# 10 Weyl's law

## 10.1 Description of the result

Let $M$ be a compact smooth manifold. Let $g$ be a Riemannian metric on $M$ and $\Delta$ the corresponding Laplace operator, equipped with a minus sign so that its principal symbol equals

$$\sigma^2_\Delta(x, \xi_x) = g_x(\xi_x, \xi_x).$$

In the final sessions of our seminar we will aim at understanding a proof of Weyl's law for the asymptotic behavior of the eigenvalues of the Laplace operator.

Our first aim will be to describe the result. For this we will look at the eigenvalues of the Laplace operator. We define $\Lambda(\Delta)$ to be the set of $\lambda \in \mathbb{C}$ such that

$$E(\lambda) := \{f \in C^\infty(M) \mid \Delta f = \lambda f\}$$

is non-trivial. We will show that $\Lambda$ is a discrete subset of $[0, \infty[$, that each $E(\lambda)$ is finite dimensional and that

$$\widehat{\oplus}_{\lambda \in \Lambda} E(\lambda) = L^2(M).$$

Here the summands are mutually orthogonal, and the hat indicates that the closure of the direct sum is taken. For each $\lambda \in \Lambda$, we put $m_\lambda := \dim E(\lambda)$ and call this the multiplicity of the eigenvalue $\lambda$. Let $0 \leq \lambda_1 \leq \lambda_2, \leq \cdots$ be an ordering of the eigenvalues, including multiplicities. For $\mu \geq 0$ we define

$$N(\mu) := \#\{j \mid \lambda_j \leq \mu\} = \dim \oplus_{\lambda \leq \mu} E(\lambda).$$

Then Weyl's law describes the asympotic behavior of $N(\mu)$, for $\mu \to \infty$.

**Theorem** (Weyl's law) The asymptotic behavior of $N(\mu)$, for $\mu \to \infty$ is given by

$$N(\mu) \sim \frac{\omega_n}{(2\pi)^n} \,\mathrm{vol(M)} \mu^{n/2},$$

where $\omega_n$ denotes the volume of the $n$-dimensional unit ball.

**Example.** The simplest example is the unit circle $S$ , with Laplace operator $-\partial^2/\partial\varphi^2$. We know that the eigenvalues are $k^2$, for $k \in \mathbb{N}$. Furthermore, the multiplicity of 0 is 1 and the multiplicity of the non-zero eigenvalues is 2, so that

$$N(\mu) = 2\lfloor \mu^{1/2} \rfloor - 1 \sim 2\mu^{1/2}.$$

On the other hand, $n = 1$, $B_1 = 2$ and $\mathrm{vol}(S) = 2\pi$, so that in this case

$$\frac{\omega_n}{(2\pi)^n} \,\mathrm{vol(S)} = 2.$$

This confirms Weyl's law for the unit circle.

**Second example.** Let $\Delta$ be the spherical Laplacian on the 2-dimensional unit sphere. Then Weyl's law predicts that

$$N(\mu) \sim \frac{\pi}{(2\pi)^2} 4\pi\mu = \mu.$$

Check this by using spherical harmonics.

Our goal is to follow the proof in a set of Lecture Notes I found on the web

[2] http://www.math.univ-toulouse.fr/~bouclet/Notes-de-cours-exo-exam/M2/cours-2012.pdf

The proof makes heavy use of Pseudo-differential operators depending on parameters, which is just within our reach. The line of reasoning is inuitively appealing.

## 10.2 Spectrum of the Laplacian

In this section we assume that $M$ is a compact manifold, $E \downarrow M$ a complex vector bundle, equipped with a Hermitian structure. Then we have the following natural sesquilinear pairing $\Gamma^\infty(E) \times \Gamma^\infty(E) \to \mathbb{C}$ given by

$$\langle f, g \rangle = \int_M \langle f(x), g(x) \rangle_x \, dx.$$

Here $dx$ denotes the Riemannian volume density on $M$. For each $s \in \mathbb{R}$ this pairing is continuous with respect to the Sobolev topology of $H_s(M, E)$ on the first factor, and the similar topology of $H_{-s}(M, E)$ on the second factor and therefore extends to a continuous sesquilinear pairing

$$H_s(M, E) \times H_{-s}(M, E) \to \mathbb{C}.$$

By a local analysis, it is seen that the pairing is perfect. In particular, for $s = 0$ we obtain a Hermitian inner product on $H_0(M, E) \simeq L^2(M, E)$ which induces the Hilbert topology on $L^2(M, E)$ and thus turns $L^2(M, E)$ into a Hilbert space.

By compactness of $M$, each section $f$ of $\Gamma^{-\infty}(M, E)$ has a finite order as a distribution, so that by local analysis it follows that $f \in H_s(M, E)$ for some $s \in \mathbb{R}$. In other words,

$$\Gamma^{-\infty}(M, E) = \cup_{s \in \mathbb{R}} H_s(M, E) = H_{-\infty}(M, E).$$

**Definition 1** A pseudo-differential operator $P \in \Psi(E, E)$ is said to be self-adjoint if

$$\langle Pf, g \rangle = \langle f, Pg \rangle$$

for all $f, g \in \Gamma^\infty(E)$.

The Laplace operator $\Delta$ may be viewed as a self-adjoint pseudodifferential operator in $\Psi^2(\mathbb{C}_M, \mathbb{C}_M)$. The Hodge-Laplacian of degree $p$ is the operator $\Delta_p : \Omega^p(M) \to \Omega^p(M)$ given by $\Delta_p = dd^* + d^*d$. It is a self-adjoint pseudodifferential operator in $\Psi^2(\wedge^p T^*M, \wedge^p T^*M)$. Moreover, these operators form prime examples of elliptic differential operators of order 2.

In the following we assume that $L$ is a self-adjoint elliptic operator in $\Psi^d(E, E)$, where $d > 0$.

**Lemma 2** *Let $s \in \mathbb{R}$. Then for the above pairing, the restricted operator $L_s : H_s(M, E) \to H_{s-d}(M, E)$ is adjoint to the restricted operator $L_{d-s} : H_{d-s}(M, E) \to H_{-s}(M, E)$.*

**Proof** Since $L$ is a pseudo-differential operator, $L_s$ is continuous linear, and so is $L_{d-s}$. It suffices to show that $\langle L_s f, g\rangle = \langle f, L_{d-s} g\rangle$ for all $f \in H_s(M, E)$ and $g \in H_{d-s}(M, E)$. By continuity and density, it suffices to show this equality for all $f, g \in \Gamma^\infty(M, E)$. This is an immediate consequence of the self-adjointness of $L$. $\square$

Let

$$H_L := \{f \in \Gamma^{-\infty}(M, E) \mid Lf = 0\};$$

here $H$ stands for 'Harmonic'. By the elliptic regularity theorem, $H_L \subset \Gamma^\infty(M, E)$. By the results of Chapter 9, the space $H_L$ is finite dimensional.

We define

$$H_L^\perp := \{f \in \Gamma^{-\infty}(M, E) \mid \forall g \in H_L : \ \langle f, g\rangle = 0\}.$$

**Lemma 3** *We have*

$$\Gamma^{-\infty}(M, E) = H_L^\perp \oplus H_L.$$

*The associated projection operator $P_L : \Gamma_{-\infty}(M, E) \to H_L$ is a smoothing operator.*

**Proof** There exists a basis $\varphi_1, \dots \varphi_n$ of $H_L$ which is orthonormal with respect to the restriction of $\langle\cdot\,,\,\cdot\rangle$ to $H_L$. We define the linear operator $T : \Gamma^{-\infty}(M, E) \to H_L$ by

$$T(f) = \sum_{j=1}^n \langle f, \varphi_j\rangle \varphi_j.$$

Then, clearly, $T^2 = T$ so that $T$ is a projection operator. Clearly $\ker T = H_L^\perp$ and $\mathrm{im}(T) = H_L^\perp$. This establishes the decomposition. Morever, $P_L = T$. We will finish the proof by showing that $T$ is a smoothing operator. Indeed, let $K : M \times M \to E \boxtimes (E^* \otimes D_M)$ be defined by

$$K(x, y)(v^* \otimes (v \otimes \mu)) = \sum_{j=1}^n v^*(\varphi_j(x))\langle v, \varphi_j(y)\rangle dm(\mu_y),$$

for $v^* \in E_x^*, v \in E_y$ and $\mu \in D^*_{My}$. Then $K$ is a smooth section of the bundle $E \boxtimes (E^* \otimes D_M) \downarrow M \times M$. Moreover, it follows from the definitions that, for $f \in \Gamma^\infty(E)$ and $g \in \Gamma^\infty(E^*)$,

$$(T_K(f))(gdx) = K(gdx \otimes f) = \sum_{j=1}^n \varphi_j(gdx) \int_M \langle f(y), \varphi_j(y) \rangle dy = T(f)(gdx).$$

Therefore, $T_K = T$ on $\Gamma^\infty(M, E)$ and by density and continuity, we find that $T_K = T$ on $\Gamma^{-\infty}(M, E)$. $\square$

**Lemma 4** *Let $s \in \mathbb{R}$. Then*

$$H_s(M, E) = (H_s(M, E) \cap H_L^\perp) \oplus H_L$$

*is a direct sum of closed subspaces. The restricted map $L_s : H_s(M, E) \to H_{s-d}(M, E)$ restricts to a topological linear isomorphism*

$$H_s(M, E) \cap H_L^\perp \xrightarrow{\simeq} H_{s-d}(M, E) \cap H_L^\perp.$$

**Proof** The projection operator $P_L : \Gamma^{-\infty}(M, E) \to \Gamma^{-\infty}(M, E)$ is smoothing, with image $H_L$. It follows that the restriction of $P_L$ defines a continuous linear operator of $H_s(M, E)$ with image equal to $H_L \cap H_s(M, E) = H_L$. This establishes the given decomposition of $H_s(M, E)$ into closed subspaces.

The map $L_s : H_s(M, E) \to H_{s-d}(M, E)$ is Fredholm (see Lecture Notes, Ch. 9) hence has finite dimensional kernel and closed image of finite codimension. Obviously the kernel equals $H_L$. It follows that $L_s$ restricts to an injective continuous linear map $H_s(M, E) \cap H_L^\perp \to H_{d-s}(M, E)$.

By adjointness of $L_s$ and $L_{s-d}$, the closure of $\mathrm{im}(L_s)$ equals the orthocomplement of $\ker(L_{s-d})$ in $H_{d-s}$. Since $L_s$ has closed image, we see that $L_s$ restricts to a bijective continuous linear map $H_s(M, E) \cap H_L^\perp \to H_{s-d}(M, E) \cap H_L^\perp$. The result now follows by application of the closed graph theorem for Banach spaces. $\square$

**Definition 5** Let $L \in \Psi^d(E, E)$ be self-adjoint elliptic. We define the Green operator $G_0 : L^2(M, E) \to H_d(M, E)$ by

(a) $G_0 = 0$ on $H_L$

(b) on $L^2(M, E) \cap H_L^\perp$, the operator $G_0$ equals the inverse of

$$L_d : H_d(M, E) \perp H_L^\perp \to H_0(M, E) \cap H_L^\perp = L^2(M, E) \cap H_L^\perp.$$

The following result shows that pseudo-differential operators naturally appear in the theory of elliptic self-adjoint differential operators.

**Theorem 6** *Suppose that $L$ is a self-adjoint elliptic operator of order $d \geq 0$. Let $G_0$ be its Green operator. Then there exists a unique pseudodifferential operator $G \in \Psi^{-d}(E, E)$ whose restriction to $L^2(M, E)$ equals $G_0$. This operator is self-adjoint and satisfies*

$$G \circ L = L \circ G = I - P_L.$$

*In particular, it is a parametrix for $L$.*

**Proof** Uniqueness follows from the fact that $\Gamma^\infty(M, E)$ is contained in $L^2(M, E)$. Existence is established as follows. For $s \in \mathbb{R}$ we define the continuous linear operator $G_s : H_s(M, E) \to H_{s+d}(M, E)$ as follows. The operator is zero on $H_L$ and on $H_s(M, E) \cap H_L^\perp$ it is the inverse of the continuous linear operator $L_{s+d}$ of Lemma 4. Since $L$ is self-adjoint, it is readily verified that the operators $G_s$ and $G_{-s+d}$ are adjoint to each other for the pairing $\langle \cdot\, , \cdot \rangle_s$.

