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THE PRINCIPAL SERIES FOR A REDUCTIVE SYMMETRIC SPACE I. H-FIXED DISTRIBUTION VECTORS

BY E. P. VAN DEN BAN

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0. Introduction

Let G be a real reductive Lie group of Harish-Chandra's class, σ an involution of G and H an open subgroup of the group G^σ of fixed points for σ. The purpose of this paper is to study the principal series of the reductive symmetric space G/H. This is a series of representations $\mathrm{Ind}_P^G(\xi \otimes e^{-\bar{\lambda}} \otimes 1)$, induced from a parabolic subgroup P, which one may expect to contribute to the "most continuous" part in the Plancherel decomposition of $L^2(G/H)$. The actual contribution to a Plancherel formula has to be described in terms of the (finite dimensional) spaces $\mathscr{D}'(P:\xi:\lambda)^H$ of H-fixed distribution vectors for $\mathrm{Ind}_P^G(\xi \otimes e^{-\bar{\lambda}} \otimes 1)$. The main result of the present paper is that the spaces $\mathscr{D}'(P:\xi:\lambda)^H$ can be provided with Hermitian inner products which are preserved by the actions of certain normalized intertwining operators (Theorem 9.2, Corollary 9.3). Since these inner products are not (and cannot be) restrictions of L^2-inner products, this result is by no means an easy consequence of unitarity of normalized intertwining operators. As we will show in a second paper ([Ba 88 II]), Corollary 9.3 has an important consequence for the asymptotics of Eisenstein integrals related to the principal series: it implies that these integrals behave asymptotically like a finite sum of vector valued plane waves, whose amplitudes have a common absolute value (when λ is imaginary). This is analogous to the situation in the case of a group $G = G \times G/\text{diagonal}$; in that case this common absolute value determines part of the Plancherel measure (*cf.* [HC 76 II]). In fact the consequence just mentioned has been one of our primary motivations for writing the present paper.

We shall now describe the contents of our paper in somewhat more detail. There exists a Cartan involution θ of G which commutes with σ. In order that the induced representation $\mathrm{Ind}_P^G(\xi \otimes e^{-\bar{\lambda}} \otimes 1)$ contributes to the Plancherel decomposition, it must have H-fixed distribution vectors. More precisely, let $C^\infty(P:\xi:-\bar{\lambda})$ be the space of C^∞-vectors for $\mathrm{Ind}_P^G(\xi \otimes e^{-\bar{\lambda}} \otimes 1)$ and $\mathscr{D}'(P:\xi:\lambda)$ its topological anti-linear dual. Then the space $\mathscr{D}'(P:\xi:\lambda)^H$ of H-fixed elements in $\mathscr{D}'(P:\xi:\lambda)$ must be non-trivial. Now this can only be true for generic λ if P is $\sigma\theta$-stable. The contributions to the "most continuous" part of the Plancherel decomposition are expected to come from *minimal* $\sigma\theta$-stable parabolic subgroups. This is known to be true firstly in the group case by Harish-Chandra's work (*cf.* [HC 58 I, II], [HC 75], [HC 76 I, II]) and secondly in a number of rank one cases ([Str 73], [Ro 78], [Fa 79], [D-P 86]). In Section 2 we classify the $K \cap H^0$-conjugacy classes of minimal $\sigma\theta$-stable parabolics. We also introduce a (finite) set $\mathscr{P}_\sigma(A_q)$ of special representatives of these conjugacy classes. Elements of $\mathscr{P}_\sigma(A_q)$ have the same MA-part in their Langlands decomposition.

In Section 3 we investigate the further conditions to be imposed on the induction data. For $\mathscr{D}'(P:\xi:\lambda)^H$ to be sufficiently rich we require that λ be contained in a linear subspace $\mathfrak{a}_{qc}^*$ of $\mathfrak{a}_c^*$ and ξ in a certain set $\hat{M}_{ps}$ of finite dimensional unitary representations of M (*cf.* Lemma 3.3). The resulting series of representations is called the principal series for G/H (*cf.* Definition 3.4). It is unitary for imaginary values of λ.

If $P_1, P_2 \in \mathscr{P}_\sigma(A_q)$, then there exists an intertwining operator $A(P_2:P_1:\xi:\lambda)$: $C^\infty(P_1:\xi:\lambda) \to C^\infty(P_2:\xi:\lambda)$ defined as in [K-S 80]. The fact that λ varies in the (generally) lower dimensional subspace $\mathfrak{a}_{qc}^*$ of $\mathfrak{a}_c^*$ forces us to study its existence and meromorphic dependence on λ in some detail. This is done in Section 4 where we also study its extension to distributions.

In Section 5 we begin the study of $\mathscr{D}'(P:\xi:\lambda)^H$. We define a finite dimensional vector space $V(\xi)$ and for $-\mathrm{Re}(\lambda) - \rho_P$ strictly P-dominant a linear map $j(P:\xi:\lambda)$ from $V(\xi)$ into $\mathscr{D}'(P:\xi:\lambda)^H$ which is bijective for generic λ and then provides a parametrization for $\mathscr{D}'(P:\xi:\lambda)^H$. Moreover, $j(P:\xi:\lambda)$ depends holomorphically on λ in the above mentioned region (Lemma 5.7). The necessity to cover imaginary values of λ (corresponding to the unitary principal series) forces us to show that the map $j(P:\xi:\lambda)$ admits a meromorphic continuation in λ (*cf.* Theorem 5.10). The existence of such a continuation was essentially announced in [Os 79], proved for symmetric spaces G/K_ε by

[O-S 80] and for spaces G/H with $H = G^\sigma$ by [Ol 86]. Both proofs depend on [B-G 69] or [B 72]. Our proof is entirely different, using an *a priori* estimate of $\dim_{\mathbb{C}} \mathscr{D}'(P:\xi:\lambda)^H$ (Corollary 5.3) and results of [S 71] and [K-S 80] on the meromorphic continuation of intertwining operators.

