Extra exercises 2016

Exercise 51. We assume that G is a Lie group, and (π, V) a finite dimensional continuous represenation of G. Show that for all $x \in G$ and $X, Y \in \mathfrak{g}$ the following identities are valid:

(a) $\pi(x)\pi_*(Y) = \pi_*(\mathrm{Ad}\,(x)Y)\pi(x)$;

(b) $\pi_*(X)\pi_*(Y) = \pi_*(Y)\pi_*(X) + \pi_*([X, Y])$.

Extra exercises 2018

Exercise 52. Let G and H be Lie groups. If (π, V) and (ρ, W) are finite dimensional continuous representations of G and H, respectively, then the exterior tensor product of π and ρ is defined to be the representation $\pi \widehat{\otimes} \rho$ of $G \times H$ in $V \otimes W$ given by

$$\pi \widehat{\otimes} \rho(g, h) = \pi(g) \otimes \rho(h), \qquad ((g, h) \in G \times H).$$

(a) Show that the representation $\pi \widehat{\otimes} \rho$ is continuous finite dimensional.

(b) Show that the character of $\pi \widehat{\otimes} \rho$ is the function $\chi_{\pi \widehat{\otimes} \rho} : G \times H \to \mathbb{C}$ given by

$$\chi_{\pi \widehat{\otimes} \rho}(g, h) = \chi_\pi(g)\chi_\rho(h).$$

(c) If G and H are compact show that the following assertions are equivalent.

(1) π is irreducible as a representation of G and ρ is irreducible as a representation of H;

(2) $\pi \widehat{\otimes} \rho$ is irreducible as a representation of $G \times H$.

Exercise 53. We assume that G and H are compact Lie groups, and that (π, V) is a finite dimensional continuous representation of $G \times H$. We identify G and H with the subgroups $G \times \{e_H\}$ and $\{e_G\} \times H$ of $G \times H$.

(a) For δ a finite dimensional irreducible representation of G, show that the projection operator $P_\delta : V \to V$ associated with $\pi|_G$ as in Exercise 33 commutes with $\pi(H)$.

We now assume that π is irreducible.

(b) Show that $P_\delta(V) = V$.

(c) Show that the canonical map $V_\delta \otimes \mathrm{Hom}_G(V_\delta, V) \to V$ is a linear isomorphism.

(d) Show that the natural representation ρ of H in $\mathrm{Hom}_G(V_\delta, V)$ is irreducible and that

$$\pi \simeq \delta \widehat{\otimes} \rho.$$