Each operator $G_s$ is continuous linear $H_s(M, E) \to \Gamma^{-\infty}(M, E)$. If $s, t \in \mathbb{R}$, then clearly the operators $G_s$ and $G_t$ coincide on $\Gamma^\infty(M, E)$. If in addition $s < t$ then $H_t(M, E) \subset H_s(M, E)$ and we see that $G_t$ equals the restriction of $G_s$ by density of $\Gamma^\infty(M, E)$ in $H_t(M, E)$. It follows that there is a unique linear operator $G_{-\infty} : \Gamma^{-\infty}(M, E) \to \Gamma^{-\infty}(M, E)$.

On the other hand all operators $G_s$ have the same restriction $G_\infty$ to $\Gamma^\infty(M, E)$. It follows that $G_\infty : \Gamma^\infty(M, E) \to H_s(M, E)$ is continuous linear for all $s \in \mathbb{R}$. By the Sobolev embedding theorem this implies that $G_\infty$ is continuous linear from $\Gamma^\infty(M, E)$ to itself. As $G_\infty$ and $G_{-\infty}$ are adjoint to each other for $\langle \cdot\, , \cdot \rangle_\infty$, it follows that $G_{-\infty}$ is a continuous linear operator from $\Gamma^{-\infty}(M, E)$ to itself. We denote this operator by $G$. Obviously,

$$G \circ L = I - P_L. \tag{10.1}$$

By ellipticity, the operator $L$ has a parametrix $Q \in \Psi^{-d}(M, E)$. Thus, $LQ = I + T$ with $T$ a smoothing operator. It now follows from (10.1) that

$$G(I + T) = GLQ = (I - P_L)Q$$

so that $G = Q - P_L Q - GT$. Now $P_L Q$ and $GT$ are continuous linear operators $\Gamma^{-\infty}(M, E) \to \Gamma^\infty(M, E)$. By the Schwartz kernel theorem, such operators have a smooth kernel, hence are smoothing operators. It follows that $G - Q \in \Psi^{-\infty}(E, E)$, hence $G$ is a pseudo-differential operator of order $-d$. The remaining assertions have been established as well. □

**Lemma 7** *Let $L \in \Psi^d(E, E)$ be elliptic and selfadjoint, $d > 0$. Let $G$ be the associated Green operator. Then $G$ restricts to a compact self-adjoint operator $L^2(M, E) \to L^2(M, E)$.*

**Proof** The embedding $i : H_d(M, \mathbb{R}) \to L^2(M, E)$ is compact by Rellich's theorem. The described restricted operator is the composition $i \circ G_0$ where $G_0 : L^2(M, E) = H_0(M, E) \to H_d(M, E)$ is continuous linear. This establishes the compactness. The self-adjointness is readily checked on the dense subspace $\Gamma^\infty(M, E)$. □

Let $L \in \Psi^d(E, E)$ be elliptic and selfadjoint, $d > 0$. Then for each $\lambda \in \mathbb{C}$ we the operator $L - zI$ belongs to $\Psi^d(E, E)$ and is elliptic, hence has finite dimensional kernel $E(L, \lambda)$, which consists of smooth functions. If $E(L, \lambda)$ is non-trivial, then $\lambda$ is called an eigenvalue for $L$. The set of eigenvalues is denoted by $\Lambda(L)$.

**Theorem 8** *Let $L \in \Psi^d(E, E)$ be elliptic and selfadjoint, $d > 0$. Then $\Lambda(\mathbb{R})$ is a subset of $\mathbb{R}$ which is discrete and has no accumulation points. For every $\lambda \in \Lambda(\mathbb{R})$ the associated eigenspace $E(\lambda)$ is finite dimensional. If $\lambda_1$ and $\lambda_2$ are distinct eigenvalues, then $E(\lambda_1) \perp E(\lambda_2)$ in $L^2(M, E)$. Finally,*

$$L^2(M, E) = \widehat{\oplus}_{\lambda \in \Lambda} E(\lambda),$$

*where the hat indicates that the $L^2$-closure of the algebraic direct sum is taken.*

**Proof** Let $G \in \Psi^{-d}(M, E)$ be the associated Green operator. We write $\Lambda(G)$ for the set of eigenvalues of $G$. It follows readily from the definitions that $E(L, 0) = E(G, 0)$ so $0 \in \Lambda(G)$ if and only if $0 \in \Lambda(L)$. Furthermore, if $\lambda \in \mathbb{C} \setminus \{0\}$, then

$$E(G, \lambda) = E(L, \lambda^{-1})$$

and we see that $\lambda \in \Lambda(G) \iff \lambda^{-1} \in \Lambda(L)$.

By elliptic regularity, $E(G, \lambda) \subset \Gamma^\infty(M, E) \subset L^2(M, E)$ and we see that $\Lambda(G)$ equals the set of eigenvalues of the compact self-adjoint restricted operator $G_0 : L^2(M, E) \to L^2(M, E)$. Moreover, $E(G, \lambda) = \ker(G_0 - \lambda I)$. By the spectral theorem of such operators, all assertions now follow. □

We define the increasing function $N = N_L : [0, \infty[ \to \mathbb{N}$ by

$$N(\mu) = \sum_{\lambda \in \Lambda(L), |\lambda| \leq \mu} \dim E(L, \lambda).$$

Then a Weyl type law for $L$ should describe the top term asymptotic behavior of $N(\mu)$, for $\mu \to \infty$. The aim of these notes is to guide the reader through the proof of Weyl's law for the case that $L$ is the scalar Laplace operator $\Delta$ for a compact Riemannian manifold $M$.

**Theorem 9** *Let $M$ be a compact Riemannian manifold of dimension $n$, $\Delta$ the associated Laplace operator and $N = N_\Delta$. Then*

$$N(\mu) \sim \frac{\omega_n}{(2\pi)^n} \mathrm{vol}(M) \mu^{n/2}, \qquad (\mu \to \infty).$$

The symbol $\sim$ indicates that the quotient of the expression on the left hand side by the expression on the right-hand side of the equation tends to 1, as $\mu \to \infty$.

# 11 Reformulation of Weyl's law

The next step in our discussion is a reformulation of Weyl's law in terms of functional calculus. For this we refer the reader to the text [2], Chapter 2, pages 12 -17.

# 12 Hilbert–Schmidt and trace class operators

## 12.1 Hilbert–Schmidt operators

In this section all Hilbert spaces are assumed to be infinite dimensional separable (i.e., of countable Hilbert dimension).

Let $A : H_1 \to T_2$ be a bounded operator of Hilbert spaces.

**Lemma 10** *Let $A^* : H_2 \to H_1$ be the adjoint of $A$, Then for all orthonormal bases $(e_j)_{j\in\mathbb{N}}$ of $H_1$ and $(f_j)_{j\in\mathbb{N}}$ of $H_2$ we have*

$$\sum_{j=0}^{\infty} \|Ae_j\|^2 = \sum_{j=0}^{\infty} \|A^* f_j\|^2.$$

*In particular, these sums are independent of the particular choices of bases $(e_j)$ and $(f_j)$.*

**Proof** Put $A_{ij} = \langle Ae_j, f_i\rangle$. Likewise, put $A^*_{ij} = \langle A^* f_j, e_i\rangle$. Then $A^*_{ij} = \overline{A}_{ji}$. Hence,

$$\sum_i \|Ae_i\|^2 = \sum_{i,j} |A_{ij}|^2 = \sum_j \|A^* f_j\|^2.$$

□

**Definition 11** The operator $A$ is said to be of Hilbert-Schmidt type if for some (hence any) orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of $H_1$ we have

$$\sum_{j=0}^{\infty} \|Ae_j\|^2 < \infty.$$

The set of these Hilbert-Schmidt operators is a linear subspace of $L(H_1, H_2)$, which we denote by $L_2(H_1, H_2)$.

Let $(e_j)$ and $(f_j)$ be orthonormal bases for $H_1$ and $H_2$ and let $A, B : H_1 \to H_2$ be Hilbert-Schmidt operators. Then by the Cauchy-Schwartz inequality, the sum

$$\sum_j \langle Ae_j, Be_j\rangle = \sum_{ij} A_{ij}\bar{B}_{ij}$$

is absolutely convergent. Clearly, it is independent of the choice of the basis $(f_j)$. We denote the value of this sum by $\langle A, B\rangle_{\mathrm{HS}}$. It is clear that $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{HS}}$ is a positive definite Hermitian inner product on $L_2(H_1, H_2)$. Moreover, it is readily verified that $L_2(H_1, H_2)$ is a Hilbert space for this inner product. The associated norm is given by

$$\|A\|_{\mathrm{HS}}^2 = \sum_j \|Ae_j\|^2$$

for any orthonormal basis $(e_j)$ of $H_1$. As this norm is independent of the choice of basis, it follows that $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{HS}}$ is independent of the choice of the basis $(e_i)$.

**Lemma 12** *Let $U_j$ be a unitary automorphism of $H_j$, for $j = 1, 2$. Then for all $A \in L_2(H_1, H_2)$ we have $U_2AU_1 \in L_2(H_1, H_2)$. Moreover, if $B \in L_2(H_1, H_2)$ then*

$$\langle U_2AU_1, U_2BU_1\rangle = \langle A, B\rangle_{\mathrm{HS}}.$$

**Proof** Straightforward. □

**Lemma 13** *On $L_2(H_1, H_2)$, the operator norm $\|\cdot\|$ is dominated by $\|\cdot\|_{HS}$.*

**Proof** Let $(e_j)$ be an orthonormal basis of $H_1$. For all $x \in H_1$, we put $x_j = \langle x, e_j\rangle$. Then by the Cauchy–Schwarz inequality,

$$\|Ax\| \leq \sum_j |x_j|\|Ae_j\| \leq \|x\|\|A\|_{\mathrm{HS}}.$$

The result follows. □

**Corollary 14** *Every $A \in L_2(H_1, H_2)$ is compact.*

**Proof** Let $(e_j)$ and $(f_j)$ be orthonormal bases for $H_1$ and $H_2$. Define the linear operator $E_{ij} : H_1 \to H_2$ by $E_{ij}(x) = \langle x, e_j\rangle f_i$. Then $E_{ij}$ is a rank one operator. Since

$$A = \sum_{ij} A_{ij}E_{ij}$$

in the Hilbert space $L_2(H_1, H_2)$, it follows that $A$ is the limit of a sequence of finite rank operators with respect to the Hilbert-Schmidt norm, hece also for the operator norm. Compactness follows. □

**Lemma 15** *If $A : H_1 \to H_2$ is Hilbert-Schmidt and $B : H_2 \to H$ a bounded operator of Hilbert spaces, then $BA : H_1 \to H$ is Hilbert-Schmidt. Likewise, if $C : H \to H_1$ is bounded, then $AC$ is Hilbert-Schmidt.*

**Proof** The first assertion follows from the observation that

$$\|BAe_j\|^2 \leq \|B\|^2\|Ae_j\|^2.$$

The second assertion now follows from $(AC)^* = C^*A^*$, by application of Lemma 10. □

## 12.2 Polar decomposition

Let $D$ be the complex unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$. We consider the holomorphic function $\rho : D \to \mathbb{C}$ given by $\rho(z) = -\sqrt{1-z}$. Here the principal value of the square root is taken, so that $\rho(0) = -1$. This function has the power series expansion

$$\rho(z) = \sum_{k=0}^{\infty} c_k z^k, \qquad (|z| < 1), \tag{12.2}$$

where $c_k = \rho^{(k)}(0)/k!$ is readely checked to be a positive real number for $k \geq 1$. The radius of convergence is 1.

