

# Exercises Lie groups

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**Exercise 1.** Let  $G$  be a group, equipped with the structure of a  $C^\infty$ -manifold. Let  $\mu : G \times G \rightarrow G$ ,  $(x, y) \mapsto xy$  be the multiplication map. We assume that  $\mu$  is smooth, i.e.,  $C^\infty$ .

- (a) Show that the tangent map  $T_{(e,e)}\mu : T_eG \times T_eG \rightarrow T_eG$  of  $\mu$  at  $(e, e)$  is given by  $(X, Y) \mapsto X + Y$ . Hint: use partial derivatives with respect to  $x$  and  $y$ .
- (b) Show that the inversion map  $\iota : G \rightarrow G$ ,  $x \mapsto x^{-1}$  is smooth in an open neighborhood of  $e$  and that the tangent map  $T_e\iota : T_eG \rightarrow T_eG$  is given by  $X \mapsto -X$ . Hint: use the implicit function theorem.
- (c) Show that  $G$  is a Lie group.

**Exercise 2.** We recall that  $SU(2)$  is the group of unitary  $2 \times 2$  matrices with determinant 1. Let  $S^3$  denote the unit sphere in  $\mathbb{R}^4$ , centered at the origin. For  $x \in S^3$  we define the complex  $2 \times 2$  matrix

$$u_x := \begin{pmatrix} x_1 + ix_2 & -x_3 + ix_4 \\ x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}$$

- (a) Show that the map  $\varphi : x \mapsto u_x$  is a bijection from  $S^3$  onto  $SU(2)$ .
- (b) We view  $S^3$  as a smooth submanifold of  $\mathbb{R}^4$  and transfer this manifold structure to  $SU(2)$  so that  $\varphi$  becomes a diffeomorphism. Show that  $SU(2)$ , equipped with this manifold structure is a Lie group.
- (c) Show that  $SU(2)$ , equipped with this manifold structure, is a smooth submanifold of  $GL(2, \mathbb{C})$ .

**Exercise 3.** We recall that  $O(n)$  is the group of real  $n \times n$  matrices  $a \in GL(n, \mathbb{R})$  such that  $aa^t = I$ .

- (a) Let  $S$  be the linear space of symmetric  $n \times n$  matrices in  $M(n, \mathbb{R})$ . Show that  $\varphi : A \mapsto AA^t$  defines a smooth map  $M(n, \mathbb{R}) \rightarrow S$  with tangent map at  $I$  given by

$$T_I\varphi : X \mapsto X + X^t, \quad M(n, \mathbb{R}) \rightarrow S.$$

(b) Show that  $\varphi$  is a submersion at  $I$ .

(c) Show that  $O(n, \mathbb{R})$  is a Lie group.

**Exercise 4.** We identify  $M(n, \mathbb{R}) \simeq \mathbb{R}^{n^2}$  and transfer the standard inner product on  $\mathbb{R}^{n^2}$  to an inner product on  $M(n, \mathbb{R})$ .

(a) Show that

$$\langle X, Y \rangle = \text{tr}(XY^t), \quad (X, Y \in M(n, \mathbb{R})).$$

(b) Show that  $O(n)$  is contained in the sphere of center 0 and radius  $\sqrt{n}$  for this inner product.

(c) Show that  $O(n)$  is compact.

**Exercise 5.** We identify  $M(n, \mathbb{C}) \simeq \mathbb{C}^{n^2}$  and transfer the standard complex inner product on  $\mathbb{C}^{n^2}$  to an inner product on  $M(n, \mathbb{C})$ .

(a) Show that

$$\langle X, Y \rangle = \text{tr}(XY^*), \quad (X, Y \in M(n, \mathbb{C})).$$

(b) Show that  $U(n)$  is contained in the sphere of center 0 and radius  $\sqrt{n}$  for this inner product.

(c) Show that  $U(n)$  is compact.

**Exercise 6.** Let now  $G$  be a Lie group. The commutator of two elements  $x, y \in G$  is the element  $c(x, y) := xyx^{-1}y^{-1}$ . Show that for all  $X, Y \in T_e G$  we have

$$[X, Y] = \left. \frac{\partial}{\partial s} \right|_{s=0} \left. \frac{\partial}{\partial t} \right|_{t=0} c(\exp sX, \exp tY).$$

Hint: use relations like  $x \exp Y x^{-1} = \exp(\text{Ad}(x)Y)$ .

**Exercise 7.** If  $u, v$  are two smooth vector fields on a smooth manifold  $M$ , we recall that their bracket  $[u, v]$  is the vector field defined by

$$[u, v] = \left. \frac{d}{dt} \right|_{t=0} (\varphi^t)^* v.$$

Here  $\varphi^t$  denotes the flow of  $u$ . Moreover, the pull-back of a vector field  $v$  by a diffeomorphism  $\varphi$  is given by  $\varphi^* v(x) = (T_x \varphi)^{-1} v(\varphi(x))$ , for  $x \in M$ . We recall that the bracket defines a bilinear map  $\mathcal{V}(M) \times \mathcal{V}(M) \rightarrow \mathcal{V}(M)$  that is anti-symmetric and satisfies the Jacobi identity, hence turns  $\mathcal{V}(M)$  into a Lie algebra.

Let  $G$  be a Lie group. The purpose of this exercise is to relate the Lie algebra structures of  $\mathcal{V}(G)$  and  $T_e G$ . We recall that for  $X \in T_e G$  the left  $G$ -invariant vector field  $v_X$  on  $G$  is defined by  $v_X(x) = T_e(l_x)X$ . Let  $\Phi_X : \mathbb{R} \times G \rightarrow G$  denote the flow of  $v_X$ .

- (a) Show that  $\Phi_X(t, x) = x \exp(tX)$  for  $t \in \mathbb{R}, x \in G$ .
- (b) Let  $Y \in T_e G, t \in \mathbb{R}$ . Show that  $(\Phi_X^t)^* v_Y$  is a left invariant vector field.
- (c) Let  $Y, t$  be as above. Show that  $(\Phi_X^t)^* v_Y = v_{\text{Ad}(\exp tX)Y}$ .
- (d) Let  $X, Y \in T_e G$ . Show that  $[v_X, v_Y] = v_{[X, Y]}$ .
- (e) Show that  $\mathcal{V}_L(G)$ , the space of left invariant vector fields, is a Lie subalgebra of  $\mathcal{V}(G)$ , and that the map  $X \mapsto v_X$  is an isomorphism from the Lie algebra  $T_e G$  onto  $\mathcal{V}_L(G)$ .

**Exercise 8.** Let  $V$  be a finite dimensional real linear space.

- (a) Show that  $\det : \text{GL}(V) \rightarrow \mathbb{R}^*$  is a homomorphism of Lie groups.
- (b) Show that  $D(\det)(I) = \text{tr}$ . Here  $\text{tr}$  denotes the linear map  $\text{End}(V) \rightarrow \mathbb{R}$ , assigning to a linear endomorphism of  $V$  its trace.
- (c) Show that for all  $A \in \text{End}(V)$  we have:

$$\det(e^A) = e^{\text{tr} A}$$

(hint: avoid computations; use a result from the text instead).

