Homework exercise HW7 = 38, revised form

Let $\mathcal{H}$ be a complex Hilbert space. We denote by $\mathrm{End}\,(\mathcal{H})$ the space of bounded linear maps $\mathcal{H} \to \mathcal{H}$. Given $T : \mathcal{H} \to \mathcal{H}$ we define the adjoint $T^* \in \mathrm{End}\,(\mathcal{H})$ by

$$\langle T^* v, w\rangle = \langle v, T^* w\rangle.$$

The operator T is said to be Hermitian if $T^* = T$. We fix an angular momentum operator on $\mathcal{H}$, i.e., a Lie algebra homomorphism $\rho : \mathfrak{su}(2) \to \mathrm{End}\,(H)$ such that $\rho(U)^* = -\rho(U)$ for all $U \in \mathfrak{su}(2)$. By complex linearity, this Lie algebra homomorphism is extended to a linear map $\rho : \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{End}\,(H)$, which is a complex Lie algebra homomorphism. It is customary to write

$$L_j = \tfrac{1}{2}\rho(\sigma_j), \qquad (j \in \{1,2,3\}),$$

where σ_j are the Pauli matrices given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

From the commutation relations among the $\sigma_1, \sigma_2, \sigma_3$ it easily follows that the operators L_1, L_2, L_3 satisfy the commutation relations

$$[L_2, L_3] = iL_1, \quad [L_3, L_1] = iL_2, \quad [L_1, L_2] = iL_3.$$

In the physics literature this is often abbreviated as

$$L \times L = iL.$$

Let H, X, Y be the standard basis of $\mathfrak{sl}(2,\mathbb{C})$. We define the following bounded, but not Hermitian, operators of $\mathcal{H}$.

$$L_{\pm} := L_1 \pm iL_2.$$

(a) Show that $L_+ = \rho(X)$, $L_- = \rho(Y)$ and $2L_3 = \rho(H)$. Conclude that $2L_3, L_+, L_-$ is a standard $\mathfrak{sl}_2$-triple.

(b) Show that $L_+^* = L_-$.

We shall now discuss the raising and lowering procedure one finds in the physics literature. We advise the reader to keep the observation about the standard triple in mind.

We define the bounded operator L^2 of $\mathcal{H}$ by

$$L^2 := L_1^2 + L_2^2 + L_3^2.$$

(c) Show that

$$L_+L_- = L^2 - L_3(L_3 - I), \qquad L_-L_+ = L^2 - L_3(L_3 + I).$$

(d) Show that L^2 is Hermitian and commutes with L_j for every $j = 1, 2, 3$.

(e) Assume that $v \in \mathcal{H}$ is an eigenvector for both L^2 and L_3 with eigenvalues λ and λ_3, respectively. Show that then L_+v is an eigenvector for L^2 and L_3 with the eigenvalues λ and $\lambda_3 + 1$, respectively. Show also that

$$\|L_+v\|^2 = [\lambda - \lambda_3(\lambda_3 + 1)]\|v\|^2$$

(f) Let $v \in \mathcal{H}$ be an eigenvector for both L^2 and L_3. Show that there exists a $k \in \mathbb{N}$ such that $L_+^k v = 0$. Show that there exists an $l \in \mathbb{N}$ such that $L_-^l v = 0$.

(g) Let $v \in \mathcal{H}$ be an eigenvector for both L^2 and L_3. Show that there exists a unique ρ-invariant linear subspace $V \subset \mathcal{H}$ containing v and such that $\rho|_V$ is a finite dimensional irreducible representation of $\mathfrak{su}(2)$.

We know that the irreducible representation $\rho|_V$ is completely determined by its dimension $n+1$, where $n \geq 0$. Put $j = \frac{n}{2}$.

(h) Show that the eigenvalues of $L_3|_V$ are $j, j-1, \ldots, -j$.

(i) Show that L^2 acts by a scalar on V. Show that this scalar is $j(j+1)$. Hint: select $v \in V \setminus \{0\}$ such that $L_+v = 0$ and compute $L^2 v$ by using (c).