Invent. math. 98, 639–657 (1989)

Inventiones mathematicae

Local boundary data of eigenfunctions on a Riemannian symmetric space

E.P. van den Ban[1] and H. Schlichtkrull[2]

[1] University of Utrecht, Mathematical Institute, P.O. Box 80010, NL-3508 TA Utrecht, The Netherlands

[2] The Royal Veterinary and Agricultural University, Department of Mathematics, Thorvaldsensvej 40, DK-1871 Frederiksberg C, Denmark

0. Introduction

Let $X = G/K$ be a Riemannian symmetric space of non-compact type, and let $\mathbf{D}(X)$ be the algebra of invariant differential operators on X. In a previous paper [1] we developed a theory of asymptotic expansions for joint eigenfunctions of $\mathbf{D}(X)$ of at most exponential growth. In the present paper we show that local asymptotic data determine the eigenfunctions completely. We also develop a theory of asymptotic expansions "along walls". Both results are of importance for the theory of the discrete series of a semisimple symmetric space.

Let $\mathfrak{a}$ be a maximal abelian split subspace of the Lie algebra $\mathfrak{g}$ of G, and denote by $\mathfrak{a}_c^*$ its complexified dual. Let Σ be the restricted root system of $\mathfrak{a}$ in $\mathfrak{g}$, and W the associated Weyl group. Then Harish-Chandra's isomorphism $\mathbf{D}(X) \simeq S(\mathfrak{a})^W$ determines a bijection $\lambda \mapsto \chi_\lambda$ from $\mathfrak{a}_c^*/W$ to the set $\mathbf{D}(X)^\wedge$ of algebra homomorphisms $\mathbf{D}(X) \to \mathbf{C}$. Given $\lambda \in \mathfrak{a}_c^*$, let $\mathscr{E}_\lambda(X)$ denote the joint eigenspace of functions $f \in C^\infty(X)$ satisfying:

$$Df = \chi_\lambda(D) f \quad (D \in \mathbf{D}(X)) .$$

A function f on X is said to be of at most exponential growth if there exist $r \in \mathbf{R}$ and $C > 0$ such that

$$|f(x)| \leq C e^{r d(x)} \quad (x \in X) .$$

Here $d(x)$ denotes the Riemannian distance of x to the origin. The space of functions in $\mathscr{E}_\lambda(X)$ of at most exponential growth is denoted by $\mathscr{E}_\lambda^*(X)$. In [1] we proved that every $f \in \mathscr{E}_\lambda^*(X)$ admits an asymptotic expansion of the form

$$f(x \exp tH) \sim \sum_{\xi \in X(\lambda)} p_\xi(x, tH) e^{t\xi(H)} \tag{1}$$

for $x \in G$, $H \in \mathfrak{a}^+$ as $t \to \infty$. Here $X(\lambda) = \{w\lambda - \rho - \mu;\ w \in W,\ \mu \in \mathbf{N}\Sigma\}$ and $H \mapsto p_\xi(.,H)$ is a polynomial function on $\mathfrak{a}$ with values in the space $\mathscr{D}'(G)$ of distributions on G. The expansion has to be interpreted in a distribution sense: after testing both sides with a compactly supported C^∞ function it becomes a genuine asymptotic expansion. Let now U be a non-empty open subset of G and suppose that $p_\xi = 0$ on $U \times \mathfrak{a}$ for all $\xi \in W\lambda - \rho$. Then a main result of this paper is that f must vanish identically on X (Corollary 4.10, see also Corollary 4.2). This uniqueness result could be expected in view of the analogous result Theorem 4.4 in [10]. The nature of the theory in [10] is different however; boundary values are defined by means of microlocal analysis à la [14] rather than by asymptotic methods. This makes it difficult to compare the results in a direct way, especially for "degenerate" values of the parameter λ.

Example. If $G = SU(1, 1)$, $K = S(U(1) \times U(1))$, then X is the unit disk D in $\mathbf{C}$ endowed with the Poincaré metric. Moreover, $\mathscr{E}_\rho(D)$ is the space of harmonic functions on D. In this case the coefficients $p_\xi(x, H)$ are independent of H. Moreover, for a function $f \in \mathscr{E}^*_\rho(D)$ which extends smoothly to $\bar{D}$, one can show that the $p_\xi(x)$ depend smoothly on x, and that (1) holds pointwise. This gives rise to an expansion

$$f\left(e^{i\theta} \tanh \frac{1}{2} t\right) \sim \sum_{n=0}^{\infty} p_n(e^{i\theta}) e^{-nt}, \tag{2}$$

for $\theta \in \mathbf{R}$, as $t \to \infty$. Here p_n is a smooth function on ∂D. A formal computation (which can be made rigorous) then shows that

$$p_0(e^{i\theta}) = f(e^{i\theta}),$$

$$p_1(e^{i\theta}) = -2 \frac{\partial f}{\partial r}(e^{i\theta}),$$

where r denotes the Euclidean distance to the origin. Now suppose that p_0 and p_1 vanish on a non-empty open subset U of ∂D. Then our uniqueness theorem asserts that f vanishes identically on D. In the present case this can also be seen as follows. By the reflection principle f can be extended to a harmonic function on a neighbourhood of U, and then it follows from the Cauchy-Kowalewski theorem that $f = 0$ on a neighbourhood of U. By real analyticity this implies that f vanishes on the entire disk D.

Sections 1–3 of this paper are devoted to the development of asymptotic techniques needed for the proof in Sect. 4 of the uniqueness result. In Sect. 5 we show that these techniques are also strong enough to replace the microlocal analysis used in Oshima and Matsuki's fundamental paper [11] on the discrete series for semisimple symmetric spaces.

Let us finally say something about the nature of our proof of the uniqueness result. The basic idea is to use induction on the rank of G/K. For the rank one case the uniqueness result can be obtained by using the Poisson transformation. The induction step is based on a property of transitivity of asymptotic expansions (Theorem 3.1), resembling the transitivity of the constant term in Harish-Chandra's work [5].

Our result on transitivity essentially amounts to the following. An eigenfunction f which is a smooth vector for the left regular representation of G in $\mathscr{E}^*_\lambda(X)$

admits an asymptotic expansion along a wall of type A_F (the usual notation for the split component of a standard parabolic). Each coefficient $p_{F,\eta}$ of this expansion is a smooth function on G, and it is annihilated by an ideal in $\mathbf{D}(M_{1F}/K_F)$ with finite codimension. By similar methods as in [1] we then obtain asymptotic expansions for $p_{F,\eta}$ along open chambers in M_{1F}. Again its coefficients are functions on G. Theorem 3.1 relates these coefficients with the p_ξ of (1).

We are grateful to Professor Oshima for suggesting the validity of Corollary 2.4, which led to a significant simplification of our original arguments.

1. Asymptotics along the walls

Let G be a real reductive Lie group of Harish–Chandra's class, K a maximal compact subgroup, and θ the associated Cartan involution. Let $\mathbf{D}(G/K)$ denote the algebra of invariant differential operators on the Riemannian symmetric space G/K. The purpose of this section is to derive asymptotic expansions along walls for functions on G/K which behave finitely under $\mathbf{D}(G/K)$. When comparing with [1], we are thus generalizing in three directions: (1) The asymptotic theory is "along walls", (2) we deal with $\mathbf{D}(G/K)$-finite functions instead of plain eigenfunctions and (3) G is of Harish–Chandra's class.

We adopt the usual notational conventions. Thus, Lie groups are denoted by Roman capitals, and their Lie algebras by the corresponding lower case Gothic letters. Moreover, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition defined by θ. We fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, and a choice Σ^+ of positive roots for the restricted root system $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ of $\mathfrak{a}$ in $\mathfrak{g}$; the associated fundamental system is denoted by Δ. Finally, $\mathfrak{n}$ denotes the sum of the positive root spaces, $\bar{\mathfrak{n}} = \theta(\mathfrak{n})$, $N = \exp(\mathfrak{n})$, $\bar{N} = \exp(\bar{\mathfrak{n}})$, and M denotes the centralizer of $\mathfrak{a}$ in K.

Given a real Lie algebra $\mathfrak{l}$, we denote the universal enveloping algebra of its complexification by $\mathscr{U}(\mathfrak{l})$, and the latter's centre by $\mathscr{Z}(\mathfrak{l})$. Elements of $\mathscr{U}(\mathfrak{l})$ will be viewed as left invariant differential operators on any Lie group with Lie algebra $\mathfrak{l}$, unless specified otherwise.

Let $I \subset \mathbf{D}(G/K)$ be a cofinite ideal (that is, an ideal of finite complex codimension). The infinitesimal right regular representation naturally induces an algebra homomorphism μ from $\mathscr{U}(\mathfrak{g})^K$ onto $\mathbf{D}(G/K)$. Let J be the left ideal of $\mathscr{U}(\mathfrak{g})$ generated by $\mathscr{U}(\mathfrak{g})\mathfrak{k}$ and the preimage $\mu^{-1}(I)$ of I in $\mathscr{U}(\mathfrak{g})^K$. Then $\mathscr{Y}_I = \mathscr{U}(\mathfrak{g})/J$ is a left $(\mathfrak{g}, K)$-module.

Fix a subset F of Δ, and let P_F denote the associated standard parabolic subgroup with Langlands decomposition $M_F A_F N_F$ (cf. [15], II, Ch. 6). We write $M_{1F} = M_F A_F$, $\bar{N}_F = \theta(N_F)$, and denote the centralizer of $\mathfrak{a}_F$ in K by K_F. Then M_{1F} is of Harish-Chandra's class, and K_F is a maximal compact subgroup.

