Geometric Analysis: Take Home Exercise 5

Let M be a manifold of dimension n. Let $P \in \Psi^d(M)$, $Q \in \Psi^e(M)$ be properly supported.

(a) Show that the commutator $R := [P, Q] = PQ - QP$ is is a properly supported operator in $\Psi^{d+e-1}(M)$.

(b) Show that the principal symbol $\sigma^{d+e-1}(R)$ of order $d+e-1$ of this operator is a bilinear function of the principal symbols $\sigma^d(P)$ and $\sigma^e(Q)$.

For smooth functions $f, g : U \times \mathbb{R}^n \to \mathbb{C}$ we define the Poisson bracket $\{f, g\}$ to be the smooth function $U \times \mathbb{R}^n \to \mathbb{C}$ given by

$$\{f, g\}(x, \xi) = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial \xi_j} \frac{\partial f}{\partial x_j} \right).$$

This bracket equals the standard Poisson bracket determined by the natural symplectic form on T^*U.

(c) Let $\varphi : U \to V$ be a diffeomorphism of open subsets of $\mathbb{R}^n$ and let $\Phi = \varphi_* : T^*U \simeq U \times \mathbb{R}^n \to T^*V \simeq V \times \mathbb{R}^n$ be the natural diffeomorphism induced by φ. Show that for functions $f, g \in C^\infty(T^*V)$ we have

$$\Phi^*(\{f, g\}) = \{\Phi^* f, \Phi^* g\}.$$

(d) Show that there exists a unique bilinear map $\{\,\cdot\,,\,\cdot\,\} : C^\infty(T^*M) \times C^\infty(T^*M) \to C^\infty(T^*M)$ such that for every chart (U, κ) of M and all smooth functions $f, g : T^*M \to \mathbb{C}$ we have

$$\psi^{-1*}(\{f, g\}|_{T^*U}) = \{\psi^{-1*}(f|_{T^*U}), \psi^{-1*}(g|_{T^*U})\}.$$

where the brackets on the right-hand side are those of $T^*(\kappa(U))$ and where ψ is the diffeomorphism $T^*U \to T^*\kappa(U)$ induced by κ.

Remark: this bilinear map is the Poisson bracket associated with the canonical symplectic structure on T^*M.

(e) Show that the bilinear map $\{\,\cdot\,,\,\cdot\,\}$ maps $S^d(M) \times S^e(M)$ into $S^{d+e-1}(M)$.

(f) Show that the above mentioned bilinear map induces a bilinear map $S^d(M)/S^{d-1}(M) \times S^e(M)/S^{e-1}(M) \to S^{e+d-1}(M)/S^{d+e-2}(M)$, which we denote by the same brackets.

(g) With notation as in (a), show that (with i the imaginary unit)

$$\sigma^{d+e-1}([P, Q]) = -i\,\{\sigma^d(P), \sigma^e(Q)\}.$$