In these notes we will indicate a fast way to calculate the symbol of the composition of two pseudo-differential operators. In the present lecture notes, the symbol is given by an exact formula and then an asympotic expansion. This is a process I learned from Hörmander's book 'The analysis of linear partial differential operators III.' Unfortunately that proof is rather long. The treatment can be shortened considerably if one is satisfied with just having an asympotic expansion for the complete symbol.

In the following text I am going to suggest this different approach, with reference to Michael Taylor's book [MT]: "Pseudodifferential Operators", Princeton Mathematical Series 34, Princeton University Press, 1981.

We start with section 7.2 in the lecture notes.

- Introduce the new space of symbols Σ^d and the associated operators Ψ_r defined by (7.2.9). These operators are actually pseudo-differential operators. The appropriate formulation of Prop. 7.2.1 is:

- **Proposition 1** *Let $r \in \Sigma^d_{\mathcal{K}}$. Then there exists a (unique) $p \in S^d(\mathbb{R}^n)$ such that $\Psi_r = \Psi_p$. Furthermore, modulo $S^{-\infty}(\mathbb{R}^n)$ the symbol p is determined by the asymptotic expansion*

$$p \sim \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} D_y^\alpha \partial_\xi^\alpha r(x, \xi, y)|_{y=x}.$$

  The first part of this proposition is given in the lecture notes. A quicker proof for the asymptotic expansion, which generalizes the analogous proof of Lemma 5.4.2, is given in [MT], Theorem 3.8, page 44. Here the function $a(x, y, \xi)$ is our $r(x, \xi, y)$.

The above result has a very nice application to the calculation of the symbol of the adjoint of a pseudo-differential operator.

- Follow the discussion in §6.2 of the lecture notes for preparation.

- The appropriate statement of Prop. 6.2.2 is now as follows.

- **Proposition 2** *Let $p \in S^d_c(U)$. Then the transpose Ψ_p is a pseudo-differential operator of the form Ψ_q for a unique symbol $q = p^t \in S^d(U)$. Up to $S^{-\infty}(U)$ this symbol is uniquely determined by the asymptotic expansion*

$$q = p^t \sim \sum_{\alpha \in \mathbb{N}^n} \frac{(-1)^{|\alpha|}}{\alpha!} D_x^\alpha \partial_\xi^\alpha p(x, -\xi).$$

- For the proof of the above proposition we refer to [MT], Thm. 4.1 and 4.2, page 45. Be aware that in Taylor the adjoint is taken with respect to the L^2-type sesquilinear pairing, while we use the bilinear pairing $C_c^\infty(\mathbb{R}^n) \times$

$C_c^\infty(\mathbb{R}^n) \to \mathbb{C}$ given by integration of the product. Accordingly, Taylor uses the symbol p^* for our p^t and he writes $p(x,\xi)^*$ for the complex conjugate of $p(x,\xi)$. The advantage of using the L^2-pairing is that there is no sign-change on ξ. The comparison is made in Corollary 6.2.6 of the lecture notes.

- discuss lecture notes, Cor. 6.2.7.

The theory of the adjoint has a very nice application to the composition of pseudo-differential operators.

- We turn to Proposition 7.1.1 in 7.1 of the lecture notes. The appropriate formulation of that proposition is

  Proposition 3 *Let p, q be as stated. Then the composition $\Psi_p \circ \Psi_q$ is a pseudo-differential operator of order $d+e$ whose symbol r has an asymptotic expansion of the form*

  $$r \sim \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial_\xi^\alpha p(x,\xi) D_x^\alpha q(x,\xi)$$

- The proof can be found in [MT], Thm. 4.3. An important ingredient in the proof is the observation that the Fourier transform following a transpose of a properly supported pseudo-differential operator admits the formula:

  $$\mathcal{F}(\Psi_q^t(f))(\xi) = \int_{\mathbb{R}^n} e^{-i\xi y} q(y,\xi) f(y)\, dy.$$

  This leads quickly to a result for the composition

  $$\Psi_p \circ \Psi_q^t.$$

- For the completion of the proof, see [MT], Thm. 4.4.

- By the way, Taylor restricts properly supported pseudo-differential operators, which is good enough. A pseudo-differential operator P is said to be properly supported if the support S of its distribution kernel $K_P \in \mathcal{D}'(U \times U)$ is properly supported along the diagonal, i.e., for every compact $C \subset U$ we have that S has compact intersections with both $C \times U$ and $U \times C$.