**Lemma 16** *The power series (12.2) converges uniformly absolutely on the closed unit disk $\bar{D}$ and defines a continuous extension of $\rho$ to it.*

**Proof** It follows from the positivity of the coefficients that for every $n \geq 1$,

$$-1 + \sum_{k=1}^{n} c_k x^k \leq -\sqrt{1-x} \qquad (0 \leq x < 1).$$

By taking limits for $x \uparrow 1$ we see that this inequality remains valid for $x = 1$. As this is true for all $n$, it follows that the series $\sum_{k=1}^{n} c_k$ converges. Therefore, the power series (12.2) converges absolutely for $z = 1$, hence uniformly absolutely on $\bar{D}$. This implies the continuity statement. □

By a positive operator on a Hilbert space $H$ we shall mean a Hermitian operator $T : H \to H$ which is positive semidefinite, i.e., $\langle Tv, v\rangle \geq 0$ for all $v \in H$. The above result allows us to define the square root of such an operator.

**Lemma 17** *Let $T : H \to H$ be a positive operator. Then there exists a unique positive operator $S : H \to H$ such that $S^2 = T$. The operator $S$ has the following properties,*

(a) $\ker S = \ker T$;

(b) *if $A \in L(H, H)$ then $A$ commutes with $S$ if and only if it commutes with $T$.*

**Proof** We may assume that $T \neq 0$ so that $\|T\| > 0$. Dividing $T$ by its norm if necessary, we may arrange $\|T\| \leq 1$. Clearly, $I - T$ is symmetric and

$$\langle (I-T)v, v\rangle = \|v\|^2 - \langle Tv, v\rangle \geq \|v\|^2 - \|T\|\|v\|^2 \geq 0,$$

so $I - T$ is positive. On the other hand,

$$\langle (I-T)v, v\rangle = \langle v, v\rangle - \langle Tv, v\rangle \leq \|v\|^2$$

and we conclude that $\|I-T\| \leq 1$. Therefore,

$$S := \sum_{k=0}^{\infty} c_n (I-T)^k$$

converges in operator norm and defines a Hermitian operator which commutes with any operator that commutes with $T$. By the usual multiplication of absolutely convergent series, we see that $\sum_{k=0}^{n} c_k c_{n-k} = 0$ for all $n \geq 2$. Applying this multiplication to the power series for $S$ we obtain

$$S^2 = c_0^2 I + 2c_0 c_1 (I-T) = I - (I-T) = T.$$

By positivity of the coefficients for $S - I$ it follows that $S - I$ is positive. On the other hand, by straightforward estimation, it follows that $\|S\| \leq \sqrt{\|I-T\|} \leq 1$. As in the above we conclude that $S = I - (I - S)$ is positive. This establishes existence and the commutant property.

We turn to uniqueness. Let $R$ be a positive operator on $H$ with square $T$. Then obviously, $\ker R \subset \ker T$. Conversely, if $v \in H$ and $Tv = 0$ then $\langle Rv, Rv \rangle = \langle Tv, v \rangle = 0$ so $v \in \ker R$. We thus see that $\ker R = \ker T$. In particular, $\ker S = \ker T$ and we see that $\ker R = \ker S$. It follows that $S = R = 0$ on $\ker T$ and $R, T, S$ preserve $(\ker T)^\perp$. Passing to the latter subspace if necessary, we may as well assume that $\ker T = 0$, so that also $\ker S = \ker R = 0$.

By the first part of the proof, $R = R_0^2$ for a positive operator $R_0$ whose kernel is zero. Thus,

$$\langle Rv, v \rangle = 0 \Rightarrow \langle R_0 v, R_0 v \rangle = 0 \Rightarrow v = 0.$$

Likewise, $\langle Sv, v \rangle = 0 \Rightarrow v = 0$. We now infer, by positivity of $S$ and $R$, that

$$(R+S)(v) = 0 \Rightarrow \langle Rv, v \rangle + \langle Sv, v \rangle = 0 \Rightarrow \langle Sv, v \rangle = 0 \Rightarrow v = 0.$$

Thus, $R + S$ has trivial kernel. As this operator is Hermitian, it follows that it has dense image.

We note that $RT = R^3 = TR$ so $R$ commutes with $S$. It follows that $S^2 - R^2 = (S-R)(S+R)$, so $S - R$ is zero on the image of $R + S$ which is dense. We conclude that $R = S$. □

**Definition 18** For $T : H \to H$ a positive operator on the Hilbert space we define the square root $\sqrt{T}$ to be the unique positive operator on $H$ whose square equals $T$.

If $A : H_1 \to H_2$ is a bounded linear operator of Hilbert spaces, then $A^*A$ is a positive operator on $H_1$.

**Definition 19** In the setting just described, we define $|A| = \sqrt{A^*A}$.

Thus, $|A|$ is a positive operator on $H_1$. We recall that a partial isometry is a bounded linear map $U : H_1 \to H_2$ of Hilbert spaces, such that $U$ restricts to an isometry $(\ker U)^\perp \to H_2$. If $U$ is a partial isometry, then so is its adjoint $U^*$. As the image of a partial isometry is closed, it follows that

$$(\ker U)^\perp = \mathrm{im}(U^*), \quad \text{and} \quad (\mathrm{im} U)^\perp = \ker(U^*).$$

**Lemma 20** *Let $U : H_1 \to H_2$ be a partial isometry. Then $U^*U : H_1 \to H_1$ is the orthogonal projection onto* $\mathrm{im}(U)$. *In particular, if $U$ is isometric, then $U^*U = I$.*

**Proof** Straightforward. □

The following may be viewed as a generalisation of the decomposition in polar coordinates for $\mathbb{C}$.

**Theorem 21** (Polar decomposition) *Let $A : H_1 \to H_2$ be a bounded operator of Hilbert spaces. Then there exists a partial isometry $U : H_1 \to H_2$ such that*

$$A = U|A|. \tag{12.3}$$

(a) *The restriction of $U$ to* $(\ker A)^\perp$ *is unique and isometric with image* $\overline{\mathrm{im}(A)}$.

(b) *The restriction of $U$ to* $\ker A$ *is a partial isometry to* $\mathrm{im}(A)^\perp$. *If $U_0 : \ker A \to \mathrm{im}(A)^\perp$ is any given partial isometry, then $U$ exists uniquely such that $U|_{\ker A} = U_0$.*

(c) *For any partial isometry $U$ such that (12.3) we have $|A| = U^*A$.*

**Proof** We start by observing that $\ker|A| = \ker A^*A = \ker A$. Since $|A|$ is Hermitian, the image $\mathrm{im}|A|$ is dense in $(\ker A)^\perp$.

We define the linear map $U_1 : \mathrm{im}|A| \to H_2$ by $U_1|A|x = A(x)$, for $x \in H_1$. This definition is unambiguous, since $\ker|A| = \ker A$. It follows that

$$\langle U_1|A|x, U_1|A|x\rangle = \langle Ax, Ax\rangle = \langle |A|^2 x, x\rangle = \langle A|x|, A|x|\rangle,$$

so $U_1$ is isometric, and uniquely extends to an isometry $U_1 : (\ker A)^\perp \to H_2$. The image of $U_1$ is closed, contains $\mathrm{im}(A)$ and is contained in the closure of $\mathrm{im}(A)$, hence equal to the latter.

Let a partial isometry $U_0 \ker A \to (\mathrm{im} A)^\perp$ be given. Let $U$ be the map $H_1 \to H_2$ that restricts to $U_0$ on $\ker A$ and to $U_1$ on $(\ker A)^\perp$. Then $U$ is a partial isometry. Furthermore, it is obvious that $U \circ |A| = A$. The first assertion and (12.3) follow.

Given any partial isometry $U' : H_1 \to H_2$ with $A = U'|A|$ we see that $U'(|A|(x)) = Ax = U_1(|A|(x))$ so that $U' = U_1$ on $\mathrm{im}(|A|)$ hence on its closure $(\ker A)^\perp$. This implies (a). Since $U'$ is a partial isometry, we see that $U'$ must restrict to a partial isometry on $\ker A$ with image contained in $U((\ker A)^\perp)^\perp = \mathrm{im} U_1)^\perp = (\mathrm{im} A)^\perp$. This proves the first statement of (b). The second statement of (b) has already been established above.

We finally turn to (c). Since $U$ is isometric when restricted to $(\ker A)^\perp$ it follows that $\ker U \subset \ker A = \ker |A|$, hence

$$\overline{\mathrm{im}|A|} = (\ker |A|)^\perp \subset (\ker U)^\perp.$$

Now $U^*U$ equals the orthogonal projection onto $\ker(U)^\perp$, hence

$$U^*A = U^*U|A| = |A|.$$

□

**Corollary 22**

(a) *Let $A : H_1 \to H_2$ a bounded operator with trivial kernel. Then there exists a unique isometry $U : H_1 \to H_2$ such that $A = U|A|$.*

(b) *Let $A : H \to H$ be a bounded self-adjoint operator. Then there exists an isometry $U : H \to H$ such that $A = U|A|$.*

**Proof** (a) is immediate from the theorem. For (b) we note that by self-adjointness, $(\ker A)^\perp = \overline{\mathrm{im}(A)}$, so $U_0 = I$ is an isometry. It now follows from Theorem 21 (b) that $U_0$ uniquely extends to a partial isometry $U : H \to H$ such that $A = U|A|$. Since $\ker U \subset \ker U_0 = 0$ it follows that $U$ is an isometry. □

As a converse to Theorem 21, we have the following.

**Lemma 23** *Let $A = US$ with $S : H_1 \to H_1$ positive and $U : H_1 \to H_2$ a partial isometry with kernel contained in* $\ker S$. *Then $S = |A|$.*

**Proof** From the assumption it follows that $A^*A = S^*U^*US$. Now $U^*U$ is the orthogonal projection onto $(\ker U)^\perp$, which contains $(\ker S)^\perp = \mathrm{im}(S)$. Hence $U^*US = S$ and we see that $A^*A = S^*S = S^2$. By positivity, it follows that $S = \sqrt{A^*A} = |A|$. □

The polar decomposition behaves well with respect to Hilbert–Schmidt operators.

**Lemma 24** *Let $A : H_1 \to H_2$ be a bounded linear operator. Then the following statements are equivalent,*

(a) *$A$ is Hilbert–Schmidt,*

(b) *$|A|$ is Hilbert–Schmidt.*

*Furthermore, if the above conditions are satisfied, then*

$$\|A\|_{\mathrm{HS}} = \|\,|A|\,\|_{\mathrm{HS}}.$$

**Proof** Let $A = U|A|$ be a polar decomposition with $U$ a partial isometry. Then $|A| = U^*A$. Since $U$ and $U^*$ are bounded, the equivalence of (a) and (b) follows. The operator norms of $U$ and $U^*$ are at most 1, hence

$$\|A\|_{\mathrm{HS}} = \|U|A|\|_{\mathrm{HS}} \leq \|\,|A|\,\|_{\mathrm{HS}} = \|U^*A\|_{\mathrm{HS}} \leq \|A\|_{\mathrm{HS}}.$$

□

## 12.3 Operators of trace class

By an orthonormal sequence in a Hilbert space $H$ we shall mean a sequence $(e_i)_{i\in\mathbb{N}}$ of unit vectors in $H$ which are mutually perpendicular. Such a sequence need not be a basis. More precisely, given such a sequence $(e_i)$ and an orthonormal basis $(f_i)_{i\in\mathbb{N}}$ of $H$ there is a unique isometry $U : H \to H$ which maps $f_i$ to $e_i$ for all $i \in \mathbb{N}$. The sequence $(e_i)$ is a basis if and only if $U$ is surjective.

