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Asymptotic expansions and boundary values of eigenfunctions on Riemannian symmetric spaces

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Introduction

Let $X = G/K$ be a Riemannian symmetric space of the noncompact type with boundary $B = K/M$, and let $\mathbb{D}(X)$ be the algebra of all G-invariant differential operators on X. In harmonic analysis on X it is of fundamental importance to study the properties of simultaneous eigenfunctions of $\mathbb{D}(X)$ on X, and in particular to determine their behavior at infinity (that is, near the boundary B). For eigenfunctions transforming finitely under K this was done by Harish-Chandra ([H-C 58] and [H-C 60], see also [War 72 II] and [C-M 82]), who obtained converging series expansions. In the present paper we adapt a technique due to Wallach ([Wal 83]) to obtain *asymptotic expansions* for a wider class of joint eigenfunctions f on X, imposing on f and its derivatives from the left a certain condition of (at most) *exponential growth* with respect to the Riemannian distance on X. The asymptotic expansions lead to a theory of *boundary values* in $C^\infty(B)$ of f, defined via leading terms in the expansion, and by means of this theory we are able to characterize the eigenfunctions which satisfy our growth condition as being precisely the functions on X which can be represented by a generalized *Poisson integral* of a C^∞-function on B.

To be more precise, let χ be a map $\mathbb{D}(X) \to \mathbb{C}$ and let $\mathscr{E}_\chi(X)$ denote the corresponding joint eigenspace

$$\mathscr{E}_\chi = \{ f \in C^\infty(X) \mid \forall D \in \mathbb{D}(X) : Df = \chi(D) f \}.$$

Let L denote the left regular representation of G on $C^\infty(X)$, and let $\mathscr{U}(\mathfrak{g})$ be the enveloping algebra of the Lie algebra $\mathfrak{g}$ of G. Via L the elements of $\mathscr{U}(\mathfrak{g})$ act as differential operators on X. The condition we impose on f is that there exists $r \in \mathbb{R}$ such that for all $u \in \mathscr{U}(\mathfrak{g})$ we have an upper bound

$$(0.1) \qquad |L_u f(x)| \leqq C e^{r \operatorname{dist}(x, eK)}$$

for all $x \in X$, with C a constant (depending on u but not on x). Here dist denotes the Riemannian distance on X. Let $\mathscr{E}_\chi^\infty(X)$ denote the space of all $f \in \mathscr{E}_\chi(X)$ satisfying (0. 1). Let $\mathfrak{a}$ be a maximal abelian split subspace of $\mathfrak{g}$, and $\mathfrak{a}^+$ an open Weyl chamber. We derive (Theorem 3. 5) for each $f \in \mathscr{E}_\chi^\infty$ and $g \in G$ an asymptotic expansion

$$(0.2) \qquad f(g \exp tH) \sim \sum_\xi p_\xi(g, tH)\, e^{t\xi(H)}$$

for $H \in \mathfrak{a}^+$, as $t \to +\infty$. Here ξ belongs to an explicitly determined discrete infinite set of linear forms on $\mathfrak{a}$, and the $p_\xi(g, \cdot)$ are polynomials on $\mathfrak{a}$, depending smoothly on g. The boundary value $\beta(f) \in C^\infty(B)$ of f is then defined by means of the top order coefficient of the polynomial p_ξ with ξ a "leading exponent" of (0. 2) (cf. Section 8). Now it is easily seen that the generalized Poisson transformation $\mathscr{P}_\chi$ from functions on B to $\mathscr{E}_\chi(X)$ maps $C^\infty(B)$ into $\mathscr{E}_\chi^\infty(X)$. We prove (Theorem 10. 1) that it is actually *onto* $\mathscr{E}_\chi^\infty(X)$, and *characterize its inverse as being the boundary value map* β (up to a constant non-zero factor).

We give two major applications of this theory, presented respectively in Parts II and III of the paper. The first is a new proof of a theorem due to Oshima and Sekiguchi ([Os 80]) which characterizes the functions on X which can be represented by a generalized Poisson integral of a distribution on B (Theorem 12. 2). Thus let $\mathscr{E}_\chi^*(X)$ denote the space of all $f \in \mathscr{E}_\chi(X)$ for which there are $r \in \mathbb{R}$ and $C > 0$ such that

$$|f(x)| \leqq C e^{r \operatorname{dist}(x, eK)}$$

(compared to (0. 1) there is no condition on the derivatives of f). It is not difficult to see that $\mathscr{P}_\chi$ maps the space $\mathscr{D}'(B)$ of distributions on B into $\mathscr{E}_\chi^*(X)$ (this was observed in [Le 78]). The theorem of [OS 80] confirms that it is actually onto $\mathscr{E}_\chi^*(X)$. The proof in [OS 80] uses advanced micro-local techniques from [SKK 73] and [KO 77] together with the solution in [KKMOOT 78] to Helgason's conjecture (which says that $\mathscr{P}_\chi$ maps the space of hyperfunctions on B onto $\mathscr{E}_\chi(X)$). (For G/K of real rank one the Oshima-Sekiguchi Theorem was proved in [Le 78] (with a restriction on χ) with a much more elementary proof, and for harmonic functions on the disk it is in [Kö 52, Satz 19]). A different proof has been given by Wallach in [Wal 83], which avoids the hyperfunction theory, but however does not explicitly show how the distribution T on B is obtained from the eigenfunction $f \in \mathscr{E}_\chi^*(X)$. Our new proof (which owes a great deal to Wallach's) simply reduces the theorem, by convolution with test functions, to the theorem mentioned above that $\mathscr{P}_\chi$ maps $C^\infty(B)$ onto $\mathscr{E}_\chi^\infty(X)$. Since this was proved by explicit construction of the inverse as a boundary value map, the inverse of $\mathscr{P}_\chi\colon \mathscr{D}'(B) \to \mathscr{E}_\chi^*(X)$ is also given by a *distribution boundary value map* (up to a constant multiple). We find this knowledge of the inverse of $\mathscr{P}_\chi$ interesting, it is certainly important in Part III of the paper.

The second application, the content of Part III, is to give a new proof of a theorem of Matsuki and Oshima [OM 84] which asserts L^2-ness of certain functions, related to the discrete series, on a non-Riemannian semisimple symmetric space G/H (Theorem 19. 1). In our opinion our proof simplifies that of [OM 84] considerably because it replaces the complicated microlocal analysis of [SKK 73], [KO 77] and [Osh 84] (prerequisites for [OM 84]) with simpler expansions from [Ban 84] (along the lines of [CM 82]) in combination with the present asymptotic techniques. In particular we obtain a new proof of Flensted-Jensen's conjecture "$C=0$" in [F-J 80], originally solved by Oshima (cf. [Schl 84], Ch. 8). In the special case of a connected real semisimple Lie group considered as a symmetric space (cf. [Schl 84], Sect. 8.4.1) this gives a new proof of the existence of a discrete series representation corresponding to each of Harish-Chandra's parameters ([H-C 66], § 41). The basic ingredient in our proof (which has some similarity with Oshima's proof of the Flensted-Jensen conjecture) consists of an argument which excludes certain coefficients in the series expansion (from [Ban 84]) of the function in question. The argument involves analytic continuation, for which the theory from Part I is crucial.

As mentioned, the asymptotic expansions for the functions in $\mathscr{E}_\chi^\infty(X)$ are derived using ideas from [Wal 83] (where asymptotic expansions for matrix coefficients of representations of G are derived). However our approach deviates somewhat from Wallach's. Thus (0.2) holds for H in $\mathfrak{a}^+$ whereas the expansions in [Wal 83] are along rank one parabolics. Moreover the fact that we are considering functions on G/K (whereas [Wal 83] assumes only K-finiteness from the right) makes it possible to get more precise knowledge of the nature of the asymptotics, such as which exponents can occur in (0.2) and uniform control when the homomorphism $\chi\colon \mathbb{D}(G/K) \to \mathbb{C}$ varies. To be more precise, the homomorphisms $\chi\colon \mathbb{D}(G/K) \to \mathbb{C}$ are parametrized (modulo the action of a finite group) by elements λ in the complex linear dual $\mathfrak{a}_c^*$ of $\mathfrak{a}$. If for each λ in an open subset of $\mathfrak{a}_c^*$ we are given a function f_λ in the corresponding space $\mathscr{E}_\chi^\infty$ $(\chi = \chi_\lambda)$, and if f_λ depends holomorphically on λ (in a proper sense) then in some explicit sense (see Theorem 3.6 and Corollary 3.7) the whole expansion (0.2) depends holomorphically on λ. This property plays an important role in the application to G/H (in the above mentioned argument involving analytic continuation).

As a byproduct we obtain in Corollary 16.6 a property of generalized Poisson integrals which is related to Fatou's theorem (the result is a generalization of a result due to Michelson [Mi 73]. It has been obtained independently by Sjögren [Sj 86]).

We would also like to point out that even in the simplest example, that of harmonic functions on the disk, our asymptotic expansions and boundary value maps seem to be new.

If this paper appears long, it is because one of our objectives has been to use only well known prerequisites (say from the books of Helgason, [He 78] and [He 84]). Apart from these the theory developed in Parts I and II is virtually self-contained (though using ideas from [Wal 83], the presentation is logically independent of it). In Part III we use some structure theory for semisimple symmetric spaces (with [Schl 84], Ch. 7 as reference) together with [Ban 84].

Part I. Asymptotic expansions

1. Notations and preliminaries

Let G be a connected real semisimple Lie group with finite center and K a maximal compact subgroup of G. Then G/K is a Riemannian symmetric space (for the following general background we refer to [He 84]). Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, $\theta \in \mathrm{Aut}(\mathfrak{g})$ the corresponding Cartan involution, $\mathfrak{p} \subset \mathfrak{g}$ the -1 eigenspace of θ, $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$, and $\Sigma \subset \mathfrak{a}^*$ the corresponding system of restricted roots with W the associated Weyl group.

Let $\mathbb{D}(G/K)$ be the algebra of invariant differential operators on G/K, and let $\mathscr{U}(\mathfrak{g})$ denote the universal enveloping algebra of the complexification $\mathfrak{g}_c$ of $\mathfrak{g}$. *Unless otherwise specified we identify elements of* $\mathscr{U}(\mathfrak{g})$ *with left invariant differential operators on* G. Then there is a natural surjective homomorphism

$$\mu: \mathscr{U}(\mathfrak{g})^K \to \mathbb{D}(G/K) \tag{1.1}$$

(where as usual, $\mathscr{U}(\mathfrak{g})^K$ means the centralizer of K in $\mathscr{U}(\mathfrak{g})$), whose kernel is $\mathscr{U}(\mathfrak{g})^K \cap \mathscr{U}(\mathfrak{g})\,\mathfrak{k}$. We also have Harish-Chandra's canonical isomorphism

$$\gamma: \mathbb{D}(G/K) \xrightarrow{\sim} \mathscr{U}(\mathfrak{a})^W. \tag{1.2}$$

From (1.2) it follows that the characters of $\mathbb{D}(G/K)$ are parametrized by the orbits of W in $\mathfrak{a}_c^*$: $\chi_\lambda(D) = \gamma(D)(\lambda)$ for $D \in \mathbb{D}(G/K)$ and $\lambda \in \mathfrak{a}_c^*$, with $\chi_\nu = \chi_\lambda$ if and only if $\nu \in W\lambda$.

Fix a basis Δ for Σ, and let $\mathbb{N} \cdot \Delta$ denote the set of all elements $\sum_{\alpha \in \Delta} n_\alpha \alpha \in \mathfrak{a}^*$ with integers $n_\alpha \geqq 0$, $\succ$ the partial order on $\mathfrak{a}_c^*$ given by $\lambda \succ \nu \Leftrightarrow \lambda - \nu \in \mathbb{N} \cdot \Delta$, Σ^+ the set of positive roots, $\mathfrak{n} \subset \mathfrak{g}$ the sum of the corresponding root spaces, $\bar{\mathfrak{n}} = \theta\mathfrak{n}$, $\mathfrak{a}^+ \subset \mathfrak{a}$ the positive open chamber, and $\varrho \in \mathfrak{a}^*$ half the sum of the roots of Σ^+ counted with their multiplicities in $\mathfrak{n}$. For $\eta \in \mathfrak{a}_c^*$ let $T_\eta: \mathscr{U}(\mathfrak{a}) \to \mathscr{U}(\mathfrak{a})$ denote the automorphism generated by

$$T_\eta(H) = H + \eta(H) \tag{1.3}$$

for $H \in \mathfrak{a}_c$, and let $'v = T_\varrho(v)$ for $v \in \mathscr{U}(\mathfrak{a})$. For $u \in \mathscr{U}(\mathfrak{g})$ let $u_\mathfrak{a} \in \mathscr{U}(\mathfrak{a})$ be defined by

$$u - u_\mathfrak{a} \in \mathfrak{n}\mathscr{U}(\mathfrak{g}) + \mathscr{U}(\mathfrak{g})\,\mathfrak{k} \tag{1.4}$$

then the isomorphism (1.2) is defined via $\gamma(\mu(u)) = {'u_\mathfrak{a}}$ for $u \in \mathscr{U}(\mathfrak{g})^K$. Equivalently

$$u - {'\gamma(\mu(u))} \in \bar{\mathfrak{n}}\mathscr{U}(\mathfrak{g}) + \mathscr{U}(\mathfrak{g})\,\mathfrak{k}. \tag{1.5}$$

For each $\lambda \in \mathfrak{a}_c^*$ let $\mathscr{E}_\lambda(G/K)$ be the corresponding joint eigenspace, consisting of those distributions $f \in \mathscr{D}'(G/K)$ which satisfy

$$Df = \chi_\lambda(D) f \tag{1.6}$$

for all $D \in \mathbb{D}(G/K)$. Then $\mathscr{E}_{w\lambda} = \mathscr{E}_\lambda$ for $w \in W$. It is well known that the elements of $\mathscr{E}_\lambda$ are real analytic functions on G/K.

Let $N = \exp\mathfrak{n}$, $\bar{N} = \exp\bar{\mathfrak{n}}$, $M = K^\mathfrak{a}$, and $P = MAN \subset G$. Then $G = KAN$ and P is a minimal parabolic subgroup. For $g \in G$ let $H(g) \in \mathfrak{a}$ be defined via $g \in K \exp H(g)\, N$, and let $e_\lambda(g) = e^{-\lambda(H(g^{-1}))}$ for $\lambda \in \mathfrak{a}_c^*$. For $a \in A$ we put $a^\lambda = e_\lambda(a)$. Notice that $L_{na} e_\lambda = a^{-\lambda} e_\lambda$ for $a \in A$ and $n \in N$, where L is the left regular representation of G. From this and (1.4) it follows that

$$u e_\lambda(nak) = (\mathrm{Ad}\, k\, u)_\mathfrak{a}(\lambda)\, a^\lambda \tag{1.7}$$

for $u \in \mathscr{U}(\mathfrak{g})$ and $k \in K$. In particular, it follows that $De_\lambda = \gamma(D)(\lambda - \varrho)\, e_\lambda$ for $D \in \mathbb{D}(G/K)$, that is, $e_{\lambda+\varrho} \in \mathscr{E}_\lambda$.

For each continuous function φ on K/M and $\lambda \in \mathfrak{a}_c^*$ the *Poisson transform* $\mathscr{P}_\lambda \varphi$ is the function on G/K given by

$$(1.8) \qquad \mathscr{P}_\lambda \varphi(gK) = \int_K \varphi(k)\, e_{\lambda+\varrho}(k^{-1}g)\, dk = \int_K \varphi(k)\, e^{-(\lambda+\varrho)H(g^{-1}k)} dk$$

for $g \in G$. Obviously $\mathscr{P}_\lambda \varphi \in \mathscr{E}_\lambda(G/K)$.

Let $C(G/P; L_\lambda)$ denote the space of continuous functions ϕ on G satisfying

$$(1.9) \qquad \phi(gman) = a^{\lambda-\varrho}\phi(g)$$

for all $g \in G$, $m \in M$, $a \in A$, and $n \in N$. The *Poisson transformation* $\mathscr{P}\colon C(G/P; L_\lambda) \to C^\infty(G/K)$ is given by

$$(1.10) \qquad \mathscr{P}\phi(gK) = \int_K \phi(gk)\, dk$$

for $g \in G$. Clearly $\mathscr{P}$ is equivariant for L. Moreover, to each $\varphi \in C(K/M)$ corresponds a unique $\varphi_\lambda \in C(G/P; L_\lambda)$ with $\varphi_\lambda(K) = \varphi(kM)$ for $k \in K$, and we have (by an easy change of variables) $\mathscr{P}\varphi_\lambda = \mathscr{P}_\lambda \varphi$.

2. Some function spaces on G

In this section we introduce a certain growth condition, which when imposed on eigenfunctions will enable us (in the following sections) to derive asymptotic expansions. The growth condition is satisfied by $\mathscr{P}_\lambda \varphi$ for $\varphi \in C^\infty(K/M)$.

For each $g \in G$ we denote by $\|g\|$ the (operator) norm of $\mathrm{Ad}\, g$ on $\mathfrak{g}$, which is a real Hilbert space when equipped with the scalar product $\langle X, Y\rangle_\theta = -B(X, \theta Y)$ (where B is the Killing form). The assignment $g \to \|g\|$ has the following properties:

Lemma 2.1. *Let $x, y \in G$. Then*

(i) $\|x\| = \|\theta x\| = \|x^{-1}\| \geqq 1$,

(ii) $\|xy\| \leqq \|x\|\,\|y\|$,

(iii) *if $x = k_1 a k_2$ with $k_1, k_2 \in K$ and $a \in A$ then*

$$\|x\| = \max_{\alpha \in \Sigma} a^\alpha,$$

(iv) *there are constants $c_1, c_2 > 0$ such that if $x = \exp Y$ with $Y \in \mathfrak{p}$ then*

$$e^{c_1|Y|} \leqq \|x\| \leqq e^{c_2|Y|}$$

where $|Y|$ denotes the Killing norm $B(Y, Y)^{\frac{1}{2}}$ of Y,

(v) $\|a\| \leqq \|an\|$ *for $a \in A$, $n \in N$.*

Proof. (i)—(iv) are elementary, and (v) follows from Kostant's convexity theorem ([He 84], IV, Thm. 10.5). □

For any function $f: G \longrightarrow \mathbb{C}$ and $r \in \mathbb{R}$ we define

$$(2.1) \qquad \|f\|_r = \sup_{g \in G} \|g\|^{-r} |f(g)|$$

and say that f increases *at most exponentially* if $\|f\|_r < +\infty$ for some $r \in \mathbb{R}$. The motivation for using this terminology is that by (iv), $\|g\|$ as a measure of the distance in G/K from gK to the base point eK, is equivalent to an exponential in the Riemannian distance (cf. [War 72 I], p. 282). Let $C_r(G)$ denote the Banach space of continuous functions f on G with $\|f\|_r < +\infty$, with norm $\|\cdot\|_r$.

Examples 2.2. (i) Let $\lambda \in \mathfrak{a}_c^*$ and let $r(\lambda) = c_1^{-1} |\mathrm{Re}\,\lambda - \varrho|$ (where c_1 is given in Lemma 2.1 (iv)). For each $\varphi \in C(K/M)$ let $\varphi_\lambda \in C(G)$ be given by $\varphi_\lambda(kan) = \varphi(k)\, a^{\lambda - \varrho}$ for $k \in K$, $a \in A$, and $n \in N$. Then Lemma 2.1 (v) shows that $\varphi_\lambda \in C_{r(\lambda)}(G)$. Clearly $\varphi \longrightarrow \varphi_\lambda$ is a bounded linear map $C(K/M) \longrightarrow C_{r(\lambda)}(G)$. Using (1.10) it follows that $\mathscr{P}_\lambda$ is a bounded linear map of $C(K/M)$ into $C_{r(\lambda)}(G)$.

(ii) A matrix coefficient of a finite dimensional representation of G has at most exponential growth.

The right and left regular representations R and L of G both preserve $C_r(G)$, indeed we have

$$(2.2) \qquad \|L_x f\|_r \leqq \|x\|^{|r|}\, \|f\|_r$$

and

$$(2.3) \qquad \|R_x f\|_r \leqq \|x\|^{|r|}\, \|f\|_r$$

for $x \in G$ and $r \in \mathbb{R}$.

Let $C_r^\infty = C_r^\infty(G)$ denote the space of C^∞-vectors for L in $C_r(G)$, that is

$$C_r^\infty(G) = \{ f \in C^\infty(G) \mid \forall u \in \mathscr{U}(\mathfrak{g}) \colon L_u f \in C_r(G) \}.$$

We endow C_r^∞ with its standard Fréchet topology, the definition of which we recall: Let $X_1, \ldots, X_p$ be a basis for $\mathfrak{g}$, and let $X^\gamma = X_1^{\gamma_1} \cdots X_p^{\gamma_p} \in \mathscr{U}(\mathfrak{g})$ for $\gamma = (\gamma_1, \ldots, \gamma_p) \in \mathbb{N}^p$. By Poincaré-Birkhoff-Witt the elements X^γ constitute a basis for $\mathscr{U}(\mathfrak{g})$. For $q \in \mathbb{N}$ and $f \in C^q(G)$ a q times continuously differentiable function we define

$$\|f\|_{q,r} = \max_{|\gamma| \leqq q} \|L_{X^\gamma} f\|_r.$$

Endowed with this norm the space

$$C_r^q = C_r^q(G) = \{ f \in C^q(G) \mid \|f\|_{q,r} < +\infty \}$$

is a Banach space. Obviously $C_r^q \subset C_r^{q'}$ if $q' \leqq q$, and $C_r^\infty = \bigcap_q C_r^q$. The topology on C_r^∞ is the projective limit for this intersection (that is, it is given by the family of norms $\|\cdot\|_{q,r}$ $(q \in \mathbb{N})$ on C_r^∞).

Example 2.3. Let $C^\infty(G/P; L_\lambda) = C^\infty(G) \cap C(G/P; L_\lambda)$ be endowed with the Fréchet topology inherited from $C^\infty(G)$. Then $C^\infty(G/P; L_\lambda) \cong C^\infty(K/M)$, and it is the space of C^∞-vectors for L in $C(G/P; L_\lambda)$. From (1.10) we infer that $\mathscr{P}_\lambda$ maps $C^\infty(K/M)$ continuously into $C_{r(\lambda)}^\infty$ (cf. Example 2.2 (i)).

We now consider for each $q \in \mathbb{N}$ the action of L and R on C_r^q. Obviously L_x $(x \in G)$ leaves C_r^q invariant. Actually it follows from (2.2) and Example 2.2(ii) (applied to the adjoint action of G on $\mathscr{U}(\mathfrak{g})_q = \{u \in \mathscr{U}(\mathfrak{g}) \mid \deg u \leqq q\}$) that there exist constants C and s in $\mathbb{R}$ (depending only on q) such that

$$\|L_x f\|_{q,r} \leqq C \|x\|^{|r|+s} \|f\|_{q,r} \tag{2.4}$$

for all $f \in C_r^q$ and $x \in G$. On the other hand it follows immediately from (2.3) that

$$\|R_x f\|_{q,r} \leqq \|x\|^{|r|} \|f\|_{q,r}. \tag{2.5}$$

Hence also R_x leaves C_r^q invariant. As for the derived actions we obviously have for $u \in \mathscr{U}(\mathfrak{g})$ and $q \geqq \deg u$ that

$$\|L_u f\|_{q-\deg u, r} \leqq C \|f\|_{q,r} \tag{2.6}$$

for all $f \in C_r^q$, where C is a constant depending on u and q. Moreover, using the relation

$$uf(g) = L_{\mathrm{Ad}\, g(u^\vee)} f(g)$$

(where $u \longrightarrow u^\vee$ is the canonical antiautomorphism of $\mathscr{U}(\mathfrak{g})$), we get for $u \in \mathscr{U}(\mathfrak{g})$, $q \geqq \deg u$, and $f \in C_r^q$ that

$$\|uf\|_{q-\deg u, r+s} \leqq C \|f\|_{q,r} \tag{2.7}$$

where $s \geqq 0$ is a constant depending only on u, and where $C > 0$ depends on u and q.

Let $\Omega \subset \mathbb{C}^n$ $(n \in \mathbb{N})$ be an open set and $\varphi: \Omega \longrightarrow C_r^\infty$ a map. We say that φ is *holomorphic* if for each $q \in \mathbb{N}$ it maps Ω holomorphically into the Banach space C_r^q, or equivalently, if for each $u \in \mathscr{U}(\mathfrak{g})$ the map $L_u \circ \varphi: \Omega \longrightarrow C_r$ is holomorphic.

Example 2.4. Let $r > 0$ and let $\Omega = \{\lambda \in \mathfrak{a}_c^* \mid r(\lambda) < r\}$ (cf. Example 2.2(i)). Then it can be seen that $\lambda \longrightarrow \varphi_\lambda$ is holomorphic $\Omega \longrightarrow C_r^\infty$ for each $\varphi \in C^\infty(K/M)$. By continuity of the map $C_r^\infty \longrightarrow C_r^\infty$ given by $f \longrightarrow \int_K R_k f\, dk$ it follows that $\lambda \longrightarrow \mathscr{P}_\lambda \varphi$ is holomorphic $\Omega \longrightarrow C_r^\infty$.

Notice that $\mathscr{E}_{\lambda,r}^\infty = \mathscr{E}_\lambda(G/K) \cap C_r^\infty(G)$ is a closed subspace of $C_r^\infty(G)$, for each $\lambda \in \mathfrak{a}_c^*$ (cf. (2.7)). We equip $\mathscr{E}_{\lambda,r}^\infty$ with the Fréchet topology inherited from C_r^∞. Let $\mathscr{E}_\lambda^\infty = \mathscr{E}_\lambda^\infty(G/K)$ denote the space $\mathscr{E}_\lambda^\infty = \bigcup_{r \in \mathbb{R}} \mathscr{E}_{\lambda,r}^\infty$, that is

$$\mathscr{E}_\lambda^\infty = \{f \in \mathscr{E}_\lambda(G/K) \mid \exists r \in \mathbb{R}\ \forall u \in \mathscr{U}(\mathfrak{g}): L_u f \in C_r(G)\}. \tag{2.8}$$

(In Section 10 we shall prove that $\mathscr{E}_\lambda^\infty = \mathscr{E}_{\lambda,r}^\infty$ for some $r \in \mathbb{R}$.)

Let $\Omega \subset \mathfrak{a}_c^*$ be an open set, and let $(f_\lambda)_{\lambda \in \Omega}$ be a family of functions on G. We say that $(f_\lambda)_{\lambda \in \Omega}$ is a *holomorphic family in* $\mathscr{E}_\lambda^\infty(G/K)$, if $f_\lambda \in \mathscr{E}_\lambda(G/K)$ for each $\lambda \in \Omega$ and moreover for each $\lambda_0 \in \Omega$ there exists $r \in \mathbb{R}$ such that $\lambda \longrightarrow f_\lambda$ maps a neighborhood of λ_0 holomorphically into $C_r^\infty(G)$. It follows from Example 2.4 that $(\mathscr{P}_\lambda \varphi)_{\lambda \in \mathfrak{a}_c^*}$ is a holomorphic family in $\mathscr{E}_\lambda^\infty(G/K)$, for each $\varphi \in C^\infty(K/M)$.

3. Asymptotic expansions

In this section we give a general definition of asymptotic expansions for functions on $\mathfrak{a}^+$, and state the main results on asymptotic expansions of eigenfunctions.

Let V be a finite dimensional complex linear space provided with a norm $|\cdot|$. By a *formal exponential expansion* (with values in V) at a point $H_0 \in \mathfrak{a}^+$, we shall mean a formal sum

$$\sum_{\xi \in X} p_\xi(H, t)\, e^{t\xi(H)} \tag{3.1}$$

where X is a subset of $\mathfrak{a}_c^*$ such that the subset $X(N)$ given by

$$X(N) = \{\xi \in X \mid \operatorname{Re} \xi(H_0) \geqq N\}$$

is finite for each $N \in \mathbb{R}$, and where each p_ξ is a continuous V-valued function, defined in some neighborhood of $\{H_0\} \times \mathbb{R}$ and polynomial in its last variable.

Let F be a function $\mathfrak{a}^+ \to V$. If $N \in \mathbb{R}$ we say that (3.1) is *asymptotic to F of order N at H_0*, if there exist a neighborhood U of H_0 in $\mathfrak{a}^+$ and constants $\varepsilon > 0$, $C > 0$ such that

$$\Big|F(tH) - \sum_{\xi \in X(N)} p_\xi(H, t)\, e^{t\xi(H)}\Big| \leqq C e^{(N-\varepsilon)t}$$

for $H \in U$, $t \geqq 0$. Moreover, we say that the expansion (3.1) is an *asymptotic expansion for F at H_0* if for every $N \in \mathbb{R}$ it is asymptotic to F of order N at H_0. We write this as

$$F(tH) \sim \sum_{\xi \in X} p_\xi(H, t)\, e^{t\xi(H)} \qquad (t \to \infty).$$

The following result shows that the p_ξ are essentially unique.

Proposition 3.1. *Let $X \subset \mathfrak{a}_c^*$, and let $\sum_{\xi \in X} p_\xi(H, t)\, e^{t\xi(H)}$ and $\sum_{\xi \in X} q_\xi(H, t)\, e^{t\xi(H)}$ be formal exponential expansions at H_0, both assumed to be asymptotic expansions for the same function $F: \mathfrak{a}^+ \to V$. Then for each $\xi \in X$ there is a neighborhood U of H_0 such that $p_\xi = q_\xi$ on $U \times \mathbb{R}$.*

Proof. An immediate consequence of the following Lemma 3.2. □

Lemma 3.2. *If (3.1) is asymptotic of order N at H_0 to the zero function $\mathfrak{a}^+ \to V$, then there exists a neighborhood U of H_0 such that $p_\xi = 0$ on $U \times \mathbb{R}$ for all $\xi \in X(N)$.*

Proof. Select an open neighborhood U of H_0 and constants $\varepsilon > 0$, $C > 0$ such that

$$\Big|\sum_{\xi \in X(N)} p_\xi(H, t)\, e^{t\xi(H)}\Big| \leqq C e^{(N-2\varepsilon)t} \tag{3.2}$$

for $H \in U$. We may assume (by shrinking U) that $\operatorname{Re} \xi(H) > N - \varepsilon$ for $H \in U$, $\xi \in X(N)$. Now let U_1 be the dense open subset of U consisting of the points $H \in U$ such that all the $\xi(H)$, $\xi \in X(N)$, are mutually different. Multiplying (3.2) with $e^{(-N+\varepsilon)t}$ and applying Lemma 3.3 below we obtain that $p_\xi = 0$ on $U_1 \times \mathbb{R}$, hence on $U \times \mathbb{R}$ by continuity. □

Lemma 3. 3. *Let $z_1, \dots, z_n$ be a finite collection of mutually different complex numbers with* $\operatorname{Re} z_i \geqq 0$, *and* $p_1, \dots, p_n$ *polynomial functions* $\mathbb{R} \to V$. *Assume that*

$$\sum_{i=1}^{n} p_i(t)\, e^{z_i t} \to 0$$

as $t \to +\infty$. *Then* $p_1 = \cdots = p_n = 0$.