Lemma 1.1. *The $(\mathfrak{m}_{1F}, K_F)$-module $\mathscr{Y}_I/\bar{\mathfrak{n}}_F\mathscr{Y}_I$ is finitely generated and admissible, for every $F \subset \Delta$. In particular, $\mathscr{Y}_I$ is an admissible $(\mathfrak{g}, K)$-module.*

Proof. By the Poincaré-Birkhoff-Witt theorem we have

$$\mathscr{U}(\mathfrak{g}) = \bar{\mathfrak{n}}_F\mathscr{U}(\mathfrak{g}) \oplus \mathscr{U}(\mathfrak{m}_{1F}) + \mathscr{U}(\mathfrak{g})\mathfrak{k}\,.$$

From this we see that $1 + \bar{\mathfrak{n}}_F \mathscr{Y}_I$ is a cyclic vector for the $(\mathfrak{m}_{1F}, K_F)$-module $\mathscr{Y}_I / \bar{\mathfrak{n}}_F \mathscr{Y}_I$.

Now observe that

$$\dim_{\mathbf{C}} \mathscr{Z}(\mathfrak{g})(1 + \bar{\mathfrak{n}}_F \mathscr{Y}_I) \leq \dim_{\mathbf{C}} \mathscr{U}(\mathfrak{g})^K / (J \cap \mathscr{U}(\mathfrak{g})^K) = \operatorname{codim}_{\mathbf{C}}(I) < \infty , \tag{3}$$

and consider the algebra homomorphism $\chi : \mathscr{Z}(\mathfrak{g}) \to \mathscr{Z}(\mathfrak{m}_{1F})$ defined by $Z - \chi(Z) \in \bar{\mathfrak{n}}_F \mathscr{U}(\mathfrak{g})$. Since $\mathscr{Z}(\mathfrak{m}_{1F})$ is a finite $\chi(\mathscr{Z}(\mathfrak{g}))$-module (cf. [15] II p. 52), we infer from (3) that

$$\dim_{\mathbf{C}} \mathscr{Z}(\mathfrak{m}_{1F})(1 + \bar{\mathfrak{n}}_F \mathscr{Y}_I) < \infty .$$

The result now follows by application of [16], Lemma 2.10. □

Consider the $\mathscr{U}(\mathfrak{m}_{1F})^{K_F}$-submodule (τ^1, Y^1) of $\mathscr{Y}_I / \bar{\mathfrak{n}}_F \mathscr{Y}_I$, generated by $1 + \bar{\mathfrak{n}}_F \mathscr{Y}_I$. From $Y^1 \subset (\mathscr{Y}_I / \bar{\mathfrak{n}}_F \mathscr{Y}_I)^{K_F}$ we see that Y^1 is finite dimensional, hence its set $S_F(I)$ of $\mathfrak{a}_F$-weights is finite. But Y^1 generates the $(\mathfrak{m}_{1F}, K_F)$-module $\mathscr{Y}_I / \bar{\mathfrak{n}}_F \mathscr{Y}_I$ and $\mathfrak{a}_F$ commutes with $\mathfrak{m}_{1F}$ and K_F. Therefore $S_F(I)$ is the set of $\mathfrak{a}_F$-weights of $\mathscr{Y}_I / \bar{\mathfrak{n}}_F \mathscr{Y}_I$ as well. We write

$$X(F, I) = S_F(I) - \mathbf{N}\Delta | \mathfrak{a}_F , \tag{4}$$

where $\mathbf{N}\Delta = \{\Sigma n_i \alpha_i ; n_i = 0, 1, 2, \dots \text{ and } \alpha_i \in \Delta\}$.

Lemma 1.2. *For $k \geq 1$, the $(\mathfrak{m}_{1F}, K_F)$-module $\mathscr{Y}_I / \bar{\mathfrak{n}}_F^k \mathscr{Y}_I$ is finitely generated and admissible. Its $\mathfrak{a}_F$-weights are contained in $X(F, I)$. The $\mathscr{U}(\mathfrak{m}_{1F})^{K_F}$-submodule (τ^k, Y^k) generated by $1 + \bar{\mathfrak{n}}_F^k \mathscr{Y}_I$ is finite dimensional. Moreover, for each $\eta \in X(F, I)$ there exists $d_\eta \in \mathbf{N}$ such that the multiplicity of η in (τ^k, Y^k) is at most d_η for every $k = 1, 2, \dots$*

Proof. Consider, for $k \geq 2$, the short exact sequence of $(\mathfrak{m}_{1F}, K_F)$-modules

$$0 \to \bar{\mathfrak{n}}_F^{k-1} \mathscr{U}(\bar{\mathfrak{n}}_F) / \bar{\mathfrak{n}}_F^k \mathscr{U}(\bar{\mathfrak{n}}_F) \otimes \mathscr{Y}_I / \bar{\mathfrak{n}}_F \mathscr{Y}_I \to \mathscr{Y}_I / \bar{\mathfrak{n}}_F^k \mathscr{Y}_I \to \mathscr{Y}_I / \bar{\mathfrak{n}}_F^{k-1} \mathscr{Y}_I \to 0 \tag{5}$$

and apply induction to infer the first two claims. Then, by admissibility,

$$\dim_{\mathbf{C}}(Y^k) \leq \dim_{\mathbf{C}}(\mathscr{Y}_I / \bar{\mathfrak{n}}_F^k \mathscr{Y}_I)^{K_F} < \infty .$$

Since the canonical map $Y^k \to Y^{k-1}$ is onto, the multiplicity of η in (τ^k, Y^k) increases with k. However, it follows from the exactness of the sequence (5) that the set of weights of the kernel of $Y^k \to Y^{k-1}$ is contained in

$$\left\{ \lambda - \sum_{j=1}^{k} \alpha_j | \mathfrak{a}_F ; \quad \lambda \in S_F(I), \alpha_1, \dots, \alpha_k \in \Sigma^+ \backslash \mathbf{N}F \right\},$$

hence does not contain η for k sufficiently large. From this the final claim follows. □

In order to generalize the theory of [1] to groups of Harish-Chandra's class, we need to define a distance function $\|\cdot\|$ on G, as in [1], Sect. 2. We equip $\mathfrak{g}$ with a $\mathrm{Ad}(K)$-invariant inner product $\langle . , . \rangle$ for which $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal, and denote the associated norm by $|\cdot|$. Let ${}^0G$ be defined as in [15], II, p. 20, and put

$$\mathfrak{a}_\Sigma = \{H \in \mathfrak{a}; \quad \alpha(H) = 0 \quad \text{for all} \quad \alpha \in \Sigma\} .$$

Then $G \simeq {}^0G \times \exp(\mathfrak{a}_\Sigma)$. We now define the function $\|\cdot\|: G \to \mathbf{R}_+$ by

$$\|x \exp H\| = \|\mathrm{Ad}(x)\|_{op}\, e^{|H|},$$

for $x \in {}^0G$ and $H \in \mathfrak{a}_\Sigma$. Here $\|\cdot\|_{op}$ denotes the operator norm. One readily checks that Lemma 2.1 in [1] is valid in the present situation as well.

For any function $f: G \to \mathbf{C}$ and $r \in \mathbf{R}$ we define

$$\|f\|_r = \sup_{x \in G} \|x\|^{-r} |f(x)|.$$

A function $f: G \to \mathbf{R}$ is said to increase at most exponentially if there exists $r \in \mathbf{R}$ such that $\|f\|_r < \infty$. The Banach space of continuous functions $f: G \to \mathbf{R}$ satisfying $\|f\|_r < \infty$ is denoted by $C_r(G)$. It is invariant under both the left regular action L and the right regular action R of G.

As in [1] we denote the Banach space of C^q-vectors, respectively the Fréchet space of C^∞-vectors, for L on $C_r(G)$ by $C_r^q(G)$, respectively $C_r^\infty(G)$. The norm on $C_r^q(G)$ is denoted $\|\cdot\|_{q,r}$. It is straightforward to check that the estimates (2.2–7) of [1] go through without change. Moreover, the crucial property that matrix coefficients of finite dimensional representations are of at most exponential growth (cf. [1], Example 2.2 (ii)), also holds for groups of Harish-Chandra's class.

Let $\mathscr{E}_I(G/K)$ denote the Fréchet space of right K-invariant smooth functions $f: G \to \mathbf{C}$ annihilated by the cofinite ideal I of $\mathbf{D}(G/K)$ (we view this space as a generalized joint eigenspace for $\mathbf{D}(G/K)$; when I is the ideal I_λ defined in [1], p. 119 ($\lambda \in \mathfrak{a}_c^*$), $\mathscr{E}_I(G/K)$ is the eigenspace $\mathscr{E}_\lambda(G/K)$ of [1]). For $r \in \mathbf{R}$, and $q \in \mathbf{N}$, we put

$$\mathscr{E}_{I,r}^q(G/K) = C_r^q(G) \cap \mathscr{E}_I(G/K).$$

Since I contains an elliptic differential operator, this intersection is a closed subspace of $C_r^q(G)$, hence a Banach space. Moreover, the space

$$\mathscr{E}_{I,r}^\infty(G/K) = C_r^\infty(G) \cap \mathscr{E}_I(G/K)$$

is a closed subspace of $C_r^\infty(G)$, hence Fréchet.

Write

$$\mathfrak{a}_F^+ = \{X \in \mathfrak{a}_F;\quad \alpha(X) > 0 \text{ for } \alpha \in \Delta \backslash F\},$$

and fix $H_0 \in \mathfrak{a}_F^+$ and $r \in \mathbf{R}$. The following is similar to [1], Proposition 6.1, but "along the walls".

Proposition 1.3 *There exist, for each* $N \in \mathbf{R}$,
(a) *an open neighbourhood* U *of* H_0 *in* $\mathfrak{a}_F^+$,
(b) *constants* $k, q \in \mathbf{N}$, $r' \geq r$ *and* $C, \varepsilon > 0$,
(c) *a continuous map* $\Psi: U \to B(C_r^q(G), Y^k \otimes C_{r'}(G))$, *and*
(d) *a linear form* $\eta \in (Y^k)^*$,
such that

(1) $\Psi(H)$ *intertwines the left actions of* G *on* C_r^q *and* $C_{r'}$, *for all* $H \in U$, *and*
(2) $\|R_{\exp tH} f - (\eta \circ \exp \tau^k(tH) \otimes 1)[\Psi(H)f]\|_{r'} \leq C \|f\|_{q,r} e^{(N-\varepsilon)t}$ *for all* $f \in \mathscr{E}_{I,r}^q(G/K)$, $H \in U$ *and* $t \geq 0$.