**Exercise 9.** First recall that a smooth manifold  $M$  is connected if and only if it is arcwise connected, i.e. for every two points  $p, q \in M$  there exists a continuous curve  $c : [0, 1] \rightarrow M$  with  $c(0) = p, c(1) = q$ . In the following we will use the notation

$$\mathfrak{so}(n) := \{A \in \text{M}(n, \mathbb{R}) \mid A^t = -A\}.$$

- (a) Consider the exponential map  $\exp : A \mapsto e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$ . Compute

$$\exp \begin{pmatrix} 0 & -\varphi \\ \varphi & 0 \end{pmatrix}$$

and show that  $\exp$  maps  $\mathfrak{so}(2)$  surjectively onto  $SO(2)$ .

- (b) Show that for every  $x \in SO(n)$  there exists a  $y \in SO(n)$  such that  $x = yby^{-1}$  with  $b$  a matrix consisting of  $2 \times 2$  matrix blocks  $B \in SO(2)$ , and  $1 \times 1$  matrix blocks  $B = (1)$  along the diagonal.
- (c) Let  $n \geq 2$ . Show that the exponential map  $\exp$  maps  $\mathfrak{so}(n)$  onto  $SO(n)$ , i.e. every  $x \in SO(n)$  may be written as  $\exp X$  with  $X \in \mathfrak{so}(n)$ .
- (d) Show that  $SO(n)$  is connected.

**Exercise 10.** Show that  $U(n)$  and  $SU(n)$  are connected. (Hint: use the method suggested in the previous exercise, but now with diagonal matrices.)

**Exercise 11.** We recall that the Lie group  $SL(n, \mathbb{R})$  has tangent space at  $I$  equal to

$$sl(n, \mathbb{R}) = \{X \in M_n(\mathbb{R}) \mid \text{tr } X = 0\}.$$

- (a) Show that the matrix  $y = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$  belongs to  $SL(2, \mathbb{R})$ , but cannot be written as  $y = \exp Y$  with  $Y \in sl(2, \mathbb{R})$ .
- (b) Let  $n \geq 2$ . Show that every  $x \in SL(n, \mathbb{R})$  can be written as  $x = \exp X_a \exp X_s$ , with  $X_a$  an antisymmetric, and  $X_s$  a symmetric matrix in  $sl(n, \mathbb{R})$ . (Hint: consider  $x^t x$ ).
- (c) Show that  $SL(n, \mathbb{R})$  is a connected Lie group.

**Exercise 12.** Let  $\varphi, \psi : H \rightarrow G$  be two homomorphisms of Lie groups, and assume that  $H$  is connected. Show that

$$\varphi = \psi \iff \varphi_* = \psi_*$$

**Exercise 13.**

- (a) Show that  $\mathbb{R}^3$  together with the exterior product  $(X, Y) \mapsto X \times Y$  is a Lie algebra.
- (b) Show that the above Lie algebra is isomorphic to the Lie algebra  $\mathfrak{o}(3)$  of the orthogonal group  $O(3)$ . Hint: Consider the map  $\psi : \mathbb{R}^3 \rightarrow M_3(\mathbb{R})$  defined by  $v \mapsto \text{mat}(v \times \cdot)$ , the matrix with respect to the standard basis of the linear map  $v \times \cdot : X \mapsto v \times X$ ,  $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ .

**Exercise 14.** Let  $H_1, H_2$  be Lie groups.

- (a) If  $\varphi : H_1 \rightarrow H_2$  is an isomorphism of groups show that  $\varphi$  is  $C^\infty$  if and only if it is  $C^\infty$  on a neighborhood of  $e$ . Hint: use left translations. Show that  $\varphi$  is a diffeomorphism if and only if it is local diffeomorphism at  $e$ .

We now assume that  $G$  is a Lie group and that  $i_j : H_j \rightarrow G$ ,  $j = 1, 2$ , are two injective Lie group homomorphisms with  $i_1(H_1) = i_2(H_2)$ .

- (b) Show that there exists a unique map  $\varphi : H_1 \rightarrow H_2$  such that  $i_2 \circ \varphi = i_1$ . Show that  $\varphi$  is an isomorphism of groups.
- (c) Show that there exists a unique map  $\tau : \mathfrak{h}_1 \rightarrow \mathfrak{h}_2$  such that  $i_{2*} \circ \tau = i_{1*}$ . Show that  $\tau$  is an isomorphism of Lie algebras.
- (d) Show that there exists an open neighborhood  $\Omega$  of 0 in  $\mathfrak{g}$  with the following properties.

- The exponential map  $\exp : \mathfrak{g} \rightarrow G$  is a diffeomorphism from  $\Omega$  onto an open subset of  $G$ .
  - For each  $j = 1, 2$ , the exponential map  $\exp_j : \mathfrak{h}_j \rightarrow H_j$  is a diffeomorphism from  $\Omega_j := i_{j*}^{-1}(\Omega)$  onto an open subset of  $H_j$ .
- (e) Show that for every  $X \in \Omega_1$  we have  $\varphi(\exp_1(X)) = \exp_2(\tau X)$ .
- (f) Show that  $\varphi$  is an isomorphism of Lie groups and that  $\varphi_* = \tau$ .
- (g) Show that every subgroup  $H$  of  $G$  carries at most one structure of a Lie subgroup.

**Exercise 15.** Let  $\mathfrak{l}$  be a Lie algebra, and  $\mathfrak{a}$  an ideal of  $\mathfrak{l}$ , that is:  $\mathfrak{a} \subset \mathfrak{l}$  is a linear subspace such that  $[\mathfrak{l}, \mathfrak{a}] \subset \mathfrak{a}$ , i.e.  $[X, Y] \in \mathfrak{a}$  for all  $X \in \mathfrak{l}$  and  $Y \in \mathfrak{a}$ .

Show that  $\mathfrak{l}/\mathfrak{a}$  has a unique structure of Lie algebra such that the canonical projection  $\pi : \mathfrak{l} \rightarrow \mathfrak{l}/\mathfrak{a}$  is a Lie algebra homomorphism.

**Exercise 16.** Recall that a subgroup  $H$  of a group  $G$  is called normal if  $xHx^{-1} \subset H$  for all  $x \in G$ . We recall that normality of  $H$  is equivalent to the existence of a group structure on the coset space  $G/H$  for which the canonical map  $\pi : G \rightarrow G/H$  is a group homomorphism. Note that  $H = \ker(\pi)$ .

Let  $\mathfrak{l}$  be a Lie algebra. By an ideal of  $\mathfrak{l}$  we mean a linear subspace  $\mathfrak{a} \subset \mathfrak{l}$  with the property that  $[\mathfrak{l}, \mathfrak{a}] \subset \mathfrak{a}$ , i.e.,  $[X, Y] \in \mathfrak{a}$  for all  $X \in \mathfrak{l}$ ,  $Y \in \mathfrak{a}$ .

- (a) Let  $\mathfrak{a} \subset \mathfrak{l}$  be an ideal. Show that the quotient (linear) space  $\mathfrak{l}/\mathfrak{a}$  has a unique structure of Lie algebra such that the canonical projection  $\pi : \mathfrak{l} \rightarrow \mathfrak{l}/\mathfrak{a}$  is a homomorphism of Lie algebras.
- (b) Let  $\varphi : \mathfrak{l} \rightarrow \mathfrak{m}$  be a surjective homomorphism of Lie algebras. Show that  $\ker \varphi$  is an ideal in  $\mathfrak{l}$  and that the induced map

$$\bar{\varphi} : \mathfrak{l}/\ker \varphi \rightarrow \mathfrak{m}$$

is an isomorphism of Lie algebras (this is the analogue of the isomorphism theorem for surjective group homomorphisms).

