L. Then $\mathcal{O} := L \setminus 0_L$ is an open subset of L hence a smooth manifold of its own right. The canonical map $p : \mathcal{O} \to M$, obtained by restriction of the projection $L \to M$ gives $\mathcal{O}$ the structure of a fiber bundle with fiber diffeomorphic to $\mathbb{R} \setminus \{0\}$.

Recall that for every $m \in M$ we have a natural map $\eta_m : L_m \setminus \{0\} \to \mathrm{or}(E_m)$ given by $\eta_m(\omega)(f) = |\omega(f)|^{-1}\omega(f)$. Let $\eta : \mathcal{O} \to \mathrm{oc}(E)$ be defined by $\eta = \eta_m$ on $\mathcal{O}_m$. Then η is surjective.

Lemma 3.1 *The set* $\mathrm{oc}(E)$ *has a unique structure of smooth manifold for which* $\eta : \mathcal{O} \to \mathrm{oc}(E)$ *is a submersion. For this manifold structure, the natural projection* $q : \mathrm{oc}(E) \to M$ *is a two-fold smooth covering projection.*

Let $\epsilon : M \to \mathrm{oc}(E)$ *a section. Then* ϵ *is smooth at* a *if and only if there exists a frame* f *of* E *defined in an open neighborhood of* a *such that* $m \mapsto \epsilon_m(f(m))$ *is locally constant.*

Remark 3.2 The last assertion implies that the new notion of smoothness of an orientation coincides with the old one.

Proof: We have to show that such a manifold structure exists. It is then necessarily unique.

First of all, we equip $\mathrm{oc}(E)$ with the quotient topology. The natural projection map $p : \mathcal{O} \to M$ factors to the natural projection map $q : \mathrm{oc}(E) \to M$. Since p is continuous, q is continuous for the quotient topology on $\mathrm{oc}(E)$. We will first show that q is a covering projection.

To see this, let $a \in M$. There exists an open neighborhood U of a in M together with a smooth local frame f of $E|_U$. Let $f^* = (f_1^*, \ldots, f_n^*)$ be the dual frame, and put $\omega = f_1^* \wedge \cdots \wedge f_n^*$. Then $\omega_1 := \omega$ is a smooth section of $\mathcal{O}|_U$. Similarly, $\omega_2 := -\omega$ is a smooth section of $\mathcal{O}|_U$.

The map $s : U \times \mathbb{R} \setminus \{0\} \to \mathcal{O}|_U$ given by $s(x,t) = t\omega(x)$ is readily seen to be a diffeomorphism. We define $V_j = \mathrm{image}(\eta \circ \omega_j)$, for $j = 1, 2$. Then the preimage of V_j in $\mathcal{O}$ equals $s(U \times \mathbb{R}^+)$ hence V_1 is open in $\mathrm{oc}(E)$. Similarly, V_2 is open in $\mathrm{oc}(E)$.

Furthermore, $q^{-1}(U)$ is the disjoint union of V_1 and V_2 and $q|_{V_j} : V_j \to U$ is a homeomorphism with inverse $\eta \circ \omega_j$. It follows that $q : \mathrm{oc}(E) \to M$ is a (two-fold) covering projection. By the lemma below, $\mathrm{oc}(E)$ has a unique structure of smooth manifold for which q becomes a local diffeomorphism. Since $p = q \circ \eta$ is a submersion $\mathcal{O} \to M$, it follows that $\eta : \mathcal{O} \to \mathrm{oc}(E)$ is a submersion.

To establish the final assertion, let $\epsilon : M \to \mathrm{oc}(E)$ be a section. Assume that ϵ is smooth at the point $a \in M$. Let f be any local frame of E defined on an open neighborhood U of a. Let f^* and ω be associated to f as above. Then $\epsilon_m(f(m)) = \eta \circ \omega(m)(f(m) = \mathrm{sign}\, \omega_m(f(m)) = +1$ and (b) follows for any local section.

Conversely, let f be a frame as in (b), defined on an open neighborhood U of a. Then replacing U by a smaller neighborhood if necessary, we may assume that $m \mapsto \epsilon_m(f(m))$ is constant on U, and replacing f by $(-f_1, f_2, \ldots, f_n)$ if necessary, we may assume that $\epsilon(f) = 1$ on U. Let f^* and ω be associated to f as before. Then $\omega(f_1, \ldots, f_n) = 1$ on U, hence $\eta_m(\omega(m)) = \epsilon_m$ on $f(m)$ from which it follows that $\epsilon = \eta \circ \omega$ is smooth. $\square$

Lemma 3.3 *Let* $p : Y \to X$ *be a continuous covering projection of topological spaces. Assume that* X *has the structure of a smooth manifold, and that* $p^{-1}(\{x\})$ *is at most countable for every* x*. Then* Y *has a unique manifold structure for which* p *is a local diffeomorphism.*

Proof: Exercise for the reader. $\square$

Remark 3.4 Recall that a continuous map $p : Y \to X$ is called a covering projection if for every $a \in X$ there exists an open neighborhood U such that $p^{-1}(U)$ is the disjoint union of open sets $V_i \subset Y$ such that $p|_{V_i} : V_i \to U$ is a homeomorphism for every i.