If $f \in \mathscr{E}_I(G/K)$ then the map $\mathscr{U}(\mathfrak{g}) \to C^\infty(G)$, $y \mapsto R_y f$ factorizes to a map $\mathscr{Y}_I \to C^\infty(G)$. For the proof of Proposition 1.3 it will be convenient to specify a representative in $\mathscr{U}(\mathfrak{g})$ for each element of $\mathscr{Y}_I = \mathscr{U}(\mathfrak{g})/I$ as follows:

Recall from [1], Proposition 5.1, that the natural linear map

$$\Gamma: \mathscr{U}(\bar{\mathfrak{n}}) \otimes E \otimes \mathbf{D}(G/K) \to \mathscr{U}(\mathfrak{g})/\mathscr{U}(\mathfrak{g})\mathfrak{k}$$

is an isomorphism of $\mathscr{U}(\bar{\mathfrak{n}})$-modules. Here E is a finite dimensional linear subspace of $\mathscr{U}(\mathfrak{a})$. The map Γ factorizes to an isomorphism of $\mathscr{U}(\bar{\mathfrak{n}})$-modules

$$\bar{\Gamma}: \mathscr{U}(\bar{\mathfrak{n}}) \otimes E \otimes \mathbf{D}(G/K)/I \to \mathscr{Y}_I .$$

In particular we see that $\mathscr{Y}_I$ is a free $\mathscr{U}(\bar{\mathfrak{n}})$-module of finite rank. Let $e_1, \ldots, e_p$ be a linear basis for E, and select finitely many elements $u_1, \ldots, u_q$ of $\mathscr{U}(\mathfrak{g})^K$ whose canonical images $\bar{u}_1, \ldots, \bar{u}_q$ constitute a linear basis for $\mathbf{D}(G/K)/I$. By the Poincaré-Birkhoff-Witt theorem, there exist $v_l \in \mathscr{U}(\bar{\mathfrak{n}}_F \oplus \mathfrak{m}_{1F})$ such that $v_l = u_l \bmod \mathscr{U}(\mathfrak{g})\mathfrak{k}$, for $1 \le l \le q$. Let

$$j: \mathscr{Y}_I \to \mathscr{U}(\bar{\mathfrak{n}}_F \oplus \mathfrak{m}_{1F})$$

be the homomorphism of $\mathscr{U}(\bar{\mathfrak{n}})$-modules defined by

$$j \circ \bar{\Gamma}(1 \otimes e_k \otimes \bar{u}_l) = e_k v_l ,$$

for all k, l. Then j followed by the canonical projection $\mathscr{U}(\mathfrak{g}) \to \mathscr{Y}_I$ yields the identity of $\mathscr{Y}_I$, that is, $j(y)$ is a representative in $\mathscr{U}(\mathfrak{g})$ of y.

If $f \in \mathscr{E}_I(G/K)$, then $yf = j(y)f$ for $y \in \mathscr{Y}_I$. We now define $R_y f$, or yf for short, for *any* $f \in C^\infty(G)$ by:

$$yf = j(y)f \quad (y \in \mathscr{Y}_I) .$$

After the above we can formulate the analogue of [1], Lemma 6.2. Define $\beta_F: \mathfrak{a}_F \to \mathbf{R}$ by:

$$\beta_F(H) = \min_{\alpha \in \Delta \setminus F} \alpha(H) .$$

Lemma 1.4. *Let $k \in \mathbf{N}$, and put*

$$\gamma(H) = |r| c_2 |H| - k\beta_F(H) ,$$

for $H \in \mathfrak{a}_F$, where c_2 is the constant of [1], Lemma 2.1 (iv).

For each $y \in \bar{\mathfrak{n}}_F^k \mathscr{Y}_I$ there exist constants $q \in \mathbf{N}$, $r' \ge r$, and $C > 0$ such that for all $H \in \mathfrak{a}_F^+$ we have:

$$\| R_{\exp H} R_y f \|_{r'} \le C \| f \|_{q,r} e^{\gamma(H)}$$

for $f \in C_r^q(G)$.

Proof. Using that $yf = j(y)f$ and $j(y) \in \bar{\mathfrak{n}}_F^k U(\bar{\mathfrak{n}}_F) U(\mathfrak{m}_{1F})$, proceed as in [1], Lemma 6.2. □

Proof of Proposition 1.3. Given $N \in \mathbf{R}$, select $k \in \mathbf{N}$ such that $\gamma(H_0) < N$, with γ as in Lemma 1.4. Fix elements $x_1, \ldots, x_p$ of $\mathscr{Y}_I$ with $x_1 = 1 + J$, such that their canonical images in Y^k constitute a basis of Y^k over $\mathbf{C}$. For $H \in \mathfrak{a}_F$, let $B(H)$ be the matrix of $\tau^k(H)$ with respect to this basis. Then

$$y_i(H) = Hx_i - \sum_{j=1}^{p} B(H)_{ji} x_j \in \bar{\mathfrak{n}}_F^k \mathscr{Y}_I ,$$

for $1 \leq i \leq p$, and $y_i(H)$ depends linearly on H. Hence the $y_i(H)$, $H \in \mathfrak{a}_F$ span a finite dimensional subspace of $\bar{\mathfrak{n}}_F^k \mathscr{Y}_I$. Now for $H \in \mathfrak{a}_F^+$, $t \geq 0$, define bounded linear maps from $C_r^q(G)$ into $C_{r'}(G)$ as in [1], p. 126 by

$$F(H, t)_i = R_{\exp tH} R_{x_i} ,$$

$$G(H, T)_i = R_{\exp tH} R_{y_i(H)} ,$$

for $1 \leq i \leq p$.

If $f \in \mathscr{E}_{I,r}^q(G/K)$ then as in [1], p. 127, one has the $C_{r'}(G)$-valued differential equation:

$$\frac{d}{dt} F(H, t) f = [B(H) F(H, t) + G(H, t)] f .$$

The proof is now completed by the same arguments as in [1]. □

Let

$$\mathscr{E}_I^\infty(G/K) = \bigcup_{r \in \mathbf{R}} \mathscr{E}_{I,r}^\infty(G/K) .$$

Then we have the following consequence of Proposition 1.3. It generalizes [1], Theorem 3.5.

Theorem 1.5. (i) *For each $f \in \mathscr{E}_I^\infty(G/K)$, $x \in G$ and $\eta \in X(F, I)$, there exists a unique continuous function $p_{F,\eta}(f, x)$ on $\mathfrak{a}_F^+$ which is radially polynomial of degree at most d_η such that*

$$f(x \exp tH) \sim \sum_{\eta} p_{F,\eta}(f, x, tH) e^{t\eta(H)} \quad (t \to \infty) \tag{6}$$

at every $H \in \mathfrak{a}_F^+$.

(ii) *Let $r \in \mathbf{R}$ and $\eta \in X(F, I)$. Then for $H_0 \in \mathfrak{a}_F^+$ there exist an open neighbourhood U of H_0 in $\mathfrak{a}_F^+$ and a constant $r' \in \mathbf{R}$ such that $(f, H) \mapsto p_{F,\eta}(f, \cdot, H)$ is a continuous map from $\mathscr{E}_{I,r}^\infty(G/K) \times U$ into $C_{r'}^\infty(G)$, which is linear and G-equivariant in f.*

Remarks. (a) That $p_{F,\eta}(f, x)$ is radially polynomial of degree at most d_η means that for each $H \in \mathfrak{a}_F^+$, the map $t \mapsto p_{F,\eta}(f, x, tH)$ $(t \in \mathbf{R}_+)$ extends to a polynomial of degree $\leq d_\eta$ on $\mathbf{R}$. Here d_η refers to the constant given in Lemma 1.2. In Corollary 3.2 we shall see that $p_{F,\eta}(f, x)$ actually extends to a polynomial on $\mathfrak{a}_F$.

(b) For the meaning of $\sim$ in (6) we refer to [1], Sect. 3. The definition given there is easily generalized to the present situation.

(c) By the definition of the topology of the spaces $C_r^\infty(G)$, the continuity of the map $(f, H) \mapsto p_{F,\eta}(f, \cdot, H)$ in (ii) amounts to the following: For each p there exists a q such that the map has a continuous extension

$$\mathscr{E}_{I,r}^q(G/K) \times U \to C_{r'}^p(G) . \tag{7}$$

Proof of Theorem 1.5. This follows from Proposition 1.3. The details present no difficulties which go beyond those of [1]. The bound on the polynomial degree of $t \mapsto p_{F,\eta}(f, x, tH)$ comes from (2) in Proposition 1.3 and Lemma 1.2. □

2. Properties of the coefficients

The purpose of this section is to study properties of the coefficients $p_{F,\eta}$ in the asymptotic expansion (6) of Theorem 1.5. We first derive differential equations for the $p_{F,\eta}(f, x, H)$ as functions of x.

For $m \in K_F$, we have $f(xm \exp tH) = f(x \exp tH)$, hence by uniqueness of asymptotics we see that

$$p_{F,\eta}(f, xm, H) = p_{F,\eta}(f, x, H) \quad (m \in K_F) .$$

Consequently the right action of $\mathscr{U}(\mathfrak{m}_{1F})^{K_F}$ on $p_{F,\eta}(f, ., H)$ induces an action of $\mathbf{D}(M_{1F}/K_F)$ on $p_{F,\eta}(f, ., H)$.