Let $A : H_1 \to H_2$ be a bounded operator between Hilbert spaces.

**Definition 25** The operator $A$ is said to be of trace class if and only if for all orthonormal sequences $(e_i)$ of $H_1$ and $(f_i)$ of $H_2$ we have

$$\sum_i |\langle Ae_i, f_i\rangle| < \infty.$$

Note that we do not assume that $(e_i)$ and $(f_i)$ are *bases* of $H_1$ and $H_2$, respectively. If $(e_i)$ is a basis and $(f_i)$ is not, then the above estimate cannot be obtained by extending $(f_i)$ to a basis, so that the present requirement with sequences is stronger than the similar requirement with bases. This seems an essential feature of the present definition.

In the literature one sees several characterisations of trace class operators. The advantage of Definition 25 is that it allows the immediate conclusion that the set of all trace class operators $H_1 \to H_2$ is a linear subspace of $L(H_1, H_2)$. It is denoted by $L_1(H_1, H_2)$.

**Lemma 26** *Let $A : H_1 \to H_2$ be a bounded operator of Hilbert spaces. Then $A$ is of trace class if and only if the adjoint $A^*$ is of trace class.*

**Proof** Immediate from the definition. □

The following result relates Hilbert–Schmidt operators to those of trace class.

**Lemma 27** *Let $A : H_1 \to H_2$ and $B : H_2 \to H_3$ be Hilbert–Schmidt operators. Then for all orthonormal sequences $(e_i)$ in $H_1$ and $(g_i)$ in $H_3$ we have*

$$\sum_i |\langle BAe_i, g_i\rangle| \leq \|A\|_{\mathrm{HS}} \|B\|_{HS}.$$

*In particular, $BA$ is of trace class.*

**Proof** Let $f_j$ be an orthonormal basis of $H_2$. Then for every $i$ we have

$$\langle BAe_i, g_i\rangle = \langle Ae_i, B^* g_i\rangle = \sum_j \langle Ae_i, f_j\rangle\langle f_j, B^* g_i\rangle.$$

By Cauchy–Schwartz, it follows that

$$\sum_i |\langle BAe_i, g_i\rangle| \leq (\sum_{i,j} |\langle Ae_i, f_j\rangle|^2)^{1/2} (\sum_{i,j} |\langle f_j, B^* g_i\rangle|^2)^{1/2} \leq \|A\|_{HS} \|B^*\|_{\mathrm{HS}}.$$

The final estimate above follows since $(e_i)$ and $(g_i)$ can be extended to full orthonormal bases of $H_1$ and $H_3$, respectively. □

We can now give the following useful characterizations of trace class operators by means of the polar decomposition.

**Theorem 28** *Let $A : H_1 \to H_2$ be a bounded operator. Then the following assertions are equivalent.*

(a) *$A$ is of trace class.*

(b) *$\sqrt{|A|}$ is a Hilbert–Schmidt operator on $H_1$.*

(c) *$A$ equals the composition $BC$ of two Hilbert–Schmidt operators $C : H_1 \to H_3$ and $B : H_3 \to H_2$.*

**Proof** By the theorem of polar decomposition there exists a partial isometry $U : H_1 \to H_2$ such that

$$A = U|A|$$

and such that $\ker U = \ker A$. Then $|A| = U^* A$. First assume that (a) is valid. Let $(e_i)$ be any orthonormal basis of $(\ker U)^\perp = (\ker |A|)^\perp$ then $(e_i)$ is an orthonormal sequence in $H_1$ and $(Ue_i)$ is an orthonormal sequence in $H_2$. We may realise $(e_i)$ as a subsequence of an orthonormal basis $(f_j)$ of $H_1$. Then the complement of $(e_i)$ in $(f_j)$ consists of vectors from $\ker |A|$. Therefore,

$$\sum_j \langle |A| f_j, f_j\rangle = \sum_i \langle |A| e_i, e_i\rangle = \sum_i |\langle Ae_i, Ue_i\rangle| < \infty.$$

It follows that

$$\sum_j \langle |A|^{1/2} f_j, |A|^{1/2} f_j \rangle = \sum_j \langle |A| f_j, f_j \rangle < \infty,$$

hence (b).

Now assume (b). Then $B := U|A|^{1/2}$ is Hilbert-Schmidt as well. Now $A$ is the composition of this operator with $C := |A|^{1/2}$ and we obtain (c) with $H_3 = H_1$ and the given $B$ and $C$.

The implication '(c) $\Rightarrow$ (a)' has been established in Lemma 27. □

The following result explains the terminology trace class operator introduced in Definition 25.

**Lemma 29** *Let $A : H \to H$ be an operator of trace class. Then there exists a unique number* $\mathrm{tr}\,(A) \in \mathbb{C}$ *such that for all orthonormal bases $(e_j)$ of $H$ we have*

$$\mathrm{tr}\,(A) = \sum_j \langle Ae_j, e_j \rangle,$$

*with absolutely convergent sum.*

**Proof** By the theorem of polar decomposition, there exists a partial isometry $U : H \to H$ such that $A = U|A|$. Let $(e_j)$ be any orthonormal basis of $H$, then for all $j$ we have

$$\langle Ae_j, e_j \rangle = \langle |A|^{1/2} e_j, |A|^{1/2} U^* e_j \rangle.$$

Since the operators $|A|^{1/2}$ and $|A|^{1/2}U^*$ are Hilbert–Schmidt, the sum over $j$ is absolutely convergent, with value given by

$$\sum_j \langle Ae_j, e_j \rangle = \langle |A|^{1/2}, |A|^{1/2} U^* \rangle_{\mathrm{HS}}.$$

□

**Corollary 30** *Let $A, B \in L(H_1, H_2)$ be Hilbert–Schmidt operators. Then $B^*A$ is of trace class, and*

$$\mathrm{tr}\,(B^*A) = \langle A, B \rangle_{\mathrm{HS}}.$$

**Proof** Let $(e_i)$ be an orthonormal basis of $H_1$. Then

$$\mathrm{tr}\,(B^*A) = \sum_i \langle Ae_i, Be_i \rangle = \langle A, B \rangle_{\mathrm{HS}}.$$

□

**Corollary 31** *Let $A : H_1 \to H_2$ be an operator of trace class, and let $H_3$ be a third Hilbert space. Then for all bounded operators $B \in L(H_2, H_3)$ and $C \in L(H_3, H_2)$, the operators $BA$ and $AC$ are of trace class.*

**Proof** By Theorem 28 there exists a Hilbert space $H$ and two Hilbert–Schmidt operators $A_1 \in L_2(H_1, H)$ and $A_2 \in L_2(H, H_2)$ so that $A = A_2A_1$. It follows that $BA_2 \in L_2(H, H_3)$ so that $BA = (BA_2)A_1 \in L_1(H_1, H_3)$. The assertion for $AC$ follows in a similar manner. □

For bounded normal on a Hilbert space, Hilbert–Schmidt and trace class may be characterized in terms of their eigenvalues as follows.

Let $A : H \to S$ be a compact self-adjoint operator on a Hilbert space. By the spectral theorem for such operators, there exists an orthonormal basis of eigenvectors $(e_i)$. Let $\lambda_j \in \mathbb{C}$ be the eigenvalues corresponding to this basis. Thus, $Ae_i = \lambda_i e_i$.

**Corollary 32** *Let $A : H \to H$ be compact normal as above.*

(a) *$A$ is a Hilbert-Schmidt if and only if $\sum_i |\lambda_i|^2 < \infty$.*

(b) *$A$ is of trace class if and only if $\sum_i |\lambda_i| < \infty$.*

**Proof** By normality, $A^*e_i = \bar{\lambda}_i e_i$. It follows that $|A|e_i = |\lambda_i|e_i$. Now (a) follows in a straightforward way. We note that $\sqrt{|A|} = \sqrt{|\lambda_i|}\, e_i$. Hence, by (a) this operator is Hilbert–Schmidt if and only if $\sum_i |\lambda_i| < \infty$. The equivalence in (b) now follows by application of Theorem 28 □

We will now show that for (separable) Hilbert spaces $H_1$ and $H_2$, the space $L_1(H_1, H_2)$ has a natural Banach norm for which the inclusion $L_1(H_1, H_2) \hookrightarrow L_2(H_1, H_2)$ is continuous. We start with a lemma.

**Lemma 33** *Let $A : H_1 \to H_2$ be of trace class. Then for all orthonormal sequences $(e_i)$ in $H_1$ and $(f_i)$ in $H_2$ we have*

$$\sum_i |\langle Ae_i, f_i\rangle| \leq \mathrm{tr}\,(|A|).$$

**Proof** We use the polar decomposition $A = U|A|$. Put $S = \sqrt{|A|}$. Then $A$ is the product of the Hilbert–Schmidt operators $US$ and $S$. It follows by Lemma 27 that

$$\sum_i |\langle Ae_i, f_i\rangle| \leq \|US\|_{\mathrm{HS}}\|S\|_{\mathrm{HS}} \leq \|U\|\|S\|_{\mathrm{HS}}^2 \leq \|S\|_{\mathrm{HS}}^2 = \mathrm{tr}\,(S^2).$$

□

In view of the lemma, we can define the norm $\|\cdot\|_1$ on $L_1(H_1, H_2)$ by

$$\|A\| = \sup_{(e_i),(f_i)} \sum_i |\langle Ae_i, f_i\rangle|,$$

where the supremum is taken over all orthonormal sequences $(e_i)$ in $H_1$ and $(f_i)$ in $H_2$. It is readily verified that $\|\cdot\|_1$ is indeed a norm. Obviously $\|A\|_1 \leq \mathrm{tr}\,(|A|)$ for all $A \in L_1(H_1, H_2)$.

**Lemma 34** *Let $A \in L_1(H_1, H_2)$. Then*

$$\|A\|_1 = \mathrm{tr}\,|A|.$$

**Proof** By the previous lemma it suffices to establish the existence of orthonormal sequences $(e_i)$ of $H_1$ and $(f_i)$ of $H_2$ such that

$$\sum_i |\langle Ae_i, f_i\rangle| = \mathrm{tr}\,(|A|). \tag{12.4}$$

For this we proceed as follows. Let $A = U|A|$ be the polar decompostion, where we have made sure that $U$ is an isometry. Let $(e_i)$ a basis in $H_1$ for which $|A|$ diagonalizes, say with eigenvalues $\lambda_i$. Since $U^*$ maps $\mathrm{im}(U) = (\ker U^*)^\perp$ isometrically onto $H_1$ we may fix an orthonormal basis $(f_i)$ in $\mathrm{im}(U)$ such that $U^* f_i = e_i$, for all $i \in \mathbb{N}$. For each $i$ we have

$$\langle Ae_i, f_i\rangle = \langle |A|e_i, U^* f_i\rangle = \langle |A|e_i, e_i\rangle = \lambda_i$$

hence (12.4). □

**Corollary 35** *Let $A : H \to H$ be a compact normal operator, and $(\mu_i)$ its sequence of non-zero eigenvalues counted with multiplicities.*

(a) *If $A$ is Hilbert–Schmid, then $\|H\|_{\mathrm{HS}}^2 = \sum_i |\mu_i|^2$;*

(b) *If $A$ is of trace class, then $\|A\|_1 = \sum_i |\mu_i|$.*

**Proof** There exists an orthonormal basis $(e_i)$ of eigenvectors for $A$ with $Ae_i = \lambda_i e_i$ such that $(\mu_i)$ is the subsequence of $(\lambda_i)$ obtained from omitting the zeros. Now (a) follows immediately from $\|A\|_{\mathrm{HS}}^2 = \sum_j \langle Ae_j, e_j\rangle$. For (b) we note that by normality, $|A|e_j = |\mu|e_j$. Hence $\|A\|_1 = \mathrm{tr}\,|A| = \sum_j |\mu_j|$ and the result follows. □

**Theorem 36** *Let $H_1, H_2$ and $H_3$ be separable Hilbert spaces. Then*

(a) $L_1(H_1, H_2) \subset L_2(H_1, H_2)$ *with continuous inclusion. More precisely,*

$$\|A\|_{\mathrm{HS}} \leq \|A\|_1 \tag{12.5}$$

*for all* $A \in L_1(H_1, H_2)$.