Proof. Use [H-C 58], p. 305, Corollary. □

We also leave to the reader to verify the following consequence of the uniqueness expressed by Proposition 3. 1.

Corollary 3. 4. *Let* $X \subset \mathfrak{a}_c^*$ *and* $F\colon \mathfrak{a}^+ \to V$. *Assume that for each* $H_0 \in \mathfrak{a}^+$ *there is given a formal exponential expansion*

$$\sum_{\xi \in X} p_{\xi, H_0}(H, t)\, e^{t\xi(H)}$$

which is an asymptotic expansion for F *at* H_0. *Then for each* $\xi \in X$ *there exists a (unique) continuous function* $p_\xi\colon \mathfrak{a}^+ \to V$ *such that each* $H_0 \in \mathfrak{a}^+$ *has a neighborhood* U *with*

$$p_{\xi, H_0}(H, t) = p_\xi(tH) \quad (H \in U,\ t > 0).$$

We shall now state our main results on asymptotics. Let Σ_0^+ be the set of positive indivisible roots and $d \in \mathbb{N}$ the number of elements in Σ_0^+ (not counting multiplicities). Let $P_d(\mathfrak{a})$ be the space of polynomials on $\mathfrak{a}$ (with complex coefficients) of degree $\leqq d$. For $\lambda \in \mathfrak{a}_c^*$ let

$$X(\lambda) = \{ w\lambda - \varrho - \mu \mid w \in W,\ \mu \in \mathbb{N} \cdot \Delta \}.$$

For $\alpha \in \Sigma$ let $\alpha^\vee \in \mathfrak{a}_c^*$ be the dual root given by $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$.

Theorem 3. 5. *Let* $\lambda \in \mathfrak{a}_c^*$.

(i) *For each* $f \in \mathscr{E}_\lambda^\infty(G/K)$, $x \in G$ *and* $\xi \in X(\lambda)$ *there exists a unique polynomial* $p_{\lambda,\xi}(f, x)$ *on* $\mathfrak{a}$ *such that*

$$f(x \exp tH) \sim \sum_{\xi \in X(\lambda)} p_{\lambda,\xi}(f, x, tH)\, e^{t\xi(H)} \qquad (t \to \infty)$$

at every $H_0 \in \mathfrak{a}^+$. *The polynomials* $p_{\lambda,\xi}(f, x)$ *have degree* $\leqq d$.

(ii) *Let* $r \in \mathbb{R}$ *and* $\xi \in X(\lambda)$. *There exists* $r' \in \mathbb{R}$ *such that* $f \to p_{\lambda,\xi}(f)$ *is a continuous linear map of* $\mathscr{E}_{\lambda,r}^\infty$ *into* $C_{r'}^\infty(G) \otimes P_d(\mathfrak{a})$, *equivariant for the left actions of* G *on* $\mathscr{E}_{\lambda,r}^\infty$ *and* $C_{r'}^\infty$.

(iii) *If for every* $\alpha \in \Sigma$ *we have* $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}$, *then all the polynomials* $p_{\lambda,\xi}(f, x)$ *on* $\mathfrak{a}$ *are constant.*

This theorem will be proved in Sections 6—7, together with the next result, which expresses a certain holomorphic dependence on λ of the asymptotic expansions. Let $\Omega_0 \subset \mathfrak{a}_c^*$ be open.

Theorem 3. 6. *Let $(f_\lambda)_{\lambda \in \Omega_0}$ be a holomorphic family in $\mathscr{E}_\lambda^\infty(G/K)$, and fix $\lambda_0 \in \Omega_0$ and $\xi_0 \in X(\lambda_0)$. Let*

$$\Xi(\lambda) = \{w\lambda - \varrho - \mu \in X(\lambda) \mid w \in W \quad \text{and} \quad \mu \in \mathbb{N} \cdot \Delta \quad \text{with} \quad w\lambda_0 - \varrho - \mu = \xi_0\}.$$

There exist an open neighborhood $\Omega \subset \Omega_0$ of λ_0 and a constant $r' \in \mathbb{R}$ such that the map

$$(\lambda, H) \to \sum_{\xi \in \Xi(\lambda)} p_{\lambda,\xi}(f_\lambda, H)\, e^{\xi(H)}$$

is continuous from $\Omega \times \mathfrak{a}^+$ into $C_{r'}^\infty$, and in addition holomorphic in λ. In particular, if for all $\alpha \in \Sigma$ we have $\langle \lambda_0, \alpha^\vee \rangle \notin \mathbb{Z}$, then for each $w \in W$ and $\mu \in \mathbb{N} \cdot \Delta$ there exists a constant $r' \in \mathbb{R}$ such that $\lambda \to p_{\lambda, w\lambda - \varrho - \mu}(f_\lambda)$ is holomorphic from a neighborhood of λ_0 into $C_{r'}^\infty$.

In terms of the asymptotic expansion, Theorem 3. 6 implies that its "head" is holomorphic in λ:

Corollary 3. 7. *Let $(f_\lambda)_{\lambda \in \Omega_0}$ be a holomorphic family in $\mathscr{E}_\lambda^\infty(G/K)$, and let $\lambda_0 \in \Omega_0$, $H_0 \in \mathfrak{a}^+$, $N \in \mathbb{R}$, and $\delta_0 > 0$ be given. There exist an open neighborhood Ω of λ_0, and constants $r' \in \mathbb{R}$ and $\delta \in \,]0, \delta_0[$ such that the map*

$$(\lambda, H) \to \sum_{\substack{\xi \in X(\lambda) \\ \operatorname{Re} \xi(H_0) > N - \delta}} p_{\lambda,\xi}(f_\lambda, H)\, e^{\xi(H)}$$

is continuous from $\Omega \times \mathfrak{a}^+$ to $C_{r'}^\infty$, and in addition holomorphic in λ.

Finally we mention that in Theorem 10. 3 we shall give some extra information on the nature of the polynomials $p_{\lambda,\xi}$ when (iii) does not hold.*)

4. Some lemmas on W-invariant differential operators

As a preparation for the proofs of the theorems stated in the previous section we recall in this section some results of Chevalley, Harish-Chandra, and Steinberg (cf. [He 84], Ch. III), and prove a related lemma.

Let $P(\mathfrak{a})$ be the ring of complex polynomial functions on $\mathfrak{a}$, and $S(\mathfrak{a})$ the symmetric algebra of $\mathfrak{a}$ (with complex coefficients). We identify $S(\mathfrak{a})$ with $P(\mathfrak{a}^*)$ and, since $\mathfrak{a}$ is abelian, also with $\mathscr{U}(\mathfrak{a})$. The subring $\mathscr{U}(\mathfrak{a})^W$ of W-invariants in $\mathscr{U}(\mathfrak{a})$ is generated by n algebraically independent homogeneous generators (where $n = \dim \mathfrak{a}$). By ∂ we denote the homomorphism of $\mathscr{U}(\mathfrak{a})$ into the algebra of constant coefficient differential operators on $\mathfrak{a}$, determined by

$$(\partial(Y) f)(X) = \left.\frac{d}{dt}\right|_{t=0} f(X + tY)$$

for $f \in C^\infty(\mathfrak{a})$ and $X, Y \in \mathfrak{a}$.

Let $\Omega \subset \mathfrak{a}$ be an open connected nonempty set, and let $\lambda \in \mathfrak{a}_c^*$. We are interested in the joint eigenspace $E_\lambda(\Omega)$ consisting of all solutions (say distributions) h on Ω to the homogeneous system

$$\text{(4. 1)} \qquad [\partial(v) - v(\lambda)]\, h = 0 \qquad (v \in \mathscr{U}(\mathfrak{a})^W).$$

There is a natural representation δ_λ of $\mathscr{U}(\mathfrak{a})$ on $E_\lambda(\Omega)$ given by $\delta_\lambda(u)\, h = \partial(u)\, h$ for $u \in \mathscr{U}(\mathfrak{a})$.

*) See also note 1 at the end of the paper.

Consider first $\mathscr{H}=E_0(\mathfrak{a})$, which is the space of *W-harmonic polynomials on* $\mathfrak{a}$. It is known that $\mathscr{H}$ has a basis consisting of m homogeneous polynomials, where m is the order of W, and also that it contains the polynomial ω given by

$$\omega(H)=\prod_{\alpha\in\Sigma_0^+}\alpha(H)$$

($H\in\mathfrak{a}$), as a cyclic vector for δ_0 (that is, $\mathscr{H}=\partial(\mathscr{U}(\mathfrak{a}))\,\omega$) (see [He 84], Sect. III 3.2).

Similarly we define the subspace $E_0(\mathfrak{a}^*)$ of $P(\mathfrak{a}^*)\cong\mathscr{U}(\mathfrak{a})$ as the space of solutions $h\in C^\infty(\mathfrak{a}^*)$ to $[\partial(v)-v(\lambda)]\,h=0$ for all $v\in S(\mathfrak{a}^*)^W$. Via the Killing form we can identify $\mathfrak{a}$ with $\mathfrak{a}^*$, and hence $\mathscr{H}=E_0(\mathfrak{a})$ with $E_0(\mathfrak{a}^*)$. The map

$$h\otimes v\longrightarrow hv\quad(h\in E_0=E_0(\mathfrak{a}^*),\ v\in\mathscr{U}(\mathfrak{a})^W)$$

extends to a linear bijection of $E_0\otimes\mathscr{U}(\mathfrak{a})^W$ onto $\mathscr{U}(\mathfrak{a})$ (cf. loc. cit.). Hence $\mathscr{U}(\mathfrak{a})$ is a free $\mathscr{U}(\mathfrak{a})^W$-module with m free homogeneous generators. For each $\lambda\in\mathfrak{a}_c^*$ let $\mathscr{I}_\lambda$ denote the maximal ideal of $\mathscr{U}(\mathfrak{a})^W$ given by

$$\mathscr{I}_\lambda=\{v\in\mathscr{U}(\mathfrak{a})^W\mid v(\lambda)=0\},\tag{4.2}$$

then it follows that $\mathscr{U}(\mathfrak{a})=E_0\oplus E_0\mathscr{I}_\lambda$ and that $E_0\mathscr{I}_\lambda$ is an ideal of $\mathscr{U}(\mathfrak{a})$.

For any $\mu\in\mathfrak{a}_c^*$ let W^μ denote the subgroup $\{w\mid w\mu=\mu\}$ of W, then W^μ is the Weyl group of the root system Σ^μ consisting of all roots orthogonal to μ (cf. [He 78], VII, Thm. 2.15). The W^μ-harmonic polynomials on $\mathfrak{a}$ are the solutions h to $\partial(v)\,h=0$ for all $v\in\mathscr{U}(\mathfrak{a})^{W^\mu}$. In particular, if μ is *regular*, that is if W^μ is trivial, then only the constants are W^μ-harmonic. Let e^μ denote the function $H\longrightarrow e^{\mu(H)}$ on $\mathfrak{a}$.

Proposition 4.1. *Let $\lambda\in\mathfrak{a}_c^*$. The space $E_\lambda(\Omega)$ has dimension m, and each element $h\in E_\lambda(\Omega)$ has a unique expression*

$$h=\Big(\sum_{w\in W/W^\lambda}q_w e^{w\lambda}\Big)\Big|_\Omega\tag{4.3}$$

with $W^{w\lambda}$-harmonic polynomials q_w on $\mathfrak{a}$. Conversely any function h of the form (4.3) *belongs to $E_\lambda(\Omega)$. Moreover $E_0\mathscr{I}_\lambda$ is the annihilating ideal in $\mathscr{U}(\mathfrak{a})$ of δ_λ, and $(\delta_\lambda, E_\lambda)$ is equivalent as an $\mathfrak{a}$-module to $\mathscr{U}(\mathfrak{a})/E_0\mathscr{I}_\lambda$.*

Proof. See [He 84], III, Thm. 3.13). □

In particular each element of $E_\lambda(\Omega)$ extends to a real analytic function on $\mathfrak{a}$.

Let (δ, V) be a finite dimensional representation of $\mathfrak{a}$ and let $\mu\in\mathfrak{a}_c^*$. The space

$$V^\mu=\{v\in V\mid\exists r\in\mathbb{N}\ \forall Y\in\mathfrak{a}:[\delta(Y)-\mu(Y)]^r v=0\}$$

is called the *generalized weight space* corresponding to μ and its dimension the *multiplicity* of μ. If the multiplicity is not zero, μ is called a weight of δ.

Corollary 4.2. *The set of weights of δ_λ is $W\lambda$, and the multiplicity of each $\mu\in W\lambda$ is the number of elements in W^λ.*

Proof. Immediate from (4.3). □

Recall that an exponential polynomial on Ω is a function of the form $H \to q(H)\, e^{\mu(H)}$ where $q \in P(\mathfrak{a})$ and $\mu \in \mathfrak{a}_c^*$. Let $Q(\Omega)$ denote the space of functions on Ω spanned by the exponential polynomials. From Proposition 4. 1 we have that the solution space for the homogeneous system (4. 1) is contained in $Q(\Omega)$. We now consider solutions to a similar inhomogeneous system. Fix any $\lambda \in \mathfrak{a}_c^*$.

Lemma 4. 3. *If* $h \in \mathscr{D}'(\Omega)$ *and* $[\partial(v) - v(\lambda)]\, h \in Q(\Omega)$ *for all* $v \in \mathscr{U}(\mathfrak{a})^W$, *then* $h \in Q(\Omega)$.

Proof. The assumption on h can be expressed as $\partial(\mathscr{I}_\lambda)\, h \subset Q(\Omega)$. Hence $\partial(E_0 \mathscr{I}_\lambda)\, h \subset Q(\Omega)$, and the lemma is a consequence of the following Lemma 4. 4. □

Lemma 4. 4. *Let* $\mathscr{J} \subset \mathscr{U}(\mathfrak{a})$ *be an ideal of finite codimension. If* $f \in \mathscr{D}'(\Omega)$ *and* $\partial(\mathscr{J}) f \subset Q(\Omega)$ *then* $f \in Q(\Omega)$.

Proof. Let $V = \partial(\mathscr{U}(\mathfrak{a})) f \subset \mathscr{D}'(\Omega)$ and $V_0 = V \cap Q(\Omega)$. The assumption on f implies that $\dim V/V_0 < +\infty$. Since differentiations leave $Q(\Omega)$ invariant, ∂ induces a representation of $\mathfrak{a}$ on V/V_0. Now if $V/V_0 \neq 0$ there exist $\mu \in \mathfrak{a}_c^*$ and $v \in V \setminus V_0$ such that

$$\partial(Y)\, v - \mu(Y)\, v \in V_0 \tag{4. 4}$$

for all $Y \in \mathfrak{a}$. Define $\tilde{v} \in \mathscr{D}'(\Omega)$ by $\tilde{v} = e^{-\mu} v$, then (4. 4) shows that $\partial(Y)\, \tilde{v} \in Q(\Omega)$ for all $Y \in \mathfrak{a}$. It follows easily (by n integrations) that $\tilde{v} \in Q(\Omega)$, and hence $v \in Q(\Omega)$, contradicting $v \notin V_0$. Thus $V_0 = V$ and $f \in Q(\Omega)$. □

5. The weights of the spherical principal series

The purpose of this section is to analyze the action of $\mathfrak{a}$, and its dependence on λ, in the right $\mathfrak{g}$-module generated by an element of $\mathscr{E}_\lambda(G/K)$. For this aim it is convenient to introduce some algebraic models for the spherical principal series, originating from Kostant.

For each $\lambda \in \mathfrak{a}_c^*$ let I_λ be the ideal $\gamma^{-1}(\mathscr{I}_\lambda)$ of $\mathbb{D}(G/K)$, that is

$$I_\lambda = \{D \in \mathbb{D}(G/K) \mid \chi_\lambda(D) = 0\},$$

and let $J_\lambda \subset \mathscr{U}(\mathfrak{g})$ be the left ideal generated by $\mathfrak{k}$ and by $\mu^{-1}(I_\lambda)$ where μ is the map (1. 1). Notice that J_λ is invariant under the adjoint representation of K. Let $\mathscr{Y}_\lambda$ denote the $(\mathfrak{g}, K)$-module $\mathscr{U}(\mathfrak{g})/J_\lambda$, which is Kostant's model for the spherical principal series (cf. [Kos 75], p. 244). Our interest in $\mathscr{Y}_\lambda$ comes from the fact that if $f \in \mathscr{E}_\lambda(G/K)$ then the map $u \to uf$ of $\mathscr{U}(\mathfrak{g})$ into $C^\infty(G)$ factors through $\mathscr{Y}_\lambda$.

Remark 5. 1. Though we do not need it in the sequel it is of interest to notice that it follows from [Kos 75], p. 323, Remark 2. 10. 2 that the $(\mathfrak{g}, K)$-module $\mathscr{Y}_\lambda$ is equivalent to the $(\mathfrak{g}, K)$-module of K-finite vectors in $C(G/P; L_{-\lambda})$, provided $\mathrm{Re}\langle\lambda, \alpha\rangle \geqq 0$ for all $\alpha \in \Sigma^+$.

Our aim is to describe, for each $k \in \mathbb{N}$, the set of weights of $\mathfrak{a}$ in $\mathscr{Y}_\lambda/\bar{\mathfrak{n}}^k \mathscr{Y}_\lambda$, and moreover to consider the dependence on λ of the corresponding generalized weight spaces.

It is convenient to have available a model for the $(\mathfrak{g}, K)$-module $\mathscr{Y}_\lambda$ in which the action of $\bar{\mathfrak{n}}$ is easily described and in which the underlying vector space is independent of λ. Let $E_0 = E_0(\mathfrak{a}^*) \subset \mathcal{U}(\mathfrak{a})$ be the m dimensional space defined in the previous section, and let $E = T_\varrho(E_0) \subset \mathcal{U}(\mathfrak{a})$, where $v \to T_\varrho(v)$, from now on denoted $v \to 'v$, is defined by (1. 3). Then $h \otimes v \to h'v$ extends to a bijection of $E \otimes \mathcal{U}(\mathfrak{a})^W$ onto $\mathcal{U}(\mathfrak{a})$. Let

$$\mathcal{Y} = \mathcal{U}(\bar{\mathfrak{n}}) \otimes E. \tag{5. 1}$$

We shall construct a linear bijection of $\mathcal{Y}$ with $\mathcal{Y}_\lambda$, and for this purpose we need the following (well known) proposition (cf. [Kos 78], Prop. 5. 2). Let

$$\Gamma: \mathcal{Y} \otimes \mathbb{D}(G/K) \to \mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})\,\mathfrak{k} \tag{5. 2}$$

be the map defined by

$$\Gamma(y \otimes \mu(u)) = yu + \mathcal{U}(\mathfrak{g})\,\mathfrak{k}$$

for $y \in \mathcal{Y}$ and $u \in \mathcal{U}(\mathfrak{g})^K$. Here (and in the following) we identify $\mathcal{Y}$ with a subspace of $\mathcal{U}(\mathfrak{g})$ via $x \otimes e \to xe$ for $x \in \mathcal{U}(\bar{\mathfrak{n}})$ and $e \in E$.

Proposition 5. 1. *The map Γ is bijective.*

Proof. We include the proof for completeness. Via the Iwasawa decomposition we have $\mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})\,\mathfrak{k} \cong \mathcal{U}(\bar{\mathfrak{n}}) \otimes \mathcal{U}(\mathfrak{a})$. Via this isomorphism the degree on $\mathcal{U}(\mathfrak{a})$ induces a degree on $\mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})\,\mathfrak{k}$, denoted $\deg_\mathfrak{a}$. Let $\mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})\,\mathfrak{k}$ be filtered according to $\deg_\mathfrak{a}$, and let $\mathcal{Y} \otimes \mathbb{D}(G/K) = \mathcal{U}(\bar{\mathfrak{n}}) \otimes E \otimes \mathbb{D}(G/K)$ be filtered according to the total degree on $E \otimes \mathbb{D}(G/K)$.

Notice that for each $D \in \mathbb{D}(G/K)$ there exists $u \in \mu^{-1}(D)$ with $\deg u = \deg D$ (cf. [He 84], II, Thm. 4. 9). It follows that Γ preserves the filtrations. Moreover, from the definition of $'\gamma$ (cf. (1. 5)) we infer that

$$\deg_\mathfrak{a}[u - {}'\gamma(D) + \mathcal{U}(\mathfrak{g})\,\mathfrak{k}] < \deg u = \deg D.$$

Since $\deg {}'\gamma(D) = \deg D$ it then easily follows that the graded map

$$\operatorname{gr}\Gamma: \mathcal{U}(\bar{\mathfrak{n}}) \otimes \operatorname{gr}(E \otimes \mathbb{D}(G/K)) \to \mathcal{U}(\bar{\mathfrak{n}}) \otimes \operatorname{gr}(\mathcal{U}(\mathfrak{a}))$$

associated to Γ, is given by $x \otimes e \otimes D \to x \otimes e'\gamma(D)$. This being bijective the proposition follows. □

Corollary 5. 2. (i) *Γ maps $\mathcal{Y} \otimes I_\lambda$ onto J_λ.*

(ii) *For each $u \in \mathcal{U}(\mathfrak{g})$ there exists a unique $y \in \mathcal{Y}$ such that $u - y \in J_\lambda$.*

Proof. (i) According to the proposition and the definition of J_λ, every element of J_λ can be written (modulo $\mathcal{U}(\mathfrak{g})\,\mathfrak{k}$) as a sum of terms $\Gamma(y \otimes D)\,v$ with $y \in \mathcal{Y}$, $D \in \mathbb{D}(G/K)$ and $v \in \mu^{-1}(I_\lambda)$. Since $\Gamma(y \otimes D)\,v = \Gamma(y \otimes D\mu(v))$, we have (i).

(ii) Let $u \in \mathcal{U}(\mathfrak{g})$. According to the proposition there is a unique element $\sum y_i \otimes D_i \in \mathcal{Y} \otimes \mathbb{D}(G/K)$ with $u = \Gamma(\sum y_i \otimes D_i)$ (modulo $\mathcal{U}(\mathfrak{g})\,\mathfrak{k}$). It follows easily that

$$u - \sum \chi_\lambda(D_i)\, y_i \in J_\lambda. \tag{5. 3}$$

Conversely if $y \in \mathcal{Y}$ and $u - y \in J_\lambda$ then from (i) we get that $\sum y_i \otimes D_i - y \otimes 1 \in \mathcal{Y} \otimes I_\lambda$, from which it follows that $y = \sum \chi_\lambda(D_i)\, y_i$. □

From the corollary we obtain a linear bijection $\P_\lambda$ of $\mathcal{Y}_\lambda$ onto $\mathcal{Y}$, defined via

$$u - \P_\lambda(u + J_\lambda) \in J_\lambda$$

for $u \in \mathcal{U}(\mathfrak{g})$. Notice that by (5. 3) we have

$$\text{(5. 4)} \qquad \P_\lambda(\Gamma(y \otimes D) + J_\lambda) = \chi_\lambda(D)\, y$$

for $y \in \mathcal{Y}$ and $D \in \mathbb{D}(G/K)$. Notice also that from the definition of $\P_\lambda$ we immediately get

$$\text{(5. 5)} \qquad uf = \P_\lambda(u + J_\lambda) f$$

for all $f \in \mathcal{E}_\lambda(G/K)$ (where, according to our convention, both u and $\P_\lambda(u + J_\lambda)$ act as left invariant differential operators).

For each $\lambda \in \mathfrak{a}_c^*$ we denote by τ_λ the unique representation of $\mathfrak{g}$ on $\mathcal{Y}$ which turns $\P_\lambda$ into a morphism of $\mathfrak{g}$-modules. Explicitly the action of $\tau_\lambda(X)$ for $X \in \mathfrak{g}$ is determined as follows. Let $y \in \mathcal{Y} \subset \mathcal{U}(\mathfrak{g})$. The element Xy of $\mathcal{U}(\mathfrak{g})$ can be written (modulo $\mathcal{U}(\mathfrak{g})\,\mathfrak{k}$) as $\Gamma(\sum y_i \otimes D_i)$ according to Proposition 5. 1, and then (cf. (5. 4)):

$$\text{(5. 6)} \qquad \tau_\lambda(X)\, y = \sum_i \chi_\lambda(D_i)\, y_i.$$

Notice that if $X \in \bar{\mathfrak{n}}$, then $\tau_\lambda(X)$ is just left multiplication by X in $\mathcal{Y}$. It follows that for every $k \in \mathbb{N}$, τ_λ induces a representation τ_λ^k of $\mathfrak{a}$ on the finite dimensional space $\mathcal{Y}/\bar{\mathfrak{n}}^k\mathcal{Y}$. In particular, τ_λ^1 is a representation of $\mathfrak{a}$ on $\mathcal{Y}/\bar{\mathfrak{n}}\mathcal{Y} \cong E$.

Lemma 5. 3. *The set of weights of τ_λ^1 is $W\lambda - \varrho$, and the multiplicity of each $\xi \in W\lambda - \varrho$ is the number of elements in W^λ.*

Proof. Let $'\mathcal{I}_\lambda = T_\varrho(\mathcal{I}_\lambda)$. Comparing with Corollary 4. 2 it suffices to construct an $\mathfrak{a}$-isomorphism of (τ_λ^1, E) onto $\mathcal{U}(\mathfrak{a})/E'\mathcal{I}_\lambda$. Obviously the map $e \to e + E'\mathcal{I}_\lambda$ is a linear bijection, we claim that it is also an $\mathfrak{a}$-morphism. Let $H \in \mathfrak{a}$, $e \in E$ and write $He = \sum_i e_i{}'\gamma(D_i)$ with $e_i \in E$, $D_i \in \mathbb{D}(G/K)$. By (1. 5) we then have

$$He - \Gamma(\sum_i e_i \otimes D_i) \in \bar{\mathfrak{n}}\,\mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\,\mathfrak{k},$$

and hence $\tau_\lambda^1(H)\, e = \sum_i \chi_\lambda(D_i)\, e_i$ by (5. 6). But then

$$He - \tau_\lambda^1(H)\, e = \sum_i e_i('\gamma(D_i) - \chi_\lambda(D_i)) \in E'\mathcal{I}_\lambda$$

as claimed. □

For $k \in \mathbb{N}$ let Λ_k be an enumeration of the weights of the finite dimensional $\mathfrak{a}$-module

$$\mathcal{M}_k = \mathcal{U}(\bar{\mathfrak{n}})/\bar{\mathfrak{n}}^k\,\mathcal{U}(\bar{\mathfrak{n}})$$

counting multiplicities. The elements of Λ_k are easily seen to be contained in $-\mathbb{N} \cdot \Delta$.

Proposition 5. 4. *Let $\lambda \in \mathfrak{a}_c^*$ and $k \in \mathbb{N}$, $k > 0$. The set of weights of τ_λ^k is $W\lambda - \varrho + \Lambda_k$. The multiplicity of a weight ξ is the number of pairs $(w, -\mu) \in W \times \Lambda_k$ such that $\xi = w\lambda - \varrho - \mu$.*

Proof. Use induction on k. For $k=1$ this is Lemma 5.3. For $k>1$ the induction step is a consequence of the following two exact sequences of $\mathfrak{a}$-modules:

$$0 \longrightarrow [\bar{\mathfrak{n}}^{k-1}\mathscr{U}(\bar{\mathfrak{n}})/\bar{\mathfrak{n}}^k\mathscr{U}(\bar{\mathfrak{n}})] \otimes [\mathscr{Y}_\lambda/\bar{\mathfrak{n}}\mathscr{Y}_\lambda] \longrightarrow \mathscr{Y}_\lambda/\bar{\mathfrak{n}}^k\mathscr{Y}_\lambda \longrightarrow \mathscr{Y}_\lambda/\bar{\mathfrak{n}}^{k-1}\mathscr{Y}_\lambda \longrightarrow 0$$

and

$$0 \longrightarrow \bar{\mathfrak{n}}^{k-1}\mathscr{U}(\bar{\mathfrak{n}})/\bar{\mathfrak{n}}^k\mathscr{U}(\bar{\mathfrak{n}}) \longrightarrow \mathscr{M}_k \longrightarrow \mathscr{M}_{k-1} \longrightarrow 0. \qquad \square$$

In the proof of the next proposition we need the following lemma.

Lemma 5.5. *Let $\lambda \in \mathfrak{a}_c^*$ and assume for all $\alpha \in \Sigma$ that $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}$. Then $w\lambda - \lambda \notin \mathbb{Z} \cdot \Delta$ for all $w \in W \backslash \{e\}$.*

Proof. See [Schl 84], Lemma 5.2.3. $\square$

Proposition 5.6. *Fix $\lambda \in \mathfrak{a}_c^*$ and assume for all $\alpha \in \Sigma$ that $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}$. Let $k \in \mathbb{N}$. Then the $\tau_\lambda^k(H)$ for $H \in \mathfrak{a}$ are simultaneously diagonalizable.*

Proof. It follows from Proposition 5.4 and Lemma 5.5 that the multiplicity of $w\lambda - \varrho$ is one in τ_λ^k for all $w \in W$ and $k \in \mathbb{N}$. In particular, τ_λ^1 is diagonalizable, and moreover the natural projection $p: \mathscr{Y}/\bar{\mathfrak{n}}^k\mathscr{Y} \longrightarrow \mathscr{Y}/\bar{\mathfrak{n}}\mathscr{Y}$ admits a section $j: \mathscr{Y}/\bar{\mathfrak{n}}\mathscr{Y} \longrightarrow \mathscr{Y}/\bar{\mathfrak{n}}^k\mathscr{Y}$ (i.e. $p \circ j = \mathrm{id}$) which is an $\mathfrak{a}$-morphism.