If $D \in \mathbf{D}(G/K)$, then D may be represented by some $u \in \mathscr{U}(\mathfrak{g})^K$ (determined modulo $\mathscr{U}(\mathfrak{g})\mathfrak{k}$). By the Poincaré-Birkhoff-Witt theorem there exist $u_F \in \mathscr{U}(\mathfrak{m}_{1F})^{K_F}$ and $w \in \bar{\mathfrak{n}}_F \mathscr{U}(\bar{\mathfrak{n}} \oplus \mathfrak{a})$ such that

$$u \in u_F + w + \mathscr{U}(\mathfrak{g})\mathfrak{k} .$$

The image of u_F in $\mathbf{D}(M_{1F}/K_F)$ only depends on D and is denoted by $\delta_F(D)$. Moreover, w only depends on D and can be written as a finite sum $w = \sum_i w_i$, with $w_i \in \mathscr{U}(\bar{\mathfrak{n}} \oplus \mathfrak{a})$ such that $\mathrm{ad}(\mathfrak{a}_F)$ acts on w_i by a non-zero weight $-\mu_i$, with $\mu_i \in \mathbf{N}\Delta|_{\mathfrak{a}_F}$.

In the following, it will be convenient to adopt the convention that $p_{F,\eta} = 0$ if $\eta \notin X(F, I)$.

Proposition 2.1. *Let $D \in I$. Then*

$$\delta_F(D) p_{F,\eta}(f, ., H) = -\sum_i w_i p_{F,\eta+\mu_i}(f, ., H) , \tag{8}$$

for all $f \in \mathscr{E}_I^\infty(G/K)$, $\eta \in X(F, I)$, $H \in \mathfrak{a}_F^+$.

Proof. By equivariance it suffices to establish this identity of functions on G at the identity element e. Let u_F represent $\delta_F(D)$ as above. Then u_F commutes with $\mathfrak{a}_F$, hence

$$u_F f(\exp tH) = L(\check{u}_F) f(\exp tH) ,$$

where L denotes the infinitesimal left regular representation, and $u \mapsto \check{u}$ the canonical anti-automorphism of $\mathscr{U}(\mathfrak{g})$. By left equivariance of the map $f \mapsto p_{F,\eta}(f, ., tH)$ we now obtain:

$$u_F f(\exp tH) \sim \sum_\eta [\delta_F(D) p_{F,\eta}(f, ., tH)](e) e^{\eta(tH)} .$$

Similarly:

$$w_i f(\exp tH) \sim \sum_{\eta} [w_i p_{F,\eta}(f, \cdot, tH)](e) e^{(\eta - \mu_i)(tH)}$$

and the identity follows by uniqueness of asymptotics. □

The following two lemmas are well known.

Lemma 2.2. *Let I be a cofinite ideal in $\mathbf{D}(G/K)$. Then $\delta_F(I)$ generates a cofinite ideal in $\mathbf{D}(M_{1F}/K_F)$.*

Proof. See [4] Lemma 15. □

Notice, that if W is a $(\mathfrak{g}, K)$-module, then there is a natural action of $\mathbf{D}(G/K)$ on the set W^K of K-fixed vectors in W, defined via the canonical map $\mathscr{U}(\mathfrak{g})^K \to \mathbf{D}(G/K)$.

Lemma 2.3. *Let I be a cofinite ideal in $\mathbf{D}(G/K)$, and let F be a finite dimensional $\mathfrak{g}$-module. There exists a cofinite ideal I' in $\mathbf{D}(G/K)$ with the following property:*

Let W be any $(\mathfrak{g}, K)$-module which is generated by W^K, and assume that W^K is annihilated by I. Then I' annihilates $(F \otimes W)^K$.

Proof. This result follows from [8], Theorem 5.1, since the canonical image of $\mathscr{Z}(\mathfrak{g})$ is cofinite in $\mathbf{D}(G/K)$. For completeness we give an independent proof.

Let I_λ be the ideal of codimension 1 defined in [1], p. 119. It is easily seen that we can assume $I = I_\lambda$ for some $\lambda \in \mathfrak{a}_c^*$ (if $I = (I_{\lambda_1})^{m_1} \cap \ldots \cap (I_{\lambda_n})^{m_n}$, then by using a suitable $\mathbf{D}(G/K)$-stable filtration of W^K it is seen that we can take $I' = (I'_{\lambda_1})^{m_1} \cap \ldots \cap (I'_{\lambda_n})^{m_n}$). Moreover, we may assume that $\dim W^K = 1$, and that the real part of λ is dominant. By a theorem of Kostant ([7]), W is equivalent to a quotient of the space $C(G/P, L_{-\lambda})_K$ of K-finite vectors in the spherical principal series $C(G/P, L_{-\lambda})$ induced from $1 \otimes e^\lambda$ (cf. [1], Remark 5.1). Hence we may assume that $W = C(G/P, L_{-\lambda})_K$. It is now easily seen that $F \otimes W$ has a filtration, of length at most $\dim F$, in which each subquotient is equivalent to a principal series representation induced from $\delta \otimes e^{\lambda+\nu}$ where $\delta \otimes e^\nu$ occurs in the MA-decomposition of F. Hence the intersection I^0 of the ideals $I_{\lambda+\nu}$, where each ν is an $\mathfrak{a}$-weight of F^M, annihilates the K-fixed vectors in each of these subquotients. Let $I' = (I^0)^{\dim F}$. □

Corollary 2.4. *Let $\eta \in X(F, I)$. There exists a cofinite ideal I_η in the algebra $\mathbf{D}(M_{1F}/K_F)$, which annihilates $p_{F,\eta}(f, \cdot, H)$ for all f and H.*

Proof. Consider the partial ordering $\prec_F$ on $\mathfrak{a}_F$ defined by

$$\eta_1 \prec_F \eta_2 \Leftrightarrow \eta_2 - \eta_1 \in \mathbf{N}\Delta|_{\mathfrak{a}_F} \setminus \{0\}\,. \tag{9}$$

We prove the corollary by downward induction along $\prec_F$. Fix $\eta \in X(F, I)$ and $D \in I$, and assume the result holds for all elements in $X(F, I)$ greater than η. Then we claim that the right hand side of equation (8) is annihilated by a cofinite ideal $I' \subset \mathbf{D}(M_{1F}/K_F)$. To see this, notice that the w_i generate a finite dimensional $\mathrm{Ad}(M_{1F})$-invariant subspace F of $\mathscr{U}(\mathfrak{g})$, and that the $p_{F,\eta+\mu_i}(f, \cdot, H)$, which by the induction hypothesis are annihilated by a cofinite ideal, generate a $(\mathfrak{m}_{1F}, K_F)$-module $W \subset C^\infty(G)$. Then the right hand side of (8) lies in the image of $(F \otimes W)^K$

under the morphism of $(\mathfrak{m}_{1F}, K_F)$-modules $F \otimes W \to C^\infty(G)$ given by $u \otimes \varphi \to R_u \varphi$, and the claim follows from the previous lemma. Moreover, I' depends only on η and D, but since I is finitely generated, we can actually choose I' independently of D. Let I_η be the product of the cofinite ideals I' and $\delta_F(I)$, then it follows from (8) that I_η annihilates $p_{F,\eta}(f, \cdot, H)$. □

Corollary 2.5. *For every $f \in \mathscr{E}_I^\infty(G/K)$, $\eta \in X(F, I)$, $H \in \mathfrak{a}_F^+$ and $x \in G$, the function $m \mapsto p_{F,\eta}(f, xm, H)$ is real analytic on M_{1F}.*

Proof. In view of Corollary 2.4 the above function is annihilated by the cofinite ideal I_η in $\mathbf{D}(M_{1F}/K_F)$, which contains an elliptic differential operator with real analytic coefficients. Now apply the elliptic regularity theorem. □

We shall now see that this real analytic function on M_{1F} has at most exponential growth. For this purpose, we need some lemmas. Let $\|\cdot\|_F$ denote the distance function on M_{1F} as defined in the previous section.

Lemma 2.6. *There exists a constant $\sigma \geq 1$, such that*

$$\|m\|_F^{1/\sigma} \leq \|m\| \leq \|m\|_F^\sigma, \quad m \in M_{1F} .$$

Proof. In view of [1], Lemma 2.1 (iii), it suffices to prove the inequalities for $m = \exp Y$, with $Y \in \mathfrak{m}_{1F} \cap \mathfrak{p}$. Both inequalities then follow from [1], Lemma 2.1 (iv). □

Lemma 2.7. *Fix constants $r > 0$ and $p \in \mathbf{N}$, and let $s = \sigma r$ (where σ is given in Lemma 2.6). There exist constants $\tilde{s} \in \mathbf{R}$ and $C > 0$ such that for all $\varphi \in C_r^p(G)$ and $x \in G$, the function $m \mapsto \varphi(xm)$ belongs to $C_s^p(M_{1F})$ with the following bound on the norm*

$$\|\varphi(x\cdot)|_{M_{1F}}\|_{p,s} \leq C\|x\|^{\tilde{s}}\|\varphi\|_{p,r}$$

Proof. It suffices to prove the bound on the norm. By [1], equation (2.4) we may assume $x = e$. Fix a basis $Y_1, \ldots, Y_k$ for $\mathfrak{m}_{1F}$, and write

$$Y^\gamma = Y_1^{\gamma_1} \ldots Y_k^{\gamma_k}$$

for $\gamma = (\gamma_1, \ldots, \gamma_k) \in \mathbf{N}^k$. Then by the definition of $\|\cdot\|_{p,s}$ on M_{1F}

$$\|\varphi|_{M_{1F}}\|_{p,s} = \max_{|\gamma| \leq p} \sup_{m \in M_{1F}} \|m\|_F^{-s} |L(Y^\gamma)\varphi(m)| ,$$

and hence by the definition of $\|\cdot\|_{p,r}$ on G

$$\|\varphi|_{M_{1F}}\|_{p,s} \leq C \sup_{m \in M_{1F}} \|m\|_F^{-s}\|m\|^r\|\varphi\|_{p,r} .$$