(b) *The space* $L_1(H_1, H_2)$ *equipped with* $\| \cdot \|_1$ *is a Banach space.*

(c) *The bilinear map* $L_2(H_1, H_2) \times L_2(H_2, H_3) \to L_1(H_1, H_3)$, $(A, B) \mapsto BA$ *is continuous. More precisely, for* $A \in L_2(H_1, H_2)$ *and* $B \in L_2(H_2, H_3)$,

$$\|BA\|_1 \leq \|A\|_{\mathrm{HS}} \|B\|_{\mathrm{HS}}.$$

(d) *Let* $A \in L_1(H_1, H_2)$. *Then the maps* $R_A : B \mapsto BA$, $L(H_2, H_3) \mapsto L_1(H_1, H_3)$ *and* $L_A : C \mapsto AC$, $L(H_3, H_1) \to L_1(H_3, H_2)$ *are continuous.*

**Proof** We begin with (a). First, consider the case that $H_2 = H_1 = H$ and that $A : H \to H$ is a self-adjoint and positive semi-definite bounded operator. Assume that $A$ is of trace class, hence compact. Let $(e_i)$ be an orthonormal basis of $H$ consisting of eigenvectors for $A$, and let $\lambda_i$ be the associated eigenvalues. Then $\lambda := (\lambda_i)$ is a sequence in $l^1(\mathbb{N})$ hence in $l^2(\mathbb{N})$ and it is well-known and easy to verify that for the associated norms on these sequence spaces we have

$$\|\lambda\|_{\mathrm{HS}} \leq \|\lambda\|_1.$$

This immediately implies the inequality12.5

Let now $A \in L_1(H_1, H_2)$ be arbitrary. Let $U|A|$ be a polar decomposition, with $U : H_1 \to H_1$ a partial isometry. Then $|A| = U^*A$, with $U^*$ a partial isometry, hence

$$\|A\|_{\mathrm{HS}} = \||A|\|_{\mathrm{HS}} \leq \||A|\|_1 = \|A\|_1.$$

We turn to (b). Then $X := L_2(H_1, H_2)$ is Hilbert space, $X_1 := L_1(H_1, H_2)$ a normed subspace such that the inclusion map is continuous.

Let $\Pi$ be the set of pairs $p = ((e_i), (f_i))$ of orthononormal sequences of $H_1$ and $H_2$, respectively. For such a $p$ we define the linear map $\xi_p : X_1 \to \mathbb{C}^{\mathbb{N}}$ by $\xi_p(A) = \langle Ae_i, f_i \rangle$. Then by definition $\xi_p$ is a continuous linear map from $X_1$ to $l^1(\mathbb{N})$. It follows that $\nu_p = \|\xi_p\|_1$ is a continuous seminorm on $X_1$. Furthermore, $\sup_{p \in \Pi} \nu_p$ equals the norm $\| \cdot \|_1$ on $X_1$.

We will now show that $X_1$ is Banach. Let $(A_k)$ be a Cauchy sequence in $X_1$. Then $(A_k)$ is Cauchy in $X$ hence has a limit $A \in X$. Furthermore, for $p = ((e_i), (f_i))$ as above, $\xi_p(A_k)$ is Cauchy in $l^1(\mathbb{N})$ hence has a limit $A_p$ in $l^1(\mathbb{N})$. For all $i$ we have, by continuity of the maps $l^1(\mathbb{N}) \to \mathbb{C}$, $b \mapsto b_i$ and $X \to \mathbb{C}$, $A \mapsto \langle Ae_i, f_i \rangle$ that $\langle Ae_i, f_i \rangle = (A_p)_i$. It follows that $\xi_p(A) = A_p \in l^1(\mathbb{N})$. As this is valid for every $p$, we conclude that $A \in X$ is trace class, hence belongs to $X_1$.

It remains to be shown that $A_k \to A$ in $\| \cdot \|_1$. Let $\epsilon > 0$. Then there exists $N$ such that $s, t > N \Rightarrow \|A_s - A_t\|_1 < \epsilon$. Fix $p \in \Pi$ as above. Then for all $s, t > N$ we have

$$\|\xi_p(A_s - A_t)\|_1 < \epsilon.$$

Now $\xi_p(A_t) \to \xi_p(A)$ in $l^1(\mathbb{N})$, for $t \to \infty$ and by taking the limit for $t \to \infty$, we conclude that

$$\|\xi_p(A_s - A_t)\|_1 \leq \epsilon, \qquad (s > N).$$

As this estimate holds for all $p \in \Pi$ we conclude that $\|A_s - A\| \leq \epsilon$ for all $s > N$. This completes the proof of (b).

Assertion (c) follows from Lemma 27 and the definition of $\| \cdot \|_1$. For assertion (d) we use the notation of the proof of Cor. 31 and consider the decomposition decompose $A = A_2 A_1$, with $A_1 \in L_2(H_1, H)$ and $A_2 \in L_2(H, H_2)$. Then

$$\|BA\|_1 \leq \|BA_2\|_{HS} \|A_1\|_{\mathrm{HS}} \leq \|B\| \|A_2\|_{\mathrm{HS}} \|A_1\|_{\mathrm{HS}}.$$

Therefore, $R_A$ is continuous as stated. The assertions about $L_A$ are proved in a similar fashion. $\square$

# 13 Smoothing operators are of trace class

In this section we will show that smoothing operators with compactly supported kernels are of trace class. We start by investigating such operators on $\mathbb{R}^n$, and will then extend the results to manifolds.

Given $p \geq 1$, we denote by $\mathcal{S}(\mathbb{N}^p)$ the space of rapidly decreasing functions on $\mathbb{N}^p$, i.e., the space of functions $c : \nu \mapsto c_\nu$, $\mathbb{N}^p \to \mathbb{C}$ such that for all $N \in \mathbb{N}$,

$$s_N(c) := \sup_{k \in \mathbb{N}^p} (1 + \|k\|)^N |c_k| < \infty.$$

Equipped with the seminorms $s_N$ this space is a Fréchet space. The space $\mathcal{S}(\mathbb{Z}^p)$ is defined similarly, with everywhere $\mathbb{N}^p$ replaced by $\mathbb{Z}^p$. Obviously, through extension by zero, $\mathcal{S}(\mathbb{N}^p)$ can be viewed as a closed subspace of $\mathcal{S}(\mathbb{Z}^p)$.

Let $H_1$ and $H_2$ be Hilbert spaces, with orthonormal basis $(e_i)$ and $(f_j)$, respectively. For $K \in \mathcal{S}(\mathbb{N}^2)$, we denote by $A_K$ the unique bounded linear operator $H_1 \to H_2$ determined by $\langle A_K(e_i), f_j \rangle = K_{i,j}$.

**Lemma 37** *If $K \in \mathcal{S}(\mathbb{N}^2)$, then $A_K$ is of trace class. The map $K \mapsto A_K, \mathcal{S}(\mathbb{N}^2) \to L_1(H_1, H_2)$ is continuous linear.*

**Proof** Let $U : H_1 \to H_1$ and $V : H_2 \to H_2$ be isometries. Put $Ue_i = \sum_k U_{ki} e_k$ and $Vf_i = \sum_l V_{li} e_l$. Then $\sum_k |U_{ki}|^2 = 1$ and $\sum_l |V_{li}|^2 = 1$ so that by the Cauchy–Schwartz inequality we have

$$\sum_i |U_{ki} V_{li}| \leq 1.$$

Then

$$\sum_i |\langle AUe_i, Vf_i\rangle| \leq \sum_i \sum_{k,l} |\langle Ae_k, f_l\rangle||U_{ki}V_{li}| \leq \sum_{k,l} \sum_i |\langle Ae_k, f_l\rangle||U_{ki}V_{li}|$$

$$\leq \sum_{k,l} |K_{k,l}| \leq s_N(K) \sum_{k,l} (1+k)^{-N/2}(1+l)^{-N/2} = C_N s_N(K)$$

with $C_N = (\sum_k (1+k)^{-N/2})^2 < \infty$ for $N > 2$. As this estimate holds for all isometries $U$ and $V$, it follows that $A_K$ is of trace class and

$$\|A_K\|_1 \leq C_N s_N(K).$$

The continuity statement follows. □

**Lemma 38** *Let $p \geq 1$. There exists a bijection $\varphi : \mathbb{N} \to \mathbb{Z}^p$ such that the induced map $\varphi^* : f \mapsto f \circ \varphi$ is a continuous linear isomorphism $\mathcal{S}(\mathbb{Z}^p) \to \mathcal{S}(\mathbb{N})$.*

**Proof** We consider the norm $\|x\|_m = \max\{|x_j| \mid 1 \leq j \leq p\}$ on $\mathbb{R}^p$. For $r \in \mathbb{N}$, let $\bar{B}(r) := \{k \in \mathbb{Z}^p \mid \|k\|_m \leq r\}$. Then $\bar{B}(r) = (\mathbb{Z} \cap [-r, r])^p$ has $(2r+1)^p$ elements. Take any bijection $\varphi : \mathbb{N} \to \mathbb{Z}^p$ such that $\varphi(\{1, \ldots, (2r+1)^p\}) \subset \bar{B}(r)$ for all $r \in \mathbb{N}$. Then for all $r \in \mathbb{Z}_{>0}$ we have for all $j \in \mathbb{N}$ that

$$\|\varphi(j)\|_m = r \iff (2r-1)^p < j \leq (2r+1)^p.$$

Hence, for all $j \in \mathbb{N}$,

$$2\|\varphi(j)\|_m \leq j + 1 \leq (2r+1)^p.$$

Let $\psi$ denote the inverse to $\varphi$. Then, by equivalence of norms

$$\|\varphi(j)\|_m = \mathcal{O}(1 + |j|), \quad \text{and} \quad \psi(k) = \mathcal{O}(1 + \|k\|_m)^p$$

for $j \in \mathbb{N}, |j| \to \infty$ and $k \in \mathbb{Z}^p, |k| \to \infty$. By equivalence of norms, these estimates are also valid with $\|\cdot\|$ in place of $\|\cdot\|_m$. It is now straightforward to check that $\varphi$ and $\psi$ induce continuous linear maps $\varphi^* : \mathcal{S}(\mathbb{Z}^p) \to \mathcal{S}(\mathbb{N})$ and $\psi^* : \mathcal{S}(\mathbb{N}) \to \mathcal{S}(\mathbb{Z}^p)$ which are each others inverses. □

**Lemma 39** *Let $dm$ a smooth positive density on $\mathbb{R}^n$. Then there exists an orthornormal basis $(\varphi_i)$ of $L^2(\mathbb{R}^n, dm)$ such that*

(a) *For each $i \in \mathbb{N}$, the function $\varphi_i : \mathbb{R}^n \to \mathbb{C}$ is smooth.*

(b) *The functions $\{\varphi_i \mid i \in \mathbb{N}\}$ are locally uniformly bounded*

(c) *The map $f \mapsto (\langle f, \varphi_i\rangle)_{i\in\mathbb{N}}$ is continuous linear from $C_c^\infty(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{N})$.*

**Proof** The proof goes by reduction to a similar result on the $n$-dimensional torus, which in turn relies on the classical theory of Fourier series.