We now define an $\mathfrak{a}$-morphism

$$\Phi: \mathscr{U}(\bar{\mathfrak{n}}) \otimes \mathscr{Y}/\bar{\mathfrak{n}}\mathscr{Y} \longrightarrow \mathscr{Y}/\bar{\mathfrak{n}}^k\mathscr{Y} \tag{5.7}$$

by $\Phi(x \otimes y) = \tau_\lambda^k(x)\, j(y)$ for $x \in \mathscr{U}(\bar{\mathfrak{n}})$ and $y \in \mathscr{Y}/\bar{\mathfrak{n}}\mathscr{Y}$. Obviously Φ induces a linear map: $\mathscr{M}_k \otimes \mathscr{Y}/\bar{\mathfrak{n}}\mathscr{Y} \longrightarrow \mathscr{Y}/\bar{\mathfrak{n}}^k\mathscr{Y}$. We claim that this is a bijection. By dimensions (cf. Proposition 5.4) it suffices to prove injectivity, that is:

$$\ker \Phi = \bar{\mathfrak{n}}^k\mathscr{U}(\bar{\mathfrak{n}}) \otimes \mathscr{Y}/\bar{\mathfrak{n}}\mathscr{Y}. \tag{5.8}$$

To prove (5.8), let $e_1, \ldots, e_m$ be a linear basis for E; and choose $y_1, \ldots, y_m \in \mathscr{Y}$ such that $j(e_i) = y_i + \bar{\mathfrak{n}}^k\mathscr{Y}$ $(i = 1, \ldots, m)$. Since $p \circ j = \mathrm{id}$ we have that $e_i - y_i \in \bar{\mathfrak{n}}\mathscr{Y}$ for $i = 1, \ldots, m$. Let $x_1, \ldots, x_m \in \mathscr{U}(\bar{\mathfrak{n}})$ and assume that $\Phi(\sum x_i \otimes e_i) = 0$, that is $\sum x_i y_i \in \bar{\mathfrak{n}}^k\mathscr{Y}$. The claim is that then $x_i \in \bar{\mathfrak{n}}^k\mathscr{U}(\bar{\mathfrak{n}})$ for each i. We will prove that $x_i \in \bar{\mathfrak{n}}^{j+1}\mathscr{U}(\bar{\mathfrak{n}})$ $(i = 1, \ldots, m)$ for $j < k$, by induction on j. Assume $x_i \in \bar{\mathfrak{n}}^j\mathscr{U}(\bar{\mathfrak{n}})$ $(i = 1, \ldots, m)$ (to get the induction started this obviously holds with $j = 0$). Then

$$\sum x_i e_i = \sum x_i y_i + \sum x_i(e_i - y_i) \in \bar{\mathfrak{n}}^k\mathscr{Y} + \bar{\mathfrak{n}}^{j+1}\mathscr{Y} = \bar{\mathfrak{n}}^{j+1}\mathscr{Y}$$

and hence $x_i \in \bar{\mathfrak{n}}^{j+1}\mathscr{U}(\bar{\mathfrak{n}})$ for all i, by the linear independence of $e_1, \ldots, e_m$. Thus (5.8) is proved.

Since the left hand side of (5.7) is diagonalizable (using Poincaré-Birkhoff-Witt on $\mathscr{U}(\bar{\mathfrak{n}})$) and Φ is bijective, the proposition follows. $\square$

Let $\lambda \in \mathfrak{a}_c^*$ and $k \in \mathbb{N}$. For each weight ξ of τ_λ^k we denote by $P_{\lambda,\xi}$ the projection map of $\mathscr{Y}/\bar{\mathfrak{n}}^k\mathscr{Y}$ onto the generalized weight space of weight ξ, along the remaining generalized weight spaces.

Corollary 5.7. *There exists for each $\lambda \in \mathfrak{a}_c^*$ and each weight ξ of τ_λ^k, a unique polynomial $q_{\lambda,\xi}$ on $\mathfrak{a}$ with coefficients in* $\mathrm{End}_{\mathbb{C}}(\mathscr{Y}/\bar{\mathfrak{n}}^k\mathscr{Y})$ *such that $P_{\lambda,\xi}q_{\lambda,\xi}(H)\,P_{\lambda,\xi}=q_{\lambda,\xi}(H)$ and*

$$\exp\tau_\lambda^k(H)=\sum_\xi e^{\xi(H)}q_{\lambda,\xi}(H) \tag{5.9}$$

for $H\in\mathfrak{a}$. If for all $\alpha\in\Sigma$ we have $\langle\lambda,\alpha^\vee\rangle\notin\mathbb{Z}$ then all the polynomials $q_{\lambda,\xi}$ are constant on $\mathfrak{a}$.

Here the operator $\exp\tau_\lambda^k(H)$ is as usual defined by $\sum\frac{1}{n!}[\tau_\lambda^k(H)]^n$. The corollary is straightforward.

We now consider the dependence on λ of τ_λ^k. It follows immediately from (5.6) that τ_λ^k depends holomorphically on λ and by Proposition 5.4 its weights are given by holomorphic expressions in λ.

Proposition 5.8. *Let $\lambda_0\in\mathfrak{a}_c^*$ and $k\in\mathbb{N}$, and fix a weight ξ_0 of $\tau_{\lambda_0}^k$. For each $\lambda\in\mathfrak{a}_c^*$ let*

$$\Xi(\lambda)=\{w\lambda-\varrho-\mu \mid w\in W,\ -\mu\in\Lambda_k,\ \text{and } w\lambda_0-\varrho-\mu=\xi_0\}$$

and let $P(\lambda)=\sum_{\xi\in\Xi(\lambda)}P_{\lambda,\xi}$. Then the projection operator $P(\lambda)$ depends holomorphically on λ in a neighborhood of λ_0.

Proof. The proposition follows at once from Lemma 5.9 below. □

Let F be an n-dimensional complex vector space, and let τ_z be a family of representations of $\mathfrak{a}$ on F, depending holomorphically on a parameter $z\in\mathbb{C}^N$. Assume that the weights of τ_z are $s_1(z),\dots,s_n(z)\in\mathfrak{a}_c^*$ (each repeated according to its multiplicity), where each s_j depends continuously on z. Fix $z_0\in\mathbb{C}^N$ and let $\sigma_1,\dots,\sigma_m$ be the mutually different elements of $\{s_1(z_0),\dots,s_n(z_0)\}$.

Lemma 5.9. *There exists a neighborhood Ω of z_0, and for each $z\in\Omega$ a basis $f_1(z),\dots,f_n(z)$ for F, with each f_i depending holomorphically on z, such that the corresponding matrix for $\tau_z(H)$ $(H\in\mathfrak{a})$ has the form*

$$\begin{pmatrix} B_z^1(H) & & 0\\ & \ddots & \\ 0 & & B_z^m(H)\end{pmatrix}$$

where each block $B_z^j(H)$ is a matrix whose eigenvalues are the $s_i(z,H)$ for which $s_i(z_0)=\sigma_j$.

Proof. Use [Ka 80], Ch. 2, § 14, formula (1.1b).

From Proposition 5.8 we immediately get the following corollary by applying $P(\lambda)$ to (5.9).

Corollary 5.10. *There is a neighborhood Ω of λ_0 such that the map*

$$(\lambda,H)\to\sum_{\xi\in\Xi(\lambda)}e^{\xi(H)}q_{\lambda,\xi}(H) \tag{5.10}$$

from $\Omega\times\mathfrak{a}$ to $\mathrm{End}(\mathscr{Y}/\bar{\mathfrak{n}}^k\mathscr{Y})$ is continuous, and in addition holomorphic in λ.

Recalling that a polynomial q on $\mathfrak{a}$ is W-harmonic if $\partial(u)\,q=0$ for all $u \in \mathscr{U}(\mathfrak{a})^W$, and that this implies $\deg q \leqq d = |\Sigma_0^+|$, we now have

Corollary 5. 11. *The polynomials* $q_{\lambda,\xi}$ *on* $\mathfrak{a}$ *are all W-harmonic. In particular,* $\deg q_{\lambda,\xi} \leqq d$.

Proof. Fix $\lambda_0 \in \mathfrak{a}_c^*$ and let ξ_0 be a weight of $\tau_{\lambda_0}^k$. For each $\lambda \in \mathfrak{a}_c^*$ let φ_λ be the $\mathrm{End}(\mathscr{Y}/\bar{\mathfrak{n}}^k\mathscr{Y})$-valued C^∞-function on $\mathfrak{a}$ given by

$$\varphi_\lambda(H) = e^{-\xi_0(H)} P(\lambda) \exp \tau_\lambda^k(H) = \sum_{\xi \in \Xi(\lambda)} e^{(\xi-\xi_0)H} q_{\lambda,\xi}(H). \tag{5. 11}$$

Let $u \in \mathscr{U}(\mathfrak{a})$. It follows from (5. 11) and the continuity of $\lambda \to P(\lambda)$ that $\lambda \to [\partial(u)\,\varphi_\lambda](H)$ is continuous near λ_0 for each $H \in \mathfrak{a}$. Now if $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}$ for all $\alpha \in \Sigma$ then it follows from the final statement of Corollary 5. 7 that

$$\partial(u)\,\varphi_\lambda = \sum_{\xi \in \Xi(\lambda)} u(\xi-\xi_0)\, e^{\xi-\xi_0} q_{\lambda,\xi}. \tag{5. 12}$$

If $\xi \in \Xi(\lambda)$ then by definition we have $\xi - \xi_0 = w(\lambda - \lambda_0)$ for some $w \in W$, hence if $u \in \mathscr{U}(\mathfrak{a})^W$ we get from (5. 12) that

$$\partial(u)\,\varphi_\lambda = u(\lambda-\lambda_0)\,\varphi_\lambda. \tag{5. 13}$$

By density of $\{\lambda \mid \forall \alpha \in \Sigma\colon \langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}\}$ in $\mathfrak{a}_c^*$, we infer from (5. 13) that $\partial(u)\,\varphi_{\lambda_0} = u(0)\,\varphi_{\lambda_0}$ for $u \in \mathscr{U}(\mathfrak{a})^W$, that is, φ_{λ_0} is W-harmonic. Since $\varphi_{\lambda_0} = q_{\lambda_0,\xi_0}$ the corollary is proved. $\square$

Remark 5. 12. For $\nu \in \mathfrak{a}_c^*$ let W_ν denote the subgroup $\{w \mid w\nu - \nu \in \mathbb{Z} \cdot \Delta\}$ of W. If $\xi_0 = w_0\lambda_0 - \varrho - \mu_0 \in X(\lambda_0)$ then it can actually be seen that q_{λ_0,ξ_0} is $W_{w_0\lambda_0}$-harmonic. In fact, we have for each $\xi \in \Xi(\lambda)$ that $\xi - \xi_0 \in w\,w_0(\lambda - \lambda_0)$ with $w \in W_{w_0\lambda_0}$. Hence if u is an $W_{w_0\lambda_0}$-invariant element of $\mathscr{U}(\mathfrak{a})$ we obtain from (5. 12) that

$$\partial(u)\,\varphi_\lambda = u(w_0(\lambda-\lambda_0))\,\varphi_\lambda$$

and the claim follows as above.

Notice that it follows from [KKMOOT 78], App. II that W_ν is actually the Weyl group of a root system, for each $\nu \in \mathfrak{a}_c^*$.

6. Existence of asymptotic expansions

In this section we derive asymptotic expansions for the functions in $\mathscr{E}_\lambda^\infty$, and use them to prove most of Theorem 3. 5. The proof will be completed in the next section. The basic result is the following technical proposition.

Fix $\lambda_0 \in \mathfrak{a}_c^*$, $H_0 \in \mathfrak{a}^+$, and $r \in \mathbb{R}$. If A_1 and A_2 are Banach spaces we denote by $B(A_1, A_2)$ the Banach space of bounded linear operators from A_1 to A_2.

Proposition 6. 1. *There exist, for each* $N \in \mathbb{R}$

(a) *open neighborhoods* Ω *of* λ_0 *in* $\mathfrak{a}_c^*$ *and* U *of* H_0 *in* $\mathfrak{a}^+$,

(b) *constants* $k, q \in \mathbb{N}$, $r' \geqq r$, *and* $C, \varepsilon > 0$,

(c) *a continuous map* $\Psi: \Omega \times U \longrightarrow B(C_r^q, \mathscr{Y}/\bar{\mathfrak{n}}^k \mathscr{Y} \otimes C_{r'})$, *holomorphic in the first variable, and*

(d) *a linear form* $\eta \in (\mathscr{Y}/\bar{\mathfrak{n}}^k \mathscr{Y})^*$

such that

(i) $\Psi(\lambda, H)$ *intertwines the left actions of* G *on* C_r^q *and* $C_{r'}$, *for all* $(\lambda, H) \in \Omega \times U$, *and*

(ii) $\|R_{\exp tH} f - (\eta \circ \exp \tau_\lambda^k(tH) \otimes 1)(\Psi(\lambda, H) f)\|_{r'} \leqq C \|f\|_{q,r} e^{(N-\varepsilon)t}$ *for all*

$$f \in \mathscr{E}_\lambda(G/K) \cap C_r^q, \; \lambda \in \Omega, \; H \in U, \quad \text{and } t \geqq 0.$$

In the proof of the proposition we need the following lemma. Define

$$\beta: \mathfrak{a} \longrightarrow \mathbb{R} \text{ by } \beta(H) = \min_{\alpha \in \Delta} \alpha(H).$$

Lemma 6. 2. *Let* $k \in \mathbb{N}$, *and put*

$$\gamma(H) = |r| \, c_2 |H| - k \beta(H) \tag{6. 1}$$

for $H \in \mathfrak{a}$, *where* c_2 *is the constant of Lemma* 2. 1 (iv). *For each* $y \in \bar{\mathfrak{n}}^k \mathscr{Y}$ *there exist constants* $q \in \mathbb{N}$, $r' \geqq r$, *and* $C > 0$ *such that for all* $H \in \mathfrak{a}^+$ *we have*

$$\|R_{\exp H} y f\|_{r'} \leqq C \|f\|_{q,r} e^{\gamma(H)} \tag{6. 2}$$

for $f \in C_r^q$.

Proof. We may assume that $y = Y_1 \cdots Y_l e$ where $e \in E$, $l \geqq k$, and Y_i belongs to the $-\alpha_i$ root space for some $\alpha_i \in \Sigma^+$. Then

$$R_{\exp H} y f = e^{-\alpha_1(H) - \cdots - \alpha_l(H)} y R_{\exp H} f$$

and the estimate (6. 2) follows from (2. 7) and (2. 5). □

Proof of Proposition 6. 1. Let $N \in \mathbb{R}$ be given, and select $k \in \mathbb{N}$ such that $\gamma(H_0) < N$, where γ is given by (6. 1). Let $S(\lambda) \subset X(\lambda)$ denote the set of weights of τ_λ^k (cf. Proposition 5. 4). The first step is to split $S(\lambda)$ in two parts, which will give different growth orders in the asymptotics.

Fix $\varepsilon > 0$ such that $\gamma(H_0) + \varepsilon < N$ and such that for $\xi \in S(\lambda_0)$ we have

$$\operatorname{Re} \xi(H_0) \notin [N - 2\varepsilon, N[.$$

We can then choose a compact connected neighborhood U of H_0 in $\mathfrak{a}^+$ such that

$$\gamma(H) + \varepsilon < N \tag{6. 3}$$

and

$$\operatorname{Re} \xi(H) \notin \left[N - 2\varepsilon, N - \frac{1}{2} \varepsilon \right] \tag{6. 4}$$

for $H \in U$ and $\xi \in S(\lambda_0)$. By compactness of U and the continuous dependence of τ_λ^k on λ we can now choose a connected bounded open neighborhood Ω of λ_0 in $\mathfrak{a}_c^*$ such that (6.4) holds for $\lambda \in \Omega$, $\xi \in S(\lambda)$, and $H \in U$. Thus for $\lambda \in \Omega$, $S(\lambda)$ is the disjoint union of the subsets

$$S_+(\lambda) = \left\{\xi \in S(\lambda) \mid \operatorname{Re}\xi(H) > N - \frac{1}{2}\varepsilon, \ \forall H \in U\right\}$$

and

$$S_-(\lambda) = \{\xi \in S(\lambda) \mid \operatorname{Re}\xi(H) < N - 2\varepsilon, \ \forall H \in U\}.$$

Let $V_+(\lambda)$ and $V_-(\lambda)$ denote the sums of the corresponding generalized weight spaces for τ_λ^k, and $Q_+(\lambda)$ (resp. $Q_-(\lambda)$) the projection onto $V_+(\lambda)$ along $V_-(\lambda)$ (resp. onto $V_-(\lambda)$ along $V_+(\lambda)$), then $Q_+(\lambda)$ and $Q_-(\lambda)$ depend holomorphically on λ by Proposition 5.8. If necessary we shrink Ω such that the operator norms of $Q_\pm(\lambda)$ are uniformly bounded for $\lambda \in \Omega$.

Fix elements $x_1, \dots, x_p$ of $\mathcal{Y}$ with $x_1 = 1$, such that their canonical images $\pi_k(x_i)$ in $\mathcal{Y}/\bar{\mathfrak{n}}^k\mathcal{Y}$ constitute a basis for $\mathcal{Y}/\bar{\mathfrak{n}}^k\mathcal{Y}$. Via this basis we identify $\mathcal{Y}/\bar{\mathfrak{n}}^k\mathcal{Y}$ with $\mathbb{C}^p$, and denote by $B(\lambda, H)$ the matrix of $\tau_\lambda^k(H)$. Let

$$y_i(\lambda, H) = \tau_\lambda(H)\, x_i - \sum_{j=1}^{p} B(\lambda, H)_{ji}\, x_j \in \bar{\mathfrak{n}}^k\mathcal{Y}$$

for $i = 1, \dots, p$, then $y_i(\lambda, H)$ depends linearly on H. From (5.6) it follows that the $y_i(\lambda, H)$ ($\lambda \in \mathfrak{a}_c^*$, $H \in \mathfrak{a}$, and $i = 1, \dots, p$) span a finite dimensional subspace of $\bar{\mathfrak{n}}^k\mathcal{Y}$, and that $\lambda \to y_i(\lambda, H)$ is holomorphic from $\mathfrak{a}_c^*$ into this space. Since $U \subset \mathfrak{a}^+$ and $\Omega \subset \mathfrak{a}_c^*$ are bounded sets, it follows from Lemma 6.2 that there exist elements $q \in \mathbb{N}$ and $r' \geqq r$, and constants $c, C \geqq 0$ such that

$$\|R_{\exp tH}\, x_i f\|_{r'} \leqq C\, \|f\|_{q,r}\, e^{ct}$$

(here we have used Lemma 6.2 with $k = 0$) and

$$\|R_{\exp tH}\, y_i(\lambda, H) f\|_{r'} \leqq C\, \|f\|_{q,r}\, e^{\gamma(H)t}$$

for all $f \in C_r^q$, $H \in U$, $\lambda \in \Omega$, $t \geqq 0$, and $i = 1, \dots, p$. Hence if we define

$$F(H, t)_i = R_{\exp tH}\, R_{x_i}$$

and

$$G_\lambda(H, t)_i = R_{\exp tH}\, R_{y_i(\lambda, H)}$$

then $F(H, t)_i$ and $G_\lambda(H, t)_i$ are bounded linear maps from C_r^q to $C_{r'}$, and their operator norms satisfy

$$\|F(H, t)_i\| \leqq C e^{ct} \tag{6.5}$$

and

$$\|G_\lambda(H, t)_i\| \leqq C e^{\gamma(H)t} \tag{6.6}$$

for all $H \in U$, $\lambda \in \Omega$, $t \geqq 0$, and $i = 1, \dots, p$.

The motivation for these definitions is that if $f \in \mathscr{E}_\lambda(G/K)$ then by (5.5)

$$Hxf = [\tau_\lambda(H)\,x]\,f$$

for $H \in \mathfrak{a}$ and $x \in \mathscr{Y}$, hence

$$Hx_i f = \sum_{j=1}^{p} B(\lambda, H)_{ji}\, x_j f + y_i(\lambda, H) f$$

and we obtain the ($C_{r'}$-valued) differential equation

$$\frac{d}{dt} F(H, t) f = [B(\lambda, H)\, F(H, t) + G_\lambda(H, t)]\, f \tag{6.7}$$

for $f \in \mathscr{E}_\lambda(G/K) \cap C_r^q$. The estimate (ii) will be obtained from manipulations with this equation (notice that $R_{\exp tH} f = F(H, t)_1 f$). First we rewrite it as the integral equation

$$F(H, t) f = \left[e^{tB(\lambda, H)} F(H, 0) + \int_0^t e^{(t-s)B(\lambda, H)}\, G_\lambda(H, s)\, ds \right] f. \tag{6.8}$$

We will use $Q_\pm(\lambda)$ to split the right hand side of this equation, but first we estimate some of the resulting terms.

We have

$$\|Q_-(\lambda)\, e^{sB(\lambda, H)}\| \leqq C(1+s)^d\, e^{s(N-2\varepsilon)}$$

for all $\lambda \in \Omega$, $H \in U$, and $s \geqq 0$, for a suitable constant $C > 0$ (cf. Corollaries 5.7 and 5.11). From this we deduce

$$\|Q_-(\lambda)\, e^{sB(\lambda, H)}\| \leqq C e^{s(N-\varepsilon)} \tag{6.9}$$

for some (new) constant $C > 0$. Similarly

$$\|Q_+(\lambda)\, e^{-sB(\lambda, H)}\| \leqq C e^{-s(N-\varepsilon)} \tag{6.10}$$

for all $\lambda \in \Omega$, $H \in U$, and $s \geqq 0$. From (6.5), (6.6), and (6.9) we infer

$$\|Q_-(\lambda)\, e^{tB(\lambda, H)}\, F(H, 0)\| \leqq C e^{t(N-\varepsilon)} \tag{6.11}$$

and

$$\|Q_-(\lambda)\, e^{(t-s)B(\lambda, H)}\, G_\lambda(H, s)\| \leqq C e^{t(N-\varepsilon)} e^{-s(N-\varepsilon-\gamma(H))}$$

for all $\lambda \in \Omega$, $H \in U$, and $0 \leqq s \leqq t$. Combining the latter estimate with (6.3) we find

$$\left\| Q_-(\lambda) \int_0^t e^{(t-s)B(\lambda, H)}\, G_\lambda(H, s)\, ds \right\| \leqq C e^{t(N-\varepsilon)} \tag{6.12}$$

with $C > 0$ a (new) constant. On the other hand from (6.6) and (6.10) we have an estimate

$$\|Q_+(\lambda)\, e^{(t-s)B(\lambda, H)}\, G_\lambda(H, s)\| \leqq C e^{t(N-\varepsilon)} e^{-s(N-\varepsilon-\gamma(H))}$$

for $0 \leqq t \leqq s$. Taking (6.3) into account, we infer that

$$\int_t^\infty Q_+(\lambda)\, e^{(t-s)B(\lambda,H)} G_\lambda(H,s)\, ds \tag{6.13}$$

converges absolutely and that

$$\left\| \int_t^\infty Q_+(\lambda)\, e^{(t-s)B(\lambda,H)} G_\lambda(H,s)\, ds \right\| \leqq C e^{t(N-\varepsilon)}. \tag{6.14}$$

Moreover, since all estimates are uniform in $\lambda \in \Omega$ and $H \in U$, (6.13) depends continuously on $(\lambda, H) \in \Omega \times U$, and holomorphically on $\lambda \in \Omega$. In particular, if we define

$$\Psi(\lambda, H) = Q_+(\lambda)\, F(H, 0) + \int_0^\infty Q_+(\lambda)\, e^{-sB(\lambda,H)} G_\lambda(H,s)\, ds \tag{6.15}$$

for $\lambda \in \Omega$ and $H \in U$, then $\Psi(\lambda, H)_i \in B(C_r^q, C_{r'})$ for $i = 1, \ldots, p$. Moreover $\Psi(\lambda, H)_i$ depends continuously on (λ, H) and holomorphically on λ, and we have $\|\Psi(\lambda, H)\| \leqq C$ (take $t = 0$ in (6.14)).

Now if $f \in \mathscr{E}_\lambda \cap C_r^q$ we obtain from (6.8) and (6.15) that

$$\begin{aligned} F(H,t)f = \Big[& e^{tB(\lambda,H)}\Psi(\lambda,H) + Q_-(\lambda)\, e^{tB(\lambda,H)} F(H,0) \\ & + \int_0^t Q_-(\lambda)\, e^{(t-s)B(\lambda,H)} G_\lambda(H,s)\, ds \\ & - \int_t^\infty Q_+(\lambda)\, e^{(t-s)B(\lambda,H)} G_\lambda(H,s)\, ds \Big] f \end{aligned}$$

and hence by (6.11), (6.12), and (6.14) that

$$\|F(H,t)f - e^{tB(\lambda,H)}\Psi(\lambda,H)f\|_{r'} \leqq C e^{t(N-\varepsilon)} \|f\|_{q,r}$$

for some constant $C > 0$ (independent of f). Let $\eta \in (\mathbb{C}^p)^*$ be defined by $\eta(z_1, \ldots, z_p) = z_1$, then we finally get that

$$\|R_{\exp tH} f - \eta\, e^{tB(\lambda,H)}\Psi(\lambda,H)f\|_{r'} \leqq C e^{t(N-\varepsilon)} \|f\|_{q,r} \tag{6.16}$$

for all $f \in \mathscr{E}_\lambda \cap C_r^q$, $\lambda \in \Omega$, $H \in U$, and $t \geqq 0$. □

Remark 6.3. It follows easily from the definition of Ψ (see (6.15)) that $\Psi(\lambda, tH) = \Psi(\lambda, H)$ if $t > 0$ and $tH \in U$. Moreover $\Psi(\lambda, H)$ maps C_r^q into $V_+(\lambda) \otimes C_{r'}$.

Remark 6.4. Let $q' \in \mathbb{N}$. Restricting from C_r^q to $C_r^{q+q'}$ we have that Ψ maps $\Omega \times U$ continuously (and holomorphically in the first variable) into $B(C_r^{q+q'}, \mathscr{Y}/\bar{\mathfrak{n}}^k\mathscr{Y} \otimes C_{r'}^{q'})$, and that

$$\|R_{\exp tH} f - (\eta \circ \exp \tau_\lambda^k(tH) \otimes 1)\, \Psi(\lambda, H)f\|_{q',r'} \leqq C \|f\|_{q+q',r}\, e^{(N-\varepsilon)t}$$

for some constant C (which depends on q'). Indeed this is a consequence of the following observation:

Lemma 6.5. *Let $r, r' \in \mathbb{R}$, $q \in \mathbb{N}$, and let T be a bounded linear map of $C_r^q(G)$ into $C_{r'}(G)$, G-equivariant for the left action and with operator norm $\|T\|$. Then T maps $C_r^{q+q'}(G)$ continuously into $C_{r'}^{q'}(G)$, with operator norm $\leqq C\|T\|$ for each $q' \in \mathbb{N}$, C being a constant depending only on r, r', q, and q'.*

Proof. Follows from the (easily established) fact that $C_r^{q+q'}$ is the set of $C^{q'}$-vectors for the left action of G on C_r^q. □

We now begin the proof of Theorem 3.5. Using Corollary 5.7 we can write

$$(\eta \exp \tau_\lambda^k(tH) \otimes 1)\, \Psi(\lambda, H) = \sum_\xi p_{\lambda,\xi}(H, t)\, e^{t\xi(H)}$$

for $\lambda \in \Omega$, $H \in U$, and $t \geqq 0$, where the summation extends over the weights ξ of τ_λ^k, and

$$p_{\lambda,\xi}(H, t) = (\eta \circ q_{\lambda,\xi}(tH) \otimes 1)\, \Psi(\lambda, H) \in B(C_r^q, C_{r'}). \tag{6.17}$$

Each $p_{\lambda,\xi}(H, t)$ is continuous in H and polynomial in t of degree $\leqq d$ (cf. Corollary 5.11). From (ii) we have

$$\Big\|R_{\exp tH} f - \sum_\xi e^{t\xi(H)} p_{\lambda,\xi}(H, t) f\Big\|_{r'} \leqq C \|f\|_{q,r}\, e^{t(N-\varepsilon)} \tag{6.18}$$

for $f \in \mathscr{E}_\lambda \cap C_r^q$. For $g \in G$ we put $p_{\lambda,\xi}(f, g, H, t) = (p_{\lambda,\xi}(H, t) f)(g)$. It then follows from (6.18) that $\sum_\xi p_{\lambda,\xi}(f, g, H, t)\, e^{t\xi(H)}$ is asymptotic to $f(g \exp tH)$ of order N at H_0.

Since N was arbitrary we deduce from Corollary 3.4 that for each $\lambda \in \mathfrak{a}_c^*$, $\xi \in X(\lambda)$, and $r \in \mathbb{R}$ there exist constants r' and q, and a continuous map $p_{\lambda,\xi}\colon \mathfrak{a}^+ \to B(\mathscr{E}_\lambda \cap C_r^q, C_{r'})$ such that

$$f(g \exp tH) \sim \sum_{\xi \in X(\lambda)} p_{\lambda,\xi}(f, g, tH)\, e^{t\xi(H)} \qquad (t \to \infty)$$

at every $H_0 \in \mathfrak{a}^+$, for $f \in \mathscr{E}_\lambda \cap C_r^\infty$ and $g \in G$. From Remark 6.4 we have that $p_{\lambda,\xi}(H)$ maps $\mathfrak{a}^+$ continuously into $B(\mathscr{E}_\lambda \cap C_{r'}^{q+q'}, C_{r'}^{q'})$ for every $q' \in \mathbb{N}$. Moreover $p_{\lambda,\xi}(H)$ intertwines the left regular representations, and the map $t \to p_{\lambda,\xi}(f, g, tH)$ is a polynomial on $\mathbb{R}_+$ of degree $\leqq d$, which is constant if $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}$ for all $\alpha \in \Sigma$ (cf. Corollary 5.7). In order to prove Theorem 3.5 it only remains to be seen that $p_{\lambda,\xi}(f, g, H)$ is polynomial in H. Postponing this to the next section we now consider the dependence on λ in order to prove Theorem 3.6.