Now the result follows from Lemma 2.6. □

Fix $\eta \in X(F, I)$, a relatively compact open subset U of $\mathfrak{a}_F^+$ and a constant $r > 0$. Then by Theorem 1.5 (see also (7)) there exists a constant $r' > 0$, and for each $p \in \mathbf{N}$ a number $q \in \mathbf{N}$ and a constant $C > 0$ such that $p_{F,\eta}(f, \cdot, H)$ belongs to $C_{r'}^p(G)$ for all $H \in U$ and $f \in \mathscr{E}_{I,r}^q(G/K)$, with the following bound on the norm:

$$\|p_{F,\eta}(f, \cdot, H)\|_{p,r'} \leq C\|f\|_{q,r} .$$

Combining this with Lemma 2.7, we obtain the following result:

Proposition 2.8. *Fix η, U and r, and let $s = \sigma r$ as above. For each $p \in \mathbf{N}$ there exist $q \in \mathbf{N}$, $\tilde{s} \in \mathbf{R}$ and $C > 0$ such that the function $m \mapsto p_{F,\eta}(f, xm, H)$ belongs to $C^p_s(M_{1F}/K_F)$ for all $H \in U$, $x \in G$ and $f \in \mathscr{E}^q_{I,r}(G/K)$, with the following bound on the norm:*

$$\| p_{F,\eta}(f, x\cdot, H)|_{M_{1F}} \|_{p,s} \leq C \|x\|^{\tilde{s}} \|f\|_{q,r} .$$

In conclusion, we have:

Theorem 2.9. *Fix $\eta \in X(F, I)$. There exists a cofinite ideal I_η in the algebra $\mathbf{D}(M_{1F}/K_F)$ such that the function $m \mapsto p_{F,\eta}(f, xm, H)$ belongs to $\mathscr{E}^\infty_{I_\eta}(M_{1F}/K_F)$ for all $f \in \mathscr{E}^\infty_I(G/K)$, $x \in G$ and $H \in \mathfrak{a}_F^+$. Moreover, for each $r \in \mathbf{R}$ and $U \subset \mathfrak{a}_F^+$ relatively compact there exists $s \in \mathbf{R}$ such that $(f, H) \mapsto p_{F,\eta}(f, x\cdot, H)|_{M_{1F}}$ maps $\mathscr{E}^\infty_{I,r}(G/K) \times U$ continuously into $\mathscr{E}^\infty_{I_\eta,s}(M_{1F}/K_F)$ for all $x \in G$.*

3. Expansions for the coefficients

Consider a function $f \in \mathscr{E}^\infty_I(G/K)$. From Theorem 1.5 with $F = \emptyset$ we know that f has an asymptotic expansion along $\mathfrak{a}^+$:

$$f(x \exp tH) \sim \sum_{\xi \in X(I)} p_\xi(f, x, tH) e^{t\xi(H)} \quad (t \to \infty) , \tag{10}$$

for $H \in \mathfrak{a}^+$, $x \in G$ (when the special choice of the empty set for F is made, we suppress the symbol $\emptyset$ in our notations, thus $X(I) = X(\emptyset, I) = S - \mathbf{N}\Delta$ where $S = S_\emptyset(I)$).

On the other hand, for any F we have the following expansion along $\mathfrak{a}_F^+$:

$$f(x \exp tH_0) \sim \sum_{\eta \in X(F,I)} p_{F,\eta}(f, x, tH_0) e^{t\eta(H_0)} \quad (t \to \infty) , \tag{11}$$

for $H_0 \in \mathfrak{a}_F^+$, $x \in G$.

Let $\Sigma_F = \Sigma \cap \mathbf{Z}F$ be the restricted root system of $\mathfrak{a}$ in $\mathfrak{m}_{1F}$. The positive system $\Sigma_F^+ = \Sigma_F \cap \Sigma^+$ determines a positive Weyl chamber in $\mathfrak{a}$:

$$\mathfrak{a}^+(F) = \{ X \in \mathfrak{a}; \quad \alpha(X) > 0 \quad \text{for} \quad \alpha \in F \} .$$

Now let $\eta \in X(F, I)$, $H_0 \in \mathfrak{a}_F^+$, and $x \in G$. Then by Theorem 2.9 the function $m \mapsto p_{F,\eta}(f, xm, H_0)$ on M_{1F} is annihilated by a cofinite ideal I_η in $\mathbf{D}(M_{1F}/K_F)$, and hence by Theorem 1.5 it has asymptotic expansions along rays in $\mathfrak{a}^+(F)$. The following theorem determines the coefficients of these expansions in terms of the coefficients p_ξ of (10). Put

$$X(I, \eta) = \{ \xi \in X(I); \xi|_{\mathfrak{a}_F} = \eta \} .$$

Theorem 3.1. *Let $f \in \mathscr{E}^\infty_I(G/K)$, $x \in G$, and $\eta \in X(F, I)$.*

(α) *For every $H_0 \in \mathfrak{a}_F^+$ and $H_1 \in \mathfrak{a}^+(F)$ the following asymptotic expansion holds:*

$$p_{F,\eta}(f, x \exp tH_1, H_0) \sim \sum_{\xi \in X(I,\eta)} p_\xi(f, x, H_0 + tH_1) e^{t\xi(H_1)} \quad (t \to \infty) . \tag{12}$$

(β) *For all* $\xi \in X(I)$ *with* $\xi|_{\mathfrak{a}_F} \notin X(F, I)$ *we have* $p_\xi(f, x) = 0$.

Notice that if the set $X(I, \eta)$ contains some element ξ, then it contains the set $\xi - \mathbf{N}F$, so that (12) is actually an infinite expansion. On the other hand if $X(I, \eta)$ is empty, we define the right hand side of (12) to be the trivial expansion with all coefficients vanishing.

Proof. Since both $p_{F,\eta}$ and p_ξ are equivariant for the left G-action we may assume that $x = e$.

By Theorem 1.5 there exists a finite set $T(\eta) = S(I_\eta) \subset \mathfrak{a}_c^*$, and for each $\varphi \in \mathscr{E}_{I_\eta}^\infty(M_{1F}/K_F)$, $m \in M_{1F}$ and $\zeta \in T(\eta) - \mathbf{N}F$ a continuous function $q_\zeta(\varphi, m)$ on $\mathfrak{a}^+(F)$ such that we have the asymptotic expansion

$$\varphi(m \exp tH_1) \sim \sum_\zeta q_\zeta(\varphi, m, tH_1) e^{t\zeta(H_1)} \quad (t \to \infty) .$$

We apply this result to the function $\varphi = p_{F,\eta}(f, \cdot, H_0)|_{M_{1F}}$ at $m = e$, and denote the coefficients $q_\zeta(\varphi, e, tH_1)$ by $q_{F,\eta,\zeta}(f, H_0, tH_1)$:

$$p_{F,\eta}(f, \exp tH_1, H_0) \sim \sum_{\zeta \in T(\eta) - \mathbf{N}F} q_{F,\eta,\zeta}(f, H_0, tH_1) e^{t\zeta(H_1)} \quad (t \to \infty) . \tag{13}$$

To establish part (α) of the theorem, it suffices to prove the following:

(i) *If* $\xi \in X(I, \eta) \backslash (T(\eta) - \mathbf{N}F)$, *then* $p_\xi(f, e) = 0$.
(ii) *If* $\zeta \in (T(\eta) - \mathbf{N}F) \backslash X(I, \eta)$, *then* $q_{F,\eta,\zeta}(f) = 0$.
(iii) *If* $\zeta \in (T(\eta) - \mathbf{N}F) \cap X(I, \eta)$, *then*

$$q_{F,\eta,\zeta}(f, H_0, tH_1) = p_\zeta(f, e, H_0 + tH_1) ,$$

for all $H_0 \in \mathfrak{a}^+$, $H_1 \in \mathfrak{a}^+(F)$ *and* $t > 0$.

We will concentrate on proving (i)–(iii); part (β) of the theorem will be obtained along the way.

Since $p_{F,\eta}(f, \cdot, H_0)$ and $q_\zeta(\varphi, e, tH_1)$ depend continuously and linearly on f and φ, respectively, the coefficient $q_{F,\eta,\zeta}(f, H_0, tH_1)$ depends continuously and linearly on f. Similarly for every $\xi \in X(I)$, and $H \in \mathfrak{a}^+$, $p_\xi(f, e, H)$ depends continuously and linearly on f. Hence by density we may assume that f is K-finite. This assumption will allow us to apply the results of [3].

For $H \in \mathfrak{a}^+$ we define $z(H) \in \mathbf{C}^\Delta$ by

$$z(H)_\alpha = e^{-\alpha(H)} \quad (\alpha \in \Delta) .$$

Let D denote the unit disk in $\mathbf{C}$, centred at the origin. Then by [3] there exists a finite set $S' \subset \mathfrak{a}_c^*$ such that the canonical map $S' \to \mathfrak{a}_c^*/\mathbf{Z}\Delta$ is injective, and moreover an integer $d \geq 0$ and finitely many holomorphic functions $\Phi_{s,m}\colon D^\Delta \to \mathbf{C}$ such that:

$$f(\exp H) = \sum_{s \in S', |m| \leq d} H^m e^{s(H)} \Phi_{s,m}(z(H)) , \tag{14}$$

for $H \in \mathfrak{a}^+$. Here the summation involves $m \in \mathbf{N}^\Delta$ with $|m| = \sum_{\alpha \in \Delta} m_\alpha \leq d$, and we have used the notation $H^m = \prod_{\alpha \in \Delta} \alpha(H)^{m_\alpha}$. Being holomorphic, $\Phi_{s,m}(z)$ has a power series expansion

$$\Phi_{s,m}(z) = \sum_{\mu \in \mathbf{N}\Delta} c_{s-\mu,m} z^{\mu} \quad (z \in D^{\Delta}),$$

where $z^{\mu} = \prod_{\alpha\in\Delta}(z_{\alpha})^{\mu_{\alpha}}$. For convenience, let $c_{\xi,m} = 0$ whenever $\xi \notin S' - \mathbf{N}\Delta$.