We consider the standard density on the $n$-dimensional torus $T = (\mathbb{R}/\mathbb{Z})^n$, equipped with the $n$-fold power $dt = dt_1 \cdots dt_n$ of the unit density on $\mathbb{R}/\mathbb{Z}$. Thus, for $f \in C(\mathbb{T}^n)$ we have

$$\int_{\mathbb{T}} f(t)\, dt = \int_0^1 \cdots \int_0^1 f(t)\, dt_1 \cdots dt_n.$$

For each $\nu \in \mathbb{Z}^n$ we consider the function $\chi_\nu : T \to \mathbb{C}$ given by

$$\chi_\nu(t) = e^{2\pi i \langle \nu, t\rangle}.$$

By the theory of Fourier series, these functions form an orthonormal basis for $L^2(T)$. For $f \in C(T)$ and $\nu \in 2\pi i\mathbb{Z}^n$ we define the Fourier coefficent $\hat{f}(\nu) = \langle f, \chi_\nu\rangle_{L^2}$ (the $L^2$-inner product). From elementary considerations, involving partial differentiation, we know that $f \mapsto \hat{f}$ defines a continuous linear map $C^\infty(T) \to \mathcal{S}(\mathbb{Z}^n)$. We now fix a bijection $\mathbb{N} \to \mathbb{Z}^n$ $j \mapsto \nu_j$ which by pull-back induces a continuous linear isomorphism $\mathcal{S}(\mathbb{Z}^n) \to \mathcal{S}(\mathbb{N})$. Put $e_j := \chi_{\nu_j}$. Then the functions $(e_j)$ form an orthonormal basis of $L^2(\mathbb{T})$. Furthermore, the map $f \mapsto (\hat{f}(\nu_j))_{j\in\mathbb{N}}$ defines a continuous linear map $f \mapsto \tilde{f}$, $C^\infty(T) \to \mathcal{S}(\mathbb{N})$.

To relate the asserted result for $\mathbb{R}^n$ to the obtained result for $T$, we fix an open embedding $\iota : \mathbb{R}^n \to T$, with image $\Omega := ]0,1[^n + \mathbb{Z}^n$, whose complement has measure zero in $T$. For instance, we may take the $n$-fold power induced by any diffeomorphism $\mathbb{R} \simeq\, ]0,1[$. The pull-back $\iota^*(dt)$ of $dt$ under $\iota$ is an everywhere positive density on $\mathbb{R}^n$.

By positivity of the densities involved, there exists a unique positive smooth function $\mu : \mathbb{R}^n \to ]0,\infty[$ such that $\iota^*(dt) = \mu dm$. Thus, the pull-back under $\iota$ defines an isometric isomorphism

$$\iota^* : f \mapsto f \circ \iota,\ L^2(T, dt) \to L^2(\mathbb{R}^n, \mu dm).$$

We define the functions $\varphi_j : \mathbb{R}^n \to \mathbb{C}$ by $\varphi_j = \mu^{1/2}\iota^*\chi_j$ and claim that these satisfy the desired properties. First of all, their $L^2$-inner products in $L^2(\mathbb{R}^n, dm)$ are given by

$$\langle \varphi_i, \varphi_j\rangle = \int_{\mathbb{R}^n} \iota^*(\chi_i))\iota^*(\overline{\chi}_j))\ \mu dm = \int_{\mathbb{R}^n} \iota^*(\chi_i))\iota^*(\overline{\chi}_j))\ \iota^*(dt) = \langle \chi_i, \chi_j\rangle,$$

from which one sees that the functions $(\varphi_j)$ form an orthonormal basis. Next, they are smooth and locally uniformly bounded, and we see that (a) and (b) are valid.

Given a function $f \in C_c^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned}\langle f, \varphi_j\rangle &= \int_{\mathbb{R}^n} f\overline{\varphi_j} dm = \int_{\mathbb{R}^n} (f\mu^{-1/2})\iota^*(\chi_j)\mu dm \\ &= \langle \iota_*(f\mu^{-1/2}), \chi_j\rangle = [\iota_*(\mu^{-1/2}f)]\tilde{}(j).\end{aligned}$$

Since $\iota_* \circ M_{\mu^{-1/2}}$ defines a continuous linear map $C_c^\infty(\mathbb{R}^n) \to C^\infty(T)$, condition (c) follows. □

**Lemma 40** *Let $dm$ and $dm'$ be a two smooth densities on $\mathbb{R}^n$ and let $K \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Then the integral operator $T : L^2(\mathbb{R}^n, dm) \to L^2(\mathbb{R}^n, dm')$ defined by*

$$T_K f(x) := \int_{\mathbb{R}^n} K(x,y) f(y)\, dm(y)$$

*is of trace class. The map*

$$K \mapsto T_K, \quad C_c^\infty(\mathbb{R}^{2n}) \to L_1(L^2(\mathbb{R}^n, dm), L^2(\mathbb{R}^n, dm'))$$

*is continuous linear. Furthermore, if $dm' = dm$, then*

$$\mathrm{tr}\,(T_K) = \int_{\mathbb{R}^n} K(x,x)\, dm(x).$$

**Proof** We fix an orthonormal basis $(\varphi_i)$ for $L^2(\mathbb{R}^n, dm)$ such that the conditions of Lemma 39 such that the induced map $C_c^\infty(\mathbb{R}^n) \to \mathcal{S}(\mathbb{N})$, $f \mapsto \langle f, \varphi_i\rangle$ is continuous linear. A similar basis $(\psi_j)$ is fixed for $L^2(\mathbb{R}^n, dm')$. Then it follows that the map

$$K \mapsto K_{j,i} := \langle K, \psi_j \otimes \bar{\varphi}_i\rangle$$

is continuous linear $C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n) \to \mathcal{S}(\mathbb{N}^2)$. We note that $\langle T_K(\varphi_i), \psi_j\rangle = K_{j,i}$. By application of Lemma 37 we now see that $K \to T_K$ is continuous linear from $C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ to $L_1(L^2(\mathbb{R}^n, dm), L^2(\mathbb{R}^n, dm'))$.

For the final statement, assume that $dm' = dm$. Then we may take $\psi_i = \varphi_i$ for all $i$, so that for a fixed $K \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ we have

$$K(x,y) = \sum_{i,j} K_{j,i}\, \varphi_j(x) \overline{\varphi_i(y)},$$

with convergence in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Since $K_{\cdot\cdot} \in \mathcal{S}(\mathbb{N}^2)$ and the functions $\varphi_i$ are uniformly locally bounded, the equality holds with uniform convergence over compact sets. This implies that

$$K(x,x) = \sum_{i,j} K_{j,i} \varphi_j(x) \overline{\varphi_i(x)}, \qquad (x \in \mathbb{R}^n).$$

By the Cauchy-Schwartz inequality the functions $x \mapsto \varphi_j(x)\overline{\varphi_i(x)}$ all have $L^1(\mathbb{R}^n, dm)$-norm bounded by 1. It follows that the above equality holds with convergence in $L^1(\mathbb{R}^n, dm)$. This implies that integration of the sum may be done termwise. Taking the orthonomality relations into account, we thus find

$$\int_{\mathbb{R}^n} K(x,x)\, dm(x) = \sum_{i,j} K_{j,i} \delta_{ij} = \mathrm{tr}\,(T_K).$$

□

The above lemma has the following interesting corollary. We first recall the idea of approximation by convolution. Let $\varphi \in C^\infty(\mathbb{R}^n)$. Given $f \in L^2(\mathbb{R}^n)$ we note that $\varphi * f \in L^2(\mathbb{R}^n)$. It is readily seen that $C(\varphi) : f \mapsto \varphi * f$ is a bounded operator on $L^2(\mathbb{R}^n)$.

By an approximation of the identity on $R^n$ we shall mean a sequence of functions $\varphi_k \in C_c^\infty(\mathbb{R}^n)$ with $\varphi_k \geq 0$, $\int_{\mathbb{R}^n} \varphi_k dx = 1$ for all $k$ and such that $\operatorname{supp} \varphi_k \to \{e\}$ for $k \to \infty$. It is well known that $C(\varphi_k) \to I$ in the strong operator topology, i.e., pointwise.

**Approximation principle.** *Let $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be an operator of trace class. Let $(\varphi_k)$ and $(\psi_k)$ be two approximations of the identity on $\mathbb{R}^n$. Then*

$$C(\varphi_k) T C(\psi_k) \to T \quad \text{in } L_1(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n)), \qquad (k \to \infty).$$

The proof of this principle will be given in an appendix.

**Corollary 41** *Let $dm$ be a smooth density on $\mathbb{R}^n$ and let $K \in C_c(\mathbb{R}^n \times \mathbb{R}^n)$. If the integral operator $T : L^2(\mathbb{R}^n, dm) \to L^2(\mathbb{R}^n, dm)$ defined by*

$$T_K f(x) := \int_{\mathbb{R}^n} K(x,y) f(y)\, dm(y)$$

*is of trace class then its trace is given by*

$$\operatorname{tr}(T_K) = \int_{\mathbb{R}^n} K(x,x)\, dm(x).$$

**Proof** We consider an approximation of the identity $\varphi_j$ on $\mathbb{R}^n$ consisting of smooth functions. Then it is clear that $(\psi_j := \varphi_j \otimes \varphi_j)$ is an approximation of the identity on $\mathbb{R}^{2n}$. It follows that the functions $K_j := \psi_j * K$ are smooth, and $K_j \to K$ in $C_c(\mathbb{R}^{2n})$. Then by the previous lemma it follows that

$$\operatorname{tr}(T_j) = \int_{\mathbb{R}^n} K_j(x,x)\, dm(x).$$

The right-hand side of this equality has limit $\int K(x,x)\, dm(x)$ for $j \to \infty$. Thus, it suffices to prove the claim that $T_{K_j} \to T_K$ in $L_1(L^2(\mathbb{R}^n, dm), L^2(\mathbb{R}^n, dm))$.

It is readily checked that $T_{K_j} = C(\varphi_j) \circ T_K \circ C_j(\varphi_j^\vee)$, where $(\varphi_j^\vee)$ is the approximation of the identity given by $\varphi_j^\vee(x) = \varphi_j(-x)$. The claim now follows by application of the above approximation principle. □

We now turn to smoothing operators on manifolds. Let $M$ be a manifold of dimension $n$ equipped with everywhere positive densites $dm$ and $E \downarrow M$ as smooth vector bundle. We equip $\Gamma_c(M, E)$ with the Hermitian inner product given by the formula

$$\langle f, g \rangle = \int_M \langle f(x), g(x) \rangle\, dm(x), \qquad (f, g \in \Gamma_c(M, E)).$$

Then the completion $L^2(M,E)$ of this space is a separable Hilbert space, which we may view as a subspace of $L^2_{\rm loc}(M,E)$.

Let now $M'$ be a second manifold, of dimension $n'$, equipped with an everywhere positive smooth density $dm'$. Let $E' \downarrow M$ be complex vector bundle, equipped with a Hermitian structure.

Let $\mathrm{pr}_1, \mathrm{pr}_2$ denote the projection maps from $M' \times M$ onto $M'$ and $M$ respectively. We briefly write $\mathrm{Hom}(E,E')$ for the vectorbundle $\mathrm{Hom}(\mathrm{pr}_2^*E, \mathrm{pr}_1^*E')$ over $M' \otimes M$, and equip it with the naturally induced Hermitian structure. Thus, for $(x,y) \in M' \times M$ and $A \in \mathrm{Hom}(E,E')_{(x,y)} = \mathrm{Hom}(E_y, E'_x)$ we have

$$\|A\|^2_{x,y} = \mathrm{tr}\,(A^*A).$$

Let $K$ be an $L^2$-section of $\mathrm{Hom}(E',E)$. Then we define the kernel operator $T_K : L^2(M,E) \to L^2(M',E')$ by

$$\langle T_K f, g\rangle = \int_M \langle K(x,y)f(y), g(x)\rangle \, dm(y), \tag{13.6}$$

for $f \in L^2(M,E)$ and $g \in L^2(M',E')$.