Let $(f_\lambda)_{\lambda \in \Omega_0}$ be a holomorphic family in $\mathscr{E}_\lambda^\infty$. Let $\lambda_0 \in \Omega_0$ and assume $f_\lambda \in C_r^\infty$ in a neighborhood Ω_1 of λ_0. It follows from (6.17), Corollary 5.10 and Remark 6.4 that the map

$$(\lambda, H) \to \sum_{\xi \in \Xi(\lambda)} e^{\xi(H)} p_{\lambda,\xi}(H)$$

is continuous from $\Omega \times U$ to $B(C_r^{q+q'}, C_{r'}^{q'})$, and in addition holomorphic in λ, for each $q' \in \mathbb{N}$. Hence

$$(\lambda, H) \longrightarrow \sum_{\xi \in \Xi(\lambda)} e^{\xi(H)} p_{\lambda,\xi}(f_\lambda, H)$$

is continuous from $(\Omega \cap \Omega_1) \times U$ to $C_{r'}^{q'}$, and in addition holomorphic in λ. The first assertion of Theorem 3.6 easily follows, and the second assertion is an immediate consequence of this and Theorem 3.5(iii) (whose proof will be completed shortly).

7. Differential equations for the expansion coefficients

In this section we derive certain differential equations for the functions $p_{\lambda,\xi}(f, g)$ on $\mathfrak{a}^+$, where $f \in \mathscr{E}_\lambda^\infty$ and $g \in G$. In particular the equations will show that these functions are polynomials.

Fix $u \in \mathscr{U}(\mathfrak{g})^K$ and let $D = \mu(u) \in \mathbb{D}(G/K)$. By (1.5) we can choose finitely many $x_i \in \bar{\mathfrak{n}}\mathscr{U}(\bar{\mathfrak{n}})$ and $v_i \in \mathscr{U}(\mathfrak{a})$ such that

$$u - {'\gamma}(D) - \sum_i x_i v_i \in \mathscr{U}(\mathfrak{g})\,\mathfrak{k} \tag{7.1}$$

and such that $\mathrm{ad}(\mathfrak{a})$ acts on x_i by a weight $-\eta_i \neq 0$ where $\eta_i \in \mathbb{N} \cdot \Delta$.

Proposition 7.1. *Let $\lambda \in \mathfrak{a}_c^*$ and $f \in \mathscr{E}_\lambda^\infty(G/K)$. The functions $p_{\lambda,\xi}(f)\, e^\xi$ on $G \times \mathfrak{a}^+$ satisfy the following recursive equations*

$$\begin{aligned}(1 \otimes [\partial({'\gamma}(D)) - \chi_\lambda(D)])\, p_{\lambda,\xi}(f)\, e^\xi \\ = - \sum_{i,\, \xi+\eta_i \in X(\lambda)} (R_{x_i} \otimes e^{-\eta_i} \partial(v_i))\, p_{\lambda,\xi+\eta_i}(f)\, e^{\xi+\eta_i}\end{aligned} \tag{7.2}$$

for all $\xi \in X(\lambda)$.

Before giving the proof of the proposition, let us use it to finish what was left in the previous section.

Proof of Theorem 3.5. It only remains to be seen that $p_{\lambda,\xi}(f, g, H)$ is polynomial in H. Since we know that $t \longrightarrow p_{\lambda,\xi}(f, g, tH)$ is polynomial on $\mathbb{R}_+$ it is easily seen that it suffices to prove that $p_{\lambda,\xi}(f, g) \in Q(\mathfrak{a}^+)$ (i.e. $p_{\lambda,\xi}(f, g)$ is a finite sum of exponential polynomials on $\mathfrak{a}^+$) for all $f \in \mathscr{E}_\lambda^\infty$, $g \in G$, and $\xi \in X(\lambda)$. By the intertwining property of $p_{\lambda,\xi}$ it suffices to take $g = e$.

If ξ is $\prec$-maximal in $X(\lambda)$ the right hand side of (7.2) vanishes and it follows that $p_{\lambda,\xi}(f, e)\, e^{\xi+\varrho} \in E_\lambda(\mathfrak{a}^+)$ (cf. (4.1)), hence $p_{\lambda,\xi}(f, e) \in Q(\mathfrak{a}^+)$ for all $f \in \mathscr{E}_\lambda^\infty$. In general evaluation of (7.2) at e yields

$$[\partial({'\gamma}(D)) - \chi_\lambda(D)]\, p_{\lambda,\xi}(f, e)\, e^\xi = - \sum_{i,\, \xi+\eta_i \in X(\lambda)} e^{-\eta_i} \partial(v_i)\, p_{\lambda,\xi+\eta_i}(L_{x_i^\vee} f, e)\, e^{\xi+\eta_i}. \tag{7.3}$$

Using this and Lemma 4.3 recursively for all $f \in \mathscr{E}_\lambda^\infty$ gives the desired result. □

The main step in the proof of Proposition 7. 1 is the following lemma.

Lemma 7. 2. *Let $\lambda \in \mathfrak{a}_c^*$, $f \in \mathscr{E}_\lambda^\infty$, and $g \in G$. The functions $p_{\lambda,\xi}(f, g)$ on $\mathfrak{a}^+$ are smooth for all $\xi \in X(\lambda)$. For each $v \in \mathscr{U}(\mathfrak{a})$ the function vf has the exponential asymptotic expansion*

$$(7.4) \qquad vf(g \exp tH) \sim \sum_{\xi \in X(\lambda)} [\partial(T_\xi v)\, p_{\lambda,\xi}(f, g)]\,(tH)\, e^{t\xi(H)}$$

at every $H_0 \in \mathfrak{a}^+$, for $t \to \infty$.

Here $T_\xi v$ is defined by (1. 3). The proof of Lemma 7. 2 follows similar lines as the previous section. We start with a technical lemma elaborating Proposition 6. 1. Fix $\lambda_0 \in \mathfrak{a}_c^*$, $H_0 \in \mathfrak{a}^+$, $r \in \mathbb{R}$, and $N \in \mathbb{R}$, and let Ω, U, k, q, r', ε, Ψ, and η be given by Proposition 6. 1.

Lemma 7. 3. *Let $l_0 \in \mathbb{N}$. There exist constants $q'' \in \mathbb{N}$ and $r'' \in \mathbb{R}$ (with $q'' \geqq q$ and $r'' \geqq r'$), such that*

(i) *The map $\Omega \times U \to B(C_r^{q''}, \mathscr{Y}/\bar{\mathfrak{n}}^k\mathscr{Y} \otimes C_{r''})$, obtained from Ψ (and denoted by the same letter) via restriction and injection, has derivatives in $H \in U$ up to order l_0. The derivatives are continuous in (λ, H) and holomorphic in λ.*

(ii) *For each $v \in \mathscr{U}(\mathfrak{a})$ of degree $\leqq l_0$ there exists $C > 0$ such that*

$$\|R_{\exp tH}\, vf - \partial(v)\,[(\eta \circ \exp \tau_\lambda^k(\cdot) \otimes 1)\, \Psi(\lambda, \cdot) f]\,(tH)\|_{r''} \leqq C\, \|f\|_{q'', r}\, e^{(N-\varepsilon)t}$$

for all $f \in \mathscr{E}_\lambda(G/K) \cap C_r^{q''}$, $\lambda \in \Omega$, $H \in U$, and $t \geqq 0$.

Proof. We use notation from the proof of Proposition 6. 1. The maps $F(H, t)$ and $G_\lambda(H, t)$ are differentiated in H as follows. Assume (without loss of generality) that $v \in \mathscr{U}(\mathfrak{a})$ is homogeneous of degree $l \leqq l_0$. Then

$$\partial(v)_H\, F(H, t)_i = (-t)^l\, R_v \circ F(H, t)_i$$

(where the subscript H indicates differentiation in the H-variable). Moreover, if $H_1, \ldots, H_n$ is a basis for $\mathfrak{a}$, we can write $v = \sum_{j=1}^{n} u_j H_j$ with $u_1, \ldots, u_n \in \mathscr{U}(\mathfrak{a})$ homogeneous of degree $l-1$ (possibly some $u_j = 0$), and then

$$\partial(v)_H\, G_\lambda(H, t)_i = (-t)^l\, R_v \circ G(H, t)_i + (-t)^{l-1} \sum_{j=1}^{n} R_{\exp tH}\, R_{u_j y_i(\lambda, H_j)}$$

for $i = 1, \ldots, p$. It follows from Lemma 6. 2 that there exist q'' and r'' such that $\partial(v)_H\, F(H, t)_i$ and $\partial(v)_H\, G_\lambda(H, t)_i$ exist as bounded linear maps from $C_r^{q''}$ to $C_{r''}$, and that their operator norms satisfy

$$\|\partial(v)_H\, F(H, t)_i\| \leqq C(1+t)^l\, e^{ct}$$

and

$$\|\partial(v)_H\, G_\lambda(H, t)_i\| \leqq C(1+t)^l\, e^{\gamma(H)t}$$

for suitable constants C and c, for all $H \in U$, $\lambda \in \Omega$, $t \geqq 0$, and $i = 1, \ldots, p$.

In analogy with (6.9) and (6.10) we also have

$$\|\partial(v)_H [Q_-(\lambda)\, e^{sB(\lambda,H)}]\| \leqq C e^{s(N-\varepsilon)}$$

and

$$\|\partial(v)_H [Q_+(\lambda)\, e^{-sB(\lambda,H)}]\| \leqq C e^{-s(N-\varepsilon)},$$

and then

$$\|\partial(v)_H [Q_-(\lambda)\, e^{tB(\lambda,H)} F(H,0)]\| \leqq C e^{t(N-\varepsilon)}$$

and (using the Leibniz rule) for $0 \leqq s \leqq t$

$$\|\partial(v)_H [Q_-(\lambda)\, e^{(t-s)B(\lambda,H)} G_\lambda(H,s)]\| \leqq C(1+s)^l\, e^{t(N-\varepsilon)} e^{-s(N-\varepsilon-\gamma(H))}$$

from which we get in analogy with (6.12)

$$\|\partial(v)_H [Q_-(\lambda) \int_0^t e^{(t-s)B(\lambda,H)} G_\lambda(H,s)\, ds]\| \leqq C e^{t(N-\varepsilon)}$$

(each time with a new constant $C > 0$). Similarly we get

$$\|\partial(v)_H [Q_+(\lambda)\, e^{(t-s)B(\lambda,H)} G_\lambda(H,s)]\| \leqq C(1+s)^l\, e^{t(N-\varepsilon)} e^{-s(N-\varepsilon-\gamma(H))}$$

for $0 \leqq t \leqq s$, and we conclude in analogy with (6.13), (6.14) that

$$\int_t^\infty \partial(v)_H [Q_+(\lambda)\, e^{(t-s)B(\lambda,H)} G_\lambda(H,s)]\, ds$$

converges absolutely and has its norm dominated by $C e^{t(N-\varepsilon)}$. In particular it follows from this that $\partial(v)_H\, \Psi(\lambda, H)$ exists as a bounded linear map from $C_r^{q''}$ to $\mathscr{Y}/\bar{\mathfrak{n}}^k\mathscr{Y} \otimes C_{r''}$, and that it depends continuously on (λ, H) and holomorphically on λ.

Combining the estimates above one obtains (cf. (6.16))

$$(7.5) \qquad \|\partial(v)_H [R_{\exp tH} f - \eta\, e^{tB(\lambda,H)}\, \Psi(\lambda,H) f]\|_{r''} \leqq C e^{t(N-\varepsilon)} \|f\|_{q'',r}$$

for $f \in \mathscr{E}_\lambda \cap C_r^{q''}$, $\lambda \in \Omega$, $H \in U$, and $t \geqq 0$. Moreover, by the homogeneity of v, the left hand side of (7.5) equals

$$t^l \|R_{\exp tH}\, vf - \partial(v)\, [\eta\, e^{B(\lambda,\cdot)}\, \Psi(\lambda,\cdot) f]\, (tH)\|_{r''}$$

(recall Remark 6.3 that $\Psi(\lambda, th) = \Psi(\lambda, H)$). The lemma follows. □

Proof of Lemma 7.2. Easy from Lemma 7.3. Notice that we are not using the polynomial property of $p_{\lambda,\xi}$ on $\mathfrak{a}^+$. □

Remark 7.4. The proof above shows that the expansion (7.4) of vf has similar properties as that of f (expressed in Theorems 3.5 and 3.6). Thus $f \to \partial(T_\xi v)_H\, p_{\lambda,\xi}(f, H)$ is continuous from $\mathscr{E}_{\lambda,r}^\infty$ to $C_{r''}^\infty$ for each $H \in \mathfrak{a}^+$, and if $(f_\lambda)_{\lambda \in \Omega_0}$ is a holomorphic family in $\mathscr{E}_\lambda^\infty$, then for each $\lambda_0 \in \Omega_0$ and $\xi_0 \in X(\lambda_0)$ there is a neighborhood Ω of λ_0 such that

$$(\lambda, H) \to \partial(v)_H \,[\sum_{\xi \in \Xi(\lambda)} p_{\lambda,\xi}(f_\lambda, H)\, e^{\xi(H)}]$$

is continuous from $\Omega \times \mathfrak{a}^+$ to $C_{r''}^\infty$, and holomorphic in λ.

Lemma 7.5. *Let $\lambda \in \mathfrak{a}_c^*$, $f \in \mathscr{E}_\lambda^\infty$, $g \in G$, $v \in \mathscr{U}(\mathfrak{a})$, $x \in \mathscr{U}(\bar{\mathfrak{n}})$, and $\eta \in \mathbb{N} \cdot \Delta$, and assume $[H, x] = -\eta(H)\, x$ for $H \in \mathfrak{a}$. Then the function xvf has the exponential asymptotic expansion*

$$xvf(g \exp tH) \sim \sum_{\xi+\eta \in X(\lambda)} [(R_x \otimes \partial(T_{\xi+\eta} v))\, p_{\lambda,\xi+\eta}(f)]\,(tH)\, e^{t\xi(H)}$$

at every $H_0 \in \mathfrak{a}^+$, for $t \to +\infty$.

Proof. We may assume $g = e$. Then

$$xvf(\exp tH) = e^{-\eta(tH)}\, v L_{x^\vee} f(\exp tH)$$

and it follows from Lemma 7.2 that

$$xvf(\exp tH) \sim \sum_{\xi \in X(\lambda)} [\partial(T_\xi v)\, p_{\lambda,\xi}(L_{x^\vee} f, e)]\,(tH)\, e^{t(\xi-\eta)H}.$$

The lemma now follows from

$$p_{\lambda,\xi}(L_{x^\vee} f, e) = (L_{x^\vee}\, p_{\lambda,\xi}(f))\,(e) = (R_x p_{\lambda,\xi}(f))\,(e). \qquad \square$$

Proof of Proposition 7.1. From (7.1) we have $Df = (v + x_i v_i) f$ where $v = {}'\gamma(D)$ and hence by Lemma 7.5 we have that $Df(g \exp tH)$ is asymptotic to

$$\sum_{\xi \in X(\lambda)} [(1 \otimes \partial(T_\xi v))\, p_{\lambda,\xi}(f) + \sum_i (R_{x_i} \otimes \partial(T_{\xi+\eta_i} v_i))\, p_{\lambda,\xi+\eta_i}(f)]\,(tH)\, e^{t\xi(H)}$$

as $t \to +\infty$ (with the convention that $p_{\lambda,\xi+\eta_i} = 0$ if $\xi + \eta_i \notin X(\lambda)$). Since $Df = \chi_\lambda(D) f$ and the asymptotics are unique, we get (7.2) from the relation

$$\partial(T_\zeta v) = e^{-\zeta} \partial(v) \circ e^\zeta \quad (\zeta \in \mathfrak{a}_c^*). \qquad \square$$

With this the proofs of Theorems 3.5 and 3.6 are finished.

8. Leading exponents and principal parts

We shall now consider the "leading exponents" ξ in the asymptotic expansion from Theorem 3.5, and show that for these ξ the $P_d(\mathfrak{a})$-valued functions $p_{\lambda,\xi}(f)$ on G satisfy a homogeneity under the right action of the minimal parabolic subgroup $P = MAN$.

Let $f \in \mathscr{E}_\lambda^\infty(G/K)$ and $g_0 \in G$. We say that an element $\xi \in X(\lambda)$ is an *exponent of f at g_0 along $g_0 A^+$* if the support of $p_{\lambda,\xi}(f)$ contains g_0. The set of exponents at g_0 along $g_0 A^+$ is denoted by $E(f, g_0)$. If V is an open subset of G we put

$$E(f, V) = \bigcup_{g \in V} E(f, g).$$

Thus an element $\xi \in X(\lambda)$ belongs to $E(f, V)$ if and only if $p_{\lambda,\xi}(f)$ is not identically zero on V. The set of $\prec$-maximal elements in $E(f, g_0)$ (resp. $E(f, V)$), called the *leading exponents*, is denoted $E_L(f, g_0)$ (resp. $E_L(f, V)$). Recall that $W^\nu = \{w \in W \mid w\nu = \nu\}$ for $\nu \in \mathfrak{a}_c^*$.

Proposition 8. 1. *Let* $\lambda \in \mathfrak{a}_c^*$, $f \in \mathscr{E}_\lambda^\infty(G/K)$, $g_0 \in G$, *and* $\sigma \in E_L(f, g_0)$. *Then* $\sigma \in W\lambda - \varrho$. *Moreover there is a neighborhood* V *of* g_0 *such that* $p_{\lambda,\sigma}(f, g)$ *is a* $W^{\sigma+\varrho}$-*harmonic polynomial on* $\mathfrak{a}$, *for all* $g \in V$.

Proof. Since σ is leading and $(\sigma + \mathbb{N} \cdot \Delta) \cap X(\lambda)$ is finite there is a neighborhood V of g_0 such that $p_{\lambda,\xi}(f) \equiv 0$ on V for all $\xi \succ \sigma$, $\xi \neq \sigma$. Therefore the right hand side of (7. 2) vanishes on V, so that $p_{\lambda,\sigma}(f, g)\, e^{\sigma+\varrho} \in E_\lambda(\mathfrak{a}^+)$ for $g \in V$. Since $p_{\lambda,\sigma}(f, g)$ is a polynomial the proposition follows from Proposition 4. 1. □

Let $V \subset G$ be open and let $g_0 \in V$. It is easily seen that

$$E_L(f, V) \cap E(f, g_0) \subset E_L(f, g_0).$$

Hence we obtain the following.

Corollary 8. 2. *We have* $E_L(f, V) \subset W\lambda - \varrho$. *Moreover* $p_{\lambda,\sigma}(f, g)$ *is* $W^{\sigma+\varrho}$-*harmonic for all* $\sigma \in E_L(f, V)$ *and* $g \in V$.

We define the *principal part* of f in V along VA^+ to be the element $P(f, V)$ of $C^\infty(V) \otimes C^\infty(\mathfrak{a})$ given by

$$P(f, V) = \sum_{\sigma \in E_L(f, V)} p_{\lambda,\sigma}(f)\, e^\sigma.$$

The following lemma is an immediate consequence of this definition.

Lemma 8. 3. *The principal part* $P(f, V)$ *of* f *in* V *along* VA^+ *vanishes if and only if* $f(g \exp tH)$ *is asymptotic to zero for all* $H \in \mathfrak{a}^+$ *and* $g \in V$ (*i.e.* $p_{\lambda,\xi}(f, g) \equiv 0$, $\forall \xi \in X(\lambda)$).

Given an element $Y \in \mathfrak{a}$ we define the linear automorphism T_Y of $C^\infty(\mathfrak{a})$ by $T_Y \psi(H) = \psi(H + Y)$ for $\psi \in C^\infty(\mathfrak{a})$, $H \in \mathfrak{a}$. The following result describes the transformation of the principal part under the action of $P = MAN$ from the right.

Theorem 8. 4. *Let* $\lambda \in \mathfrak{a}_c^*$, $f \in \mathscr{E}_\lambda^\infty(G/K)$, *and* V *an open subset of* G. *Then* $E_L(f, V) = E_L(f, VP)$, *and the principal part* $P(f, VP)$ *of* f *in* VP *satisfies*

$$(R_{man} \otimes 1)\, P(f, VP) = (1 \otimes T_{\log a})\, P(f, VP)$$

for all $m \in M$, $a \in A$, $n \in N$.

The proof is preceded by a series of lemmas in which we describe the transformation under P of $p_{\lambda,\xi}(f)$ for each $\xi \in X(\lambda)$.

Lemma 8. 5. *Let* $\xi \in X(\lambda)$, $m \in M$ *and* $a \in A$. *Then*

$$(R_{ma} \otimes 1)\, p_{\lambda,\xi}(f) = a^\xi (1 \otimes T_{\log a})\, p_{\lambda,\xi}(f).$$

Proof. Clearly $f(gm \exp tH) = f(g \exp tH)$ so that R_m leaves $p_{\lambda,\xi}(f)$ fixed. We have

$$f(ga \exp tH) = f(g \exp(tH + \log a)).$$

Let $H_0 \in \mathfrak{a}^+$ and $N \in \mathbb{R}$, and let U be a neighborhood of H_0 in $\mathfrak{a}^+$ such that

$$|f(g \exp tH) - \sum_{\operatorname{Re} \xi(H_0) \geqq N} p_{\lambda,\xi}(f, g, tH)\, e^{t\xi(H)}| \leqq C e^{t(N-\varepsilon)}$$

for all $H \in U$. Since $tH + \log a = t(H + t^{-1} \log a)$ and $H + t^{-1} \log a \in U$ for t sufficiently large, it follows that for H near H_0 we have

$$|f(g\,a \exp tH) - \sum_{\mathrm{Re}\,\xi(H_0) \geqq N} p_{\lambda,\xi}(f, g, tH + \log a)\, e^{\xi(tH + \log a)}| \leqq C_1\, e^{t(N-\varepsilon)}$$

for some constant C_1. The lemma now follows from the uniqueness of the asymptotic expansion. □

Before considering the effect of N on $p_{\lambda,\xi}(f)$ we need the following simple lemma. We define $\boldsymbol{\alpha}: A \to \mathbb{R}^\Delta$ by $\boldsymbol{\alpha}(a)_\gamma = a^{-\gamma}$ for $a \in A$ and $\gamma \in \Delta$.

Lemma 8. 6. *Fix $n \in N$. There exists a unique real analytic map $z: \mathbb{R}^\Delta \to \bar{N}A$, such that $na \in z(\boldsymbol{\alpha}(a))\, aK$ for $a \in A$. In particular, $z(0) = e$.*

Proof. By easy root space calculations there exist real analytic maps $\psi: \mathbb{R}^\Delta \times NA \to NA$ and $\bar{\psi}: \mathbb{R}^\Delta \times \bar{N}A \to \bar{N}A$ such that $\psi(\boldsymbol{\alpha}(a), y) = a^{-1} y a$ and $\bar{\psi}(\boldsymbol{\alpha}(a), \bar{y}) = a \bar{y} a^{-1}$ for all $y \in NA$, $\bar{y} \in \bar{N}A$, and $a \in A$. Let $\tau: NA \to \bar{N}A$ be the real analytic diffeomorphism defined by $y \in \tau(y)\, K$, and define

$$z(t) = \bar{\psi}(t, \tau(\psi(t, n))).$$

Then $z(\boldsymbol{\alpha}(a)) = a\tau(a^{-1} n a)\, a^{-1} = \tau(na)\, a^{-1}$, and the first assertion follows. It follows easily from the definition of z that $z(0) = e$. □

Let $n \in N$ and define z according to this lemma. For each $f \in C_r^\infty$ the function $x \to L(z(x)^{-1}) f$ from $\mathbb{R}^\Delta$ to the Fréchet space C_r^∞ has a Taylor expansion at 0

$$L(z(x)^{-1}) f = \sum_{\substack{\mu \in \mathbb{N} \cdot \Delta \\ |\mu| < k}} x^\mu f_\mu + F_k(x) \tag{8. 1}$$

where $F_k(x)$ is the remainder term ($k \in \mathbb{N}$). Here we have used the multiindex notations $|\mu| = \sum \mu_\alpha$ and $x^\mu = \prod (x_\alpha)^{\mu_\alpha}$ for $\mu = \sum_{\alpha \in \Delta} \mu_\alpha \alpha \in \mathbb{N} \cdot \Delta$ and $x \in \mathbb{R}^\Delta$. The remainder is estimated by

$$\|F_k(x)\|_{q-k,r} \leqq C_{q,k} |x|^k \, \|f\|_{q,r} \tag{8. 2}$$

for $|x| \leqq 1$, for each $q \in \mathbb{N}$ ($q \geqq k$), with $C_{q,k}$ a constant independent of x and f. Notice that $f_0 = f$ and $f_\mu \in L(\mathscr{U}(\mathfrak{g})) f$.

Lemma 8. 7. *Assume $f \in \mathscr{E}_{\lambda,r}^\infty$ and $\xi \in X(\lambda)$. Then*

$$p_{\lambda,\xi}(f, n) = \sum_{\substack{\mu \in \mathbb{N} \cdot \Delta \\ \xi + \mu \in X(\lambda)}} p_{\lambda,\xi+\mu}(f_\mu, e).$$

Proof. Let $N \in \mathbb{R}$ and $H_0 \in \mathfrak{a}^+$, and choose k, ε, and U as in the proof of Proposition 6. 1 so that $\gamma(H) < N - \varepsilon$ for $H \in U$ (cf. (6. 3)). Fix $H \in U$ and let $x_t = \boldsymbol{\alpha}(\exp tH)$, then

$$f(n \exp tH) = f(z(x_t) \exp tH).$$

From (8. 1) we then obtain

$$f(n \exp tH) - \sum_{\substack{\mu \in \mathbb{N} \cdot \Delta \\ |\mu| < k}} f_\mu(\exp tH)\, e^{-t\mu(H)} = F_k(x_t)\,(\exp tH)$$

and in view of (8.2) there exists a constant C such that

$$|F_k(x_t)(\exp tH)| \leqq C e^{(N-\varepsilon)t}$$

for all $t \geqq 0$, because $|x_t| \leqq e^{-t\beta(H)}$. Now substitute the asymptotic approximations for each f_μ in the above equation, then the result follows from Lemma 3.2. □

Proof of Theorem 8.4. From Lemma 8.5 we find that $E(f, V) = E(f, VMA)$ and hence $E_L(f, V) = E_L(f, VMA)$. Moreover

$$(R_{ma} \otimes 1)(p_{\lambda,\xi}(f) e^{\xi}) = (1 \otimes T_{\log a})(p_{\lambda,\xi}(f) e^{\xi})$$

for any $\xi \in X(\lambda)$. From Lemma 8.7 and the equivariance for L of $p_{\lambda,\xi}$ we infer that

$$p_{\lambda,\sigma}(f, gn) = p_{\lambda,\sigma}(f, g)$$

for $g \in G$, $\sigma \in E_L(f, g)$, and $n \in N$. From these identities the theorem easily follows. □

9. Boundary values

Using the asymptotic expansion we assign in this section to each function in $\mathscr{E}_\lambda^\infty$ a boundary value (in general actually several boundary values), which is a C^∞-function on K/M.

Recall that $\alpha^\vee = 2\langle\alpha, \alpha\rangle^{-1}\alpha$ for $\alpha \in \Sigma$ and let

(9.1) $$\mathscr{A} = \{\lambda \in \mathfrak{a}_c^* \mid \forall \alpha \in \Sigma^+ : \langle\lambda, \alpha^\vee\rangle \notin -\mathbb{N}^*\}$$

where $\mathbb{N}^* = \mathbb{N}\setminus\{0\}$. In particular, if $\operatorname{Re}\lambda$ is dominant then $\lambda \in \mathscr{A}$. Hence $W\mu \cap \mathscr{A} \neq \emptyset$ for all $\mu \in \mathfrak{a}_c^*$.

Lemma 9.1. *Let $\lambda \in \mathfrak{a}_c^*$. Then $\lambda \in \mathscr{A}$ if and only if $w\lambda \succ \lambda$ implies $w\lambda = \lambda$ for all $w \in W$.*

Proof. See [KKMOOT 78], Appendix II, Prop. 2(2). □

From this lemma it follows that if $\lambda \in \mathscr{A}$ then $\lambda - \varrho$ is $\prec$-maximal in $X(\lambda)$, and hence from Corollary 8.2 that $p_{\lambda,\lambda-\varrho}(f, g)$ is a W^λ-harmonic polynomial on $\mathfrak{a}$ for all $g \in G$. Let $\mathscr{A}'$ denote the set of regular elements in $\mathscr{A}$, that is

(9.2) $$\mathscr{A}' = \{\lambda \in \mathfrak{a}_c^* \mid \forall \alpha \in \Sigma^+ : \langle\lambda, \alpha^\vee\rangle \notin -\mathbb{N}\},$$

then it follows that $p_{\lambda,\lambda-\varrho}(f, g)$ is constant on $\mathfrak{a}$ if $\lambda \in \mathscr{A}'$. For $\lambda \in \mathscr{A}'$ we define the *boundary value* $\beta_\lambda(f) \in C^\infty(G)$ of $f \in \mathscr{E}_\lambda^\infty(G/K)$ by $\beta_\lambda(f)(g) = p_{\lambda,\lambda-\varrho}(f, g)$, and call $\beta_\lambda : \mathscr{E}_\lambda^\infty \to C^\infty$ a *boundary value map*. Notice that $w\lambda$ may be contained in $\mathscr{A}'$ for several $w \in W$, in this case we get several boundary value maps $\beta_{w\lambda} : \mathscr{E}_\lambda^\infty \to C^\infty$. Let $\pi \in \mathscr{U}(\mathfrak{a})$ be the element defined via the polynomial function

(9.3) $$\pi(v) = \prod_{\alpha \in \Sigma_0^+} \langle v, \alpha^\vee\rangle$$

on $\mathfrak{a}_c^*$.