**Exercise 17.** Suppose that  $H$  is a Lie subgroup of a Lie group  $G$ . As usual we denote the Lie algebras of  $G$  and  $H$  by  $\mathfrak{g}$  and  $\mathfrak{h}$ , respectively.

- (a) Show that if  $H$  is normal in  $G$  then  $\mathfrak{h}$  is an ideal in  $\mathfrak{g}$ . Hint: this exercise is more subtle than it may seem at first. The reason is that  $H$  need not be a smooth submanifold of  $G$ . Use a suitable characterization of  $\mathfrak{h}$ .

Now assume that  $G$  and  $H$  are connected.

- (b) Show: if  $\mathfrak{h}$  is an ideal in  $\mathfrak{g}$ , then  $H$  is normal in  $G$ .

**Exercise 18.** We define the center  $Z(G)$  of a Lie group  $G$  by

$$Z(G) = \{ x \in G \mid \forall y \in G : xy = yx \}.$$

Now assume that  $G$  is connected.

- (a) Show that  $Z(G) = \ker \text{Ad}$ .
- (b) Show that  $Z(G)$  has Lie algebra equal to  $\ker \text{ad}$ .

If  $\mathfrak{l}$  is a Lie algebra, we define its center by

$$Z(\mathfrak{l}) = \{ X \in \mathfrak{l} \mid [X, Y] = 0 \quad \forall Y \in \mathfrak{l} \} = \ker \text{ad}.$$

- (c) Show that  $Z(\mathfrak{l})$  is an ideal in  $\mathfrak{l}$ .
- (d) Show that the Lie algebra of  $Z(G)$  equals  $Z(\mathfrak{g})$ .

**Exercise 19.** We consider the Lie algebra  $\mathfrak{so}(4)$  of the group  $\text{SO}(4)$ . It consists of the matrices of the form

$$\begin{pmatrix} A & -C^t \\ C & B \end{pmatrix},$$

with  $A, B \in \mathfrak{so}(2)$  and  $C \in M(2, \mathbb{R})$ .

Let  $D_1$  be a matrix as above with  $A = B \neq 0$  and  $C = 0$ . Let  $D_2$  be a matrix as above with  $A = -B \neq 0$  and  $C = 0$ . Let  $\mathfrak{v} \subset \mathfrak{so}(4)$  be the linear subspace of matrices of the above form with  $A = B = 0$ .

- (a) Show that  $\mathfrak{v}$  is invariant under both  $\text{ad } D_1$  and  $\text{ad } D_2$ , and compute  $\mathfrak{v}_2 := \ker \text{ad } D_1 \cap \mathfrak{v}$  and  $\mathfrak{v}_1 := \ker \text{ad } D_2$ .
- (b) Show that  $\mathfrak{v} = \mathfrak{v}_1 \oplus \mathfrak{v}_2$  and that  $[\mathfrak{v}_1, \mathfrak{v}_2] = 0$ , i.e.,  $[X_1, X_2] = 0$  for all  $X_1 \in \mathfrak{v}_1$  and  $X_2 \in \mathfrak{v}_2$ .
- (c) Put  $\mathfrak{a}_1 = \mathbb{R}D_1 \oplus \mathfrak{v}_1$  and  $\mathfrak{a}_2 = \mathbb{R}D_2 \oplus \mathfrak{v}_2$ . Show that  $[\mathfrak{a}_1, \mathfrak{a}_2] = 0$ .
- (d) Show that  $\mathfrak{a}_1$  and  $\mathfrak{a}_2$  are ideals in  $\mathfrak{so}(4)$  and show that  $\mathfrak{so}(4) = \mathfrak{a}_1 \oplus \mathfrak{a}_2$  as Lie algebras.
- (e) Determine a basis  $R_{j1}, R_{j2}, R_{j3}$  of  $\mathfrak{a}_j$ , for  $j = 1, 2$ , such that the linear map  $T_j : \mathfrak{so}(3) \rightarrow \mathfrak{a}_j$  determined by  $T_j R_i = R_{ji}$  is an isomorphism of Lie algebras.
- (f) Given  $A \in \text{End}(\mathfrak{so}(4))$ , let  $\text{mat } A$  denote its matrix with respect to the basis  $R_{11}, R_{12}, \dots, R_{23}$  of  $\mathfrak{so}(4)$ . Show that  $\varphi = \text{mat} \circ \text{ad}$  determines a Lie algebra embedding from  $\mathfrak{so}(4)$  into  $M(6, \mathbb{R})$  with image isomorphic to  $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ .
- (g) Show that  $\Phi = \text{mat} \circ \text{Ad}$  determines a Lie group homomorphism from  $\text{SO}(4)$  into  $\text{GL}(6, \mathbb{R})$  with image isomorphic to  $\text{SO}(3) \times \text{SO}(3)$ . Hint: follow the same reasoning as for  $\text{SU}(2) \rightarrow \text{SO}(3)$  in the lecture notes.
- (h) Determine the kernel of  $\Phi$ .

**Exercise 20.**

(a) Show that the matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

generate the Lie algebra  $sl(2, \mathbb{R})$  of  $SL(2, \mathbb{R})$ .

(b) Express the commutator brackets  $[H, X]$ ,  $[H, Y]$  and  $[X, Y]$  as linear combinations of  $H, X, Y$ . *Remark:* a triple of elements  $H, X, Y$  of an arbitrary Lie algebra satisfying the same commutator relations, is called a standard  $sl(2, \mathbb{R})$ -triple.

(c) Compute the matrix of  $\text{ad } A$  with respect to  $H, X, Y$ , for  $A = H, X, Y$ .

**Exercise 21.** This exercise gives an introduction to the non-commutative field  $\mathbb{H}$  of quaternions. We introduce  $\mathbb{H}$  as the  $\mathbb{R}$ -algebra of complex  $2 \times 2$ -matrices of the form

$$m(a, b) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

The map  $m$  is a real linear isomorphism from  $\mathbb{R}^4 \simeq \mathbb{C}^2 \rightarrow \mathbb{H}$ . The images of the standard basis vectors  $e_1, \dots, e_4$  of  $\mathbb{R}^4$  are denoted by  $1, i, j, k$ , respectively. Thus,  $1 = I$ ,

$$i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

(a) Show that  $i^2 = j^2 = k^2 = -1$  and  $ij = -ji = k$ . Determine  $ik, ki, jk, kj$ .

We embed  $\mathbb{C}$  into  $\mathbb{H}$  by  $x + yi \mapsto m(x + yi, 0)$ . Thus  $x + yi$  is mapped to the matrix  $xI + yi$ . In particular, we view  $\mathbb{R}$  as a subspace of  $\mathbb{H}$  via the map  $a \mapsto aI$ .

(b) Show that, accordingly,  $\mathbb{H}$  is a two dimensional vector space over  $\mathbb{C}$  with basis  $1, j$ . Show that for all  $z \in \mathbb{C}$  we have  $zj = j\bar{z}$ .

We define the conjugation  $\iota$  on  $\mathbb{H}$  by

$$\iota : x_1 + x_2i + x_3j + x_4k \mapsto x_1 - (x_2i + x_3j + x_4k).$$

(c) Considering the elements of  $\mathbb{H}$  as matrices, show that  $\iota(h)$  equals  $h^*$ , the complex conjugate of  $h$ . Show that  $\iota(\alpha\beta) = \iota(\beta)\iota(\alpha)$  for all  $\alpha, \beta \in \mathbb{H}$ . Show that  $\iota^2 = I_{\mathbb{H}}$ .