**Theorem 42** *With notation as above, let $K \in L^2(M' \times M, \mathrm{Hom}(E,E'))$. Then the operator $T_K : L^2(M,E) \to L^2(M',E')$ is Hilbert-Schmidt, and*

$$\|T_K\|^2_{\rm HS} = \int_{M\times M} \|K(x,y)\|^2 \, dm(x)\, dm(y). \tag{13.7}$$

**Proof** We select a locally finite collection $(U_\beta)$ of disjoint open sets of $M$, so that the union has a complement of measure zero, such that each $U_\alpha$ has compact closure and is contained in an open coordinate chart diffeomorphic to $\mathbb{R}^n$ on which $E$ allows a trivialisation.

Likewise, we select a locally finite collection $(U'_\alpha)$ of disjoint open subsets of $M'$ with similar properties relative to the bundle $E'$.

The characteristic functions $\chi_\beta = 1_{U_\beta}$ are mutually perpendicular and add up to 1 in $L^2_{\rm loc}(M)$. Similar remarks are valid for $\chi'_\alpha = 1_{U'_\alpha}$. Put $K_{\alpha,\beta}(x,y) = \chi'_\alpha(x)\chi_\beta(y)K(x,y)$. Then $K$ is the $L^2(M' \times M, E' \boxtimes E)$-orthogonal sum of the functions $K_{\alpha,\beta} \in L^2(U'_\alpha \times U_\beta)$ and it suffices to prove the result for each $K_{\alpha,\beta}$.

In other words, we have reduced to the situation that $M' = \mathbb{R}^{n'}, M = \mathbb{R}^n$, $E = M \times \mathbb{C}^k$ and $E' = M \times \mathbb{C}^{k'}$, and $K$ is a compactly supported $L^2_{\rm loc}$-function on $\mathbb{R}^n \times \mathbb{R}^n$ with values in $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^{k'})$.

By using Gramm-Schmidt orthogonalisation, we may change the trivialisations of $E$ and $E'$ such that the Hermitian structure becomes the standard Hermitian inner products on $\mathbb{C}^k$ and $\mathbb{C}^{k'}$.

Let $e_1, \ldots, e_k$ be the standard basis of $\mathbb{C}^k$, and $e_1, \ldots, e_{k'}$ the similar basis of $\mathbb{C}^{k'}$. Let $(E_{s,t} \mid 1 \le t \le k, 1 \le s \le k')$ be the standard basis for $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^{k'})$. We write

$$K = \sum_{s,t} K^{s,t} E_{s,t}$$

with scalar compactly supported functions $K^{s,t} \in L^2_{\text{loc}}(\mathbb{R}^n)$. Then

$$\|K(x,y)\|^2 = \sum_{s,t} |K^{s,t}(x,y)|^2.$$

Let $(\varphi_i)$ be an orthonormal basis for $L^2(\mathbb{R}^{n'}, dm')$ and $\psi_j$ a similar basis for $L^2(\mathbb{R}^n, dm)$. Then $\varphi_i \otimes e_s$ and $\psi_j \otimes e_t$ form orthonormal bases for $L^2(M, dm') \otimes \mathbb{C}^{k'}$ and $L^2(M, dm) \otimes \mathbb{C}^k$, respectively. We thus see that

$$\begin{aligned}
\|T_K\|^2_{\text{HS}} &= \sum_{i,j,s,t} \left|\langle T(\psi_j \otimes e_t), \varphi_i \otimes e_s \rangle_{L^2(\mathbb{R}^n, dm')}\right|^2 \\
&= \sum_{i,j,s,t} \left| \int_{\mathbb{R}^{n'}} \int_{\mathbb{R}^n} K^{s,t}(x,y) \varphi_i(x) \overline{\psi_j(y)} dm(y) dm'(x) \right|^2 \\
&= \sum_{i,j,s,t} |\langle K^{s,t}, \varphi_i \otimes \overline{\psi_j} \rangle_{L^2(\mathbb{R}^{n'+n}, dm' dm)}|^2 \\
&= \sum_{s,t} \|K^{s,t}\|^2_{L^2(\mathbb{R}^{n'+n})},
\end{aligned}$$

where the equality follows from the fact that the functions $\varphi_i \otimes \overline{\psi_j}$ form an orthonormal basis for $L^2(\mathbb{R}^{n'+n}, dm'dm)$. It follows that $T_K$ is Hilbert–Schmidt, and that (13.7). □

**Theorem 43** *With notation as in Theorem 42 let $K \in \Gamma_c(M' \times M, \text{Hom}(E, E'))$.*

(a) *If $K$ is smooth, then $T_K$ is of trace class.*

(b) *Let $M = M', E = E'$ and $dm = dm'$. Let $K \in \Gamma_c(M \times E, \text{End}(E))$ and assume that $T_K : L^2(M, E) \to L^2(M, E)$ is of trace class, then*

$$\text{tr}\,(T_K) = \int_M \text{tr}\,(K(x,x))\, dm(x). \tag{13.8}$$

**Proof** Let $(U_\beta)$ be a finite open cover of $\text{pr}_2(\text{supp}\,(K)$ and let $(\chi_\beta)$ be a partition of unity subordinate to it. Likewise, let $(U'_\alpha)$ be a finite open cover of $\text{pr}_1(\text{supp}\, K)$ and let $(\chi'_\alpha)$ be a partition of unity subordinate to it. We may assume that each $U'_\alpha$ is diffeomorphic to $\mathbb{R}^{n'}$ and that $E'$ has a trivialisation over it. Likewise, we may assume that $U_\beta$ is diffeomorphic to $\mathbb{R}^n$ and that $E$ has a trivialisation over it. Define $K_{\alpha,\beta}(x,y) = \chi'_\alpha(x)\chi_\beta(y)K(x,y)$. Then $T_K$ is the finite sum of the operators $T_{\alpha,\beta} := T_{K_{\alpha,\beta}}$ and for (a) it suffices to show that each $T_{\alpha,\beta}$ is of trace class. Now this follows by application of Lemma 40.

We now turn to (b) and assume that $M = M'$, $E = E'$. Then we may assume that the covers $(U_\alpha)$ and $(U_\beta)$ are equal finite covers of $\text{pr}_1(\text{supp}\, K) \cup \text{pr}_2(\text{supp}\, K)$ and have the additional property that $U_\alpha \cap U_\beta \neq \emptyset$ implies that $U_\alpha \cup U_\beta$ is contained in an open subset $\Omega_{\alpha,\beta}$ of $M$ which is diffeomorphic to $\mathbb{R}^n$, and such that $E$ allows a trivialisation over $\Omega_{\alpha,\beta}$.

Define $K_{\alpha,\beta}(x,y) = \chi_\alpha(x)\chi_\beta(y)K(x,y)$. Since $T_K$ is of trace class, and $T_{K_{\alpha,\beta}} = M_{\chi_\alpha} \circ T_K \circ M_{\chi_\beta}$, where $M_{\chi_\alpha} : f \mapsto \chi_\alpha f$ are bounded operators on $L^2(M, E, dm)$, it follows from Corollary 31 that each of the operators $T_{\alpha,\beta}$ is of trace class. Thus, by linearity it suffices the result for each $K_{\alpha,\beta}$.

We will first deal with the case that $U_\alpha \cap U_\beta = \emptyset$. Then $K_{\alpha,\beta}$ is zero on the diagonal of $M$ so that the integral on the right-hand side of (13.8) vanishes.

On the other hand, we may use an orthonormal basis of $L^2(U_\alpha, E)$, one of $L^2(U_\beta, E)$ and a basis of $L^2(M \setminus U_\alpha \cap U_\beta)$. Together these form an orthonormal basis of $L^2(M, E, dm)$. Moreover, it is clear that for each $\varphi$ in this basis, we have that $(\varphi \otimes \langle \cdot , \varphi\rangle) \perp K_{\alpha,\beta}$. This implies that $\langle T_{\alpha,\beta}(\varphi), \varphi\rangle = 0$. From this we see that $\mathrm{tr}\,(T_{\alpha,\beta}) = 0$ in this case.

Thus, we have reduced to the situation that $\alpha = \beta$. This is our original setting, with the additional assumption that $M = \mathbb{R}^n$ that $E$ allows a trivialisation over $M$. By applying Gramm-Schmidt orthonormalisation to a choice of global frame, we see that $E$ allows a smooth trivialisation on which the Hermitian structure attains the standard form. Thus, we may assume that $E = M \times \mathbb{C}^k$, equipped with the standard Hermitian form of $\mathbb{C}^k$. Let $e_1, \ldots, e_k$ be the standard basis for $\mathbb{C}^k$, and for $1 \leq s \leq k$ define $i_s : \mathbb{C} \to \mathbb{C}^k, z \mapsto ze_s$ and $p_s : \mathbb{C}^k \to \mathbb{C}$, $w \mapsto w_s$.For $1 \leq s, t \leq k$ we define the compactly supported continuous function $K^{s,t} : M \times M \to \mathbb{C}$ by

$$K_{s,t}(x,y) = p_s \circ K(x,y) \circ i_s.$$

Associated to this function we define the kernel operator $T_{s,t} : L^2(M) \to L^2(M)$ by

$$T_{s,t}(f)(x) = \int_M K_{s,t}(x,y) f(y)\; dm(y).$$

Then it is readily seen that

$$T_{s,t} f = p_s \circ T(i_t \circ f).$$

Now $f \mapsto i_s \circ f$ is a bounded operator $L^2(M) \to L^2(M, \mathbb{C}^k)$. Likewise, $g \mapsto p_s \circ g$ is a bounded operator $L^2(M, \mathbb{C}^k) \to L^2(M)$. It follows by application of Corollary 31 that each of the operators $T_{s,t}$ is of trace class. Hence, by .... we find that

$$\mathrm{tr}\,(T_{s,t}) = \int_M K_{s,t}(x,x)\; dx.$$

Let now $(\varphi_j)$ be an orthonormal basis for $L^2(M, dm)$, then $(\varphi_i \otimes e_s \mid i \in \mathbb{N}, 1 \leq$

$s \leq k$) is an orthonormal basis for $L^2(M) \otimes \mathbb{C}^k$. It follows that

$$\begin{aligned}
\mathrm{tr}\,(T_K) &= \sum_{i,s} \langle T(\varphi_i \otimes e_s), \varphi_i \otimes e_s \rangle \\
&= \sum_s \sum_i \langle T_{s,s}\varphi_i, \varphi_i \rangle = \sum_s \mathrm{tr}\,(T_{s,s}) \\
&= \sum_s \int_M K_{s,s}(x,x)\, dm(x) \\
&= \int_M \mathrm{tr}\,(K(x,x))\, dm(x).
\end{aligned}$$

□

# 14 Pseudo-differential operators of trace class

Let $V$ and $W$ be infinite topological linear spaces, whose topologies are separable Hilbert. This means that there exist topological linear isomorphisms $S : H \to V$ and $T : H \to W$, where $H = l^2(\mathbb{N})$ with the standard Hilbert structure. A continuous linear map $A : V \to W$ gives rise to a continuous linear map $A_H : H \to H$ such that the following diagram commutes:

$$\begin{array}{ccc}
V & \xrightarrow{A} & W \\
S \uparrow & & \uparrow T \\
H & \xrightarrow{A_H} & H.
\end{array}$$

The operator $A$ is said to be Hilbert–Schmidt or of trace class, if $A_H$ is Hilbert–Schmidt or of trace class. This definition is independent of the choice of $S, T$ as it should. For assume that $S'$ and $T'$ are similar topological linear isomorphisms $H \to V$ and $H \to W$ and $A'_H : H \to H$ the similarly associated map, then $A'_H = (T')^{-1}TA_HS^{-1}S'$ with $(T')^{-1}T$ and $S^{-1}S'$ bounded endomorphisms of $H$.

We are now ready for the following result, for $M$ a compact manifold of dimension $n$, and $E$ and $F$ vector bundles of rank $k$ and $l$ on $M$.

The spaces $L^2(M,E)$ and $L^2(M,F)$ are well defined, with a Hilbert topology.

