**Inleiding Topologie - Retake (March 21, 2016)**

**Note:** All the questions are worth 0.5 points (adding up to a total of 10.5 points). Please motivate all your answers. For instance, in Exercise 1, point e) do not just answer with yes or no, but provide a proof. Also, in d) of the same exercise, please do not just write down the final result, but also explain how you found it.

You can write either in English or Dutch.

**Exercise 1.** (*3 points*) Consider $\mathbb{R}^2$ endowed with the product topology $\mathcal{T} = \mathcal{T}_l \times \mathcal{T}_{\text{Eucl}}$, where $\mathcal{T}_l$ is the lower limit topology on $\mathbb{R}$ (i.e., the topology generated by the right-open intervals $[a, b)$) and $\mathcal{T}_{\text{Eucl}}$ is the Euclidean topology on $\mathbb{R}$. Consider also

$$A = [0, 1] \times [0, 1].$$

a. Show that any subset $U \subset \mathbb{R}^2$ which is open with respect to the Euclidean topology is open also with respect to $\mathcal{T}$.

b. Give an example of a sequence in $\mathbb{R}^2$ which is convergent with respect to the Euclidean topology but is not convergent in $(\mathbb{R}^2, \mathcal{T})$.

c. Show that the interior of $A$ in $(\mathbb{R}^2, \mathcal{T})$ is $B := [0, 1) \times (0, 1)$.

d. Compute the closure of $B$ in $(\mathbb{R}^2, \mathcal{T})$.

e. Is $A$, with the topology induced from $(\mathbb{R}^2, \mathcal{T})$, connected?

f. Is $A$, with the topology induced from $(\mathbb{R}^2, \mathcal{T})$, compact?

**Exercise 2.** (*2 points*) Consider

$$X = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, 0 \leq z < 1\}$$

$$Y = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, 0 < z < 1\}$$

both endowed with the Euclidean topology. Compute their one point compactifications and then show that $X$ and $Y$ are not homeomorphic.

**Exercise 3.** (*3.5 points*) Let $A$ be the set of polynomials in one variable, with real coefficients, which take the same values at 0 and 1:

$$A = \{P \in \mathbb{R}[t] : P(0) = P(1)\} = \{a + t(t-1)Q : a \in \mathbb{R}, Q \in \mathbb{R}[t]\}.$$

a. Show that, with the usual operations between polynomials, $A$ is an algebra.

We want to show that any character on $A$ is of type

$$\chi_r : A \to \mathbb{R}, \quad \chi_r(P) = P(r)$$

for some real number $r$. Start with an arbitrary character $\chi : A \to \mathbb{R}$ and define

$$u_p := \chi(t^p(t-1)),$$

for each $p \geq 1$ integer. Please be aware that one can talk about $\chi(P)$ only for $P \in A$ (since we assume that $\chi$ is defined only on $A$); i.e. it does not make sense to talk about $\chi(t)$ or $\chi(t^2)$!). Show that

b. $u_p \cdot u_q = u_{p'} \cdot u_{q'}$ whenever $p + q = p' + q'$.

c. $(u_1)^3 = (u_2)^2 - u_1 \cdot u_2$.

d. If $u_1 = 0$ then $u_p = 0$ for any $p$ and $\chi = \chi_0$.

e. If $u_1 \neq 0$ then $\chi = \chi_r$ where $r = \frac{u_2}{u_1}$.

Denoting by $X(A)$ the spectrum of $A$,

f. show that the map $\mathbb{R} \to X(A)$, $r \mapsto \chi_r$ is a continuous surjection.

g. Finally, describe a subspace of $\mathbb{R}^2$ that is homeomorphic to $X(A)$.

**Exercise 4.** (*2 points*) Consider the group

$$\Gamma := \mathbb{Z} \times \mathbb{Z},$$

with the group operation denoted by $\oplus$ and defined by

$$(m, n) \oplus (m', n') = (m + m', n + n').$$

Find all the real numbers $\lambda \in \mathbb{R}$ with the property that

$$\phi_{m,n}(x, y) = (m, n) \cdot (x, y) := (x + n + \lambda m y + \frac{\lambda m(m - \lambda)}{2}, y + \lambda^2 m)$$

defines an action of $\Gamma$ on $\mathbb{R}^2$. Then, in each case that you find, show that the quotient space $\mathbb{R}^2/\Gamma$ is homeomorphic to either a torus or a cylinder.