(d) Via  $m$  we transfer the Euclidean norm  $\|\cdot\|$  on  $\mathbb{R}^4$  to a norm on  $\mathbb{H}$ . Show that

$$h \iota(h) = \iota(h) h = \|h\|^2 \quad (h \in \mathbb{H}).$$

(e) Show that every element  $h \in \mathbb{H} \setminus \{0\}$  is invertible, with inverse  $h^{-1} = \|h\|^{-2}\iota(h)$  (this extends the similar formula for  $\mathbb{C}$ ). Thus,  $\mathbb{H}$  has all properties of a field, except for commutativity of the multiplication.

(f) Show that

$$\|\alpha\beta\| = \|\alpha\|\|\beta\| \quad (\alpha, \beta \in \mathbb{H}).$$

(g) Show that the unit sphere  $S$  in  $\mathbb{H}$ , is a group for the  $\mathbb{H}$ -multiplication. Show that  $S = \text{SU}(2)$ . Show that the Lie algebra of  $\text{SU}(2)$  coincides with the space  $\mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k \simeq \mathbb{R}^3$ ; this space is called the space of pure quaternions. Show in a direct fashion that the adjoint action of  $\text{SU}(2)$  on its Lie algebra is by means of transformations from  $\text{SO}(3)$ .

**Exercise 22.** We retain the notation of the previous exercise. We denote by  $\text{GL}_{\mathbb{R}}(\mathbb{H})$  the group of invertible real linear transformations of  $\mathbb{H}$ ; thus,  $\text{GL}_{\mathbb{R}}(\mathbb{H}) \simeq \text{GL}(4, \mathbb{R})$ . We denote by  $\text{SO}(\mathbb{H})$  the subgroup corresponding to  $\text{SO}(4)$ .

(a) Show that for all  $A, B \in \text{SU}(2) \times \text{SU}(2)$  the map  $\varphi(A, B) : \mathbb{H} \rightarrow \mathbb{H}, h \mapsto AhB^{-1}$  belongs to  $\text{SO}(\mathbb{H}) \simeq \text{SO}(4)$ .

(b) Show that the kernel  $K$  of the map  $\varphi$  consists of  $(I, I)$  and  $(-I, -I)$ .

(c) Show that  $\varphi$  factors to a Lie group isomorphism  $\bar{\varphi} : \text{SU}(2) \times \text{SU}(2)/K \xrightarrow{\simeq} \text{SO}(4)$ . Hint: try to use a minimal amount of computation. Be inspired by the lecture notes on  $\text{SU}(2)$  and  $\text{SO}(3)$ . First consider the derivative of  $\bar{\varphi}$ . What is its kernel?

(d) Let  $H = \{I, -I\} \subset \text{SU}(2)$ . Let  $\pi$  denote the canonical projection  $\text{SU}(2) \times \text{SU}(2)/K \rightarrow \text{SU}(2)/H \times \text{SU}(2)/H$ . Show that the map  $\psi := \pi \circ (\bar{\varphi})^{-1}$  is a surjective Lie group homomorphism from  $\text{SO}(4)$  onto  $\text{SU}(2)/H \times \text{SU}(2)/H$ .

(e) Show that  $\mathfrak{so}(4) \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(3)$  and that  $\text{SO}(3) \times \text{SO}(3) \simeq \text{SO}(4)/\{\pm I\}$ .

**Exercise 23. Grassmannian manifold.** Let  $\mathbb{K}$  be the field  $\mathbb{R}$  or  $\mathbb{C}$ . As a set, the Grassmannian manifold  $G_{n,k} = G_{n,k}(\mathbb{K})$  consists of all  $k$ -dimensional linear subspaces of  $\mathbb{K}^n$ . We consider the linear space  $\text{Hom}(\mathbb{K}^k, \mathbb{K}^n)$  of linear maps  $\mathbb{K}^k \rightarrow \mathbb{K}^n$ . Given a sequence  $i = (i_1, \dots, i_k)$  of integers  $i_j$  with  $1 \leq i_1 < i_2 < \dots < i_k \leq n$  and an element  $A \in \text{Hom}(\mathbb{K}^k, \mathbb{K}^n)$ , we denote by  $D_i(A)$  the determinant of the  $k \times k$  submatrix of  $A$  determined by the columns with numbers  $i_1, \dots, i_k$ . The set  $H_i$  of matrices  $A$  with  $D_i(A) \neq 0$  is open in  $\text{Hom}(\mathbb{K}^k, \mathbb{K}^n)$  (why?). We denote by  $\text{Hom}_0(\mathbb{K}^k, \mathbb{K}^n)$  the collection of  $A \in \text{Hom}(\mathbb{K}^k, \mathbb{K}^n)$  with  $\ker A = \{0\}$ . Then  $\text{Hom}_0(\mathbb{K}^k, \mathbb{K}^n)$  is the union of the sets  $H_i$ , hence open in  $\text{Hom}(\mathbb{K}^k, \mathbb{K}^n)$ .

We define the map  $p : \text{Hom}_0(\mathbb{K}^k, \mathbb{K}^n) \rightarrow G_{n,k}$  by  $\varphi(A) = A(\mathbb{R}^k)$ .

(a) Show that  $p$  is surjective.



It is known that the set  $G_{n,k}$  has a structure of smooth manifold such that  $p$  is submersive. Put  $G = \text{GL}(N, \mathbb{K})$ . We define the map  $\alpha : G \times \text{Hom}_0(\mathbb{K}^k, \mathbb{K}^n) \rightarrow \text{Hom}_0(\mathbb{K}^k, \mathbb{K}^n)$  by  $\alpha(g, A) = g \circ A$ . We define the map  $\beta : G \times G_{n,k} \rightarrow G_{n,k}$  by  $\beta(g, V) = g \cdot V := g(V)$ .

- (b) Show that  $\alpha$  and  $\beta$  are actions of  $G$  on  $\text{Hom}_0(\mathbb{K}^k, \mathbb{K}^n)$  and  $G_{n,k}$  respectively, and that  $p(\alpha(g, A)) = \beta(g)p(A)$ , for all  $A \in \text{Hom}_0(\mathbb{K}^k, \mathbb{K}^n)$  and all  $g \in G_{n,k}$ .
- (c) Show that  $\alpha$  is smooth.
- (d) Show that  $\beta$  is smooth.
- (e) Show that  $G_{n,k} \simeq \text{GL}(n)/P$ , where  $P$  is the subgroup of matrices  $g \in \text{GL}(n)$  of the form

$$g = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

with  $A \in \text{GL}(k)$ ,  $B \in \text{GL}(n - k)$  and  $C$  a  $k \times n$  matrix with entries in  $\mathbb{K}$ .

- (f) Show that  $G_{n,k}$  is compact. Hint: treat the fields  $\mathbb{R}$  and  $\mathbb{C}$  separately.

**Exercise 24. Flag manifold.** Let  $\mathbb{K}$  be one of the fields  $\mathbb{R}$  or  $\mathbb{C}$ . Let  $n \geq 2$  and let  $d = (d_1, \dots, d_k)$  be a sequence of positive integers with  $\sum_{j=1}^k d_j = n$ . We define a *flag* of type  $d$  in  $\mathbb{K}^n$  as an ordered sequence  $F = (F_0, F_1, \dots, F_{k-1}, F_k)$  of linear subspaces of  $\mathbb{K}^n$  with  $0 = F_0 \subset F_1 \subset \dots \subset F_k = \mathbb{K}^n$  and with  $\dim(F_j/F_{j-1}) = d_j$ , for all  $1 \leq j \leq k$ . The collection of all flags of type  $d$ , denoted by  $\mathcal{F} = \mathcal{F}_d$ , is called a flag manifold.