By uniqueness of the coefficients in (10) we draw the following two conclusions:

(a) If $\xi \in X(I) = S - \mathbf{N}\Delta$ then

$$p_{\xi}(f, e, H) = \sum_{|m| \le d} c_{\xi,m} H^m . \tag{15}$$

(b) If $\xi \in (S' - \mathbf{N}\Delta)\backslash(S - \mathbf{N}\Delta)$, then $c_{\xi,m} = 0$ for all m.

Now fix $H_1 \in \mathfrak{a}^+(F)$ for the moment. If $R > 0$ we put

$$\mathfrak{a}_F^+(R) = \{H_0 \in \mathfrak{a}_F; \alpha(H_0) > R \text{ for all } \alpha \in \Delta\backslash F\} .$$

We may fix $R > 0$ such that

$$H_0 \in \mathfrak{a}_F^+(R) \Rightarrow H_1 + H_0 \in \mathfrak{a}^+ .$$

For $H_0 \in \mathfrak{a}_F^+(R)$ we write $H = H_1 + H_0$. Then in view of conclusion (b) above, the expansion (14) for f can be rewritten as follows:

$$\begin{aligned} f(\exp H) &= \sum_{\substack{\xi\in S-\mathbf{N}_{\Delta} \\ |m|\le d}} c_{\xi,m} H^m e^{\xi(H)} \\ &= \sum_{\substack{\xi\in S-\mathbf{N}(\Delta\backslash F) \\ |m|\le d}} \left[\sum_{\mu\in\mathbf{N}F} c_{\xi-\mu,m} e^{-\mu(H_1)} \right] H^m e^{\xi(H_1)} e^{\xi(H_0)} . \end{aligned} \tag{16}$$

Since the functions $\Phi_{s,m}$ are holomorphic on D^{Δ}, the series between brackets in (16) converges absolutely. Moreover, again by holomorphy we obtain an asymptotic expansion

$$f(\exp H_1 \exp tH_0) \sim \sum_{\substack{\xi\in S-\mathbf{N}(\Delta\backslash F) \\ |m|\le d}} \left[\sum_{\mu\in\mathbf{N}F} c_{\xi-\mu,m} e^{-\mu(H_1)} \right] (H_1 + tH_0)^m e^{\xi(H_1)} e^{t\xi(H_0)}$$

as $t \to \infty$.

Now put

$$X(\eta) = \{\xi \in S - \mathbf{N}(\Delta\backslash F); \xi|_{\mathfrak{a}_F} = \eta\} . \tag{17}$$

By uniqueness of the coefficients in (11) we infer the following.

(c) For all $H_0 \in \mathfrak{a}_F^+(R)$ one has

$$p_{F,\eta}(f, \exp H_1, H_0) = \sum_{\substack{\xi\in X(\eta) \\ |m|\le d}} \left[\sum_{\mu\in\mathbf{N}F} c_{\xi-\mu,m} e^{-\mu(H_1)} \right] (H_1 + H_0)^m e^{\xi(H_1)} . \tag{18}$$

(d) If $\xi \in X(I) = S - \mathbf{N}\Delta$ and $\xi|_{\mathfrak{a}_F} \notin X(F, I)$ then $c_{\xi,m} = 0$ for all m.

In view of (15) part (β) of the theorem now follows from (d).

Clearly (17) is a finite set, so that the summation in (18) over the ξ's is finite. Since $p_{F,\eta}(f, \exp H_1, H_0)$ is continuous and radially polynomial in the variable H_0

it follows that (18) holds for all $H_1 \in \mathfrak{a}^+(F)$ and all $H_0 \in \mathfrak{a}_F^+$. Moreover, we have an asymptotic expansion:

$$p_{F,\eta}(f, \exp tH_1, H_0) \sim \sum_{\substack{|m| \leq d \\ \zeta \in X(\eta) - \mathbf{N}F}} c_{\zeta,m}(H_0 + tH_1)^m e^{t\zeta(H_1)}, \tag{19}$$

as $t \to \infty$, for $H_0 \in \mathfrak{a}_F^+$, $H_1 \in \mathfrak{a}^+(F)$.

Finally uniqueness of asymptotics, this time in (13), allows us to conclude:

(1) If $\xi \in (X(\eta) - \mathbf{N}F) \setminus (T(\eta) - \mathbf{N}F)$ then $c_{\xi,m} = 0$ for all m. Hence $p_\xi = 0$ by (15).
(2) If $\zeta \in (T(\eta) - \mathbf{N}F) \setminus (X(\eta) - \mathbf{N}F)$, then $q_{F,\eta,\zeta}(f) = 0$.
(3) If $\zeta \in (T(\eta) - \mathbf{N}F) \cap (X(\eta) - \mathbf{N}F)$, then

$$q_{F,\eta,\zeta}(f, H_0, tH_1) = \sum_m c_{\zeta,m}(H_0 + tH_1)^m,$$

for $H_0 \in \mathfrak{a}_F^+$, $H_1 \in \mathfrak{a}^+(F)$. Hence $q_{F,\eta,\zeta}(f, H_0, tH_1) = p_\zeta(f, e, H_0 + tH_1)$ by (15). Since $X(\eta) - \mathbf{N}F = X(I, \eta)$, this finishes the proof of (i), (ii) and (iii). □

Corollary 3.2. *Let $f \in \mathscr{E}_I^\infty(G/K)$, $x \in G$ and $\eta \in X(F, I)$. The function $p_{F,\eta}(f, x)$ on $\mathfrak{a}_F^+$ extends to a polynomial on $\mathfrak{a}_F$ of degree at most d_η, where d_η is the constant of Lemma 1.2. Moreover, if $\eta \notin X(I)|_{\mathfrak{a}_F}$, this polynomial vanishes identically.*

Proof. By equivariance we may as well assume that $x = e$. For K-finite f we know already that $p_{F,\eta}(f, e)$ is polynomial (see (18)), but also that it is radially polynomial of degree at most d_η (cf. Theorem 1.5, (i)). Therefore $p_{F,\eta}(f, e)$ belongs to the finite dimensional linear space P of polynomial functions on $\mathfrak{a}_F$ of degree at most d_η. It follows by density that $p_{F,\eta}(f, e)$ belongs to P for f arbitrary.

If $\eta \notin X(I)|_{\mathfrak{a}_F}$, then the set (17) is empty so that (18) vanishes for every K-finite function f. Now we again apply density. □

4. Local boundary data

This section contains the main result about asymptotics. As in [1] Sect. 8, we define the set of exponents of $f \in \mathscr{E}_I^\infty(G/K)$ at $x \in G$ along xA^+ by

$$E(f, x) = \{\xi \in X(I);\quad x \in \operatorname{supp} p_\xi(f, .)\}. \tag{20}$$

Here we have regarded $p_\xi(f)$ as a function on G with values in a finite dimensional space of polynomials (cf. Corollary 3.2 with $F = \emptyset$).

Theorem 4.1. *There exists a finite set $R(I) \subset X(I)$ such that the following holds: Let $f \in \mathscr{E}_I^\infty(G/K)$. If $E(f, x) \cap R(I) = \emptyset$ for some $x \in G$, then $f = 0$. In particular, if $I = I_\lambda$ for some $\lambda \in \mathfrak{a}_c^*$, then we can take $R(I) = W\lambda - \rho$.*

Corollary 4.2. *Let $f \in \mathscr{E}_I^\infty(G/K)$ and assume that there exists an open nonempty set $U \subset G$ such that $p_\xi(f, x) = 0$ for all $\xi \in X(I)$ and $x \in U$. Then $f = 0$.*

The corollary is an immediate consequence of the theorem. Thus a function in $\mathscr{E}_I^\infty(G/K)$ is uniquely determined by its asymptotic coefficients on any fixed nontrivial open subset of G. This uniqueness result is of a similar nature as Proposition 2.15 in [12] and Theorem 4.4 in [10], the proofs of which are based on Holmgren's uniqueness principle for hyperfunctions as formulated in [14], Ch. III, Theorem 2.2.1.

The proof of Theorem 4.1 proceeds by induction on the rank of the root system Σ, but first we construct $R(I)$ and show that it suffices to prove the result for $I = I_\lambda$, that is, for simultaneous eigenfunctions of $\mathbf{D}(G/K)$.

Lemma 4.3. *For each $D \in \mathbf{D}(G/K)$ and $f \in \mathscr{E}_I^\infty(G/K)$ we have $Df \in \mathscr{E}_I^\infty(G/K)$. Moreover, there exists a finite set $T \subset \mathbf{N}\Delta$ such that*

$$E(Df, x) \subset \{\xi - \mu;\quad \xi \in E(f, x), \mu \in T\} \tag{21}$$

for all $D \in \mathbf{D}(G/K)$, $f \in \mathscr{E}_I^\infty(G/K)$ and $x \in G$.

