Take home exercise 3

Let S^1 be the unit circle in the complex plane. For $w = e^{i\varphi} \in S^1$ we define the following rotations about the z-axis and about the y-axis in $\mathbb{R}^3$,

$$R_w := \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad r_w := \begin{pmatrix} \sin\varphi & 0 & \cos\varphi \\ 0 & 1 & 0 \\ \cos\varphi & 0 & -\sin\varphi \end{pmatrix}.$$

Let

$$X = S^1 \times [-1,1],$$

equipped with the restriction of the Euclidean topology on $\mathbb{C} \times \mathbb{R}$. We define the map $f : S^1 \times [-1,1] \to \mathbb{R}^3$ by

$$f(w,t) = R_{w^2}(2e_1 + r_w(te_3)),$$

where e_j denotes the j-th standard basis vector in $\mathbb{R}^3$.

(a) Argue that the image M of f is a geometric realization of the Möbius band in $\mathbb{R}^3$. See also Exercise 1.12.

(b) Determine the equivalence relation R on X which turns $f : X \to M$ into a quotient modulo R. Determine an action of the group $\mathbb{Z}_2 = \{1, -1\}$ on X whose orbits are precisely the equivalence classes of R.

(c) Show that there exists a continuous bijection $F : X/\mathbb{Z}_2 \to M$. (Later we will see that by compactness of X this implies that F is a homeomorphism).

(d) We consider the continuous map

$$h : [0,1] \times [-1,1] \to X = S^1 \times [-1,1], \quad h(s,t) = (e^{i\pi s}, t).$$

Let $p : X \to X/\mathbb{Z}_2$ be the natural projection. Show that $p \circ h$ is a continuous surjection from $[0,1] \times [-1,1]$ onto $X/\mathbb{Z}^2$.

(e) Describe the gluing relation G on $[0,1] \times [-1,1]$ for which $p \circ h$ is a quotient modulo G. Show that $([0,1] \times [-1,1])/G$ is homeomorphic to $X/\mathbb{Z}_2$.