Let  $G = \text{GL}(n, \mathbb{K})$  and let  $\alpha : G \times \mathcal{F} \rightarrow \mathcal{F}$  be defined by  $\alpha(g, F) = g \cdot F := (g(F_j) \mid 0 \leq j \leq k)$ .

- (a) Show that  $\alpha$  is a transitive action of  $G$  on  $\mathcal{F}$ .

Let the standard flag  $E$  of type  $d$  be defined by  $E_0 = 0$  and  $E_j = \text{span}\{e_1, \dots, e_{d_1+\dots+d_j}\}$ , for  $1 \leq j \leq k$ .

- (b) Determine the stabilizer  $P = P_d$  of  $E$  in  $G$ . Show that  $P$  is a closed subgroup of  $G$ .
- (c) Show that  $\varphi : G \rightarrow \mathcal{F}$  induces a bijection  $\bar{\varphi} : G/P \rightarrow \mathcal{F}$ . Accordingly, we equip  $\mathcal{F}$  with the structure of a smooth manifold such that  $\bar{\varphi}$  is a diffeomorphism.
- (d) Put  $K = \text{O}(n)$  if  $\mathbb{K} = \mathbb{R}$  and  $K = \text{U}(n)$  if  $\mathbb{K} = \mathbb{C}$ . In both cases show that  $\varphi(K) = \mathcal{F}$ . Put  $H = K \cap P$  and show that  $\mathcal{F} \simeq K/H$ . Conclude that  $\mathcal{F}$  is compact.
- (e) With notation as in (d), show that  $m : K \times P \rightarrow G$ ,  $(k, p) \mapsto kp$  is a surjective map. Hint: use (d) and (b). Moreover, show that  $m$  is a smooth submersion. Hint: use homogeneity.
- (f) Determine  $d$  such that  $\mathcal{F}_d \simeq \mathbb{P}^{n-1}(\mathbb{K})$ . More generally, let  $1 \leq k < n$ . Determine  $d$  such that  $\mathcal{F}_d \simeq G_{n,k}(\mathbb{K})$ .

**Exercise 25.** Let  $A$  and  $B$  be commuting endomorphisms of a linear space  $V$  over the ground field  $\mathbf{k} = \mathbb{R}$  or  $\mathbb{C}$ . Show that for each  $\lambda \in \mathbf{k}$  and all  $k \in \mathbb{N}$  the endomorphism  $B$  leaves the space  $\ker(A - \lambda I)^k$  invariant. We define the generalized eigenspace of  $A$  for the eigenvalue  $\lambda$  to be the space  $E_\lambda$  of vectors  $v \in V$  for which there exists a  $k \in \mathbb{N}$  such that  $(A - \lambda I)^k v = 0$ . Show that  $B$  leaves the space  $E_\lambda$  invariant.

**Exercise 26.** Let  $G$  be a commutative Lie group, and let  $(\pi, V)$  be a unitary representation of  $G$ .

- Show that  $\pi$  is irreducible if and only if  $\dim V = 1$ . Hint: use Schur's lemma.
- Show that there exist mutually orthogonal one dimensional invariant linear subspaces  $V_1, \dots, V_n$  of  $V$ , such that  $V = V_1 \oplus \dots \oplus V_n$ .
- Show that the natural representation of  $\text{SO}(2)$  in  $\mathbb{C}^2$  is not irreducible.
- Show that the natural representation of  $\text{SO}(2)$  in  $\mathbb{R}^2$  is irreducible.

**Exercise 27.** Let  $n \geq 3$ , and let  $\pi$  be the natural representation of  $\text{SO}(n)$  in  $\mathbb{C}^n$ . We consider the centralizer  $\mathfrak{c}$  of  $\text{SO}(n)$  in  $M(n, \mathbb{C})$ .

- Let  $v_1, v_2$  be a pair of orthonormal vectors in  $\mathbb{R}^n$  and  $w_1, w_2$  a second such pair. Show that there exists an element  $g \in \text{SO}(n)$  such that  $gv_j = w_j$ , for  $j = 1, 2$ .
- Let  $v_1, v_2$  and  $w_1, w_2$  be two pairs as in item (a). Show that for all  $T \in \mathfrak{c}$  we have  $\langle Tv_1, v_2 \rangle = \langle Tw_1, w_2 \rangle$ .
- Show that  $\pi$  is irreducible.

**Exercise 28.** Let  $(\delta, V)$  be an irreducible finite dimensional representation of the Lie group  $G$ .

- Let  $\pi$  be the  $n$ -fold direct product of  $\delta$  in  $W = V \oplus \dots \oplus V$  ( $n$  summands). Show that  $\text{Hom}_G(V, W)$  is a linear space of dimension  $n$ .
- Show that  $\text{End}_G(V \oplus V)$  is 4-dimensional.
- Show that the space  $V^2 = V \oplus V$  has a direct sum decomposition  $V^2 = U_1 \oplus U_2$  into  $G$ -invariant non-trivial subspaces different from  $V \oplus \{0\}$  and  $\{0\} \oplus V$ . From this we draw the conclusion that  $V^2$  has no canonical decomposition into irreducibles.

**Exercise 29.** Let  $(\delta_1, V_1), (\delta_2, V_2)$  be two irreducible finite dimensional representations of the Lie group  $G$ . Show that

- $\dim \text{Hom}_G(V_1, V_2) = 1 \iff \delta_1 \sim \delta_2$  (hint: use the previous exercise).
- $\dim \text{Hom}_G(V_1, V_2) = 0 \iff \delta_1 \not\sim \delta_2$ .

In the following exercises,  $(\pi, V)$  and  $(\rho, W)$  will be finite dimensional continuous representations of a compact Lie group  $G$ . We recall that the character of the representation  $\pi$  is defined to be the function  $\chi_\pi : G \rightarrow \mathbb{C}$  given by

$$\chi_\pi(x) = \text{tr}(\pi(x)), \quad (x \in G).$$

**Exercise 30.** Show that:

- (a)  $\chi_\pi(e) = \dim V$ ;
- (b)  $\chi_\pi(xy x^{-1}) = \chi_\pi(y)$ , for all  $x, y \in G$ ;
- (c)  $\chi_\pi(x^{-1}) = \overline{\chi_\pi(x)}$ , for  $x \in G$ .

**Exercise 31.** Let  $G = \text{SU}(2)$ . For  $\varphi \in \mathbb{R}$  we define  $t_\varphi \in G$  to be the diagonal matrix with entries  $e^{i\varphi}$  in the upper left corner, and  $e^{-i\varphi}$  in the lower right corner.