Theorem 9. 2. (i) *Let $\lambda \in \mathscr{A}'$. The boundary value map β_λ maps $\mathscr{E}^\infty_{\lambda,r}(G/K)$ linearly, continuously, and G-equivariantly into $C^\infty(G/P; L_\lambda)$ for each $r \in \mathbb{R}$.*

(ii) *Let $\Omega \subset \mathfrak{a}_c^*$ be open, and let $(f_\lambda)_{\lambda\in\Omega}$ be a holomorphic family in $\mathscr{E}^\infty_\lambda$. Then $\lambda \longrightarrow \beta_\lambda(f_\lambda)|_K$ is holomorphic from $\Omega \cap \mathscr{A}'$ to $C^\infty(K/M)$ (that is, holomorphic into the Banach space $C^q(K/M)$ for each $q \in \mathbb{N}$), and $\lambda \longrightarrow \pi(\lambda)\,\beta_\lambda(f_\lambda)|_K$ has removable singularities in $\Omega \cap (\mathscr{A}\backslash\mathscr{A}')$.*

Proof. (i) follows immediately from Theorems 3. 5 and 8. 4, and the first statement in (ii) follows from Theorem 3. 6 (using continuity of the restriction map from $C_r^\infty(G)$ to $C^\infty(K)$). Hence $\lambda \longrightarrow \pi(\lambda)\,\beta_\lambda(f_\lambda)|_K$ is holomorphic in $\Omega \cap \mathscr{A}'$. To justify the final statement in (ii) it suffices by a classical theorem (cf. [Osg 29], p. 187) to prove that all singularities have codimension $\geqq 2$, that is, to prove that the function extends holomorphically to the set

$$\bigcup_{\alpha\in\Sigma_0^+} \{\lambda \in \Omega \cap \mathscr{A} \mid \langle\lambda, \gamma\rangle \neq 0 \text{ for all } \gamma \in \Sigma_0^+\backslash\{\alpha\}\}.$$

Let $\lambda_0 \in \Omega \cap \mathscr{A}$ and assume $\langle\lambda_0, \alpha\rangle = 0$ for some $\alpha \in \Sigma_0^+$ and $\langle\lambda, \gamma\rangle \neq 0$ for all $\gamma \in \Sigma_0^+\backslash\{\alpha\}$. It follows from Theorem 3. 6 that for each $a \in A^+ = \exp\mathfrak{a}^+$ the $C_{r'}(G)^\infty$-valued function (with a suitable $r' \in \mathbb{R}$)

$$\phi_a(\lambda) = \beta_\lambda(f_\lambda)\, a^{\lambda-\varrho} + \beta_{s_\alpha\lambda}(f_\lambda)\, a^{s_\alpha\lambda-\varrho}$$

on $\Omega \cap \mathscr{A}'$, extends holomorphically to a neighborhood of λ_0. We choose $a_1, a_2 \in A^+$ such that $a_1^\alpha \neq a_2^\alpha$ and denote by $A(\lambda)$ the matrix

$$A(\lambda) = \begin{pmatrix} a_1^{\lambda-\varrho} & a_1^{s_\alpha\lambda-\varrho} \\ a_2^{\lambda-\varrho} & a_2^{s_\alpha\lambda-\varrho} \end{pmatrix}. \tag{9. 4}$$

Put

$$\Phi(\lambda) = \begin{pmatrix} \phi_{a_1}(\lambda) \\ \phi_{a_2}(\lambda) \end{pmatrix}, \quad \beta(\lambda) = \begin{pmatrix} \beta_\lambda(f_\lambda) \\ \beta_{s_\alpha\lambda}(f_\lambda) \end{pmatrix},$$

then $\Phi(\lambda) = A(\lambda)\,\beta(\lambda)$ for $\lambda \in \Omega \cap \mathscr{A}'$. A simple computation shows that $\lambda \longrightarrow \langle\lambda, \alpha^\vee\rangle^{-1} \det A(\lambda)$ extends to a never vanishing holomorphic function on a neighborhood of λ_0. Hence $\langle\lambda, \alpha^\vee\rangle\, A(\lambda)^{-1}$ extends holomorphically to a neighborhood of λ_0, and hence so does

$$\pi(\lambda)\,\beta(\lambda) = \pi(\lambda)\,A(\lambda)^{-1}\,\Phi(\lambda). \qquad \square$$

Corollary 9. 3. *Let $\lambda_0 \in \mathscr{A}$, $\Omega \subset \mathfrak{a}_c^*$ a neighborhood of λ_0, and $(f_\lambda)_{\lambda\in\Omega}$ a holomorphic family in $\mathscr{E}^\infty_\lambda$. Then*

$$\lim_{\lambda\to\lambda_0} \pi(w\lambda)\,\beta_{w\lambda}(f_\lambda) = \lim_{\lambda\to\lambda_0} \pi(\lambda)\,\beta_\lambda(f_\lambda) \tag{9. 5}$$

for all $w \in W^{\lambda_0}(=\{w \mid w\lambda_0 = \lambda_0\})$.

Proof. It is easily seen that (9. 5) is equivalent to

(9. 6) $$\lim_{\lambda \to \lambda_0} \pi(\lambda)\, \beta_\lambda(f_{s\lambda}) = \lim_{\lambda \to \lambda_0} \pi(\lambda)\, \beta_\lambda(f_\lambda)$$

for all $s \in W^{\lambda_0}$ (let $s = w^{-1}$ and substitute $\lambda = s\nu$ on the left side of (9. 5)). To prove (9. 6) we may assume that s is reflection in a root α with $\langle \lambda_0, \alpha \rangle = 0$. Let $h_\lambda = \langle \lambda, \alpha \rangle^{-1} (f_\lambda - f_{s\lambda})$, then $(h_\lambda)_{\lambda \in \Omega}$ is a holomorphic family in $\mathscr{E}_\lambda^\infty$. Theorem 9. 2 (ii) shows that $\pi(\lambda)\, \beta_\lambda(h_\lambda)$ has a removable singularity at λ_0. Hence $\langle \lambda, \alpha \rangle\, \pi(\lambda)\, \beta_\lambda(h_\lambda)$ is zero at λ_0. □

We now consider the non-regular elements λ in $\mathscr{A}$. For any $\lambda \in \mathfrak{a}_c^*$ let $\Sigma^\lambda = \{\alpha \in \Sigma \mid \langle \alpha, \lambda \rangle = 0\}$ and define $\omega^\lambda \in P(\mathfrak{a})$ and $\pi^\lambda \in \mathscr{U}(\mathfrak{a})$ by

(9. 7) $$\omega^\lambda(H) = \prod_{\alpha \in \Sigma^\lambda \cap \Sigma_0^+} \alpha(H)$$

for $H \in \mathfrak{a}$, and

(9. 8) $$\pi^\lambda(\mu) = \prod_{\alpha \in \Sigma^\lambda \cap \Sigma_0^+} \langle \alpha^\vee, \mu \rangle$$

for $\mu \in \mathfrak{a}_c^*$. Notice that $\partial(\pi^\lambda)\, \omega^\lambda$ is a positive constant (cf. [Va 77], part I, p. 59, Cor. 7).

Let $\lambda \in \mathscr{A} \setminus \mathscr{A}'$ and $f \in \mathscr{E}_\lambda^\infty$, then $p_{\lambda, \lambda - \varrho}(f, g)$ is a W^λ-harmonic polynomial on $\mathfrak{a}$ for each $g \in G$. In particular, $p_{\lambda, \lambda - \varrho}(f, g) \in \partial(\mathscr{U}(\mathfrak{a}))\, \omega^\lambda$ (cf. Section 4). It follows that we can define $\beta_\lambda(f, g) \in \mathbb{C}$ uniquely by $\deg(p_{\lambda, \lambda - \varrho}(f, g) - \beta_\lambda(f, g)\, \omega^\lambda) < \deg \omega^\lambda$ (thus $\beta_\lambda(f, g)$ is the coefficient of $p_{\lambda, \lambda - \varrho}(f, g)$ to ω^λ in any homogeneous basis for the W^λ-harmonic polynomials).

Notice that we have

(9. 9) $$\beta_\lambda(f, g) = [\partial(\pi^\lambda)\, \omega^\lambda]^{-1}\, \partial(\pi^\lambda)\, p_{\lambda, \lambda - \varrho}(f, g)$$

for each $g \in G$. The function $\beta_\lambda(f) \in C^\infty(G)$ is called a *boundary value* of f, and $\beta_\lambda \colon \mathscr{E}_\lambda^\infty \to C^\infty$ a *boundary value map.*

For each $\lambda \in \mathfrak{a}_c^*$, let $a(\lambda)$ denote the following non-zero constant

(9. 10) $$a(\lambda) = |W^\lambda|^{-1}\, [\partial(\pi^\lambda)\, \omega^\lambda] \prod_{\alpha \in \Sigma_0^+ \setminus \Sigma^\lambda} \langle \lambda, \alpha^\vee \rangle.$$

Theorem 9. 4. (i) *Let $\lambda \in \mathscr{A} \setminus \mathscr{A}'$. The boundary value map β_λ maps $\mathscr{E}_{\lambda, r}^\infty(G/K)$ linearly, continuously, and G-equivariantly into $C^\infty(G/P; L_\lambda)$ for each $r \in \mathbb{R}$.*

(ii) *In the setting of Theorem* 9. 2 (ii) *we have*

(9. 11) $$\lim_{\lambda \to \lambda_0,\, \lambda \in \mathscr{A}'} \pi(\lambda)\, \beta_\lambda(f_\lambda) = a(\lambda_0)\, \beta_{\lambda_0}(f_{\lambda_0})$$

for $\lambda_0 \in \mathscr{A} \cap \Omega$.

Proof. (i) That $\beta_\lambda(f) \in C^\infty(G/P; L_\lambda)$ follows from Theorem 8. 4 because $T_{\log a}\omega^\lambda$ equals ω^λ plus some lower order terms. The continuity of β_λ follows from Theorem 3. 5.

(ii) By Theorem 3. 6 we have for $H \in \mathfrak{a}^+$ that

(9. 12) $$\sum_{w \in W^{\lambda_0}} \beta_{w\lambda}(f_\lambda)\, e^{w\lambda(H)} \to p_{\lambda_0, \lambda_0 - \varrho}(f_{\lambda_0}, H)\, e^{\lambda_0(H)}$$

as $\lambda \longrightarrow \lambda_0$, $\lambda \in \mathscr{A}'$. By Remark 7.4 we still have convergence of (9.12) after applying the differential operator $\partial(\pi^{\lambda_0})$ on $\mathfrak{a}^+$ to both sides. Now

$$\partial(\pi^{\lambda_0})\, e^{w\lambda} = \pi^{\lambda_0}(w\lambda)\, e^{w\lambda} = \operatorname{sgn} w \left[\prod_{\alpha \in \Sigma_0^+ \setminus \Sigma^{\lambda_0}} \langle \lambda, \alpha^\vee \rangle\right]^{-1} \pi(\lambda)\, e^{w\lambda}$$

and hence

$$\lim_{\lambda \to \lambda_0} \partial(\pi^{\lambda_0}) \sum_{w \in W^{\lambda_0}} \beta_{w\lambda}(f_\lambda)\, e^{w\lambda} = |W^{\lambda_0}| \left[\prod_{\alpha \in \Sigma_0^+ \setminus \Sigma^{\lambda_0}} \langle \lambda_0, \alpha^\vee \rangle\right]^{-1} \lim_{\lambda \to \lambda_0} \pi(\lambda)\, \beta_\lambda(f_\lambda)\, e^{\lambda_0}$$

by Corollary 9.3. On the other hand, using (9.9)

$$\partial(\pi^{\lambda_0})\, [p_{\lambda_0, \lambda_0 - \varrho}(f_{\lambda_0})\, e^{\lambda_0}] = [\partial(\pi^{\lambda_0})\, \omega^{\lambda_0}]\, \beta_{\lambda_0}(f_{\lambda_0})\, e^{\lambda_0}$$

and (9.11) follows. □

Finally we notice that for certain λ we can obtain the boundary value $\beta_\lambda(f)$ from $f \in \mathscr{E}_\lambda^\infty$ by a simple limiting procedure. More precisely we have

Lemma 9.5. *Let $\lambda \in \mathfrak{a}_c^*$ and assume that* $\operatorname{Re}\langle \lambda, \alpha \rangle > 0$ *for all $\alpha \in \Sigma^+ \setminus \Sigma^\lambda$. Then*

$$\beta_\lambda(f, g) = \lim_{t \to +\infty} [\omega^\lambda(tH)\, e^{(\lambda - \varrho)(tH)}]^{-1} f(g \exp tH)$$

for $f \in \mathscr{E}_\lambda^\infty$, $g \in G$, and $H \in \mathfrak{a}^+$.

Notice that for $\lambda \in \mathfrak{a}^*$ the assumption is that λ is dominant (that is, $\langle \lambda, \alpha \rangle \geqq 0$ for all $\alpha \in \Sigma^+$).

Proof. The condition on λ implies that $\operatorname{Re} \xi(H) < \operatorname{Re}(\lambda - \varrho)(H)$ for all $\xi \in X(\lambda)$ with $\xi \neq \lambda - \varrho$. Then the result follows from Theorem 3.5 and the definition of β_λ. □

10. Inversion of the Poisson transformation

In this section we give the relation of the boundary value map β_λ to the Poisson transformation $\mathscr{P}_\lambda$. Essentially they are the inverse of each other.

Recall from Section 2 that $\mathscr{P}_\lambda$ maps $C^\infty(K/M)$ continuously into $\mathscr{E}_{\lambda, r(\lambda)}^\infty(G/K)$ for each $\lambda \in \mathfrak{a}_c^*$ ($r(\lambda) \in \mathbb{R}$ given in Example 2.2(i)). On the other hand, when $\lambda \in \mathscr{A}$ (and in particular, cf. (9.1), when $\operatorname{Re}\langle \lambda, \alpha \rangle \geqq 0$ for all $\alpha \in \Sigma^+$) we have seen in the previous section that β_λ maps $\mathscr{E}_{\lambda, r}^\infty$ continuously into $C^\infty(K/M)$ for each $r \in \mathbb{R}$ (with the identification $C^\infty(K/M) \cong C^\infty(G/P; L_\lambda)$).

Let c denote Harish-Chandra's c-function, which is the meromorphic function on $\mathfrak{a}_c^*$ given by the well known integral formula

$$c(\lambda) = \int_{\bar{N}} e^{(-\lambda - \varrho)(H(\bar{n}))}\, d\bar{n}$$

when $\mathrm{Re}\langle\lambda, \alpha\rangle > 0$ for all $\alpha \in \Sigma^+$ (cf. [He 84], Sect. IV 6. The Haar measure $d\bar{n}$ on $\bar{N}$ is normalized such that $c(\varrho)=1$). Explicitly $c(\lambda)$ is given by the expression ([He 84], IV, Thm. 6.14) due to Gindikin and Karpelevic. It follows from this expression that the meromorphic function $\lambda \to \pi(\lambda)\, c(\lambda)$ on $\mathfrak{a}_c^*$ is nonzero and without poles in $\mathscr{A}$. For $\lambda_0 \in \mathscr{A}$ we define the nonzero constant $k(\lambda_0) \in \mathbb{C}$ by

$$k(\lambda_0) = a(\lambda_0)^{-1}\,[\pi(\lambda)\, c(\lambda)]_{\lambda=\lambda_0} \tag{10.1}$$

where $a(\lambda_0)$ is given by (9.10). In particular if $\lambda_0 \in \mathscr{A}'$ then $k(\lambda_0) = c(\lambda_0)$.

Theorem 10.1. *Let $\lambda \in \mathscr{A}$. Then $\mathscr{P}_\lambda$ is a topological isomorphism of $C^\infty(K/M)$ onto $\mathscr{E}^\infty_{\lambda, r(\lambda)}(G/K)$ whose inverse is $k(\lambda)^{-1}\,\beta_\lambda$. Moreover $\mathscr{E}^\infty_{\lambda, r(\lambda)}(G/K) = \mathscr{E}^\infty_\lambda(G/K)$.*

Proof. Assume first that $\mathrm{Re}\langle\lambda, \alpha\rangle > 0$ for all $\alpha \in \Sigma^+$. From Lemma 9.5 we then obtain that

$$\beta_\lambda(\mathscr{P}_\lambda \varphi, k) = \lim_{t \to +\infty} e^{-(\lambda-\varrho)(tH)}\, \mathscr{P}_\lambda \varphi(k \exp tH) \tag{10.2}$$

for all $\varphi \in C^\infty(K/M)$, $k \in K$, and $H \in \mathfrak{a}^+$. A simple argument involving a transformation of the integral over K to an integral over $\bar{N}$ shows that the right hand side of (10.2) equals $c(\lambda)\,\varphi(k)$ (cf. [He 70], p. 130 or [Schl 84], Thm. 5.1.4). Hence for any $\varphi \in C^\infty(K/M)$

$$\beta_\lambda(\mathscr{P}_\lambda \varphi) = c(\lambda)\, \varphi. \tag{10.3}$$

Assuming only $\lambda \in \mathscr{A}'$ it follows from Theorem 9.2 (ii) that (10.3) still holds, because $\mathscr{P}_\lambda \varphi$ is a holomorphic family in $\mathscr{E}^\infty_\lambda$. Finally, for arbitrary $\lambda \in \mathscr{A}$ we infer that

$$\beta_\lambda(\mathscr{P}_\lambda \varphi) = k(\lambda)\, \varphi \tag{10.4}$$

for $\varphi \in C^\infty(K/M)$, using Theorem 9.4 (ii). In particular this proves that β_λ is surjective from $\mathscr{E}^\infty_{\lambda, r(\lambda)}$ onto $C^\infty(K/M)$.

Let $r \in \mathbb{R}$ with $r \geqq r(\lambda)$, and let $V \subset \mathscr{E}^\infty_{\lambda, r}$ be the kernel of β_λ, then V is a closed invariant subspace of $\mathscr{E}^\infty_{\lambda, r}$. Now for any such subspace V we have either $V=0$ or $\phi_\lambda \in V$, where $\phi_\lambda = \mathscr{P}_\lambda 1$ is the spherical function (because if $f \in V$ and $f(e) \neq 0$ then $\int_K L_k f\, dk \in V \setminus \{0\}$). However (10.4) shows that $\beta_\lambda(\phi_\lambda) = k(\lambda) \neq 0$, hence $V=0$ and β_λ is injective.

Since both $\mathscr{P}_\lambda$ and β_λ are known to be continuous the theorem easily follows (of course actually the continuity of one of them would suffice, by the closed graph theorem). □

In the rest of this section we give some simple applications of Theorem 10.1.

Corollary 10.2. *Let $\lambda_0 \in \mathfrak{a}_c^*$ and $f \in \mathscr{E}^\infty_{\lambda_0}$. There exists a holomorphic family $(f_\lambda)_{\lambda \in \mathfrak{a}_c^*}$ in $\mathscr{E}^\infty_\lambda$ such that $f = f_{\lambda_0}$.*

Proof. Choose $w \in W$ such that $w\lambda_0 \in \mathscr{A}$, let $T = k(w\lambda_0)^{-1}\,\beta_{w\lambda_0}(f)$, and take $f_\lambda = \mathscr{P}_{w\lambda} T$. □

Using this corollary we can now describe in more detail which polynomials $p_{\lambda,\xi}$ can occur in the asymptotic expansion of $f \in \mathscr{E}_\lambda^\infty$. For $\nu \in \mathfrak{a}_c^*$ we have earlier defined $W_\nu = \{w \in W \mid w\nu - \nu \in \mathbb{Z} \cdot \Delta\}$ (cf. Remark 5.12).

Theorem 10.3. *Let $\lambda_0 \in \mathfrak{a}_c^*$, $f \in \mathscr{E}_{\lambda_0}^\infty(G/K)$, and $g \in G$, and let*

$$\xi = w\lambda_0 - \varrho - \mu \in X(\lambda_0)$$

where $w \in W$ and $\mu \in \mathbb{N} \cdot \Delta$. Then $p_{\lambda_0,\xi}(f, g)$ is a $W_{w\lambda_0}$-harmonic polynomial on $\mathfrak{a}$, hence in particular a W-harmonic polynomial.

Proof. We may assume that f is the f_{λ_0} of a holomorphic family f_λ in $\mathscr{E}_\lambda^\infty$. For $\lambda \in \mathfrak{a}_c^*$ let $\psi_\lambda \in C^\infty(\mathfrak{a})$ be given by

$$\psi_\lambda = \sum_{\xi \in \Xi(\lambda)} p_{\lambda,\xi}(f_\lambda, g)\, e^{\xi - \xi_0}.$$

It follows from Remark 7.4 that $\lambda \to [\partial(u)\,\psi_\lambda](H)$ is continuous near λ_0 for each $u \in \mathscr{U}(\mathfrak{a})$ and $H \in \mathfrak{a}^+$. The argument from the proof of Corollary 5.11 can now be repeated and gives the theorem (cf. Remark 5.12). □

Let $f \in \mathscr{E}_\lambda^\infty$. By definition, all the left derivatives of f belong to the same space $C_r(G)$. Actually, the differential equations satisfied by f force the right derivatives to behave similarly (considering f as a function on G):

Theorem 10.4. *Let $\lambda \in \mathfrak{a}_c^*$. There exists $r \in \mathbb{R}$ such that $L_v R_u f \in C_r(G)$ for all $f \in \mathscr{E}_\lambda^\infty$ and $u, v \in \mathscr{U}(\mathfrak{g})$.*

Proof. Since $L_v f \in \mathscr{E}_\lambda^\infty$ for all $v \in \mathscr{U}(\mathfrak{g})$ it suffices to take $v = 1$. Conjugating with W if necessary we may assume $\lambda \in \mathscr{A}$. By Theorem 10.1 we have $f = \mathscr{P}_\lambda \varphi$ for some $\varphi \in C^\infty(K/M)$, and then

$$uf(g) = \int_K \varphi(k)\,(u e_{\lambda+\varrho})(k^{-1} g)\, dk$$

for $g \in G$. Then the theorem follows from Lemma 10.5 below. □

Lemma 10.5. *Fix $\lambda \in \mathfrak{a}_c^*$. There exists $r \in \mathbb{R}$ such that $u e_\lambda \in C_r(G)$ for all $u \in \mathscr{U}(\mathfrak{g})$.*

Proof. Follows from (1.7) and Lemma 2.1. □

Finally we remark that $\mathscr{P}_\lambda$ is a topological isomorphism of $C^\infty(K/M)$ onto $\mathscr{E}_{\lambda,r(\lambda)}^\infty(G/K)$ for λ in a slightly bigger set than $\mathscr{A}$. Let $e(\lambda)^{-1}$ be the "denominator of c" as defined in [Schl 84], eq. (5.17).

Theorem 10.6. *Let $\lambda \in \mathfrak{a}_c^*$ and assume $e(\lambda) \neq 0$. Then $\mathscr{P}_\lambda$ is a topological isomorphism of $C^\infty(K/M)$ onto $\mathscr{E}_{\lambda,r(\lambda)}^\infty$.*

Proof. This is a simple consequence of Theorem 10.1 and the theory of intertwining operators. We omit the details ([KKMOOT 78], p. 27—30 is similar). □

If $e(\lambda) = 0$ then $\mathscr{P}_\lambda$ is neither surjective nor injective (cf. [He 84], p. 279).

Part II. Distribution boundary values

11. Eigenfunctions of weak moderate growth

In this section we introduce a "weak" growth condition for eigenfunctions on G/K, which is much weaker than that satisfied by the functions in $\mathscr{E}_\lambda^\infty(G/K)$. The weak growth condition is satisfied by the Poisson transforms of all distributions on K/M.

Recall from Section 2 that

$$C_r^q = \{f \in C^q(G) \mid \|f\|_{q,r} < +\infty\}$$

for $q \in \mathbb{N}$ and $r \in \mathbb{R}$, and $C_r^\infty = \bigcap_q C_r^q$. Following [Wal 83] we define $\mathscr{S} = \mathscr{S}(G)$ to be the space

$$\mathscr{S} = \bigcap_r C_r^\infty = \bigcap_{q,r} C_r^q$$

endowed with the projective limit topology for the intersection over q and r (that is, the topology given by the family of all the norms $\|\cdot\|_{q,r}$).

By a standard argument (involving Ascoli), the injection of C_r^q into $C_{r'}^{q'}$, is a compact mapping if $q' < q$ and $r' > r$. Hence $\mathscr{S}$ is a Fréchet-Schwartz space (in the sense of Grothendieck [Gr 54], p. 117), which means exactly that it is the projective limit of a compact sequence of Banach spaces (cf. [Kom 67], Thm. 17). In particular, $\mathscr{S}$ is a Fréchet space.

It follows from (2.4)—(2.7) that L and R leave the space $\mathscr{S}$ invariant and act smoothly on it. It is easily seen that the space $C_c^\infty(G)$ of compactly supported C^∞-functions on G is contained in $\mathscr{S}$ as a dense subspace, and that the injection $C_c^\infty \hookrightarrow \mathscr{S}$ is continuous.

Let $\mathscr{S}' = \mathscr{S}'(G)$ be the space dual to $\mathscr{S}$, equipped with the strong dual topology. It follows from the above that $\mathscr{S}'$ is a subspace of the space $\mathscr{D}'(G)$ of distributions on G. For any $T \in \mathscr{S}'$ and $q \in \mathbb{N}$, $r \in \mathbb{R}$ we denote

$$\|T\|'_{q,r} = \sup\{|T(\varphi)| \mid \varphi \in \mathscr{S}, \|\varphi\|_{q,r} \leqq 1\}.$$

The space $(C_r^q)' = \{T \in \mathscr{S}' \mid \|T\|'_{q,r} < +\infty\}$ with this norm is the dual space of C_r^q. Moreover we have $\mathscr{S}' = \bigcup_{q,r} (C_r^q)'$ and $(C_r^q)' \subset (C_{r'}^{q'})'$ if $q \leqq q'$ and $r' \leqq r$. By duality (cf. [Kom 67], Thm. 11) $\mathscr{S}'$ is isomorphic to the inductive limit of these spaces.

Using Lemma 2.1 (iii) one proves easily (cf. [War 72 II], Lemma 8.15.4) that

$$\int_G \|g\|^b \, dg < +\infty \tag{11.1}$$

for some $b \in \mathbb{R}$. From this it follows that there is a continuous injection of $C_r^0 = C_r(G)$ into $(C_{b-r}^0)'$ (hence also into $\mathscr{S}'$) defined via integration on G (with respect to some fixed normalization of the Haar measure).

Let $q' \leqq q$ and $r \in \mathbb{R}$. For each $T \in (C_r^q)'$ and $\varphi \in C_r^{q'}$ we define a $C^{q'-q}$-function $L^\vee(\varphi)\,T$ on G by

$$[L^\vee(\varphi)\,T](x) = T(R_{x^{-1}}\varphi).$$

Notice that if $f \in C_r(G)$ and $\varphi \in C_{b-r}(G)$ then

$$[L^\vee(\varphi) f](x) = \int_G \varphi^\vee(g)\, L_g f(x)\, dg$$

where $\varphi^\vee(g) = \varphi(g^{-1})$.

Lemma 11. 1. *Let $q, q' \in \mathbb{N}$ with $q' \geqq q$. There exist $s \geqq 0$ and $C > 0$ such that*

$$\|L^\vee(\varphi)\,T\|_{q'-q,|r|} \leqq C\,\|T\|'_{q,r}\,\|\varphi\|_{q',r-s}$$

for all $r \in \mathbb{R}$, $T \in (C_r^q)'$, and $\varphi \in C_{r-s}^{q'}$.

Differently put, $L^\vee$ is a bounded linear operator from $C_{r-s}^{q'}$ to $B((C_r^q)', C_{|r|}^{q'-q})$. In particular, it follows for each $T \in (C_r^q)'$ that $\varphi \to L^\vee(\varphi)\,T$ is a continuous linear map from C_{r-s}^∞ to $C_{|r|}^\infty$.

Proof. It follows from (2. 5) that

$$\|L^\vee(\varphi)\,T\|_{|r|} \leqq \|T\|'_{q,r}\,\|\varphi\|_{q,r} \tag{11. 2}$$

for all $T \in (C_r^q)'$ and $\varphi \in C_r^q$. We choose s and C such that

$$\|R_{X^\gamma}\varphi\|_{q,r} \leqq C\,\|\varphi\|_{q',r-s} \tag{11. 3}$$

for all $|\gamma| \leqq q' - q$ and $\varphi \in C_{r-s}^{q'}$ (cf. (2. 7)). Since $L_{X^\gamma} L^\vee(\varphi)\,T = L^\vee(R_{X^\gamma}\varphi)\,T$ we get from (11. 2) and (11. 3) that

$$\|L^\vee(\varphi)\,T\|_{q'-q,|r|} = \max_{|\gamma| \leqq q'-q} \|L_{X^\gamma} L^\vee(\varphi)\,T\|_{|r|} \leqq C\,\|T\|_{q,r}\,\|\varphi\|_{q',r-s}$$

for $T \in (C_r^q)'$ and $\varphi \in C_{r-s}^{q'}$. □

Let $\mathscr{E}_\lambda^*(G/K)$ denote the closed (cf. (2. 7)) subspace $\mathscr{E}_\lambda(G/K) \cap \mathscr{S}'(G)$ of $\mathscr{S}'(G)$. We call the elements of $\mathscr{E}_\lambda^*$ *eigenfunctions of weak moderate growth.* Notice that if $T \in \mathscr{E}_\lambda^*$ and $\varphi \in \mathscr{S}$ then $L^\vee(\varphi)\,T \in \mathscr{E}_\lambda^\infty$ by Lemma 11. 1.

For each distribution $T \in \mathscr{D}'(K/M)$ and $\lambda \in \mathfrak{a}_c^*$ the *Poisson transform* $\mathscr{P}_\lambda T$ is the function on G/K whose value at the coset gK is obtained by applying T to the C^∞-function $kM \to e_{\lambda+\varrho}(k^{-1}g)$ on K/M. Then $\mathscr{P}_\lambda$ maps $\mathscr{D}'(K/M)$ into $\mathscr{E}_\lambda(G/K)$, and identifying $C(K/M)$ with a subspace of $\mathscr{D}'(K/M)$ via the normalized Haar measure we see that this definition extends (1. 8).