Any pseudo-differential operator $P \in \Psi^r(E \otimes \mathcal{D}_M, F)$ with $r \leq 0$ defines a continuous linear operator $P_0 : L^2(M,E) \to L^2(M,E)$. may be viewed as a continuous

**Theorem 44** *Let $P \in \Psi^r(E,F)$, $r \leq 0$, and let $P_0 : L^2(M,E) \to L^2(M,F)$ be the associated continous linear operator.*

(a) *If $r < -n/2$, then $P_0$ is Hilbert–Schmidt.*

(b) *If $r < -n$, then $P_0$ is of trace class.*

**Proof** We first prove (a). For this it suffices to show that the kernel of $P_0$ is in $L^2(M \times M, E^\vee \boxtimes F)$. As the kernel of $P_0$ is smooth outside the diagonal, it suffices to show that $(M_\varphi \circ P \circ M_\varphi)_0$ is Hilbert-Schmidt for any $\varphi \in C_c^\infty(M)$ with support in an open coordinate patch over which $E$ and $F$ trivialize. This reduces the result to the lemma below.

We now turn to (b). Fix $s < 0$ such that $r < 2s < -n$. Then there exists an elliptic operator $Q \in \Psi^s(E, E)$. The operator $Q$ has a parametrix $R \in \Psi^{-s}(E, E)$. Now $PQ \in \Psi^{r-s}(E, F)$ with $r - s < s < -n/2$. Hence $(PQ)_0$ is Hilbert-Schmidt. Likewise, $R_0$ is Hilbert-Schmidt, and we conclude that $(PQ)_0 R_0$ is of trace class. Now $QR = I + T$, with $T$ a smoothing operator, hence $PQR = P + PT$. It follows that

$$P_0 + (PT)_0 = (PQR)_0 = (PQ)_0 R_0$$

is of trace class. Since $PT$ is smoothing, $(PT)_0$ is of trace class, and we conclude that $P_0$ is of trace class. $\square$

**Lemma 45** *Let $P \in \Psi^r(\mathbb{R}^n)$ with $r < -n/2$. Then for all $\varphi, \psi \in C_c^\infty(\mathbb{R}^n)$ the operator $M_\varphi \circ P \circ M_\psi$ has kernel contained in $L^2_{\text{comp}}(\mathbb{R}^n \times \mathbb{R}^n)$.*

**Proof** Without loss of generality we may assume that $P = \Psi_p$, with $p \in S^r(\mathbb{R}^n)$. The kernel $K_P$ of $P$ is then given by

$$K_P(x, y) = \mathcal{F}_2 p(x, y - x)$$

(to be interpreted in distribution sense). Since $r < -m/2$, it follows that $x \mapsto p(x, \cdot\,)$ is a continuous function, with values in $L^2(\mathbb{R}^n)$. It follows that $\mathcal{F}_2 p \in L^2_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)$. By substitution of variables, it follows that $K_P \in L^2_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}^n)$.

The kernel $K$ of $M_\varphi \circ P \circ M_\psi$ is given by

$$K(x, y) = \varphi(x) K_P(x, y) \psi(y),$$

hence belongs to $L^2_{\text{comp}}(\mathbb{R}^n \times \mathbb{R}^n)$. $\square$

# 15 Appendix: approximation by convolution

In order to avoid repetitions, we first work in the general setting of a Hilbert space $H$. Let $U(H)$ denote the group of unitary automorphisms of $H$. We consider $\mathbb{R}^n$ as a group for the addition, and assume that a group homomorphism $\pi : \mathbb{R}^n \to U(H), x \mapsto \pi(x)$ is given such that $\tau(0) = I$.

This map $\pi$ is said to be strongly continuous at 0 if $\lim_{x \to 0} \tau_x v = v$, for all $v \in H$. This readily seen to be equivalent to the condition that $\pi : \mathbb{R}^n \to U(H)$ is continuous for the strong operator topology. Such a group homorphism is called a unitary representation of $\mathbb{R}^n$ in $H$.

**Lemma 46** *Assume that there exists a dense subset $D \subset H$ such that*

$$\lim_{x \to 0} \pi(x)v = v$$

*for all $v \in D$. Then $\pi$ is strongly continuous at* 0.

**Proof** Let $v \in H$. Let $\epsilon > 0$. There exists an element $v_0 \in D$ such that

$$\|v - v_0\| < \epsilon/3.$$

There exists a $\delta > 0$ such that

$$x \in B(0;\delta) \Rightarrow \|\pi(x)v_0 - v_0\| < \epsilon/3.$$

Then for $x \in B(0;\delta)$ we have

$$\begin{aligned} \|\pi(x)v - v\| &\leq \|\pi(x)(v - v_0)\| + \|\pi(x)v_0 - v_0\| + \|v_0 - v\| \\ &= \|\pi(x)v_0 - v_0\| + 2\|v_0 - v\| < \epsilon. \end{aligned}$$

□

The group homomorphism $\pi : \mathbb{R}^n \to U(H)$ gives rise to the group homomorphism $L_\pi : \mathbb{R}^n \to U(L_2(H,H))$ given by

$$L_\pi(x)(A) = \pi(x)A, \qquad (x \in \mathbb{R}^n, A \in L_2(H,H).$$

**Lemma 47** *If $\pi$ is strongly continuous at* 0 *then so is $L_\pi$.*

**Proof** Let $(e_j)$ be an orthonomormal basis for $H$. Given $i, j$ we define $e_i \otimes e_j^* : H \to H$ by $v \mapsto \langle v, e_j \rangle e_i$. Then the span $F$ of the operators $e_i \otimes e_j^*$ is a dense subspace of $L_2(H,H)$. Let $A \in F$. Then it suffices to show that $\pi(x)A \to A$ in $L_2(H,H)$, for $x \to 0$. We may write

$$A = \sum_{i,j} A_{ij} e_i \otimes e_j^*,$$

with finite sum. Then

$$\|\pi(x)A - A\|_{\mathrm{HS}}^2 = \sum_i \|\pi(x)Ae_j - Ae_j\|^2$$

with finite sum. Since $\pi$ is strongly continuous at 0, this sum tends to zero for $x \to 0$. □

Assume that $\pi$ is a unitary representation of $\mathbb{R}^n$ in a Hilbert space $H$. Given $\varphi \in C_c(\mathbb{R}^n)$, the map $x \mapsto \varphi(x)\pi(x), \mathbb{R}^n \to L(H,H)$ is compactly supported and continuous for the strong operator topology. We define $\pi(\varphi) : H \to H$ by the Riemann-integral

$$\pi(\varphi)v = \int_{\mathbb{R}^n} \varphi(x)\pi(x)v \, dx.$$

Clearly, $\pi(\varphi)$ is bounded with operator norm dominated by the $L^1$-norm of $\varphi$.

**Lemma 48** *Let $(\varphi_k)$ be an approximation of the identity on $\mathbb{R}^n$. Then for every $v \in H$ we have*

$$\pi(\varphi_k)v \to v, \quad (k \to \infty).$$

**Proof** Fix $v \in H$. We note that

$$\pi(\varphi_k)v - v = \int_{\mathbb{R}^n} \varphi_j(x)(\pi(x)v - v) \, dx.$$

Let $\epsilon > 0$. There exists $\delta > 0$ such that $\|x\| < \delta \Rightarrow \|\pi(x)v - v\| < \epsilon/2$. Fix $K$ such that $k > K \Rightarrow \operatorname{supp} \varphi_k \subset B(0;\delta)$. Then for $k > K$ we have

$$\|\pi(\varphi_k)v - v\| \leq \int_{B(0;\delta)} \varphi_k(x)\|\pi(x)v - v)\| \, dx \leq \frac{1}{2}\epsilon \int_{B(0;\delta)} \varphi_k(x) \, dx < \epsilon.$$

□

**Corollary 49** *Let $\pi$ be a unitary representation of $\mathbb{R}^n$ in $H$. Let $(\varphi_k)$ be an approximation of the identity on $\mathbb{R}^n$. Then for every $A \in L_2(H,H)$,*

$$\pi(\varphi_k) \circ A \to A \quad \text{in} \quad L_2(H,H).$$

**Proof** The representation $L_\pi$ of $\mathbb{R}^n$ in $L_2(H,H)$ is unitary by Lemma 47. Hence $L_\pi(\varphi_k)A \to A$ in $L_2(H,H)$ by the previous lemma. Now use that $L_\pi(\varphi_k)A = \pi(\varphi_k) \circ A$. □

**Corollary 50** *Let $\pi$ be a unitary representation of $\mathbb{R}^n$ in $H$. Let $(\varphi_k)$ and $(\psi_k)$ be approximations of the identity on $\mathbb{R}^n$. Then for all $A \in L_1(H,H)$ we have*

$$\pi(\varphi_k) \circ A \circ \pi(\psi_k) \to A \quad \text{in} \quad L_1(H,H), \tag{15.9}$$

*for $k \to \infty$.*

**Proof** There exist $B, C \in L_2(H,H)$ such that $A = BC^*$. Define $\psi_k^\vee : x \mapsto \psi_k(-x)$. Then it is readily seen that $(\psi_k^\vee)$ is an approximation of the identity on $\mathbb{R}^n$. Moreover, $\pi(\psi_k)^* = \pi(\psi_k^\vee)$. It follows from Corollary 49 that $\pi(\varphi_k)B \to B$ in $L_2(H,H)$, and that

$$C\pi(\psi_k) = (\pi(\psi_k^\vee)C^*)^* \to C^{**} = C$$

in $L_2(H,H)$. The assertion (15.9) now follows by application of Theorem 36. □

Given $x \in \mathbb{R}^n$ we define $T_x : \mathbb{R}^n \to \mathbb{R}^n$, $y \mapsto y+x$. Then $T_{-x}$ induces a unitary map $\tau(x) = \tau_x = T_{-x}^* : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $f \mapsto f \circ T_{-x}$. Clearly $\tau$ defines a group homomorphism $\mathbb{R}^n \to U(L^2(\mathbb{R}^n))$.

**Lemma 51** *$\tau$ is a unitary representation of $\mathbb{R}^n$ in $L^2(\mathbb{R}^n)$.*

**Proof** We fix $f$ in the dense subspace $C_c(\mathbb{R}^n)$ of $L^2(\mathbb{R}^n)$. Then by Lemma 46 it suffices to show that $\tau_x(f) \to f$ in $L^2(\mathbb{R}^n)$ for $x \to 0$. For this we first observe that $f$ is uniformly continuous.

Let $\epsilon > 0$. There exists a $\delta > 0$ such that $|f(y-x) - f(y)| < \epsilon$ for all $x \in B(0;\delta)$ and $y \in \mathbb{R}^n$, hence

$$\sup |\tau_x f - f| < \epsilon, \qquad (\|x\| < \delta).$$

It follows that $\tau_x f \to f$ uniformly for $x \to 0$. Since $\operatorname{supp} \tau_x f = x + \operatorname{supp} f$, it follows that $\tau_x f$ is supported in the compact set $\operatorname{supp} f + \bar{B}(0;1)$, for $\|x\| \leq 1$. Hence, $\tau_x f \to f$ in $L^2(\mathbb{R}^n)$ for $x \to 0$. □

It is readily checked that for $\varphi \in C_c(\mathbb{R}^n)$ and $f \in L^2(\mathbb{R}^n)$ we have

$$C(\varphi)f = \varphi * f = \tau(\varphi)f.$$

We now obtain the desired approximation result in $L_1(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$.

**Lemma 52** *Let $(\varphi_k)$ and $(\psi_k)$ be approximations of the identity on $\mathbb{R}^n$ and let $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be an operator of trace class. Then*

$$C(\varphi_k) T C(\psi_k) \to T \quad \text{in} \quad L_1(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n)),$$

*for $k \to \infty$.*

**Proof** This follows from Corollary 50 applied to $H = L^2(R^n)$, $\pi = \tau$ and $A = T$. □