- (a) Show that  $T = \{t_\varphi \mid \varphi \in \mathbb{R}\}$  is a commutative compact subgroup of  $G$ . Show that every element of  $G$  that commutes with  $T$  belongs to  $T$ . (Such a group is called a maximal torus of  $G$ .)
- (b) Let  $\chi_n$  be the character of the irreducible representation  $\pi_n$  of  $G$ , for  $n \in \mathbb{N}$ . Show that

$$\chi_n(t_\varphi) = \frac{\sin(n+1)\varphi}{\sin \varphi}.$$

- (c) Prove the following Clebsch-Gordon formula, for  $m, n \in \mathbb{N}$  with  $m \leq n$  :

$$\pi_n \otimes \pi_m \sim \pi_{n+m} \oplus \pi_{n+m-2} \oplus \cdots \oplus \pi_{n-m}.$$

Hint: establish an identity of characters.

In the following exercises,  $G$  will be a compact Lie group, and  $dx$  normalized Haar measure on  $G$ . For two continuous functions  $f, g : G \rightarrow \mathbb{C}$  we define the convolution product  $f * g : G \rightarrow \mathbb{C}$  by

$$f * g(x) = \int_G f(xy^{-1})g(y) dy.$$

**Exercise 32.** Let  $\delta_1, \delta_2 \in \widehat{G}$  be inequivalent representations.

- (a) Show that  $f * g = 0$  for all  $f \in C(G)_{\delta_1}$  and  $g \in C(G)_{\delta_2}$ .
- (b) Show that  $\chi_{\delta_1} * \chi_{\delta_2} = 0$ .
- (c) Let  $\delta \in \widehat{G}$ . Show that  $\chi_\delta * \chi_\delta = \frac{1}{\dim \delta} \chi_\delta$

If  $(\pi, V)$  is a continuous finite dimensional representation of  $G$  and  $f \in C(G)$ , we define the endomorphism  $\pi(f)$  of  $V$  by

$$\pi(f) = \int_G f(x)\pi(x) dx.$$

**Exercise 33.** Let  $(\pi, V)$  be a finite dimensional unitary representation of  $G$ .

(a) Show that, for all  $f, g \in C(G)$  :

$$\pi(f * g) = \pi(f) \circ \pi(g).$$

For  $\delta \in \widehat{G}$  we put  $P_\delta = \dim(\delta) \pi(\bar{\chi}_\delta)$ , where the bar indicates that the complex conjugate of the character is taken.

(b) Show that  $P_\delta$  is  $G$ -intertwining.

(c) Show that  $P_\delta$  is an orthonormal projection, i.e.  $P_\delta$  is symmetric and  $P_\delta^2 = P_\delta$ .

(d) Show that  $P_{\delta_1} \circ P_{\delta_2} = 0$  if  $\delta_1, \delta_2 \in \widehat{G}$  are inequivalent.

(e) Show that the set  $S(\pi)$  of  $\delta \in \widehat{G}$  with  $P_\delta \neq 0$  is finite.

(f) Assume that  $W$  is an irreducible  $G$ -submodule of  $V$ , and let  $\pi|_W \simeq \delta \in \widehat{G}$ . Show that  $P_\delta = I$  on  $W$ . Hint: first show that  $P_\delta$  is a scalar on  $W$ .

(g) Show that

$$V = \bigoplus_{\delta \in S(\pi)} P_\delta(V)$$

is an orthogonal direct sum of invariant subspaces of  $V$ . Moreover, show that the restriction of  $\pi$  to  $P_\delta(V)$  is equivalent to a finite direct sum of copies of  $\delta$ .

(h) Assume that  $V = V_1 \oplus \cdots \oplus V_m$  is a decomposition of  $V$  into irreducibles. Show that for every  $\delta \in \widehat{G}$  we have

$$P_\delta(V) = \sum_{i: \pi|_{V_i} \simeq \delta} V_i$$

Show that the number of terms in the above sum equals the  $L^2$ -inner product of  $\chi_\pi$  with  $\chi_\delta$ .

## Angular momentum operators

The purpose of the following set of exercises is to clarify the connection between the angular momentum operators in quantum mechanics, and the representation theory of the Lie algebra  $\mathfrak{su}(2)$ .

In the following,  $e_1, e_2, e_3$  will denote the standard basis in  $\mathbb{R}^3$ . Matrices will be taken with respect to this basis. We will use the notation  $a \times b$  for the exterior product of two vectors  $a, b \in \mathbb{R}^3$ . Recall that this product is determined by the requirement

$$\langle x, a \times b \rangle = \det(x, a, b)$$

for all  $x \in \mathbb{R}^3$ . By substituting the elements of the standard basis for  $x$  one obtains the usual determinant formulas for the components of  $a \times b$ .

We start with a general exercise on smooth group actions and the associated representations on function spaces.

**Exercise 34.** Let  $G$  be a Lie group,  $\mathfrak{g}$  its Lie algebra. Let  $\tau : G \times M \rightarrow M, (g, m) \mapsto gm$  be a smooth action of  $G$  on a smooth manifold  $M$ . There is an associated representation  $\pi$  of  $G$  on the space  $C^\infty(M)$  of smooth functions on  $M$ . It is given by the formula:

$$\pi(g)\varphi(m) = \varphi(g^{-1}m),$$

for  $\varphi \in C^\infty(M), g \in G, m \in M$ .

- (a) Check that indeed  $\pi$  is a representation of  $G$  in  $C^\infty(M)$ ; we assert nothing on continuity here.

Given  $X \in \mathfrak{g}$  and  $\varphi \in C^\infty(M)$  we define the function  $\pi_*(X)\varphi : M \rightarrow \mathbb{C}$  by

$$\pi_*(X)\varphi = \left. \frac{d}{dt} \right|_{t=0} \pi(\exp tX)\varphi.$$

- (b) Show that  $\pi_*(X)$  is a smooth first order linear partial differential operator on  $M$ .
- (c) Show that  $\pi(x)\pi_*(X)\pi(x)^{-1} = \pi_*(\text{Ad}(x)X)$  for all  $X \in \mathfrak{g}$  and  $x \in G$ .
- (d) Show that  $\pi_*([X, Y]) = \pi_*(X)\pi_*(Y) - \pi_*(Y)\pi_*(X)$  for all  $X, Y \in \mathfrak{g}$ .

Thus,  $X \mapsto \pi_*(X)$  is a Lie algebra homomorphism from  $\mathfrak{g}$  into the algebra  $\text{DO}(M) \subset \text{End}(C^\infty(M))$  of smooth linear partial differential operators on  $M$ . In particular,  $\pi_*$  is a representation of the Lie algebra  $\mathfrak{g}$  in  $C^\infty(M)$ .

**Exercise 35.**

- (a) Show that for  $a \in \mathbb{R}^3$  the matrix  $R_a$  of the linear map  $x \mapsto a \times x$  belongs to  $\mathfrak{so}(3)$ , the Lie algebra of  $\text{SO}(3)$ ; thus  $\mathfrak{so}(3)$  consists of all real anti-symmetric  $3 \times 3$  matrices.
- (b) Show that for  $a, b \in \mathbb{R}^3$  we have  $R_{a \times b} = [R_a, R_b]$ .
- (c) Show that  $(\mathbb{R}^3, \times)$  is a Lie algebra isomorphic to  $\mathfrak{so}(3)$ ; here  $a \mapsto R_a$  is the isomorphism. Observe that  $R_{e_j} = R_j$ , the infinitesimal generating rotation around the  $x_j$ -axis.

In the following we shall apply both of the above exercises to the natural smooth action of the group  $SO(3)$  on  $\mathbb{R}^3$  by the usual matrix multiplication.