Let $\mathscr{D}'(G/P; L_\lambda)$ denote the space of distributions S on G satisfying

$$R_{man} S = a^{\lambda-\varrho} S \tag{11. 4}$$

for all $m \in M$, $a \in A$, and $n \in N$. For $T \in \mathscr{D}'(K/M)$ let $T_\lambda \in \mathscr{D}'(G)$ be defined by $T_\lambda(\varphi) = T(\varphi^\lambda)$ for $\varphi \in C_c^\infty(G)$ where $\varphi^\lambda \in C^\infty(K/M)$ is given by

$$\varphi^\lambda(k) = \int_M \int_A \int_N \varphi(kman)\, a^{\lambda+\varrho}\, dm\, da\, dm \tag{11. 5}$$

then $T_\lambda \in \mathscr{D}'(G/P; L_\lambda)$, and $T \to T_\lambda$ is a bijection of $\mathscr{D}'(K/M)$ with $\mathscr{D}'(G/P; L_\lambda)$. Let the *Poisson transformation* $\mathscr{P}: \mathscr{D}'(G/P; L_\lambda) \to \mathscr{D}'(G/K)$ be the restriction of the adjoint of the trivial map $C_c^\infty(G/K) \to C_c^\infty(G)$. Then clearly $\mathscr{P}$ is equivariant for L. Moreover $\mathscr{P} T_\lambda$ equals $\mathscr{P}_\lambda T$ (viewed as a distribution via Haar measure on G, suitably normalized) for $T \in \mathscr{D}'(K/M)$.

Let $\mathscr{D}'(K/M)$ be equipped with the strong topology.

Lemma 11. 2. *Let $\lambda \in \mathfrak{a}_c^*$. We have $\mathscr{D}'(G/P; L_\lambda) \subset \mathscr{S}'(G)$, and $T \to T_\lambda$ is continuous from $\mathscr{D}'(K/M)$ into $\mathscr{S}'(G)$.*

Proof. It follows from (11. 5) that $\varphi \to \varphi^\lambda$ extends to a continuous map from $\mathscr{S}(G)$ to $C^\infty(K/M)$ (use Lemma 2. 1). The lemma follows by duality. □

Corollary 11. 3. *Let $\lambda \in \mathfrak{a}_c^*$. We have $\mathscr{P}_\lambda T \in \mathscr{E}_\lambda^*(G/K)$ for all $T \in \mathscr{D}'(K/M)$, and $\mathscr{P}_\lambda: \mathscr{D}'(K/M) \to \mathscr{E}_\lambda^*$ is continuous.*

Proof. Immediate from $\mathscr{P}_\lambda T = \mathscr{P} T_\lambda$. □

Since T_λ and $\mathscr{P}_\lambda T$ belong to $\mathscr{S}'$ for $T \in \mathscr{D}'(K/M)$ it makes sense to form the functions $L^\vee(\varphi)\, T_\lambda$ and $L^\vee(\varphi)\, \mathscr{P}_\lambda T$ on G for $\varphi \in \mathscr{S}$.

Lemma 11. 4. *Let $T \in \mathscr{D}'(K/M)$ and $\varphi \in \mathscr{S}(G)$. Then*

$$L^\vee(\varphi)\, \mathscr{P}_\lambda T = \mathscr{P}(L^\vee(\varphi)\, T_\lambda) \tag{11. 6}$$

for all $\lambda \in \mathfrak{a}_c^*$.

Proof. By continuity of $L^\vee$ (Lemma 11. 1) it suffices to take $\varphi \in C_c^\infty(G)$, and then (11. 6) follows from the left equivariance of $\mathscr{P}$. □

By a similar argument we also have

Lemma 11. 5. *Let $f \in \mathscr{E}_\lambda^\infty(G/K)$ and $\varphi \in \mathscr{S}(G)$. Then*

$$L^\vee(\varphi)\, \beta_\lambda f = \beta_\lambda(L^\vee(\varphi) f).$$

12. Distribution boundary values

We shall now assign boundary values on K/M to the functions in $\mathscr{E}_\lambda^*(G/K)$. The boundary values will be distributions on K/M, and as before the boundary value map turns out essentially to be the inverse of the Poisson transformation.

Let $\lambda \in \mathscr{A}$ (cf. (9. 1)) and $f \in \mathscr{E}_\lambda^\infty$. It follows from Lemma 11. 5 that when $\beta_\lambda(f)$ is considered as a distribution, its value at a given $\varphi \in \mathscr{S}$ is identical to the value of the C^∞-function $\beta_\lambda(L^\vee(\varphi) f)$ at the identity element. This consideration motivates the following definition.

Let $\lambda \in \mathscr{A}$ and $f \in \mathscr{E}_\lambda^*$. For each $\varphi \in \mathscr{S}$ we have $L^\vee(\varphi) f \in \mathscr{E}_\lambda^\infty$ (cf. Lemma 11. 1), and hence the boundary value $\beta_\lambda(L^\vee(\varphi) f)$ exists in $C^\infty(G/P; L_\lambda)$. We define a linear form $\beta_\lambda(f)$ on $\mathscr{S}$, called the *boundary value* of f, by

$$\beta_\lambda(f)(\varphi) = \beta_\lambda(L^\vee(\varphi) f, e)$$

for $\varphi \in \mathscr{S}$. The map $f \to \beta_\lambda(f)$ is called *the boundary value map.* The consideration above shows that though $\mathscr{E}_\lambda^\infty \subset \mathscr{E}_\lambda^*$ there is no ambiguity in the use of the symbol β_λ.

Theorem 12. 1. *Let $\lambda \in \mathscr{A}$ and $f \in \mathscr{E}_\lambda^*(G/K)$. Then $\beta_\lambda(f) \in \mathscr{S}'(G)$ and the boundary value map $\beta_\lambda \colon \mathscr{E}_\lambda^* \to \mathscr{S}'$ is linear, continuous, and commutes with L. Moreover we have*

$$\beta_\lambda(f) \in \mathscr{D}'(G/P; L_\lambda).$$

Proof. Fix $r \in \mathbb{R}$ and consider the linear form on $\mathscr{E}_{\lambda, |r|}^\infty$ obtained from composing $\beta_\lambda \colon \mathscr{E}_{\lambda, |r|}^\infty \to C^\infty(G/P; L_\lambda)$ with evaluation at e. This being continuous there exists $q' \in \mathbb{N}$ such that it extends to a continuous linear form on $C_{|r|}^{q'} \cap \mathscr{E}_\lambda$ (by definition of the topology on $C_{|r|}^\infty$).

For each $q \in \mathbb{N}$ it follows from Lemma 11. 1 that $f \to L^\vee(\cdot) f$ is a bounded linear operator from $(C_r^q)' \cap \mathscr{E}_\lambda$ to $B(C_{r-s}^{q'+q}, C_{|r|}^{q'} \cap \mathscr{E}_\lambda)$. By composition with the linear form above we see that $f \to \beta_\lambda(L^\vee(\cdot) f, e)$ is a bounded operator from $(C_r^q)' \cap \mathscr{E}_\lambda$ to $(C_{r-s}^{q+q'})'$, hence continuous into $\mathscr{S}'$. It follows that β_λ is a continuous linear map from $\mathscr{E}_\lambda^*$ to $\mathscr{S}'$. The intertwining property is obvious from $L^\vee(L_{g^{-1}}\varphi) f = L^\vee(\varphi) L_g f$ for $g \in G$.

In analogy with Lemmas 11. 4 and 11. 5 we now have

$$L^\vee(\varphi)\, \beta_\lambda(f) = \beta_\lambda(L^\vee(\varphi) f) \tag{12. 1}$$

for $f \in \mathscr{E}_\lambda^*$ and $\varphi \in \mathscr{S}$. Evaluating both sides of (12. 1) at $x \in G$ we obtain

$$\beta_\lambda(f)\,(R_{x^{-1}}\varphi) = \beta_\lambda(L^\vee(\varphi) f, x).$$

Using this with $x \in P$ the last assertion follows from $\beta_\lambda(L^\vee(\varphi) f) \in C^\infty(G/P; L_\lambda)$. □

It follows from the last assertion of Theorem 12. 1 that we may consider $\beta_\lambda(f)$ as an element of $\mathscr{D}'(K/M)$ via the isomorphism $T \to T_\lambda$ of $\mathscr{D}'(K/M)$ onto $\mathscr{D}'(G/P; L_\lambda)$.

Theorem 12. 2. *Let $\lambda \in \mathscr{A}$. Then $\mathscr{P}_\lambda$ is a topological isomorphism of $\mathscr{D}'(K/M)$ onto $\mathscr{E}_\lambda^*(G/K)$, and its inverse is $k(\lambda)^{-1}\beta_\lambda$.*

Here $k(\lambda) \in \mathbb{C} \setminus \{0\}$ is given by (10. 1).

Proof. We have already seen (Corollary 11. 3 and Theorem 12. 1) that $\mathscr{P}_\lambda$ and β_λ are continuous.

From Lemma 11. 4, (12. 1) and Theorem 10. 1 we get that

$$L^\vee(\varphi)\, \mathscr{P} \beta_\lambda(f) = \mathscr{P} \beta_\lambda(L^\vee(\varphi) f) = k(\lambda)\, L^\vee(\varphi) f$$

for $f \in \mathscr{E}_\lambda^*$ and $\varphi \in \mathscr{S}$. Similarly

$$L^\vee(\varphi)\, \beta_\lambda \mathscr{P}_\lambda T = \beta_\lambda \mathscr{P}(L^\vee(\varphi)\, T_\lambda) = k(\lambda)\, L^\vee(\varphi)\, T_\lambda$$

for $T \in \mathscr{D}'(K/M)$ and $\varphi \in \mathscr{S}$. Evaluating these two identities at e we get the theorem. □

The first assertion in Theorem 12. 2 is due to Oshima and Sekiguchi [OS 80] (and partially to Lewis [Le 78]). As in Theorem 10. 6 (and with the same argument as there) it actually holds under the weaker condition that $e(\lambda) \neq 0$.

In analogy with Theorem 10. 4 we shall now prove that the functions in $\mathscr{E}_\lambda^*$ satisfy stronger growth conditions than those used to define the space. Following [Wal 83] an eigenfunction $f \in \mathscr{E}_\lambda$ is said to have *moderate growth* if there exists $r \in \mathbb{R}$ such that $uf \in C_r(G)$ for all $u \in \mathscr{U}(\mathfrak{g})$.

Theorem 12.3. *Let $\lambda \in \mathfrak{a}_c^*$ and let f be an eigenfunction of weak moderate growth (i.e. $f \in \mathscr{E}_\lambda^*$). Then f has moderate growth.*

Proof. We may assume (conjugating with W) that $\lambda \in \mathscr{A}$, and then $f = \mathscr{P}_\lambda T$ for some $T \in \mathscr{D}'(K/M)$ by Theorem 12.2. The result now follows from [Wal 83], p. 365 but for completeness we give a simple proof.

There exist a finite set $I \subset \mathscr{U}(\mathfrak{k})$ and a constant $A \geqq 0$ such that

$$|T(\varphi)| \leqq A \sup_{\substack{x \in I \\ k \in K}} |x\varphi(k)|$$

for $\varphi \in C^\infty(K)$. From this it follows that

$$|u\mathscr{P}_\lambda T(g)| \leqq A \sup_{\substack{x \in I \\ k \in K}} |L_x u e_{\lambda+\varrho}(k^{-1}g)|$$

for $g \in G$ and $u \in \mathscr{U}(\mathfrak{g})$. Hence the theorem is a consequence of Lemma 12.4 below.

Lemma 12.4. *Fix $x \in \mathscr{U}(\mathfrak{g})$ and $\lambda \in \mathfrak{a}_c^*$. There exists $r \in \mathbb{R}$ such that $L_x u e_\lambda \in C_r(G)$ for all $u \in \mathscr{U}(\mathfrak{g})$.*

Proof. It follows from Example 2.2(ii) and the relation

$$L_x u e_\lambda(g) = \operatorname{Ad} g^{-1}(x^\vee)\, u e_\lambda(g)$$

that we may assume $x = 1$. Now see Lemma 10.5. $\square$

Remark 12.5. Let $\mathscr{E}_{\lambda,r} = \mathscr{E}_\lambda \cap C_r(G)$ be equipped with the Banach space topology inherited from $C_r(G)$. Then the injections $\mathscr{E}_{\lambda,r} \to \mathscr{E}_\lambda^*$ are continuous and hence it follows from Theorem 12.3 and the closed graph theorem ([B 66], II, § 4, n° 6, Prop. 10) that the topology on $\mathscr{E}_\lambda^*$ is identical with the limit topology for the union $\mathscr{E}_\lambda^* = \bigcup_r \mathscr{E}_{\lambda,r}$.

13. Distributional asymptotic expansions

From the theory developed in Part I for the functions in $\mathscr{E}_\lambda^\infty$ we draw a few consequences for the functions in $\mathscr{E}_\lambda^*$. The most important is:

Theorem 13.1. *Let $\lambda \in \mathfrak{a}_c^*$, $f \in \mathscr{E}_\lambda^*(G/K)$, and $\varphi \in \mathscr{S}(G)$.*

(i) *For each $\xi \in X(\lambda)$ there exists a unique polynomial $p_{\lambda,\xi}(f,\varphi)$ on $\mathfrak{a}$ such that*

$$L^\vee(\varphi) f(\exp tH) \sim \sum_{\xi \in X(\lambda)} p_{\lambda,\xi}(f,\varphi,tH)\, e^{t\xi(H)} \qquad (t \to \infty)$$

at every $H_0 \in \mathfrak{a}^+$.

(ii) *Let $\xi \in X(\lambda)$. Then $f \to p_{\lambda,\xi}(f)$ is a continuous linear map of $\mathscr{E}_\lambda^*(G/K)$ into $\mathscr{S}'(G) \otimes P_d(\mathfrak{a})$, equivariant for the left actions of G on $\mathscr{E}_\lambda^*$ and $\mathscr{S}'$.*

(iii) *Let $\xi = w\lambda - \varrho - \mu \in X(\lambda)$ where $w \in W$, $\mu \in \mathbb{N} \cdot \Delta$. Then $p_{\lambda,\xi}(f,\varphi)$ is a $W_{w\lambda}$-harmonic polynomial on $\mathfrak{a}$. In particular if for all $\alpha \in \Sigma$ we have $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}$, then all the polynomials $p_{\lambda,\xi}(f,\varphi)$ on $\mathfrak{a}$ are constants.*

Proof. (i) is immediate from Theorem 3.5(i), and (ii) follows from Theorem 3.5(ii) (cf. also the proof of Theorem 12.1). (iii) follows from Theorem 10.4. □

Let $f \in \mathscr{E}_\lambda^*$ and $x_0 \in G$. As in Section 8 we define the set of *exponents* of f at x_0 along $x_0 A^+$ by

$$E(f, x_0) = \{\xi \in X(\lambda) \mid x_0 \in \operatorname{supp} p_{\lambda,\xi}(f)\}$$

(that is, $\xi \in E(f, x_0)$ if and only if for each neighborhood U of e there exists $\varphi \in C_c^\infty(U)$ with $\xi \in E(L^\vee(\varphi) f, x_0)$). Also as before we put $E(f, V) = \bigcup_{x \in V} E(f, x)$ for $V \subset G$, V open. Then $\xi \in E(f, V)$ if and only if $p_{\lambda,\xi}(f, \varphi) \neq 0$ for some $\varphi \in C_c^\infty(V)$. Moreover we define the *leading exponents* of f in V along VA^+ (resp. at x_0 along $x_0 A^+$) as the $\prec$-maximal elements in $E(f, V)$ (resp. $E(f, x_0)$), and denote the set of these by $E_L(f, V)$ (resp. $E_L(f, x_0)$). It follows easily from Proposition 8.1 and Corollary 8.2 that $E_L(f, V)$ (resp. $E_L(f, x_0)$) is contained in $W\lambda - \varrho$, and that $p_{\lambda,\sigma}(f, \varphi)$ is $W^{\sigma+\varrho}$-harmonic for all $\sigma \in E_L(f, V)$ and $\varphi \in C_c^\infty(V)$ (resp. for all $\sigma \in E_L(f, x_0)$ and $\varphi \in C_c^\infty(U)$, for some neighborhood U of x_0).

We define the *principal part* $P(f, V)$ of f in V to be the element of $\mathscr{D}'(V) \otimes C^\infty(\mathfrak{a})$ given by

$$P(f, V)(\varphi) = \sum_{\sigma \in E_L(f, V)} p_{\lambda,\sigma}(f, \varphi)\, e^\sigma = P(L^\vee(\varphi) f, V)(e)$$

for $\varphi \in C_c^\infty(V)$. From Theorem 8.4 we easily obtain that $E_L(f, V) = E_L(f, VP)$ and

$$(R_{man} \otimes 1)\, P(f, VP) = (1 \otimes T_{\log a})\, P(f, VP)$$

for $f \in \mathscr{E}_\lambda^*$ and $man \in P = MAN$.

Part III. Converging expansions

14. *H*-finite functions

In this section we introduce extra assumptions on the eigenfunction $f \in \mathscr{E}_\lambda(G/K)$, which — as will be seen in the following sections — ensure that the asymptotic expansion actually converges in certain directions to the boundary. Basically the assumption is that f transforms finitely under the group of fixed points of some involution of G, and the main result in the present section is that this implies $f \in \mathscr{E}_\lambda^*(G/K)$ (cf. Section 11), so that the theory of Part II applies to f. An important example is that of K-finite eigenfunctions.

Let σ be an involution of G, G^σ the subgroup of G consisting of the σ-fixed elements, and G_0^σ the identity component of G^σ. Let H be any subgroup of G satisfying

$$G_0^\sigma \subset H \subset G^\sigma.$$

Throughout we assume that σ and θ commute mutually (which can always be obtained through conjugation, cf. [Schl 84], Prop. 7.1.1). We say that a function f on G is H-finite, if the set of all its left translates $L_h f$ by elements $h \in H$, spans a finite dimensional linear space. The following basic result was proved jointly by Flensted-Jensen, Oshima and the second author, using the theory of [Osh 84] (cf. [F-J 86], Ch. IV, Prop. 8). We shall give another proof based on [Ban 84].

Theorem 14. 1. *Let $\lambda \in \mathfrak{a}_c^*$ and $f \in \mathscr{E}_\lambda(G/K)$, and assume that f is H-finite. Then f has at most exponential growth (that is, $f \in \mathscr{E}_\lambda^*(G/K)$).*

Remark 14. 2. Let $\sigma = \theta$. Then the theorem says that K-finite eigenfunctions have at most exponential growth. In this case the details of the proof below are simpler, and reference to [Ban 84] can be replaced by reference to [CM 82]. Notice that combining Theorems 14. 1 and 12. 2 we obtain a new proof of Helgason's theorem [He 76], Cor. 7. 4 that $\mathscr{P}_\lambda$ gives a bijection of the K-finite functions on K/M onto the K-finite functions in $\mathscr{E}_\lambda(G/K)$ (for $\lambda \in \mathscr{A}$). (Alternatively, Helgason's result implies Theorem 14. 1 for $\sigma = \theta$.)

Before giving the proof we establish some notation. Let $\mathfrak{h}$ be the Lie algebra of H, and $\mathfrak{q}$ the orthocomplement (with respect to the Killing form) of $\mathfrak{h}$ in $\mathfrak{g}$. We assume that the maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ is chosen σ-stable and $\mathfrak{q}$-*maximal*, that is, if $\mathfrak{a}_q = \mathfrak{a} \cap \mathfrak{q}$, then $\mathfrak{a}_q$ is maximal abelian in $\mathfrak{p} \cap \mathfrak{q}$. Let Σ_q denote the set of all non-zero restrictions to $\mathfrak{a}_q$ of roots in Σ. Let $N_K(\mathfrak{a})$ and $N_K(\mathfrak{a}_q)$ (resp. $Z_K(\mathfrak{a})$ and $Z_K(\mathfrak{a}_q)$) denote the normalizers (resp. centralizers) of $\mathfrak{a}$ and $\mathfrak{a}_q$ in K. Then $W \cong N_K(\mathfrak{a})/Z_K(\mathfrak{a})$, and we also have the following result, due to Rossmann [Ro 79].

Lemma 14. 2. *The set Σ_q is a root system on $\mathfrak{a}_q$, and its Weyl group W_q is naturally identified with $N_K(\mathfrak{a}_q)/Z_K(\mathfrak{a}_q)$. Moreover, each element $w \in W_q$ has a representative in $N_K(\mathfrak{a}_q) \cap N_K(\mathfrak{a})$.*

Proof. See [Schl 84], Section 7. 2. □

Let Σ_q^+ be the set of all non-zero restrictions to $\mathfrak{a}_q$ of roots in Σ^+. We assume (as we may) that Σ^+ is $\mathfrak{q}$-*compatible*, that is, Σ_q^+ forms a positive set for Σ_q. It follows that the set HP of all elements hp, where $h \in H$ and $p \in P$, is an open subset of G (cf. [Schl 84], Prop. 7. 1. 8 (ii)). Let Δ_q be the set of simple roots for Σ_q^+, $\mathfrak{a}_q^+$ the corresponding open chamber in $\mathfrak{a}_q$, $A_q = \exp \mathfrak{a}_q$, and $A_q^+ = \exp \mathfrak{a}_q^+$.

Let τ be a representation of H in a finite dimensional complex linear space E. An E-valued function F on G/K which satisfies

$$F(hx) = \tau(h)\, F(x) \tag{14. 1}$$

for all $h \in H$ and $x \in G/K$, will be called τ-*spherical*. Notice that if F is τ-spherical and $\eta \in E^*$ then the function $\eta \circ F$ is H-finite. Conversely, let f be an H-finite $\mathbb{C}$-valued function on G/K, and let V denote the finite dimensional space of functions on G/K spanned by the H-translates $L_h f$ of f. Let E denote the dual space V^*, τ the representation of H on E dual to L on V, and $F\colon G/K \to E$ the function given by $F(x)\, v = v(x)$ for $v \in V$. Then F is τ-spherical and $f = \eta \circ F$ for a suitable linear form η on E ($\eta = f$ when E^* is identified with V). This consideration allows us to switch between H-finite functions and τ-spherical functions, which is often convenient. We say that F is associated to f.

Let $\mathscr{E}_{\lambda,\tau}(G/K)$ denote the space of all τ-spherical C^∞-functions F on G/K satisfying the system of differential equations

$$DF = \chi_\lambda(D)\, F \qquad (D \in \mathbb{D}(G/K)).$$

We recall from [Ban 84] that if $F \in \mathscr{E}_{\lambda,\tau}(G/K)$ then F admits certain converging series expansions on A_q. To be more precise, let $\boldsymbol{\alpha}_q\colon A_q \to \mathbb{R}^{\Delta_q}$ be the map given by

$$\boldsymbol{\alpha}_q(a) = (a^{-\gamma})_{\gamma \in \Delta_q} \tag{14. 2}$$

where $a^\nu = e^{\nu(\log a)}$ for $\nu \in \mathfrak{a}_q^*$. Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$, and let $P(\mathfrak{a}_q)$ be the space of complex polynomial functions on $\mathfrak{a}_q$. We then have from [Ban 84], Thm. 3. 5:

Proposition 14. 3. *Let $\lambda \in \mathfrak{a}_c^*$ and $F \in \mathscr{E}_{\lambda,\tau}(G/K)$. There exist finite sets $S \subset \mathfrak{a}_{q,c}^*$ and $M \subset P(\mathfrak{a}_q)$, and for each pair $(\nu, p) \in S \times M$ a holomorphic function $F_{\nu,p}$ from D^{Δ_q} to E such that*

$$F(a) = \sum_{\substack{\nu \in S \\ p \in M}} F_{\nu,p}(\boldsymbol{\alpha}_q(a))\, p(\log a)\, a^\nu \tag{14. 3}$$

for all $a \in A_q^+$.

Proof of Theorem 14. 1. Let $F \in \mathscr{E}_{\lambda,\tau}(G/K)$ and fix any norm $|\cdot|$ on E. Since τ is finite dimensional there exist constants C_1 and r_1 such that

$$\|\tau(h)\| \leqq C_1 \|h\|^{r_1} \tag{14. 4}$$

for all $h \in H$, where $\|\tau(h)\|$ is the operator norm of $\tau(h)$ on E.

The expression (14. 3) shows that for each $\delta < 1$ there exist constants C_2 and r_2 such that

$$|F(a)| \leqq C_2 \|a\|^{r_2} \tag{14. 5}$$

for all $a \in A_q^+$ with $\boldsymbol{\alpha}_q(a) \in]0, \delta]^{\Delta_q}$. Actually this holds for $\delta = 1$ too, as follows from the statement for "asymptotics along the walls" similar to Proposition 14. 3, cf. [Ban 84], Lemma 8. 2 and Thm. 8. 3. Hence (14. 5) holds for all a in the closure of A_q^+, and since any positive set Σ_q^+ for Σ_q corresponds to some compatible choice of Σ^+ we get (14. 5) for all $a \in A_q$.

Combining (14. 4) and (14. 5) with Lemma 14. 4 (ii) below we have

$$|F(ha)| \leqq C_1 C_2 \|ha\|^{r_1 + r_2},$$

and the theorem is proved because $G = HA_qK$ (cf. [Schl 84], Prop. 7. 1. 3). □

Lemma 14. 4. *Let σ be an involution of G, commuting with θ. Then*

(i) *$\|\sigma g\| = \|g\|$ for all $g \in G$, and*

(ii) *if $h, a \in G$ with $\sigma h = h$ and $\sigma a = \theta a$ then*

$$\|a\| \leqq \|ha\| \quad \text{and} \quad \|h\| \leqq \|ha\|.$$

Proof. (i) is obvious since the Killing form is σ-invariant. Using the Cauchy-Schwartz inequality on the definition of $\|\cdot\|$ on G, we easily get the following inequality for all $x, y \in G$:

$$\|y\|^2 \leqq \|xy\|\ \|x(\theta y)\|. \tag{14. 6}$$

Replacing (x, y) by (y^{-1}, x^{-1}) we also have

$$(14.7) \qquad \|x\|^2 \leqq \|xy\| \, \|x(\theta y)\|.$$

With $(x, y) = (h, a)$ we get (ii) from (14.6) and (14.7) because $\|h(\theta a)\| = \|ha\|$ by (i). □

Remark 14.5. With the same arguments as in the proof of Theorem 14.1 we actually get from [Ban 84] the following more general result: A function $f \in C^\infty(G)$ has at most exponential growth if it is left H-finite, right K-finite, and transforms finitely under the center of the enveloping algebra $\mathcal{U}(\mathfrak{g})$.

Let $\lambda \in \mathfrak{a}_c^*$ and let $f \in \mathcal{E}_\lambda(G/K)$ be H-finite. We now know that the asymptotic theory of Theorem 13.5 can be applied to f. In the following we will relate the converging expansion (14.3) to the asymptotic expansion. The first step (redundant if $\mathfrak{a}_q = \mathfrak{a}$, e.g. if $H = K$) is to extend (14.3) from $a \in A_q^+$ to $a \in A^+$.

For each $a \in A$ we write $a = a_h a_q$ for the unique decomposition where $a_h \in A_h = \exp(\mathfrak{a} \cap \mathfrak{h})$ and $a_q \in A_q$. Let T denote the set of weights of $\mathfrak{a}_h$ in τ. For each $\mu \in T$ there exists an E-valued polynomial q_μ on $\mathfrak{a}_h$ such that

$$\tau(a_h) = \sum_{\mu \in T} q_\mu(\log a_h)\, a_h^\mu$$

for all $a_h \in A_h$. Combining this with (14.3) we immediately get

$$(14.8) \qquad F(a) = \sum_{\nu, \mu, p} F_{\nu, p}(\boldsymbol{\alpha}_q(a_q))\, q_\mu(\log a_h)\, p(\log a_q)\, a_h^\mu\, a_q^\nu$$

for $a \in A_h A_q^+$. However, it is more convenient to replace (14.8) with an expression involving $\boldsymbol{\alpha}(a)$ instead of $\boldsymbol{\alpha}_q(a_q)$, where $\boldsymbol{\alpha}\colon A \to \mathbb{R}^\Delta$ is defined by

$$(14.9) \qquad \boldsymbol{\alpha}(a) = (a^{-\gamma})_{\gamma \in \Delta}.$$

This we will do by means of the following lemma.

Lemma 14.6. *There exists a polynomial map $\psi\colon \mathbb{C}^\Delta \to \mathbb{C}^{\Delta_q}$ with $\psi(0) = 0$ and $\psi(D^\Delta) \subset D^{\Delta_q}$ such that*

$$\boldsymbol{\alpha}_q(a_q^2) = \psi(\boldsymbol{\alpha}(a))$$

for all $a \in A$. In particular, $A^+ \subset A_h A_q^+$.

Proof. For each $\beta \in \Delta$ we have

$$(14.10) \qquad a_q^{-2\beta} = a^{-\beta - \theta\sigma\beta}$$

for each $a \in A$. If $\beta | \mathfrak{a}_q \neq 0$ then $\theta\sigma\beta \in \Sigma^+$, so $a \in A^+$ implies $a_q \in A_q^+$. The existence of ψ follows easily from (14.10) and [Schl 84], Lemma 7.2.3. □

Let $\tilde{D} = \psi^{-1}(D^{\Delta_q})$, then $D^\Delta \subset \tilde{D} \subset \mathbb{C}^\Delta$ and $\boldsymbol{\alpha}(a) \in \tilde{D}$ for all $a \in A_h A_q^+$.

Proposition 14.7. *Let $\lambda \in \mathfrak{a}_c^*$ and $F \in \mathcal{E}_{\lambda,\tau}(G/K)$. There exist finite sets $S' \subset \mathfrak{a}_c^*$ and $M' \subset P(\mathfrak{a})$, and for each pair $(\nu, p) \in S' \times M'$ a holomorphic function $F'_{\nu,p}$ from $\tilde{D}$ to E, such that*

$$(14.11) \qquad F(a) = \sum_{\substack{\nu \in S' \\ p \in M'}} F'_{\nu,p}(\boldsymbol{\alpha}(a))\, p(\log a)\, a^\nu$$

for all $a \in A_h A_q^+$.