Lemma 3.5 *Let E be a rank n-vector bundle on a connected manifold M. Then the orientation cover* $\mathrm{oc}(E)$ *has either one or two connected components. Moreover, the following two assertions are equivalent.*

(a) *The bundle E is orientable.*

(b) *The manifold* $\mathrm{oc}(M)$ *is not connected.*

Proof: Let $q : \mathrm{oc}(E) \to M$ be the canonical projection. Then q is a two-fold smooth covering projection.

Fix a point $a \in M$. Then the fiber of $q^{-1}(a)$ consists of two points α_1, α_2. Let O_j be the connected component of $\mathrm{oc}(E)$ containing α_j. Let β be any point of $\mathrm{oc}(M)$. Then there exist a continuous curve $\gamma : [0, 1] \to M$ with initial point $b := q(\beta)$ and end point a. By the lifting theorem, the curve has a unique lift to a curve $\tilde{\gamma} : [0, 1] \to \mathrm{oc}(E)$ with $\tilde{\gamma}(0) = \beta$. The end point $\tilde{\gamma}(1)$ belongs to $q^{-1}(a)$ hence equals α_1 or α_2. Therefore, $\mathrm{oc}(E)$ is the union of $\mathcal{O}_1$ and $\mathcal{O}_2$. As $\mathcal{O}_1, \mathcal{O}_2$ are connected components, it follows that either $\mathcal{O}_1 = \mathcal{O}_2$ or $\mathcal{O}_1 \neq \mathcal{O}_2$. This establishes the first assertion.

Assume (a). Then there exists a smooth section $\epsilon_1 : M \to \mathrm{oc}(E)$. Now $\epsilon_2 = -\epsilon_1$ is a smooth section as well, and $\mathrm{oc}(E)$ is the disjoint union of $\epsilon_1(M)$ and $\epsilon_2(M)$. As q is a local diffeomorphism, each ϵ_j is a local diffeomorphism, hence $\epsilon_j(M)$ are non-empty connected open subsets of $\mathrm{oc}(E)$. This implies that $\mathrm{oc}(E)$ has two connected components.

Now assume (b). Then $\mathrm{oc}(E)$ has two connected components, $\mathcal{O}_1$ and $\mathcal{O}_2$. By the reasoning of the first part of the proof, the components $\mathcal{O}_1$ and $\mathcal{O}_2$ intersect each fiber of q in different points. Moreover, since the sets have union $o(E)$ and q is a two-fold covering, it follows that $\mathcal{O}_j$ intersects each fiber of q in precisely one point. It follows that $q_j := q|_{\mathcal{O}_j}$ is local diffeomorphism which is both injective and surjective, hence a global diffeomorphism onto M. Let $\epsilon_j : M \to o(E)$ be the inverses to q_j. Then ϵ_1 is a global smooth section of $\mathrm{oc}(E)$. Note that $\epsilon_2 = -\epsilon_1$. $\square$

Lemma 3.6 *Let E be a non-orientable vector bundle on M. Let $q : \mathrm{oc}(E) \to M$ be the associated covering projection. Then the pull-back bundle q^*E on* $\mathrm{oc}(E)$ *is orientable.*

Proof: Put $\tilde{M} = \mathrm{oc}(E)$. We recall that q^*E may be realized as the submanifold of $\tilde{M} \times E$ consisting of (m, ξ) with $\xi \in E_{q(\tilde{m})}$.

Accordingly, it is readily checked that $\mathrm{oc}(q^*(E))$ is the submanifold of $\tilde{M} \times \mathrm{oc}(E)$ consisting of the points $(\tilde{m}, \epsilon)$ with $q(\tilde{m}) = q(\epsilon)$. The identity map $I : \tilde{M} \to \mathrm{oc}(E)$ is thus seen to be a global smooth section of $\mathrm{oc}(q^*(E))$. Hence, $q^*(E)$ is orientable. $\square$

Corollary 3.7 *Let M be a non-orientable connected manifold. Then* $\mathrm{oc}(TM) \to M$ *is a double smooth covering of M by an orientable connected manifold.*

Exercise 3.8 Let M be a non-orientable smooth connected manifold, and let $q : \tilde{M} := \mathrm{oc}(TM) \to M$ be the associated two-fold covering by an orientable connected manifold.

For every $m \in M$ we define the map $S_m : \mathrm{oc}(T_mM) \to \mathrm{oc}(T_mM)$ by $S_m(\epsilon) = -\epsilon$. Moreover, we define the map $S : \tilde{M} \to \tilde{M}$ by $S = S_m$ on $\mathrm{oc}(T_mM)$. Show that S is a diffeomorphism of order 2 (i.e., $S^2 = 2$) without fixed points. Show that S reverses every choice of orientation for M.

Conversely, let N be an oriented smooth connected manifold, and let S be a diffeomorphism of N of order 2, which reverses orientation and has no fixed points. Let $\sim$ be the relation on N defined by $x \sim y \iff y \in \{x, Sx\}$. Show that $\sim$ is an equivalence relation and that $M = N/\sim$ has a unique structure of smooth manifold for which the canonical projection $N \to M$ is a submersion. Show that M is not orientable and that $N \simeq \mathrm{oc}(M)$.