Modified Retake Inl Top, WISB243 2017-04-18, 13:30 – 16:30

- Write your **name** on every sheet, and on the first sheet your **student number** and the total **number of sheets** handed in.
- You may use the lecture notes, the extra notes and personal notes, but no worked exercises.
- Do not just give answers, but also justify them with complete arguments. If you use results from the lecture notes, always **refer to them by number**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, do **continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 52. The final grade will be obtained from your total score through division by 5.
- You are free to write the solutions either in English, or in Dutch.

Succes !

14 pt total **Exercise 1.** Let $\mathscr{T}$ be the collection of sets $U \subset \mathbb{R}$ such that $U = \emptyset$ or $\mathbb{R} \setminus U$ is finite.

2 pt (a) Show that $\mathscr{T}$ is a topology on $\mathbb{R}$.

2 pt (b) Is $(\mathbb{R}, \mathscr{T})$ Hausdorff?

1 pt (c) Determine the closure of $\mathbb{Z}$ in $\mathbb{R}$ with respect to $\mathscr{T}$.

2 pt (d) Determine the interior of $[0,1]$ with respect to $\mathscr{T}$.

3 pt (e) Show that every subset S of $\mathbb{R}$ is compact for the topology induced by $\mathscr{T}$.

4 pt (f) Let $A \subset \mathbb{R}$ be a subset of at least two elements. Show that A is not connected with respect to the topology induced by $\mathscr{T}$ if and only if A is finite.

9 pt total **Exercise 2.** Let X and Y be topological spaces, and $f : X \to Y$ a continuous map. We assume that X is compact and that for every $x \in X$ there exists an open neighborhood U of x such that the restriction $f|_U : U \to Y$ is injective.

3 pt (a) Show that there exists an open covering $\{U_i \mid i \in I\}$ of X such that for every $y \in Y$ and $i \in I$ the intersection $U_i \cap f^{-1}(\{y\})$ consists of at most one point.

6 pt (b) Show that there exists an $N > 0$ such that for every $y \in Y$ the fiber $f^{-1}(\{y\})$ consists of at most N elements.

11 pt total **Exercise 3.** Let X be a topological space, and let $\{U_i \mid i \in I\}$ be an open cover of X such for each $i \in I$ the set U_i is connected for the induced topology. Let $A \subset X$ be open and closed.

2 pt (a) Show that for each $i \in I$ we have $A \cap U_i = \emptyset$ or $A \cap U_i = U_i$.

2 pt (b) Let $i, j \in I$ be such that $U_i \cap U_j \neq \emptyset$. Show that either A contains both U_i and U_j or is disjoint from both.

If $i, j \in I$, then we will write $i \sim j$ to indicate that there exists a sequence $i_0, \ldots i_n$ in I such that $i_0 = i$ and $i_n = j$ and $U_{i_{k-1}} \cap U_{i_k} \neq \emptyset$ for all $k \in \{1, \ldots, n\}$. It is readily seen that $\sim$ defines an equivalence relation on I. You may use this without proof.

2 pt (c) If $i, j \in I$ are $\sim$-equivalent, show that either A contains both U_i and U_j or is disjoint from both.

2 pt (d) Show that if all elements of I are $\sim$-equivalent then X is connected.

3 pt (e) Conversely, if X is connected, show that all elements of I are $\sim$-equivalent.

8 pt total **Exercise 4.** We consider the closed unit disk in $\mathbb{C}$ given by

$$\bar{D} = \{Z \in \mathbb{C} \mid |z| \leq 1\}.$$

We consider the map $\varphi : \bar{D} \to \mathbb{C}^2$ given by

$$\varphi(z) = ((1 - \|z\|)z, z^2).$$

We define the equivalence relation R on $\bar{D}$ by $zRw :\iff \varphi(z) = \varphi(w)$.

4 pt (a) Show that $\bar{D}/R$ is homeomorphic to $\mathbb{P}^2(\mathbb{R})$.

4 pt (b) Show that there exists a topological embedding of $\mathbb{P}^2(\mathbb{R})$ into $\mathbb{R}^4$.

10 pt total **Exercise 5.** Let X and Y be locally compact Hausdorff spaces. Their one-point compactifications are denoted by $\widehat{X} = X \cup \{\infty_X\}$ and $\widehat{Y} = Y \cup \{\infty_Y\}$. Let $f : X \to Y$ be a continuous map. We define $\widehat{f} : \widehat{X} \to \widehat{Y}$ by $\widehat{f} = f$ on X and $\widehat{f}(\infty_X) = \infty_Y$.

4 pt (a) Show: if $\widehat{f} : \widehat{X} \to \widehat{Y}$ is continuous then for every compact set $K \subset Y$ the set $f^{-1}(K)$ is compact in X.

6 pt (b) Show that the converse implication is also valid: if for every compact set $K \subset Y$ the set $f^{-1}(K)$ is compact in X, then $\widehat{f} : \widehat{X} \to \widehat{Y}$ is continuous.