Proof. By an easy rearrangement of terms in (14.8) we obtain finitely many holomorphic functions $'F_{\nu,p}: D^{\Delta_q} \longrightarrow E$ such that

$$F(a) = \sum_{\nu,\mu,p} {'F_{\nu,p}}(\boldsymbol{\alpha}_q(a_q^2))\, p_\mu(\log a_h)\, p(\log a_q)\, a_h^\mu\, a_q^\nu .$$

Combining this with Lemma 14.6 we obtain (14.11). □

15. The action of H near the boundary of G/K

The purpose of this section is to prove the following lemma which may appear technical at first sight, but is encountered naturally if one tries to relate the converging expansion (14.11) to the asymptotic expansion of Theorem 13.1. Let H be any closed subgroup of G with the property that the map $(h, p) \longrightarrow hp$ from $H \times P$ to G is submersive at (e, e) (that is, $\mathfrak{g} = T_e H + T_e P$), so that the image HP is open in G.

Lemma 15.1. *There exist an open neighborhood Ω_0 of $(e, 0)$ in $G \times \mathbb{R}^\Delta$ and real analytic maps $h: \Omega_0 \longrightarrow H$ and $a: \Omega_0 \longrightarrow A$ such that:*

(i) *For all $g \in G$ and $b \in A$ with $(g, \boldsymbol{\alpha}(b)) \in \Omega_0$*

$$gbK = h(g, \boldsymbol{\alpha}(b))\, a(g, \boldsymbol{\alpha}(b))\, bK .$$

(ii) *If $(g, 0) \in \Omega_0$ and $x = man \in P$ then $(gx, 0) \in \Omega_0$, $h(gx, 0) = h(g, 0)$, and $a(gx, 0) = a(g, 0)\, a$.*

(iii) *For $t \in \mathbb{R}^\Delta$ near 0 we have $h(e, t) = a(e, t) = e$.*

Proof. We shall use Oshima's compactification $\tilde{X}$ of X. For its definition we refer to [Schl 84], Ch. 4.

Let n be the number of elements in Δ, and let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. We identify $\mathbb{R}^\Delta$ with $\mathbb{R}^n$ and denote by π the projection $G \times \mathbb{R}^n \longrightarrow X^\sim$ (cf. loc. cit. p. 64), which is a real analytic map.

By the assumption on H there exists a subspace $\mathfrak{s}$ of $\mathfrak{h}$ which is complementary in $\mathfrak{g}$ to the Lie algebra of P. Since $\pi(x, 0) = \pi(e, 0)$ for all $x \in P$, a dimension argument shows that the differential of π at $(e, 0)$ maps $\mathfrak{s} \times \mathbb{R}^n$ bijectively onto the tangent space $T_{\pi(e,0)}\tilde{X}$. Let U be an open neighborhood of 0 in $\mathfrak{h}$ such that $\exp: \mathfrak{h} \longrightarrow H$ maps U diffeomorphically into H, and put $Y = \exp(\mathfrak{s} \cap U)$. It follows that the restriction of π is a real analytic diffeomorphism of an open neighborhood Ω_1 of $(e, 0)$ in $Y \times \mathbb{R}^n$ onto an open neighborhood of $\pi(e, 0)$ in $\tilde{X}$.

We now define $\Omega_0 \subset G \times \mathbb{R}^n$ by $\Omega_0 = \pi^{-1}(\pi(\Omega_1))$. Moreover we define real analytic maps $h: \Omega_0 \longrightarrow H$ and $s: \Omega_0 \longrightarrow \mathbb{R}^n$ by $(h(g, t), s(g, t)) \in \Omega_1$ and

$$\pi(h(g, t), s(g, t)) = \pi(g, t) \qquad (15.1)$$

for $(g, t) \in \Omega_0$.

It follows immediately from these definitions that if $(g, 0) \in \Omega_0$ and $x \in P$ then $(gx, 0) \in \Omega_0$, $h(gx, 0) = h(g, 0)$, and $(s(gx, 0) = s(g, 0) = 0$, and also that $h(e, t) = e$ and $s(e, t) = t$ for t near 0.

From (15. 1) and the definition of π it follows for all $j=1,\dots,n$ and $(g,t)\in\Omega_0$ that $s(g,t)_j=0$ if and only if $t_j=0$. Hence there is a real analytic map $\sigma\colon \Omega_0\to\mathbb{R}^n$ such that $s(g,t)_j=t_j\sigma(g,t)_j$ for $j=1,\dots,n$. In particular $\sigma(g,0)_j=\frac{\partial s_j}{\partial t_j}(g,0)$ and $\sigma(e,t)=(1,\dots,1)$. In order to finish the proof we need the following.

Lemma 15. 2. *Let* $(g,0)\in\Omega_0$ *and* $x=man\in P$. *Then* $\sigma(gx,0)_j=a^{-\alpha_j}\sigma(g,0)_j$ $(1\leqq j\leqq n)$.

Proof. First assume that $x=n\in N$. Let $z\colon\mathbb{R}^n\to\bar{N}A$ be given by Lemma 8. 6 (with the given n), then

$$\pi(gn,t)=\pi(gz(t),t) \tag{15. 2}$$

for all $(g,t)\in G\times\mathbb{R}^n$ (for $t\in\,]0,\infty[^n$ this is immediate from $na\in z(\boldsymbol{\alpha}(a))\,aK$, and by analytic continuation it is valid for all t). From (15. 1) and (15. 2) it follows that $s(gn,t)=s(gz(t),t)$. Applying the chain rule and noticing that the derivatives in the first coordinate of s vanish at $(g,0)$ (because $s(\cdot,0)\equiv 0$) we get that $\frac{\partial s_j}{\partial t_j}(gn,0)=\frac{\partial s_j}{\partial t_j}(g,0)$, for all $(g,0)\in\Omega_0$. Hence $\sigma(gn,0)=\sigma(g,0)$. Next we assume $x=a\in A$. From the definition of π it follows that

$$\pi(ga,t)=\pi(g,\boldsymbol{\alpha}(a)\,t) \tag{15. 3}$$

for all $(g,t)\in G\times\mathbb{R}^n$ (here the product of two elements $s,t\in\mathbb{R}^n$ is defined by $(st)_j=s_jt_j$). Hence $s(ga,t)=s(g,\boldsymbol{\alpha}(a)\,t)$ for t near 0 and $(g,0)\in\Omega_0$, which shows that $\sigma(ga,0)=\boldsymbol{\alpha}(a)\,\sigma(g,0)$. Finally if $x\in M$ it is obvious from (15. 1) that $s(gx,t)=s(g,t)$ and hence $\sigma(gx,0)=\sigma(g,0)$. □

Completion of the proof of Lemma 15. 1. In particular it follows from the above lemma that $\sigma(x,0)=\boldsymbol{\alpha}(a)$ for $x=man\in P$. Shrinking Ω_0 if necessary we may then assume that $\sigma_j>0$ on Ω_0 for $j=1,\dots,n$. We now define $a(g,t)\in A$ by $\boldsymbol{\alpha}(a(g,t))=\sigma(g,t)$ for $(g,t)\in\Omega_0$. Obviously $a\colon\Omega_0\to A$ is real analytic, and by the above lemma $a(gx,0)=a(g,0)\,a$. Hence (ii) and (iii) hold. Finally it follows from (15. 3) and (15. 1) that

$$\pi(h(g,t)\,a(g,t),t)=\pi(h(g,t),\boldsymbol{\alpha}(a(g,t))\,t)=\pi(g,t).$$

Applying this to $t=\boldsymbol{\alpha}(b)$ and using (15. 3) we infer that

$$\pi(h(g,\boldsymbol{\alpha}(b))\,a(g,\boldsymbol{\alpha}(b))\,b,(1,\dots,1))=\pi(gt,(1,\dots,1))$$

whence (i). □

16. Converging expansions

Resuming the notation of Section 14 we shall now prove that an H-finite eigenfunction admits converging expansions in the directions towards the open subset HP of the boundary G/P, and relate these expansions to the asymptotic expansions (the existence of which follows from Theorems 14. 1 and 13. 1).

In the following we denote by $\mathscr{H}$ the (finite dimensional) space of W-harmonic polynomials on $\mathfrak{a}$.

Theorem 16. 1. *There exists an open subset Ω of $G \times \mathbb{R}^{\Delta}$ containing $HP \times \{0\}$, with the following property. Let $\lambda \in \mathfrak{a}_c^*$ and let $f \in \mathscr{E}_\lambda(G/K)$ be H-finite. Then there exists for each $v \in W\lambda$ a real analytic function $\Phi_v\colon \Omega \longrightarrow \mathscr{H}$, such that*

$$f(gaK) = \sum_{v \in W\lambda} \Phi_v(g, \alpha(a))(\log a)\, a^{v-\varrho} \tag{16. 1}$$

for all $g \in G$ and $a \in A$ with $(g, \alpha(a)) \in \Omega$.

Proof. Let Ω_0, h and a be given by Lemma 15. 1. Since a is continuous and $a(e, 0) = e$ we may assume (by shrinking Ω_0 if necessary) that

$$\alpha(a(g, t))_\gamma \leqq |t_\gamma|^{-1}$$

for all $\gamma \in \Delta$ and $(g, t) \in \Omega_0$ with $t_\gamma \neq 0$. From this it follows that $\alpha(a(g, \alpha(b))\, b) \in D^\Delta$ for all $g \in G$ and $b \in A$ with $(g, \alpha(b)) \in \Omega_0$.

Let $F \in \mathscr{E}_{\lambda, \tau}(G/K)$. If we combine Proposition 14. 7 and Lemma 15. 1 we obtain

$$F(gbK) = \tau(h) \sum_{\substack{v \in S' \\ p \in M'}} F'_{v,p}(\alpha(ab))\, p(\log(ab))\,(ab)^v \tag{16. 2}$$

with $h = h(g, \alpha(b))$ and $a = a(g, \alpha(b))$, for all $g \in G$, $b \in A$ with $(g, \alpha(b)) \in \Omega_0$. From (16. 2) it easily follows that there exist finite sets $S'' \subset \mathfrak{a}_c^*$ and $M'' \subset P(\mathfrak{a})$, and for each pair $(v, p) \in S'' \times M''$ a real analytic function $F''_{v,p}$ from Ω_0 to E, such that

$$F(gbK) = \sum_{\substack{v \in S'' \\ p \in M''}} F''_{v,p}(g, \alpha(b))\, p(\log b)\, b^v. \tag{16. 3}$$

Moreover, after some obvious rearrangements we may assume that the elements of M'' are linearly independent and that $M'' \cap \mathscr{H}$ spans $\mathscr{H}$. Then, expanding each $F''_{v,p}(g, \cdot)$ in a power series at 0 and inserting this into (16. 3), we find for each $\xi \in S'' - \mathbb{N}\Delta$ and $p \in M''$ a uniquely determined real analytic E-valued function $c_{\xi,p}$ on $\{g \in G \mid (g, 0) \in \Omega_0\}$ such that:

$$F(gbK) = \sum_{p \in M''} p(\log b) \sum_{\xi \in S'' - \mathbb{N}\Delta} c_{\xi,p}(g)\, b^\xi \tag{16. 4}$$

when $(g, \alpha(b)) \in \Omega_0$. The sum converges absolutely, and locally uniformly with respect to g. Let $v_1, \ldots, v_r$ be the (mutually different) elements of S'' and put $\Gamma_1 = v_1 - \mathbb{N} \cdot \Delta$ and $\Gamma_i = (v_i - \mathbb{N} \cdot \Delta) \backslash (\Gamma_1 \cup \cdots \cup \Gamma_{i-1})$ recursively for $i = 2, \ldots, r$. Redefining the $F''_{v,p}$ we may assume that $F''_{v_i,p}(g, \alpha(b)) = \sum_{\xi \in \Gamma_i} c_{\xi,p}(g)\, b^\xi$ for $i = 1, \ldots, p$, and under these assumptions $F''_{v_i,p}$ is uniquely determined by F and the choices of M'', S'' and the ordering of S''.

In particular it follows that if $(g, t) \in \Omega_0$, $h \in H$, and $(hg, t) \in \Omega_0$ then

$$F''_{v,p}(hg, t) = \tau(h)\, F''_{v,p}(g, t). \tag{16. 5}$$

Hence if we put $\Omega = \{(hg, t) \in G \times \mathbb{R}^\Delta \mid h \in H, (g, t) \in \Omega_0\}$ and define $F''_{v,p}\colon \Omega \longrightarrow E$ by (16. 5), then $F''_{v,p}$ is real analytic and we have (16. 3) for all $g \in G$ and $b \in A$ with $(g, \alpha(b)) \in \Omega$.

It remains to be seen that we may take $S''=W\lambda-\varrho$ and $M''\subset\mathscr{H}$ in (16. 3).

Let $(g, 0)\in\Omega$, and let V and U be neighborhoods of g and 0, respectively, such that $V\times U\subset\Omega$. For each $\varphi\in C_c^\infty(V)$ let $L^\vee(\varphi)F$ be the E-valued function given by

$$[L^\vee(\varphi)F](x)=\int_G F(gx)\,\varphi(g)\,dg$$

for $x\in G$. It follows from (16. 3) and (16. 4) that

$$[L^\vee(\varphi)F](b)=\sum_{p\in M''} p(\log b)\sum_{\xi\in S''-\mathbb{N}\cdot\Delta} c_{\xi,p}(\varphi)\,b^\xi \tag{16. 6}$$

for $b\in A$ with $\boldsymbol{\alpha}(b)\in U$, where $c_{\xi,p}(\varphi)\in E$ is given by

$$c_{\xi,p}(\varphi)=\int_G c_{\xi,p}(g)\,\varphi(g)\,dg. \tag{16. 7}$$

The sum (16. 6) converges absolutely.

On the other hand, by Theorem 14. 1, each component of the vector function F belongs to $\mathscr{E}_\lambda^*$, and hence the asymptotic theory of Part II can be applied to F (component wise). Thus we have

$$[L^\vee(\varphi)F](\exp tY)\sim\sum_\xi p_\xi(F,\varphi,tY)\,e^{t\xi(Y)} \tag{16. 8}$$

as $t\longrightarrow+\infty$, for $Y\in\mathfrak{a}^+$ and $\varphi\in\mathscr{S}$. Here $p_\xi(F,\varphi,tY)\in E$ and p_ξ is a W-harmonic polynomial in its third variable (cf. Theorem 13. 1), and the summation extends over $\xi\in X(\lambda)=W\lambda-\varrho-\mathbb{N}\cdot\Delta$.

By the uniqueness of asymptotic expansions it follows from (16. 6) and (16. 8) that $\xi\in X(\lambda)$ whenever $c_{\xi,p}\neq0$ for some $p\in M''$. Hence redefining the $F''_{v,p}$ we may assume that $S''\subset W\lambda-\varrho$. Moreover, if $\xi\in X(\lambda)$ it follows that

$$p_\xi(F,\varphi)=\sum_{p\in M''} c_{\xi,p}(\varphi)\,p \tag{16. 9}$$

for all $\varphi\in C_c^\infty(V)$. Since $p_\xi(F,\varphi)\in\mathscr{H}$, the assumptions on M'' imply that $c_{\xi,p}=0$ if $p\in M''\backslash\mathscr{H}$. Therefore we may replace M'' by $M''\cap\mathscr{H}$ in (16. 3). □

In the course of the above proof we obtained the following (cf. (16. 4), (16. 7) and (16. 9)).

Corollary 16. 2. *Let $\lambda\in\mathfrak{a}_c^*$ and let $f\in\mathscr{E}_\lambda(G/K)$ be H-finite. The $\mathscr{H}$-valued distributions $p_\xi(f)$ in the asymptotic expansion for f restrict to real analytic $\mathscr{H}$-valued functions on HP, and the sum*

$$f(gaK)=\sum_{\xi\in X(\lambda)} p_\xi(f,g,\log a)\,a^\xi \tag{16. 10}$$

converges absolutely for all $g\in G$ and $a\in A$ with $(g,\boldsymbol{\alpha}(a))\in\Omega$. In particular, if $\lambda\in\mathscr{A}$ (see (9. 1)), then $p_{\lambda-\varrho}(f,g)=\Phi_\lambda(g,0)$ for $g\in HP$.

Remark 16. 3. Notice that in general the Φ_v $(v\in W\lambda)$ are not uniquely determined by (16. 1). However, if $\lambda\in\mathscr{A}$ then $\Phi_\lambda(g,0)$ is uniquely determined for $g\in HP$. Also, if $\langle\lambda,\alpha^\vee\rangle\notin\mathbb{Z}$ for all $\alpha\in\Sigma$, then the Φ_v $(v\in W\lambda)$ are uniquely determined and scalar valued.

In general there are other open H-orbits on G/P than HP, and for each of these one gets expansions similar to (16.1) and (16.10). More explicitly if $x \in N_K(\mathfrak{a}_q) \cap N_K(\mathfrak{a})$, then HxP is open in G (cf. [Ma 79] or [Schl 84], Prop. 7.1.8 (ii). Every open orbit can be represented in this fashion.) One may extend Ω to contain $HxP \times \{0\}$ without disturbing the validity of the assertions of Theorem 16.1. Moreover, Corollary 16.2 remains valid if HP is replaced everywhere by HxP. This is easily seen by applying the original statements to the xHx^{-1}-finite function $L_{x^{-1}}f$.

In the remainder of this section we draw some interesting consequences of Theorem 16.1 which are related to the classical "Fatou-theorem", and which improve on Proposition 9.6. They will not be used in the following sections.

Fix $\lambda \in \mathfrak{a}_c^*$ and assume (as in Proposition 9.6) that

$$\text{Re}\langle \lambda, \alpha\rangle > 0 \quad (\forall \alpha \in \Sigma^+ \backslash \Sigma^\lambda). \tag{16.11}$$

In the following the notation $a \to \infty$ means that $a \in A$ and $\boldsymbol{\alpha}(a) \to 0$ in $\mathbb{R}^\Delta$. Let ω^λ be given by (9.7).

Corollary 16.4. *Assume* (16.11) *and let $f \in \mathscr{E}_\lambda(G/K)$ be H-finite. Then*

$$[\omega^\lambda(\log a)\, a^{\lambda-\varrho}]^{-1} f(gaK) \to \beta_\lambda(f)(g) \tag{16.12}$$

as $a \to \infty$, for all $g \in HP$ (or $g \in HxP$, $x \in N_K(\mathfrak{a}) \cap N_K(\mathfrak{a}_q)$). The convergence is uniform in g when g belongs to a compact subset of HP.

Proof. The condition on λ implies that $a^{\nu-\lambda} \to 0$ as $a \to \infty$, for all $\nu \in W\lambda$ with $\nu \neq \lambda$. Hence it follows from (16.1) and the first statement of Corollary 16.2 that

$$\lim_{a\to\infty} [\omega^\lambda(\log a)\, a^{\lambda-\varrho}]^{-1} f(gaK) = \lim_{a\to\infty} [\omega^\lambda(\log a)]^{-1}\, p_{\lambda-\varrho}(f, g, \log a).$$

The corollary follows immediately from the definition of β_λ. □

Integration of (16.12) against a function $\varphi \in C_c^\infty(HP)$ of course yields a similar convergence property of $L^\vee(\varphi)f$ towards $\beta_\lambda(f, \varphi)$, for f H-finite. The following theorem shows that this holds even without H-finiteness.

Theorem 16.5. *Assume* (16.11) *and let $f \in \mathscr{E}_\lambda^*(G/K)$. Then*

$$[\omega^\lambda(\log a)\, a^{\lambda-\varrho}]^{-1}\, L^\vee(\varphi) f(a) \to \beta_\lambda(f, \varphi) \tag{16.13}$$

as $a \to \infty$, for all $\varphi \in \mathscr{S}$.

Proof. If φ is (left) K-finite this follows from Corollary 16.4 with $H = K$, applied to $L^\vee(\varphi)f$. Since the space of K-finite functions in $\mathscr{S}$ is a dense subspace of $\mathscr{S}$, it now suffices to prove the existence of a continuous seminorm ν on $\mathscr{S}$ such that

$$|[\omega^\lambda(\log a)\, a^{\lambda-\varrho}]^{-1}\, [L^\vee(\varphi) f]\,(a)| \leqq \nu(\varphi) \tag{16.14}$$

for all $\varphi \in \mathscr{S}$, $a \in \boldsymbol{\alpha}^{-1}(]0, \tfrac{1}{2}]^\Delta)$. By Theorem 12.2 there exists a $T \in \mathscr{D}'(K/M)$ such that $f = \mathscr{P}_\lambda T$, whence $L^\vee(\varphi) f = \mathscr{P}(L^\vee(\varphi)\, T_\lambda)$ (cf. (10.3)). There exists a continuous seminorm ν' on $\mathscr{S}$ such that $\sup_K |L^\vee(\varphi)\, T_\lambda| \leqq \nu'(\varphi)$ for all $\varphi \in \mathscr{S}$. It follows that $|L^\vee(\varphi) f| \leqq \nu'(\varphi)\, \phi_{\text{Re}\,\lambda}$, where $\phi_{\text{Re}\,\lambda}$ is the spherical function in $\mathscr{E}_{\text{Re}\,\lambda}(G/K)$ (use (1.8)). Applying (16.12) to the K-finite function $\phi_{\text{Re}\,\lambda}$ and using that $\omega^{\text{Re}\,\lambda}(\log a)/\omega^\lambda(\log a)$ is uniformly bounded on $\boldsymbol{\alpha}^{-1}(]0, \tfrac{1}{2}]^\Delta)$, the estimate (16.14) follows. □

Corollary 16. 6. *Assume* (16. 11) *and let* ψ *be a continuous function on* K/M. *Then*

$$(16.15) \qquad [\omega^\lambda(\log a)\, a^{\lambda-\varrho}]^{-1}\, \mathscr{P}_\lambda \psi(ya) \longrightarrow k(\lambda)\, \psi(y)$$

as $a \longrightarrow \infty$, *for all* $y \in K$.

Here $k(\lambda)$ is the non-zero constant given by (10. 1).

Proof. The proof is similar to that of Theorem 16. 5, using Theorem 12. 2, (16. 12) and density of the K-finite functions in $C(K/M)$. □

Remark 16. 7. Notice that the proofs of Corollaries 16. 5 and 16. 6 require only (16. 12) for K-finite functions.

Remark 16. 8. Assume $\langle \lambda, \alpha \rangle \geqq 0$ for all $\alpha \in \Sigma^+$, and let P_λ denote the kernel on $G/K \times K/M$ given by

$$P_\lambda(gK, kM) = \phi_\lambda(g)^{-1}\, e^{\langle -\lambda-\varrho,\, H(g^{-1}k)\rangle}$$

for $g \in G$ and $k \in K$. Then (16. 15) implies that $P_\lambda(aK, \cdot)$ is an approximate identity on K/M for $a \in A$, $a \longrightarrow \infty$. For regular λ (and $\lambda = 0$ when $\operatorname{rank} G/K = 1$) this was proved in [Mi 73]. For $\lambda = 0$ this gives the answer to a question of [Sj 81], p. 244, and in this case it has also been obtained by Sjögren [Sj 86]. In particular, it follows that Harish-Chandra's upper bound [He 84], p. 483 for the spherical function ϕ_0, is the best possible (cf. also [Ban 82]).

17. Leading exponents and L^p-estimates

In this section we relate to each other the leading exponents of the expansions on A_q and on A. The purpose is to express a natural L^p-condition on A_q in terms of the leading exponents on A.

Let $f \in \mathscr{E}_\lambda(G/K)$ be H-finite and let F be the associated E-valued τ-spherical function. We first consider exponents on A. Recall that for each $x \in G$, $E(f, x)$ is the set of $\xi \in X(\lambda)$ for which the distribution $p_\xi(f)$ is not identically zero near x, and $E_L(f, x)$ is the set of $\prec$-maximal elements in $E(f, x)$. Define $E(F, x)$ and $E_L(F, x)$ similarly for the expansion (16. 8) then $E(F, x) = \bigcup_{\eta \in E^*} E(\eta \circ F, x)$. Notice that $E(f, x) = E(f, e)$ and $E(F, x) = E(F, e)$ for all x in the identity component of HP because the p_ξ's are real analytic functions on HP (cf. Corollary 16. 2).

Lemma 17. 1. *Assume* H *is connected. Then* $E(F, x) = E(f, x)$ (*and hence* $E_L(F, x) = E_L(f, x)$) *for all* $x \in G$.

Proof. Obviously $E(f, x) \subset E(F, x)$. Let V be the H-invariant space generated by f, then it suffices to prove that $E(\tilde{f}, x) \subset E(f, x)$ for all $\tilde{f} \in V$. Since left differentiations do not create new exponents this follows from the fact that $V = \{L_u f \mid u \in \mathscr{U}(\mathfrak{h})\}$. □

Turning to A_q, we infer from Proposition 14.3 that F admits a converging expansion

(17.1) $$F(a) = \sum_{\nu \in S - \mathbb{N} \cdot \Delta_q} P_\nu(\log a)\, a^\nu$$

for $a \in A_q^+$, where each P_ν is a polynomial on $\mathfrak{a}$ (which is a linear combination of elements from M). We say that $\nu \in \mathfrak{a}_c^*$ is an *exponent* of F on A_q^+ if $P_\nu \neq 0$ in (14.1), and denote by $E_q(F)$ the set of these exponents. Introducing an ordering $\prec_q$ on $\mathfrak{a}_{q,c}^*$ by

$$\nu \prec_q \mu \Leftrightarrow \mu - \nu \in \mathbb{N} \cdot \Delta_q$$

we call the $\prec_q$-maximal elements in $E_q(F)$ the *leading exponents* of F on A_q^+, and denote the set of these by $E_{q,L}(F)$.

Theorem 17.2. *Let $\lambda \in \mathfrak{a}_c^*$ and $F \in \mathscr{E}_{\lambda,\tau}(G/K)$. Then $E_{q,L}(F)$ is the collection of $\prec_q$-maximal elements in the set*

(17.2) $$\{\xi|_{\mathfrak{a}_q} \mid \xi \in E_L(F, e)\}.$$

Before giving the proof we identify $E_q(F)$ in the following lemma.

Lemma 17.3. *For $\xi \in X(\lambda)$ and $g \in HP$ let $p_\xi(F, g)$ be the E-valued polynomial on $\mathfrak{a}$ from* (16.8). *Then*

$$E_q(F) = \{\xi|_{\mathfrak{a}_q} \mid \xi \in E(F, e),\ p_\xi(F, e) \neq 0\}.$$

Proof. Writing $p_\xi = p_\xi(F, e)$ for short, we have

(17.3) $$F(a) = \sum_{\xi \in X(\lambda)} p_\xi(\log a)\, a^\xi$$

for $a \in A_h A_q^+$ (cf. Proposition 14.7 and formula (16.10)). For $\nu \in \mathfrak{a}_{q,c}^*$ let

(17.4) $$\mathscr{E}(\nu) = \{\xi \in X(\lambda) \mid \xi|_{\mathfrak{a}_q} = \nu \text{ and } p_\xi \neq 0\}.$$

Then the lemma amounts to the following claim: $P_\nu \neq 0$ if and only if $\mathscr{E}(\nu) \neq \emptyset$.

Comparing (17.3) with (14.8) we see that $p_\xi \neq 0$ implies $\xi|_{\mathfrak{a}_h} \in T$, the set of weights of $\tau|_{\mathfrak{a}_h}$, from which it follows that $\mathscr{E}(\nu)$ is finite for all ν. Comparing (17.3) with (17.1) we then have that

(17.5) $$P_\nu(Y) = \sum_{\xi \in \mathscr{E}(\nu)} p_\xi(Y)$$

for $Y \in \mathfrak{a}_q$, since the P_ν are unique in (17.1). From this we immediately derive that $P_\nu \neq 0$ implies $\mathscr{E}(\nu) \neq \emptyset$.

The spherical property of F combined with the uniqueness of p_ξ in (17.3) implies that

(17.6) $$\tau(b)\, p_\xi(Y) = p_\xi(Y + \log b)\, b^\xi$$

for all $Y \in \mathfrak{a}$ and $b \in A_h$. Assuming that $P_\nu = 0$ we get from (17.5) and (17.6) that

$$\sum_{\xi \in \mathscr{E}(\nu)} p_\xi(Y + \log b)\, b^\xi = 0$$

for all $Y \in \mathfrak{a}_q$ and $b \in A_h$. From this it easily follows that each term in the sum must vanish, that is, $\mathscr{E}(\nu) = \emptyset$. □

Notice that in general we cannot rule out that $p_\xi(F, e) = 0$ for some $\xi \in E(F, e)$, because $\xi \notin E(F, e)$ means $p_\xi(F, g) = 0$ for all g near e. For leading exponents this is different, though, because then $p_\xi(F)$ is uniquely determined on HP by its value at e, by the spherical property and the distribution version of Theorem 8.4.

Proof of Theorem 17.2. If $\xi \in E_L(F, e)$, then $p_\xi(F, e) \neq 0$ by the above observation, hence $\xi|_{\mathfrak{a}_q} \in E_q(F)$ by Lemma 17.3. It follows that the set X defined by (17.2) is contained in $E_q(F)$. On the other hand, if $\nu \in E_{q,L}(F)$, then the set $\mathscr{E}(\nu)$ is nonempty and finite (proof of Lemma 17.4). Select a $\prec$-maximal element $\xi \in \mathscr{E}(\nu)$. Then $\xi \in E(F, e)$. Fix $\zeta \in E_L(F, e)$ with $\xi \prec \zeta$. Then $\nu \prec_q \zeta|_{\mathfrak{a}_q}$. The latter element being contained in $E_q(F)$, it follows that $\zeta|_{\mathfrak{a}_q} = \nu$, whence $\zeta \in \mathscr{E}(\nu)$. By maximality we must have $\zeta = \xi$, or $\xi \in E_L(F, e)$. It follows that $E_{q,L}(F) \subset X \subset E_q(F)$, whence the result. □

Let $1 \leqq p < \infty$ and let da be a Haar measure on A_q.

Theorem 17.4. *For each $F \in \mathscr{E}_{\lambda,\tau}(G/K)$ the following two conditions are equivalent*:

(i) *The function $a \longrightarrow \|F(a)\|$ belongs to $L^p(A_q^+; a^{2\varrho}\, da)$.*

(ii) *For all $w \in W$ with $w\lambda - \varrho \in E_L(F, e)$ and all $Y \in \mathrm{cl}(\mathfrak{a}_q^+)\backslash\{0\}$ we have*

$$\operatorname{Re} w\lambda(Y) < \frac{p-2}{p}\, \varrho(Y).$$

Proof. In [Ban 84], proof of Thm. 9.4 it is proved that (i) holds if and only if $\operatorname{Re}\nu + \frac{2}{p}\varrho$ is negative on $\mathrm{cl}(\mathfrak{a}_q^+)\backslash\{0\}$ for all $\nu \in E_{q,L}(F)$. By Theorem 17.2 this holds if and only if $\operatorname{Re}\xi + \frac{2}{p}\varrho$ is negative on $\mathrm{cl}(\mathfrak{a}_q^+)\backslash\{0\}$ for all $\xi \in E_L(F, e)$, which is (ii). □

Define $\varrho_b\colon \mathfrak{a}_q \longrightarrow [0, +\infty[$ by $\varrho_b(vY) = \varrho(Y)$ for $Y \in \mathrm{cl}\,\mathfrak{a}_q^+$ and v in the Weyl group W_q of Σ_q^+.

