Extra Exercise 6

By a 'semi-metric' on a set X we mean a map $d : X \times X \to [0,\infty)$ such that for all $x,y,z \in X$,

(1) $d(x,x) = 0$;

(2) $d(x,y) = d(y,x)$;

(3) $d(x,z) \leq d(x,y) + d(y,z)$.

(Thus, we do not assume that $d(x,y) = 0 \Rightarrow x = y$.) For such a semi-metric we define the open balls:

$$B_d(a;r) = \{x \in X \mid d(x,a) < r\}.$$

(a) Show that the balls $B_d(a;r)$, for $a \in X$ and $r > 0$ form the basis for a topology $\mathscr{T}_d$.

(b) Let d be a semi-metric on X. Show that $\hat{d} : (x,y) \mapsto \min\{1, d(x,y)\}$ is also a semi-metric. Show that the associated topology $\mathscr{T}_{\hat{d}}$ is equal to $\mathscr{T}_d$. Show that d is a metric if and only if $\hat{d}$ is a metric.

(c) Suppose that $\{d_k \mid k \in \mathbb{N}\}$ is a collection of semi-metrics on X with the property that for all $x,y \in X$ we have:

$$[\forall k : d_k(x,y) = 0] \;\Rightarrow\; x = y.$$

Let $\mathscr{T}$ be the smallest topology on X containing $\cup_{k\in\mathbb{N}}\mathscr{T}_{d_k}$. Show that $\mathscr{T}$ is metrisable.

Extra Exercise 7

For a compact space K we equip the space $C(K)$ with the uniform metric $d_K : (f,g) \mapsto \sup\{|f(x) - g(x)| \mid x \in K\}$.

We consider a locally compact Hausdorff space X which is second countable. For each compact subset $K \subset M$ we consider the restriction map $\rho_K : C(M) \to C(K)$, $f \mapsto f|_K$. We equip $C(M)$ with the smallest topology $\mathscr{T}$ for which all restriction maps ρ_K ($K \subset M$ compact), become continuous.

Show that $\mathscr{T}$ is metrizable. Hint: use the previous exercise.

Extra Exercise 8

Let M be a compact topological manifold of dimension n.

(a) Show that there exists a finite open covering $\mathscr{U} = \{U_1, \ldots, U_k\}$ of M such that for every j there exists a homeomorphism $\varphi_j : U_j \to \mathbb{R}^n$.

(b) Show that there exist finitely many continuous functions $\xi_1, \ldots, \xi_k \in C(M)$ such that $\mathrm{supp}(\xi_j) \subset U_j$ for every j and such that for every $m \in M$ there exists a j such that $\xi_j(m) \neq 0$.

(c) For $1 \leq j \leq k$ we define the map $f_j : M \to \mathbb{R}^{n+1}$ by

$$f_j(m) = \xi_j(m)(\varphi_j(m), 1) \quad \text{for } m \in U_j; \qquad f_j = 0 \ \text{ on } \ X \setminus U_j.$$

Show that f_j is continuous.

(d) Show that the map $f = (f_1, \ldots, f_k) : M \to \mathbb{R}^{k(n+1)}$ is an embedding.