Let  $\pi$  denote the associated representation of  $SO(3)$  in  $C^\infty(\mathbb{R}^3)$ , and let  $\pi_*$  be the representation of  $\mathfrak{so}(3)$  in  $\text{End}(C^\infty(\mathbb{R}^3))$ , defined as above. Thus, if  $R \in \mathfrak{so}(3)$ , then  $\pi_*(R)$  is a first order linear partial differential operator on  $\mathbb{R}^3$ .

**Exercise 36.** Show that for every  $a \in \mathbb{R}^3$ ,  $\varphi \in C^\infty(\mathbb{R}^3)$ ,

$$\pi_*(R_a)\varphi(x) = -a \cdot (x \times \nabla\varphi(x)) \quad (x \in \mathbb{R}^3).$$

In the following we shall write  $\bar{R}_a$  for  $\pi_*(R_a)$ . By linearity of the map  $a \mapsto \bar{R}_a$ , the formula of the above exercise is equivalent to:

$$\begin{aligned} \bar{R}_1 &= -(x_2\partial_3 - x_3\partial_2) \\ \bar{R}_2 &= -(x_3\partial_1 - x_1\partial_3) \\ \bar{R}_3 &= -(x_1\partial_2 - x_2\partial_1) \end{aligned}$$

In classical mechanics, angular momentum of a particle tested with a vector  $a \in \mathbb{R}^3$  is given by

$$L_a := a \cdot (r \times p).$$

The quantum mechanical analogue of momentum  $p$  is the momentum operator  $\frac{\hbar}{i}\nabla$ . The analogue of angular momentum is then the operator:

$$L_a := \frac{\hbar}{i}a \cdot (x \times \nabla).$$

Comparing with the above we see that  $L_a = i\hbar\bar{R}_a$ , for every  $a \in \mathbb{R}^3$ . In the following we shall replace  $L/\hbar$  by  $L$ , so that

$$L_a = i\bar{R}_a \quad (a \in \mathbb{R}^3).$$

Since  $a \mapsto R_a$  and  $R \mapsto \bar{R}$  are Lie algebra homomorphisms, we have that

$$\bar{R}_{a \times b} = [\bar{R}_a, \bar{R}_b].$$

For the angular momentum operators this means that we have the commutation rule:

$$[L_a, L_b] = iL_{a \times b} \quad (a, b \in \mathbb{R}^3).$$

For the components  $L_j = L_{e_j}$  this means

$$\begin{aligned} L_1L_2 - L_2L_1 &= iL_3 \\ L_2L_3 - L_3L_2 &= iL_1 \\ L_3L_1 - L_1L_3 &= iL_2. \end{aligned}$$

In the physics literature this set of equations is often briefly written as

$$L \times L = iL.$$

We thus see that the assertion expressed by the latter formula is equivalent to the assertion that the map  $a \mapsto i^{-1}L_a$  is a Lie algebra homomorphism from  $\mathbb{R}^3$ , equipped with the exterior product, to  $\text{End}(C^\infty(\mathbb{R}^3))$ , equipped with the commutator bracket. One readily checks that the map  $a \mapsto L_a$  is injective. Hence the operators  $i^{-1}L_1, i^{-1}L_2, i^{-1}L_3$  generate a sub Lie algebra of  $\text{End}(C^\infty(\mathbb{R}^3))$  isomorphic to  $(\mathbb{R}^3, \times)$ , hence to  $\mathfrak{so}(3)$ .

In quantum mechanics one now calls any observable  $L$  with the property  $L \times L = iL$  an angular momentum operator. In our language we may translate this statement as follows. To avoid technicalities we assume here that an observable corresponds to a *bounded* Hermitian operator in a Hilbert space (which in general is too strong a restriction). Thus, an angular momentum operator is by definition a map  $L$  from  $\mathbb{R}^3$  to the space of bounded Hermitian operators of  $\mathcal{H}$  such that  $a \mapsto i^{-1}L_a$  is a non-trivial homomorphism of the Lie algebra  $(\mathbb{R}^3, \times)$  into  $\text{End}(\mathcal{H})$ .

**Assumption:** In the following two exercises we assume such a general angular momentum operator  $L : \mathbb{R}^3 \rightarrow \text{End}(\mathcal{H})$  to be fixed.

**Exercise 37.**

- (a) Show that  $\mathfrak{so}(3)$  contains no ideals different from 0 and  $\mathfrak{so}(3)$ .
- (b) Show that the map  $a \mapsto L_a$  is injective.
- (c) Show (with minimum of computation) that the operators  $L_1, L_2, L_3$  satisfy the same commutation relations as  $iR_1, iR_2$  and  $iR_3$ .
- (d) Show (with minimum of computation) that the operators  $L_j$  satisfy the same commutation relations as  $\frac{\sigma_j}{2}$ , where  $\sigma_j$  are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- (e) Show that the map  $\frac{\sigma_j}{2i} \mapsto \frac{1}{i}L_j$  extends to a representation  $\rho$  of  $\mathfrak{su}(2)$  in  $\mathcal{H}$ .

The representation  $\rho$  extends to a complex linear representation of the complexification  $sl(2, \mathbb{C})$  of  $\mathfrak{su}(2)$  in  $\mathcal{H}$ . Let  $H, X, Y$  be the standard basis of  $sl(2, \mathbb{C})$ .

We define the following bounded, but not Hermitian, operators of  $\mathcal{H}$ .

$$L_\pm := L_1 \pm iL_2.$$

- (f) Show that  $L_+ = \rho(X)$ ,  $L_- = \rho(Y)$  and  $2L_3 = \rho(H)$ . Conclude that  $2L_3, L_+, L_-$  is a standard  $sl_2$ -triple.
- (g) Show that  $L_+^* = L_-$ .

**Exercise 38.** Let  $L_j, L_{\pm}$  be as above. We shall now discuss the raising and lowering procedure one finds in the physics literature. We advise the reader to keep the observation about the standard triple in mind.

We define the bounded operator  $L^2$  of  $\mathcal{H}$  by

$$L^2 := L_1^2 + L_2^2 + L_3^2.$$

(a) Show that  $L^2$  is Hermitian and commutes with  $L_j$  for every  $j = 1, 2, 3$ .

(b) Show that

$$L_+L_- = L^2 - L_3(L_3 - I), \quad L_-L_+ = L^2 - L_3(L_3 + I).$$

(c) Assume that  $v \in \mathcal{H}$  is an eigenvector for both  $L^2$  and  $L_3$  with eigenvalues  $\lambda$  and  $\lambda_3$ , respectively. Show that then  $L_+v$  is an eigenvector for  $L^2$  and  $L_3$  with the eigenvalues  $\lambda$  and  $\lambda_3 + 1$ , respectively. Show also that

$$\|L_+v\|^2 = [\lambda - \lambda_3(\lambda_3 + 1)]\|v\|^2$$

(d) Let  $v \in \mathcal{H}$  be an eigenvector for both  $L^2$  and  $L_3$ . Show that there exists a  $k \in \mathbb{N}$  such that  $L_+^k v = 0$ . Show that there exists a  $l \in \mathbb{N}$  such that  $L_-^l v = 0$ .

(e) Let  $v \in \mathcal{H}$  be an eigenvector for both  $L^2$  and  $L_3$ . Show that there exists a unique linear  $\rho$ -invariant linear subspace  $V \subset \mathcal{H}$  containing  $v$  and such that  $\rho|_V$  is a finite dimensional irreducible representation of  $\mathfrak{su}(2)$ .

We know that the irreducible representation  $\rho|_V$  is completely determined by its dimension  $n+1$ , where  $n \geq 0$ . Put  $j = \frac{n}{2}$ .