Corollary 17.5. *For each $F \in \mathscr{E}_{\lambda,\tau}(G/K)$ the following two conditions are equivalent*:

(i) *The function $a \longrightarrow \|F(a)\|$ belongs to $L^p(A_q; a^{2\varrho_b}\, da)$.*

(ii) *For all $x \in N_K(\mathfrak{a}) \cap N_K(\mathfrak{a}_q)$ and all $w \in W$ with $w\lambda - \varrho \in E_L(F, x)$ we have*

$$\operatorname{Re} w\lambda(Y) < \frac{p-2}{p}\, \varrho(Y) \textit{ for } Y \in \mathrm{cl}(\mathfrak{a}_q^+)\backslash\{0\}.$$

Proof. Let $x \in N_K(\mathfrak{a}) \cap N_K(\mathfrak{a}_q)$. It follows easily from Theorem 17.4 that the function $a \longrightarrow \|F(xa)\|$ belongs to $L^p(A_q^+; a^{2\varrho}\, da)$ if and only if $\operatorname{Re} w\lambda < \frac{p-2}{p}$ on $\mathrm{cl}(\mathfrak{a}_q^+)\backslash\{0\}$ for all $w\lambda - \varrho \in E_L(F, x)$ (replace H by $x^{-1}Hx$ and F by $L_{x^{-1}}F$). The equivalence of (i) and (ii) now follows from Lemma 14.2. □

18. Distributions supported on closed orbits

In relation to H the distributions $T \in \mathscr{D}'(G/P; L_\lambda)$ which are supported on *closed* H-orbits on G/P play a distinguished role. In this section we will study the impact of this support condition on the asymptotic expansion of $\mathscr{P}_\lambda T$.

We impose on H (or on σ) the extra condition that

$$\text{rank}\, G/K = \text{rank}\, H/H \cap K, \tag{18. 1}$$

that is, there is a maximal abelian subspace $\mathfrak{a}_1$ of $\mathfrak{p}$, contained in $\mathfrak{h} \cap \mathfrak{p}$. Our aim is to prove:

Theorem 18. 1. *Assume* (18. 1). *Let* $\lambda \in \mathfrak{a}_c^*$ *and assume* $\text{Re}\langle\lambda, \alpha\rangle > 0$ *for all* $\alpha \in \Sigma^+$. *Let* $T \in \mathscr{D}'(G/P; L_\lambda)$ *and assume that* supp T *is contained in a union of closed H-orbits on* G/P. *Let* $x \in N_K(\mathfrak{a}) \cap N_K(\mathfrak{a}_q)$ *and* $g \in HxP$. *If* $w\lambda - \varrho \in E_L(\mathscr{P}_\lambda T, g)$ *then* $\text{Re}\, w\lambda(Y) < 0$ *for all* $Y \in \text{cl}(\mathfrak{a}_q^+) \setminus \{0\}$.

Notice the similarity of the conclusion with (ii) of Corollary 17. 5 (cf. Corollary 18. 6 below).

The proof of Theorem 18. 1 consists of a series of lemmas. The proof of the key lemma 18. 5 is similar to a main ingredient in Oshima's proof of Flensted-Jensen's conjecture (cf. [Schl 84], p. 136).

Lemma 18. 2. (i) *For each* $c \in K$ *with* $\text{Ad}\, c(\mathfrak{a}) = \mathfrak{a}_1$ *we have* $HcP = (H \cap K)cP$, *hence* HcP *is closed.*

(ii) *There are only finitely many closed H-orbits on* G/P, *and each is obtained by* (i).

Proof. See [Ma 79]. ((i) follows easily from the Iwasawa decomposition of H.) □

From this lemma it follows that we may assume supp $T \subset HcP$ for some (fixed) $c \in K$ with $\text{Ad}\, c(\mathfrak{a}) = \mathfrak{a}_1$. For each continuous function γ on $K \cap H$ we define a distribution T_γ on K/M by

$$T_\gamma(\varphi) = \int_{K \cap H} \varphi(kcM)\, \gamma(k)\, dk$$

then supp $T_\gamma \subset (K \cap H)\, cM$.

Lemma 18. 3. *Let* $T \in \mathscr{D}'(K/M)$ *with* supp $T \subset (H \cap K)\, cM$. *There exist finitely many* $\gamma_1, \ldots, \gamma_m \in C(K \cap H)$ *and* $u_1, \ldots, u_m \in \mathscr{U}(\mathfrak{k})$ *such that*

$$T = \sum_{i=1}^{m} L_{u_i} T_{\gamma_i}.$$

Proof. Follows from [Schw 57], Thm. 37 (cf. also Thm. 26). □

From this lemma it follows that we may assume $T = L_u T_\gamma$ for some $\gamma \in C(K \cap H)$ and $u \in \mathscr{U}(\mathfrak{k})$ (this idea is due to Flensted-Jensen [F-J 86]). Actually, since $E(L_u \mathscr{P}_\lambda T_\gamma, g) \subset E(\mathscr{P}_\lambda T_\gamma, g)$ we may even further assume that $T = T_\gamma$.

Lemma 18. 4. *Let $\lambda \in \mathfrak{a}_c^*$ and assume $\operatorname{Re}\langle \lambda + \varrho, \alpha \rangle \geqq 0$ for all $a \in \Sigma^+$. There exist constants r and $\delta > 0$ with the following property: Let $\gamma \in C(K \cap H)$, $x \in N_K(\mathfrak{a}) \cap N_K(\mathfrak{a}_q)$, and let $L \subset HxP$ be a compact set. Then there exists $C \geqq 0$ such that*

$$|\mathscr{P}_\lambda T_\gamma(g a)| \leqq C \|a_h\|^r$$

for all $g \in L$ and $a \in A$ with $\boldsymbol{\alpha}(a) \in]0, \delta[^\Delta$.

Proof. Notice that

$$\mathscr{P}_\lambda T_\gamma(g) = \int_{K \cap H} \gamma(k)\, e_{\lambda+\varrho}(c^{-1} k^{-1} g)\, dk$$

from which it immediately follows that we may assume $\gamma = 1$ and $\lambda \in \mathfrak{a}^*$ (that is, λ is real valued on $\mathfrak{a}$). Let $\psi_\lambda = \mathscr{P}_\lambda T_1$, then $\psi_\lambda > 0$, and it follows from [Schl 84], Lemmas 7. 3. 1 and 7. 6. 1 that there exists $r \geqq 0$ such that

$$\psi_\lambda(ha) \leqq \|h\|^r \tag{18. 2}$$

for $h \in H$ and $a \in A_q$. From (18. 2) we get

$$\psi_\lambda(hxa) \leqq \|hxa_h\|^r \leqq \|h\|^r \, \|a_h\|^r \tag{18. 3}$$

for $h \in H$ and $a \in A$ (use that $x a_h x^{-1} \in A_h$). From Lemma 15. 1 (applied to the subgroup $x^{-1}Hx$) we have that there exists a $\delta > 0$ such that

$$xybK = h(y, \boldsymbol{\alpha}(b))\, x a(y, \boldsymbol{\alpha}(b))\, bK \tag{18. 4}$$

for $y \in P$ and $b \in A$ with $\boldsymbol{\alpha}(b) \in]0, \delta[^\Delta$. Combining (18. 3) and (18. 4) we easily get

$$\psi_\lambda(gb) \leqq C \|b_h\|^r$$

for $g \in L$. $\square$

Let $\mathfrak{a}_+^* = \{\lambda \in \mathfrak{a}^* \mid \langle \lambda, \alpha \rangle > 0 \text{ for all } \alpha \in \Sigma^+\}$ be the positive Weyl chamber in $\mathfrak{a}^*$, and let

$$W' = \{w \in W \mid \exists \lambda_0 \in \mathfrak{a}_+^* \, \exists Y_0 \in \mathrm{cl}(\mathfrak{a}_q^+) \setminus \{0\} : w\lambda_0(Y_0) \geqq 0\}.$$

Lemma 18. 5. *Assume that $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}$ for all $\alpha \in \Sigma$. Let $\gamma \in C(K \cap H)$ and $x \in N_K(\mathfrak{a}) \cap N_K(\mathfrak{a}_q)$. Then*

$$\beta_{w\lambda}(\mathscr{P}_\lambda T_\gamma) \equiv 0 \tag{18. 5}$$

on HxP, for all $w \in W'$.

Proof. Fix $w \in W'$ and let $w\lambda_0(Y_0) \geqq 0$, where $\lambda_0 \in \mathfrak{a}_+^*$ and $Y_0 \in \mathrm{cl}(\mathfrak{a}_q^+)\backslash\{0\}$. Perturbing first λ_0 and then Y_0 we may assume $w\lambda_0(Y_0) > 0$ and $Y_0 \in \mathfrak{a}_q^+$. Scaling λ_0 with a positive constant we may even assume $w\lambda_0(Y_0) > \varrho(Y_0)$, and perturbing λ_0 once more also that $\langle \lambda_0, \alpha^\vee \rangle \notin \mathbb{Z}$ for any $\alpha \in \Sigma$. Let Ω be a neighborhood of λ_0 in $\mathfrak{a}_c^*$ such that

$$\mathrm{Re}\, w\lambda(Y_0) > \varrho(Y_0)$$

and

$$\mathrm{Re}\langle \lambda, \alpha^\vee \rangle \in \,]0, \infty[\backslash \mathbb{Z}$$

for all $\alpha \in \Sigma$ and $\lambda \in \Omega$. By holomorphic continuation it suffices to prove (18.5) for $\lambda \in \Omega$ (use Theorem 9.2(ii) after evaluation against test functions $\varphi \in C_c^\infty(H x P)$ as in Section 13). Fix $Y_1 \in \mathfrak{a}_h$ such that $\alpha(Y_1) > 0$ for all $\alpha \in \Sigma^+$ with $\alpha|_{\mathfrak{a}_q} = 0$. Let $\lambda \in \Omega$ and choose r according to Lemma 18.4, and let $\zeta > 0$ be such that

$$(18.6) \qquad \mathrm{Re}(w\lambda - \varrho)(Y_0) > \zeta(r\|Y_1\| - \mathrm{Re}(w\lambda - \varrho)(Y_1)).$$

Put $X = Y_0 + \zeta Y_1$, then $X \in \mathfrak{a}^+$, and (18.6) implies that

$$r\|X_h\| = \mathrm{Re}(w\lambda - \varrho)(X) - \varepsilon$$

for some $\varepsilon > 0$. Hence by Lemma 18.4

$$|\mathscr{P}_\lambda[L^\vee(\varphi)\, T_\gamma](\exp tX)| \leqq C e^{\mathrm{Re}(w\lambda - \varrho)(tX) - \varepsilon t}$$

for $\varphi \in C_c^\infty(H x P)$, from which (18.5) follows. □

Proof of Theorem 18.1. The proof goes essentially by applying holomorphic continuation to Lemma 18.5. Let $\lambda_0 \in \mathfrak{a}_c^*$ with $\mathrm{Re}\langle \lambda_0, \alpha \rangle > 0$ for all $\alpha \in \Sigma^+$, let $w_0 \in W$, and assume that $\mathrm{Re}\, w_0\lambda_0(Y) \geqq 0$ for some $Y \in \mathrm{cl}(\mathfrak{a}_q^+)\backslash\{0\}$. As we have already seen below Lemma 18.3, it suffices to prove that then $p_{\lambda_0, w_0\lambda_0 - \varrho}(\mathscr{P}_{\lambda_0} T_\gamma) \equiv 0$ on $H x P$ for $\gamma \in C(K \cap H)$.

For each $\lambda \in \mathfrak{a}_c^*$ let

$$\Xi(\lambda) = \{\xi = w\lambda - \varrho - \mu \mid w \in W \text{ and } \mu \in \mathbb{N} \cdot \Delta \text{ with } w\lambda_0 - \mu = w_0\lambda_0\}.$$

Assume that $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}$ for all $\alpha \in \Sigma$ and let $\xi = w\lambda - \varrho - \mu \in \Xi(\lambda)$. By Theorem 3.6 (and Section 13) it suffices to prove that then $p_{\lambda, \xi}(\mathscr{P}_\lambda T_\gamma) \equiv 0$ on HxP. Assume the latter not to be the case. Then $\xi \in E(\mathscr{P}_\lambda T_\gamma, g)$ for some $g \in H x P$. Hence $\sigma \in E_L(\mathscr{P}_\lambda T_\gamma, g)$ for some $\sigma \in W\lambda - \varrho$ (cf. Proposition 8.1) with $\xi \prec \sigma$. Since the non-integrality condition on λ implies $\sigma = w\lambda - \varrho$ (cf. Lemma 5.5), we obtain $\beta_{w\lambda}(\mathscr{P}_\lambda T_\gamma) \not\equiv 0$ near g. However, $\xi \in \Xi(\lambda)$ implies that

$$\mathrm{Re}\, w\lambda_0(Y) = \mathrm{Re}\, w_0\lambda_0(Y) + \mu(Y) \geqq 0$$

and hence $w \in W'$, contradicting Lemma 18.5. □

Corollary 18.6. *Assume* (18.1) *and* $\mathrm{Re}\langle \lambda, \alpha \rangle > 0$ *for all* $\alpha \in \Sigma^+$, *and let* $F \in \mathscr{E}_{\lambda, \tau}(G/K)$. *If* $\mathrm{supp}\, \beta_\lambda(F)$ *is contained in a union of closed H-orbits on G/P, then the function* $a \to \|F(a)\|$ *belongs to* $L^2(A_q; a^{2\varrho_b} da)$.

Proof. Combine Theorem 18.1 (applied componentwise to F) with Corollary 17.5. □

19. Discrete series for semisimple symmetric spaces

In this final section, notation differs from the previous sections. We will indicate the construction of discrete series via Corollary 18.6.

Let G be any connected semisimple real Lie group, $\mathfrak{g}$ its Lie algebra, σ an involution of $\mathfrak{g}$, $\mathfrak{h}$ the fixed point set of σ, and H the corresponding analytic subgroup of G. Let Z denote the center of G (notice that we are not assuming Z to be finite). Let θ be a Cartan involution of $\mathfrak{g}$ commuting with σ, $\mathfrak{k}$ its fixed point set, and K the corresponding analytic subgroup of G.

Let $(\mathfrak{g}^d, \mathfrak{h}^d, \mathfrak{k}^d)$ be the symmetric triple which is dual to $(\mathfrak{g}, \mathfrak{k}, \mathfrak{h})$, that is

$$\mathfrak{g}^d = \mathfrak{k} \cap \mathfrak{h} + i(\mathfrak{k} \cap \mathfrak{q}) + i(\mathfrak{p} \cap \mathfrak{h}) + \mathfrak{p} \cap \mathfrak{q}$$

and

$$\mathfrak{h}^d = \mathfrak{g}^d \cap \mathfrak{k}_c, \qquad \mathfrak{k}^d = \mathfrak{g}^d \cap \mathfrak{h}_c.$$

(In [F-J 80] and [Schl 84], $\mathfrak{g}^d$, $\mathfrak{h}^d$ and $\mathfrak{k}^d$ are called, respectively $\mathfrak{g}^0$, $\mathfrak{k}^0$ and $\mathfrak{h}^0$.) Let G_c be a complex connected Lie group with Lie algebra $\mathfrak{g}_c$, and let G^d, H^d, and K^d denote the analytic subgroups corresponding to $\mathfrak{g}^d$, $\mathfrak{h}^d$, and $\mathfrak{k}^d$ respectively. The space $X^r = G^d/K^d$ is a Riemannian symmetric space, called the noncompact Riemannian form of $X = G/H$. According to Flensted-Jensen's duality there is a linear injection $f \to f^r$ of the space $C_K^\infty(G/H)$ of K-finite C^∞-functions f on X into the space $C_{H^d}^\infty(G^d/K^d)$ of H^d-finite C^∞-functions on X^r (cf. [Schl 84], Thm. 8.2.4). If X is simply connected this is a bijection.

Let $\mathbb{D}(G/H)$ denote the algebra of invariant differential operators on G/H, then there is a natural identification of $\mathbb{D}(G/H)$ with $\mathbb{D}(G^d/K^d)$, which is respected by the map $f \to f^r$ (that is, $(Df)^r = D(f^r)$ for $D \in \mathbb{D}(G/H) \cong \mathbb{D}(G^d/K^d)$).

Assume

$$\text{(19.1)} \qquad \operatorname{rank} G/H = \operatorname{rank} K/K \cap H$$

and let $\mathfrak{t}$ be a maximal abelian subspace of $\mathfrak{k} \cap \mathfrak{q}$, then (19.1) amounts to $\mathfrak{t}$ being maximal abelian in $\mathfrak{q}$. Let $\mathfrak{a}^d = i\mathfrak{t}$, then $\mathfrak{a}^d$ is a maximal abelian subspace of $\mathfrak{p}^d$ (where $\mathfrak{g}^d = \mathfrak{k}^d \oplus \mathfrak{p}^d$ is the Cartan decomposition of $\mathfrak{g}^d$), and $\mathfrak{a}^d \subset \mathfrak{p}^d \cap \mathfrak{h}^d$. Let Σ denote the root system of $\mathfrak{t}_c = \mathfrak{a}_c^d$ in $\mathfrak{g}_c$, and pick a positive set Σ^+. For each $\lambda \in \mathfrak{t}_c^*$ we have the canonical homomorphism χ_λ from $\mathbb{D}(G/H)$ to $\mathbb{C}$ (defined via (1.2) for G^d/K^d). Let $\mathscr{E}_\lambda(G/H)$ denote the corresponding eigenspace in $C^\infty(G/H)$, then $f \to f^r$ maps the K-finite elements in $\mathscr{E}_\lambda(G/H)$ into H^d-finite elements in $\mathscr{E}_\lambda(G^d/K^d)$.

Finally let κ be a unitary character of Z, and let $L_\kappa^2(G/H)$ denote the Hilbert space of functions $f\colon G/H \to \mathbb{C}$ satisfying

$$\text{(19.2)} \qquad L_z f = \kappa(z) f \quad (z \in Z)$$

and

$$\text{(19.3)} \qquad |f| \in L^2(G/(ZH)).$$

(Notice that $|f|$ makes sense as a function on $G/(ZH)$ because of the unitary of κ.)

Theorem 19. 1 (Oshima and Matsuki [OM 84]). *Assume* (19. 1), *let* $\lambda \in \mathfrak{t}_c^*$ *and assume* $\operatorname{Re}\langle\lambda, \alpha\rangle > 0$ *for all* $\alpha \in \Sigma^+$. *Let* $f \in \mathscr{E}_\lambda(G/H)$ *be K-finite, and assume* (19. 2). *If* $\operatorname{supp}\beta_\lambda(f^r)$ *is a union of closed* H^d*-orbits in* G^d/P^d, *then* $f \in L^2_\kappa(G/H)$.

Proof. To f corresponds a vector valued function F on G/H satisfying

$$F(kx) = \tau(k)\, F(x)$$

for $k \in K$ and $x \in G/H$, for a suitable representation τ of K (same argument as after (14. 1)). Via the duality (which is readily extended to τ-spherical functions, cf. [Schl 82], Thm. 2. 4) we obtain a function F^r, which is exactly the vector valued function associated to f^r. Since $\operatorname{supp}\beta_\lambda(F^r) = \operatorname{supp}\beta_\lambda(f^r)$ we obtain from Corollary 18. 6 that $a \to \|F(a)\|$ belongs to $L^2(A_q; a^{2\varrho_b}da)$ (on A_q we have $F = F^r$). This easily implies the Theorem, cf. [Schl 84], Thm. 8. 11. □

In particular, this theorem implies Flensted-Jensen's conjecture "$C=0$" of [F-J 80], first proved by Oshima (cf. [Schl 84], Thm. 8. 3. 1).

Notice that the main ingredient in our proof, Corollary 17. 5, gives a forceful necessary and sufficient condition, in terms of the asymptotic expansion of f^r, for f to be in $L^p_\kappa(G/H)$. For $p=2$ this result is similar to [OM 84], Prop. 2.

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Notes (added in proof). 1. It is possible to improve slightly Theorems 3. 5 and 13. 1 as follows: The set of exponents

$$X(\lambda) = \{w\lambda - \varrho - \mu \mid w \in W, \mu \in \mathbb{N} \cdot \Delta\}$$

can be replaced by the smaller set

$$X_1(\lambda) = \{w\lambda - \varrho - \mu \mid w \in W, \mu \in 2\mathbb{N} \cdot \Delta\}.$$

We sketch the proof: Let $\Omega \in U(\mathfrak{g})$ be the Casimir element, and $D_0 = \mu(\Omega)$ the Laplace-Beltrami operator on G/K. A straightforward computation shows that

$$\Omega - {}'\gamma(D_0) \in \sum_{\substack{\alpha \in \Sigma^+ \\ 1 \leqq i \leqq m(\alpha)}} X^2_{-\alpha i} + U(\mathfrak{g})\,\mathfrak{k}$$

where $X_{-\alpha i}$ $(1 \leqq i \leqq m(\alpha))$ is a suitably normalized basis for $\mathfrak{g}^{-\alpha}$. If $\langle\lambda, \alpha^\vee\rangle \notin \mathbb{Z}$ for all $\alpha \in \Sigma$ (so that $p_{\lambda,\xi}$ is constant on $\mathfrak{a}$) it follows from Proposition 7. 1 that

$$([\partial({}'\gamma(D_0)) - \chi_\lambda(D_0)]\, e^\xi)\, p_{\lambda,\xi}(f) = -\sum_{\alpha, i} (R_{X^2_{-\alpha i}}\, p_{\lambda,\xi+2\alpha}(f))\, e^\xi.$$

Now $\partial('\gamma(D_0))\, e^{\xi} = \langle \xi, \xi + 2\varrho \rangle\, e^{\xi}$ and $\chi_\lambda(D_0) = \langle \lambda, \lambda \rangle - \langle \varrho, \varrho \rangle$, and hence if $\xi = w\lambda - \varrho - \mu$ we get that

$$\langle \mu - 2w\lambda, \mu \rangle\, p_{\lambda,\xi}(f) = -\sum_{\alpha, i} R_{X^2_{-\alpha i}}\, p_{\lambda, \xi + 2\alpha}(f).$$

By recursion this shows that $p_{\lambda,\xi} = 0$ if $\xi \in X(\lambda) \setminus X_1(\lambda)$, except possibly for λ in a countable family of hyperplanes (where $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ or $\langle \mu - 2w\lambda, \mu \rangle = 0$). For general λ, the statement now follows from Corollary 10. 2 and Theorem 3. 6.

As a consequence of this, Theorem 10. 3 can also be improved: $p_{\lambda_0,\xi}$ is actually $W_{\frac{1}{2}w\lambda_0}$-harmonic on $\mathfrak{a}$. In particular if $\forall \alpha \in \Sigma$: $\langle \lambda_0, \alpha^\vee \rangle \notin 2\mathbb{Z}$ then $W_{\frac{1}{2}w\lambda_0} = \{e\}$ (cf. Lemma 5. 5) and hence $p_{\lambda_0,\xi}$ is a constant polynomial on $\mathfrak{a}$ for all ξ (this holds for instance in the case of $G = \mathrm{SL}(2, \mathbb{R})$ and $\lambda = \varrho$).

2. Let $V \subset G$ be open and let $f \in \mathscr{E}^\infty_\lambda(G/K)$. If $P(f, V) = 0$ (cf. Lemma 8. 3) then $f = 0$. Thus f is uniquely determined by its set of coefficients $p_{w\lambda - \varrho}(f)$ on V ($w \in W$). This result is contained in our forthcoming paper: *Local boundary data of eigenfunctions on Riemannian symmetric spaces.*

References

[Ban 82] *E. P. van den Ban*, Asymptotic expansions and integral formulas for eigenfunctions on a semisimple Lie group, Thesis, Utrecht 1982.

[Ban 84] *E. P. van den Ban*, Asymptotic behaviour of matrix coefficients related to a reductive symmetric space, to appear in Indag. Math. (1987).

[B 66] *N. Bourbaki*, Éléments de mathématiques. Espaces vectoriels topologiques. I–II, 2nd ed., Paris 1966.

[CM 82] *W. Casselman* and *D. Miličić*, Asymptotic behaviour of matrix coefficients of admissible representations, Duke Math. J. **49** (1982), 869—930.

[F-J 80] *M. Flensted-Jensen*, Discrete series for semisimple symmetric spaces, Ann. Math. (2) **111** (1980), 253—311.

[F-J 86] *M. Flensted-Jensen*, Analysis on Non-Riemannian Symmetric Spaces, Conf. Board Math. Sci. Series **61**, Providence, R.I. 1986.

[Gr 54] *A. Grothendieck*, Sur les espaces (*F*) et (*DF*), Summa Brasil. Math. **3** (1954), 57—123.

[H-C 58] *Harish-Chandra*, Spherical functions on a semisimple Lie group, Amer. J. Math. **80** (1958), 241—310 & 553—613.

[H-C 60] *Harish-Chandra*, Differential equations and semisimple Lie groups (unpublished 1960), Collected papers. 3, Berlin-Heidelberg-New York 1984, 57—120.

[H-C 66] *Harish-Chandra*, Discrete series for semisimple Lie groups. II, Acta Math. **116** (1966), 1—111.

[He 70] *S. Helgason*, A duality for symmetric spaces with applications to groups representations. Adv. in Math. 5 (1970), 1—154.

[He 76] *S. Helgason*, A duality for symmetric spaces with applications to group representations, II. Differential equations and eigenspace representations. Adv. in Math. 22 (1976), 187—219.

[He 78] *S. Helgason*, Differential geometry, Lie groups and symmetric spaces. Academic Press 1978.

[He 84] *S. Helgason*, Groups and geometric analysis. Academic Press 1984.

[KKMOOT 78] *M. Kashiwara*, *A. Kowata*, *K. Minemura*, *K. Okamoto*, *T. Oshima* and *M. Tanaka*, Eigenfunctions of invariant differential operators on a symmetric space, Ann. Math. (2) **107** (1978), 1—39.

[KO 77] *M. Kashiwara* and *T. Oshima*, Systems of differential equations with regular singularities and their boundary value problems, Ann. Math. (2) **106** (1977), 145—200.

[Ka 80] *T. Kato*, Perturbation theory for linear operators, Berlin-Heidelberg-New York 1980.

[Kom 67] *H. Komatsu*, Projective and injective limits of weakly compact sequences of locally convex spaces, J. Math. Soc. Japan **19** (1967), 366—383.

[Kos 75] *B. Kostant*, On the existence and irreductibility of certain series of representations. Lie groups and their representations (I. M. Gelfond, ed.), Budapest 1975, 231—329.

[Kos 78] *B. Kostant*, On Whittaker vectors and representation theory, Invent. Math. **48** (1978), 101—184.

[Kö 52] *G. Köthe*, Die Randverteilungen analytischer Funktionen, Math. Z. **57** (1952), 13—33.

[Le 78] *J. B. Lewis*, Eigenfunctions on symmetric spaces with distribution valued boundary forms, J. Funct. Anal. **29** (1978), 287—307.

[Ma 79] *T. Matsuki*, The orbits of affine symmetric spaces under the action of minimal parabolic subgroups, J. Math. Soc. Japan **31** (1979), 331—357.

[Mi 73] *H. L. Michelson*, Fatou theorems for eigenfunctions of the invariant differential operators on symmetric spaces, Trans. Amer. Math. Soc. **177** (1973), 257—274.

[Osg 29] *W. F. Osgood*, Lehrbuch der Funktionentheorie. 2, Leipzig 1929.

[Osh 84] *T. Oshima*, Boundary value problems for systems of linear partial differential equations with regular singularities, Advanced Studies in Pure Math. **4** (1984), 433—497.

[OM 84] *T. Oshima* and *T. Matsuki*, A description of discrete series for semisimple symmetric spaces, Advanced Studies in Pure Math. **4** (1984), 331—390.

[OS 80] *T. Oshima* and *J. Sekiguchi*, Eigenspaces of invariant differential operators on an affine symmetric space, Invent. Math. **57** (1980), 1—81.

[Ro 79] *W. Rossmann*, The structure of semisimple symmetric spaces, Can. J. Math. **31** (1979), 157—180.

[SKK 73] *M. Sato*, *T. Kawai* and *M. Kashiwara*, Microfunctions and pseudo-differential equations, LNM **287,** Berlin-Heidelberg-New York 1973, 265—529.

[Schl 82] *H. Schlichtkrull*, A series of unitary irreducible representations induced from a symmetric subgroup of a semisimple Lie group, Invent. Math. **68** (1982), 497—516.

[Schl 84] *H. Schlichtkrull*, Hyperfunctions and harmonic analysis on symmetric spaces, Boston 1984.

[Schw 57] *L. Schwartz*, Théorie des distributions. I, Paris 1957.

[Sj 81] *P. Sjögren*, Characterizations of Poisson integrals on symmetric spaces, Math. Scand. **49** (1981), 229—249.

[Sj 86] *P. Sjögren*, Convergence for the square root of the Poisson kernel, preprint 1986.

[Va 77] *V. S. Varadarajan*, Harmonic analysis on real reductive Lie groups, LNM **576,** Berlin-Heidelberg-New York 1977.

[Wal 83] *N. Wallach*, Asymptotic expansions of generalized matrix entries of representations of real reductive groups, LNM **1024,** Berlin-Heidelberg-New York 1983, 287—369.

[War 72 I] *G. Warner*, Harmonic analysis on semi-simple Lie groups. 1, Berlin-Heidelberg-New York 1972.

[War 72 II] *G. Warner*, Harmonic analysis on semi-simple Lie groups. 2, Berlin-Heidelberg-New York 1972.

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