Proof. Let $f \in \mathscr{E}_{I,s}^\infty(G/K)$. Since $\mathbf{D}(G/K)f$ is of finite dimension, it is contained in $\mathscr{E}_{I,s'}^\infty(G/K)$ for some $s' \geq s$ by [1], formula (2.7). Let $g = Df$ where $D \in \mathbf{D}(G/K)$. We can represent D by a finite sum $\sum_{j=1}^k v_j$ in $\mathscr{U}(\mathfrak{g})$, where each $v_j \in \mathscr{U}(\bar{\mathfrak{n}} \oplus \mathfrak{a})$ is a weight vector for $\mathrm{ad}(\mathfrak{a})$ with weight $-\mu_j \in -\mathbf{N}\Delta$. It follows that

$$\begin{aligned} g(\exp tH) &= \sum_{j=1}^k e^{-t\mu_j(H)}[L(\check{v}_j)f](\exp tH) \\ &\sim \sum_{j=1}^k \sum_{\xi \in X(I)} p_\xi([L(\check{v}_j)f], e, tH)e^{t(\xi - \mu_j)(H)} \quad (t \to \infty) \end{aligned}$$

and since $p_\xi(L(\check{v}_j)f, e) = L(\check{v}_j)[p_\xi(f)](e) = [R(v_j)p_\xi(f)](e)$ we get by uniqueness of asymptotics that $p_\xi(g, e) = \sum_j [R(v_j)p_{\xi+\mu_j}(f)](e)$. By left equivariance we now infer that

$$p_\xi(g, x) = \sum_{j=1}^k [R(v_j)p_{\xi+\mu_j}(f)](x) \quad (x \in G),$$

whence

$$E(g, e) \subset \{\xi - \mu_j;\quad \xi \in E(f, e),\quad j = 1, \ldots, k\}.$$

Since I is cofinite, it suffices to consider finitely many D's and the lemma follows. □

Since I is cofinite, $\mathbf{D}(G/K)/I$ is a finite dimensional module for the algebra $\mathbf{D}(G/K)$. Let $\Upsilon(I)$ be the finite set of all homomorphisms $\mathbf{D}(G/K) \to \mathbf{C}$ to which there corresponds a simultaneous eigenvector in $\mathbf{D}(G/K)/I$. Via Harish-Chandra's isomorphism, $\Upsilon(I)$ corresponds to a W-invariant subset $\Lambda(I)$ of $\mathfrak{a}_c^*$. Let $T \subset \mathbf{N}\Delta$ be as described in Lemma 4.3 and let

$$R(I) = \{\lambda - \rho + \mu; \lambda \in \Lambda(I), \mu \in T\}.$$

Notice that if $I = I_\lambda$, then $\Lambda(I) = W\lambda$, and we can take $T = \varnothing$ in Lemma 4.3. Hence $R(I) = W\lambda - \rho$.

The reduction to eigenfunctions will now be a consequence of the following lemmas.

Lemma 4.4. *Let $f \in \mathscr{E}_I^\infty(G/K)$, $x \in G$ and suppose that $E(f, x) \cap R(I) = \emptyset$. If $g \in \mathbf{D}(G/K) f \cap \mathscr{E}_\lambda^\infty(G/K)$, $\lambda \in \mathfrak{a}_c^*$, then there exists an open neighbourhood U of x such that $E(g, y) = \emptyset$ for all $y \in U$.*

Proof. If $g = 0$ there is nothing to prove. Assume $g \neq 0$. Then $\lambda \in \Lambda(I)$, and by definition of $R(I)$ we have $W\lambda - \rho + T \subset R(I)$. Hence $E(f, x) \cap (W\lambda - \rho + T) = \emptyset$, and it follows from Lemma 4.3 that $E(g, x) \cap (W\lambda - \rho) = \emptyset$.

Since $W\lambda - \rho$ is finite, $E(g, y) \cap (W\lambda - \rho) = \emptyset$ for all y in a neighbourhood U of x, and the result now follows from [1], Corollary 8.2. □

Lemma 4.5. *Let $f \in \mathscr{E}_I^\infty(G/K)$. If $\mathbf{D}(G/K) f \cap \mathscr{E}_\lambda^\infty(G/K) = 0$ for every $\lambda \in \mathfrak{a}_c^*$, then $f = 0$.*

Proof. If $f \neq 0$, then the finitely generated commutative algebra $\mathbf{D}(G/K)$ has a non-trivial joint eigenvector in the finite dimensional space $\mathbf{D}(G/K) f$. □

As a corollary we can now obtain a 'global' version of Theorem 4.1.

Corollary 4.6. *Let $f \in \mathscr{E}_I^\infty(G/K)$. If $p_\xi(f, x) = 0$ for all $\xi \in R(I)$ and all $x \in G$, then $f = 0$.*

Proof. Let $g \in \mathbf{D}(G/K) f \cap \mathscr{E}_\lambda^\infty(G/K)$, where $\lambda \in \mathfrak{a}_c^*$, $\operatorname{Re}\lambda$ dominant. According to Lemma 4.5 it suffices to show that $g = 0$. By Lemma 4.4 we have that $E(g, x) = 0$ for all $x \in G$. Hence the boundary value $\beta_\lambda(f)$ (cf. [1], p. 136) vanishes identically, and the result follows from [1], Theorem 10.1. □

Proof of Theorem 4.1 (induction step). Let $n = \operatorname{rank}(\Sigma) > 1$, and assume that the theorem holds for all spaces whose root system is of strictly smaller rank. By the same reasoning as in the proof of Corollary 4.6 we may reduce to the case that $f \in \mathscr{E}_\lambda^\infty(G/K)$ for some $\lambda \in \mathfrak{a}_c^*$, and that the set $\{y \in G: E(f, y) = \emptyset\}$ has a non-empty interior U.

Let F be a proper subset of Δ, and fix $x_0 \in U$. Select an open subset V of M_{1F} such that $x_0 V \subset U$. Now let $\xi \in X(I)$. Then $p_\xi(f, x_0 m) = 0$ for $m \in V$. We claim that actually $p_\xi(f, x_0 m) = 0$ for all $m \in M_{1F}$. Put $\eta = \xi|_{\mathfrak{a}_F}$. By Theorem 3.1 part ($\beta$) we may as well assume that $\eta \in X(F, I)$. By Theorem 3.1 part (α) the function $m \mapsto p_{F,\eta}(f, x_0 m, H_0)$ on M_{1F} has an asymptotic expansion in which all coefficients vanish on V, for all $H_0 \in \mathfrak{a}_F^+$. By the induction hypothesis (apply Corollary 4.2 to the space $\mathscr{E}_{I_\eta}^\infty(M_{1F}/K_F)$) it now follows that

$$p_{F,\eta}(f, x_0 m, H_0) = 0$$

for *all* $m \in M_{1F}$. Applying Theorem 3.1 part (α) again, we obtain our claim by uniqueness of asymptotics. Thus we infer that $U = UM_{1F}$. In view of [1], Theorem 8.4, it even follows that $U = UM_{1F}P = UP_F$.

Since F was an arbitrary proper subset of Δ, it follows from the lemma below that $U = G$, and then the theorem follows from Corollary 4.6 □

Lemma 4.7. *Let $F_1, F_2 \subset \Delta$ be such that $F_1 \cup F_2 = \Delta$. Then G is the closure of the set*

$$\Omega = \{x_1 \ldots x_k; \quad k \in \mathbf{N}, x_1, \ldots, x_k \in P_{F_1} \cup P_{F_2}\} .$$

Proof. The closure $\bar{\Omega}$ of Ω is a closed subgroup of G containing the minimal parabolic subgroup P. Therefore it equals a standard parabolic subgroup P_E, $E \subset \Delta$. We must have $F_i \subset E$ for $i = 1, 2$, hence $E = \Delta$. □

To prove Theorem 4.1 it remains to deal with the case of $\operatorname{rank}(\Sigma) = 1$. Before doing this, we prove the following result, which actually holds for arbitrary rank. Let $\kappa: G \to K$ and $H: G \to \mathfrak{a}$ denote the maps defined by the formula

$$x \in \kappa(x)\exp(H(x))N ,$$

for $x \in G$. Fix a representative $w \in N_K(\mathfrak{a})$ for the longest element in the Weyl group W of Σ. Define $\underline{\alpha}: A \to \mathbf{C}^\Delta$ by $\underline{\alpha}(a) = (a^{-\alpha}; \alpha \in \Delta)$, and for $\varepsilon > 0$ let $D(\varepsilon) = \{z \in \mathbf{C}; |z| < \varepsilon\}$ and $A(\varepsilon) = \{a \in A; \underline{\alpha}(a) \in D(\varepsilon)^\Delta\}$. Fix $\lambda \in \mathfrak{a}_c^*$, and let $\mathscr{P}_\lambda$ denote the Poisson transform $C(K/M) \to \mathscr{E}_\lambda(G/K)$ (cf. [1], (1.8)).

Theorem 4.8. *Let* $\varphi \in C(K/M)$ *and assume that* $\operatorname{supp}\varphi \subset w\kappa(\bar{N})M$. *Then there exists* $\varepsilon > 0$ *and a holomorphic function* $\psi: D(\varepsilon)^\Delta \to \mathbf{C}$ *such that for* $a \in A(\varepsilon)$ *we have*

$$(\mathscr{P}_\lambda \varphi)(a) = a^{w\lambda - \rho}\psi(\underline{\alpha}(a)) .$$

Proof. We have

$$\begin{aligned}\mathscr{P}_\lambda\varphi(a) &= \int_K e^{\langle -\lambda-\rho, H(a^{-1}k)\rangle}\varphi(k)\,dk \\ &= \int_{\bar{N}} e^{\langle -\lambda-\rho, H(a^{-1}w\bar{n}) - H(\bar{n})\rangle}\varphi(w\kappa(\bar{n}))e^{-2\rho H(\bar{n})}\,d\bar{n} \\ &= \tilde{a}^{-\lambda-\rho}\int_{\bar{N}} e^{\langle -\lambda-\rho, H(\tilde{a}\bar{n}\tilde{a}^{-1})\rangle}e^{\langle\lambda-\rho, H(\bar{n})\rangle}\varphi(w\kappa(\bar{n}))\,d\bar{n} ,\end{aligned} \tag{22}$$

where $\tilde{a} = (a^w)^{-1}$. Since $\operatorname{supp}\varphi \subset w\kappa(\bar{N})M$, the function $\bar{n} \mapsto \varphi(w\kappa(\bar{n}))$ is supported by a compact subset C of $\bar{N}$. Fix a bounded open subset Ω of $\bar{N}$ containing C. From a straightforward calculation of the action of $\operatorname{Ad}(\tilde{a})$ on $\bar{\mathfrak{n}}$ involving the root space decomposition, one sees that there exists an $\varepsilon > 0$ and a real analytic function $\chi: D(\varepsilon)^\Delta \times \Omega \to \mathbf{C}$, holomorphic in the $D(\varepsilon)^\Delta$-variable such that

$$e^{\langle -\lambda-\rho, H(\tilde{a}\bar{n}\tilde{a}^{-1})\rangle} = \chi(\underline{\alpha}(a), \bar{n}) .$$

This implies the result. □

Proof of Theorem 4.1 (start of the induction). Let $\operatorname{rank}(\Sigma) = 1$. By the same reasoning as in the proof of Corollary 4.6 we can reduce to the case that $f \in \mathscr{E}_\lambda^\infty(G/K)$ for some $\lambda \in \mathfrak{a}_c^*$ with $\operatorname{Re}\lambda$ dominant and that $E(f, x) = \varnothing$ for all x in an open nonempty set $U \subset G$.