(f) Show that the eigenvalues of  $L_3|_V$  are  $j, j-1, \dots, -j$ .

(g) Show that  $L^2$  acts by a scalar on  $V$ . Show that this scalar is  $j(j+1)$ . Hint: select  $v \in V \setminus \{0\}$  such that  $L_+v = 0$  and compute  $L^2v$  by using (b).

**Exercise 39.** Let  $H, X, Y$  be the standard triple for  $\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{su}(2)_{\mathbb{C}}$ , and let  $\alpha$  be the root of  $\mathfrak{t} = i\mathbb{R}H$  in  $\mathfrak{g}_{\mathbb{C}}$  determined by  $\alpha(H) = 2$ . Let  $(\pi, V)$  be a finite dimensional continuous representation of  $\mathrm{SU}(2)$ . The collection of weights of  $\pi_*|_{\mathfrak{t}}$  is denoted by  $\Lambda(\pi_*)$ . Show that

(a)  $\Lambda(\pi_*) \subset \frac{1}{2}\mathbb{Z}\alpha$ .

(b) If  $0 \notin \Lambda(\pi_*)$ , then  $\frac{1}{2}\alpha \in \Lambda(\pi_*)$ .

(c) If  $\pi$  is irreducible, then  $0$  and  $\frac{1}{2}\alpha$  do not both belong to  $\Lambda(\pi_*)$ .

(d) The representation  $\pi$  is the direct sum of  $\dim V_0 + \dim V_{\alpha/2}$  irreducibles (among which equivalent ones may occur; they are all counted).

(e)  $\pi$  is irreducible if and only if  $\dim V_0 + \dim V_{\alpha/2} = 1$ .



**Exercise 40.** We consider the connected compact Lie group  $G = \mathrm{SU}(n)$ ,  $n \geq 2$ .

(a) Show that the complexification of its Lie algebra equals

$$\mathfrak{g}_{\mathbb{C}} = \{X \in M_n(\mathbb{C}) : \mathrm{tr} X = 0\} = \mathfrak{sl}(n, \mathbb{C}).$$

Show that the linear space  $\mathfrak{t}$  consisting of all diagonal matrices in  $\mathfrak{g}$  is a maximal torus.

(b) For every  $1 \leq k \leq n$  we define  $\epsilon_k \in i\mathfrak{t}^*$  to be the real linear map  $\mathfrak{t} \rightarrow i\mathbb{R}$  that assigns to  $X \in \mathfrak{t}$  the  $k$ -th diagonal element  $X_{kk}$ .

If  $1 \leq i, j \leq n$  then we denote by  $E_{ij}$  the matrix whose entries are zero, except on the  $i$ -th row and the  $j$ -th column, where the entry is 1.

Show that the linear subspaces  $\mathbb{C}E_{ij}$  ( $i \neq j$ ) are root spaces of  $\mathfrak{g}_{\mathbb{C}}$ . Determine the set  $R = R(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t})$  of roots in terms of the  $\epsilon_k$ ,  $1 \leq k \leq n$ .

(c) Put  $E = i\mathfrak{t}^*$ . Show that  $(E, R)$  is a root system, by verifying the conditions of the definition of a root system. Determine, for every  $\alpha \in R$ , the reflection  $s_{\alpha} : E \rightarrow E$ .

(d) Show that the Weyl group  $W$  of  $(E, R)$  is isomorphic to the permutation group  $S_n$  of  $n$ -elements. More precisely, define an explicit map  $S_n \rightarrow W$ , and show that this map is an isomorphism of groups.

(e) Determine a fundamental system  $S$  for  $R$ .

(f) Prove that the reflections  $s_{\alpha}$  ( $\alpha \in S$ ) already generate  $W$ .

(g) Determine explicitly a  $W$ -invariant inner product on  $E$ .

(h) Determine the Cartan integers associated with  $S$ .

(i) Determine the Dynkin diagram of  $\mathrm{SU}(n)$ .

In the following we assume that  $n = 3$ .

(j) Let  $\alpha = \epsilon_1 - \epsilon_2$  and  $\beta = \epsilon_2 - \epsilon_3$ . Show that  $R = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta)\}$ . Show that the angle between  $\alpha$  and  $\beta$  equals  $2\pi/3$ . Make a picture of  $R$ .

**Exercise 41. Induced representation.** Let  $G$  be a Lie group,  $H$  a closed subgroup, and  $(\xi, V)$  a finite dimensional representation of  $H$ .

(a) Show that the action of  $H$  on  $G \times V$  given by  $h \cdot (g, v) = (gh, \xi(h)^{-1}v)$  is proper and free, hence of PFB type.

It follows from the above that the quotient space  $\mathcal{V} = G \times_H V := (G \times V)/H$  is a smooth manifold.

(b) Show that the map  $p : \mathcal{V} \rightarrow G/H$  induced by the projection  $G \times V \rightarrow G$  is smooth.

- (c) Show that for each  $g \in G$  the map  $\varphi_g : v \mapsto [(g, v)]$  defines a bijection from  $V$  onto the fiber  $p^{-1}(gH)$ .
- (d) Show that  $p : \mathcal{V} \rightarrow G/H$  has a unique structure of vector bundle such that all maps  $\varphi_g$  ( $g \in G$ ) are linear. This vector bundle is said to be associated with the representation  $\xi$ .
- (e) Show that the action of  $G$  on  $G \times V$  given by  $g \cdot (x, v) = (gx, v)$  factorizes to a smooth action of  $G$  on  $\mathcal{V}$ . Show that for each  $x \in G/H$  the action by  $g \in G$  maps the fiber  $\mathcal{V}_x$  linearly and bijectively onto the fiber  $\mathcal{V}_{gx}$  (a vector bundle over  $G/H$  with this property is said to be homogeneous).
- (f) Let  $C(\mathcal{V})$  denote the space of continuous sections of the vectorbundle  $\mathcal{V}$ . For  $g \in G$  and  $s \in C(\mathcal{V})$  we define  $\pi(g)s : G/H \rightarrow \mathcal{V}$  by  $\pi(g)s(x) = g \cdot s(g^{-1}x)$ . Show that  $\pi$  defines a representation of  $G$  in  $C(\mathcal{V})$ . This representation is said to be induced from  $\xi$ . Notation  $\pi = \text{Ind}_H^G(\xi)$ .
- (g) Let  $C(G, V, \xi)$  denote the space of continuous functions  $\varphi : G \rightarrow V$  such that

$$\varphi(gh) = \xi(h)^{-1}\varphi(g) \quad (g \in G, h \in H).$$

For  $\varphi$  in this space we define  $\tilde{s}_\varphi : G \rightarrow \mathcal{V}$  by  $\tilde{s}_\varphi(x) = [(x, \varphi(x))]$ . Show that  $\tilde{s}_\varphi$  factorizes to a section  $s_\varphi$  of  $\mathcal{V}$ . Show that  $\varphi \mapsto s_\varphi$  is a linear bijection  $C(G, V, \xi) \simeq C(\mathcal{V})$ . Via this bijection we may realize the representation  $\pi$  on the space  $C(G, V, \xi)$ . Show that it is then given by the formula:

$$\pi(g)\varphi(x) = \varphi(g^{-1}x) \quad (\varphi \in C(G, V, \xi), g, x \in G).$$

This realization of the induced representation is called ‘the induced picture.’