We will prove that $x \in U$ implies $f(x) = 0$. Since f is real analytic it will then follow that $f = 0$. It suffices to prove our claim for $x = e$, because by equivariance we can then apply it to $L_{x^{-1}}f$. Thus we assume $e \in U$.

By [1], Theorem 10.1, $f = \mathscr{P}_\lambda\varphi$, where $\varphi \in C^\infty(K/M)$ is a constant times the boundary value $\beta_\lambda(f)$. Identifying $K/M \simeq G/P$ canonically it follows from the definition of $\beta_\lambda(f)$ that $\operatorname{supp}\varphi \cap UP = \varnothing$.

The crucial feature of the rank 1 case needed here is the two cell Bruhat decomposition $G = P \cup w\bar{N}P$. This allows us to conclude that $\operatorname{supp}\varphi \subset w\kappa(\bar{N})M$.

From Theorem 4.8 it now follows that the function $t \mapsto f(\exp tH)$ has an exponential asymptotic expansion, which actually converges to $f(\exp tH)$ for t sufficiently big. By uniqueness of asymptotics this must be the zero expansion. Hence $f(\exp tH) = 0$ for t sufficiently big. Since f is real analytic it follows that $f(e) = 0$. □

Remark 4.9. Instead of using the above argument to prove Theorem 4.1 for the rank one case, we could also have referred to [2]. In the rank one case there is essentially only one differential equation, which comes from the Casimir operator. From [13], Sect. 4.2, it follows that this equation is such that Theorem 4 of [2] is applicable.

Let

$$\mathscr{E}_I^*(G/K) = \bigcup_{r \in \mathbf{R}} \mathscr{E}_I(G/K) \cap C_r(G) .$$

By the same arguments as in [1], Sect. 13, for the case $I = I_\lambda$, the functions in $\mathscr{E}_I^*(G/K)$ have distributional asymptotic expansions. For $f \in \mathscr{E}_I^*(G/K)$ we also define the set of exponents at $x \in G$ by (20).

Corollary 4.10. *Let $f \in \mathscr{E}_I^*(G/K)$ and assume $E(f, x) \cap R(I) = \emptyset$ for some $x \in G$. Then $f = 0$.*

Proof. Since $R(I)$ is finite there is an open neighborhood U of x such that $E(f, y) \cap R(I) = \emptyset$ for all $y \in U$. Choose a non-empty open subset U_1 of U and a neighbourhood V of e in G such that $VU_1 \subset U$. Let $\varphi \in C_c^\infty(G)$ with $\operatorname{supp} \varphi \subset V$, and let $L^\vee(\varphi)f$ be the function obtained by left convolution of f with φ:

$$L^\vee(\varphi)f(x) = \int_G \varphi(y) f(yx)\, dy .$$

Then $L^\vee(\varphi)f$ is contained in $\mathscr{E}_I^\infty(G/K)$ (cf. [1] Lemma 11.1) and satisfies $\operatorname{supp} p_\xi(L^\vee(\varphi)f) \cap U_1 = \emptyset$ for all $\xi \in R(I)$. From Corollary 4.2 it then follows that $L^\vee(\varphi)f = 0$. This implies that $f = 0$. □

5. Existence of certain exponents

Let $f \in \mathscr{E}_I^*(G/K)$, $x \in G$ and assume that $E(f, x) \neq \emptyset$. Hence $f \neq 0$ and it follows from Corollary 4.10 that $E(f, y) \neq \emptyset$ for any $y \in G$. The following Proposition is a generalization of this statement.

Proposition 5.1. *Let $\xi \in E(f, x)$, $F \subset \Delta$ and $m_0 \in M_{1F}$. Then there exists an element $\zeta \in E(f, xm_0)$ such that $\zeta|_{\mathfrak{a}_F} = \xi|_{\mathfrak{a}_F}$.*

Proof. Let U_i be a fundamental system of neighbourhoods at e in G. For each i there exists a function $\varphi_i \in C_c^\infty(U_i)$ such that

$$p_\xi(L^\vee(\varphi_i)f, x) \neq 0 . \tag{23}$$

Let $\eta = \xi|_{\mathfrak{a}_F}$, and put $\psi_i(m) = p_{F,\eta}(L^\vee(\varphi_i)f, xm)$ $(m \in M_{1F})$. By Theorem 2.9 the function ψ_i belongs to $\mathscr{E}_{I_\eta}^\infty(M_{1F}/K_F)$. Hence this function has an asymptotic expansion along $\mathfrak{a}^+(F)$ at every $m \in M_{1F}$. According to Theorem 3.1 its exponents are contained in the set $X(I, \eta) = \{\zeta \in X(I);\ \zeta|_{\mathfrak{a}_F} = \eta\}$ and its coefficients are

$p_\zeta(L^\vee(\varphi_i)f, xm)$. In view of (23) the expansion for ψ_i is non-vanishing at $m = e$. Hence $\psi_i \neq 0$. Let $R(I_\eta)$ be the set of Theorem 4.1 applied to $\mathscr{E}^\infty_{I_\eta}(M_{1F}/K_F)$. Then we conclude that $E(\psi_i, m_0) \cap R(I_\eta) \neq \emptyset$. Hence

$$xm_0 \in \operatorname{supp} p_{\zeta_i}(L^\vee(\varphi_i)f)$$

for some $\zeta_i \in R(I_\eta) \cap X(I, \eta)$. The latter being a finite set, we may pass to a subsequence and assume that the ζ_i are all equal to a fixed $\zeta \in X(I, \eta)$. It follows that $xm_0 \in \operatorname{supp} p_\zeta(f)$. □

Remark 5.2. Proposition 5.1 is similar to Lemma 1 in [11], p. 354. By combining it with [1] Corollary 17.5 and the purely geometrical Lemma 3 of [11], p. 360 (see also [9], Lemma 1.2), one obtains a proof of Theorem 1 of [11], p. 359 which is independent of the microlocal analysis of [14, 6] and [10]. In particular this gives a proof of the necessity of the rank condition "rk$(G/H) =$ rk$(K/K \cap H)$" for the existence of the discrete series for a semisimple symmetric space.

References

1. Ban, van den, E.P., Schlichtkrull, H.: Asymptotic expansions and boundary values of eigenfunctions on a Riemannian symmetric space. J. Reine Angew. Math. **380**, 108–165 (1987)
2. Baouendi, M.S., Goulaouic, G.: Cauchy problems with characteristic initial hypersurface. Commun. Pure Appl. Math. **26**, 455–475 (1973)
3. Casselman, W., Miličić, D.: Asymptotic behaviour of matrix coefficients of admissible representations. Duke Math. J. **49**, 869–930 (1982)
4. Harish-Chandra: Spherical functions on a semisimple Lie group. Am. J. Math. **80**, 241–310, 553–613 (1958)
5. Harish-Chandra: Harmonic analysis on real reductive groups I. The theory of the constant term. J. Funct. Anal. **19**, 103–204 (1975)
6. Kashiwara, M., Oshima, T.: Systems of differential equations with regular singularities and their boundary values. Ann. Math. **106**, 145–200 (1977)
7. Kostant, B.: On the existence and irreducibility of certain series of representations. In: Lie groups and their representations (I.M. Gelfand, ed.) Proc. Budapest 1971, pp. 231–329. London: Hilger 1975
8. Kostant, B.: On the tensor product of a finite and an infinite dimensional representation. J. Funct. Anal. **20**, 257–285 (1975)
9. Matsuki, T.: A description of discrete series for semisimple symmetric spaces II. Adv. Stud. Pure Math. **14**, 531–540 (1988)
10. Oshima, T.: Boundary value problems for systems of linear partial differential equations with regular singularities. Adv. Stud. Pure Math. **4**, 433–497 (1984)
11. Oshima, T., Matsuki, T.: A description of discrete series for semisimple symmetric spaces. Adv. Stud. Pure Math. **4**, 331–390 (1984)
12. Oshima, T., Sekiguchi, J.: Eigenspaces of invariant differential operators on a semisimple symmetric space. Invent. Math. **57**, 1–81 (1980)
13. Schlichtkrull, H.: Hyperfunctions and harmonic analysis on symmetric spaces. Boston: Birkhäuser 1984
14. Sato, M., Kawai, T., Kashiwara, M.: Microfunctions and pseudo-differential equations. (Lect. Notes Math. Vol. **287**, pp. 265–529). Berlin-Heidelberg-New York: Springer 1973
15. Varadarajan, V.S.: Harmonic analysis on real reductive groups. (Lect. Notes Math. Vol. **576**). Berlin-Heidelberg-New York: Springer 1977
16. Wallach, N.: Asymptotic expansions of generalized matrix entries of representations of real reductive groups. (Lect. Notes Math. Vol. **1024**, pp. 287–369). Berlin-Heidelberg-New York: Springer 1983

Oblatum 27-XII-1988 & 26-V-1989