In Section 6 we are finally prepared to study the actions of intertwining operators on H-fixed distributions. More precisely, if $P_1, P_2 \in \mathscr{P}_\sigma(A_q)$, then the equation $A(P_2:P_1:\xi:\lambda) \circ j(P_1:\xi:\lambda) = j(P_2:\xi:\lambda) \circ B(P_2:P_1:\xi:\lambda)$ uniquely determines an endomorphism $B(P_2:P_1:\xi:\lambda)$ of $V(\xi)$ which depends meromorphically on λ. The basic result of our paper is that $V(\xi)$ carries a Hermitian inner product (independent of λ) such that $B(P_2:P_1:\xi:\lambda)^* = B(P_1:P_2:\xi:-\bar{\lambda})$ (Theorem 6.3). It is proved in the course of Sections 7, 8 by means of a σ-split rank one reduction. In the final Section 9 results are reformulated in terms of normalized operators.

In a slightly different form the endomorphism B has for the first time been introduced by Oshima and Sekiguchi for spaces G/K_ε (*cf.* [O-S 80]). Then $\hat{M}_{ps}$ consists of only the trivial representation and the matrix $B(P_2:P_1:1:\lambda)$ admits an explicit computation (*cf.* [*loc. cit.* Lemma 4.14]). Moreover, it plays a crucial role in the theory of the Poisson transformation for the above mentioned spaces.

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1. Preliminaries, root systems and Weyl groups

In this paper, G will always be a real reductive Lie group of Harish-Chandra's class, σ an involution of G, and θ a Cartan involution commuting with σ (for its existence, *cf.* [Be 57], [Ba 87 II]). Let H be an open subgroup of the group G^σ of fixed points for σ. We call G/H a reductive symmetric space of the Harish-Chandra's class (*cf.* [Ba 87 II]).

In the course of the proof of our main result, Theorem 6.3, we shall also need the following assumption on G:

(A) Every Cartan subgroup of G is abelian.

[Vo 81] works under the same assumption; notice that we do not require G to be linear. Recall that (A) is inherited by Levi components of parabolic subgroups of G. Therefore the usual induction arguments may be applied to our class of groups. It should be noted that much of the theory of this paper holds for groups of the Harish-Chandra's class not satisfying (A); in fact it is not until Lemma 6.16 that we do require (A) to hold permanently. The above assumption is explicitly used only in the proofs of Lemmas 5.4 and 6.16. Its necessity for Lemma 5.4 to be valid was pointed out to me by H. Schlichtkrull. This is illustrated by the following example, due to him and R. Lipsman.

Let $\mathbb{F}=\{-1,0,1\}$ be the field of three elements. Then, being a finite group, $G=SL(2,\mathbb{F})$ is of Harish-Chandra's class; however, G is not abelian, hence does not satisfy (A). Let $\sigma: G \to G$ be conjugation by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then its group of fixed points is $H=\{-I, I\}$. Let $\mathscr{H}_\xi$ be the 3-dimensional complex linear space of functions $\mathbb{P}^1(\mathbb{F}) \to \mathbb{C}$ whose average value is zero. Then the natural representation ξ of G on $\mathscr{H}_\xi$ is irreducible, and all of its vectors are H-fixed. In the notations of Lemma 5.4 we have: $M=G$, $\xi \in \hat{M}_{fu}$, and $\dim \mathscr{H}_\xi^{M \cap H}=3$.

Notice that the above also provides an example of a reductive symmetric space of Harish-Chandra's class whose discrete series are not multiplicity free.

Put $K=G^\theta$, and let $\mathfrak{k}$ and $\mathfrak{p}$ ($\mathfrak{h}$ and $\mathfrak{q}$) denote the $+1$ and -1 eigenspaces of $\theta(\sigma)$ in $\mathfrak{g}$ respectively (as usual groups are denoted by Roman capitals; their Lie algebras by the corresponding lower case German letters). We extend the Killing form on $\mathfrak{g}_1=[\mathfrak{g},\mathfrak{g}]$ to a non-degenerate bilinear form B on $\mathfrak{g}$ which is positive definite on $\mathfrak{p}$, negative definite on $\mathfrak{k}$ and for which $\operatorname{center}(\mathfrak{g}) \cap \mathfrak{h}$ and $\operatorname{center}(\mathfrak{g}) \cap \mathfrak{q}$ are orthogonal. Then the joint eigenspace decomposition

$$\mathfrak{g}=(\mathfrak{k} \cap \mathfrak{q}) \oplus (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{q}) \oplus (\mathfrak{p} \cap \mathfrak{h})$$

for θ and σ is B-orthogonal. Fix a maximal abelian subspace $\mathfrak{a}_{0q}$ of $\mathfrak{p} \cap \mathfrak{q}$ and extend it to a σ-stable maximal abelian subspace $\mathfrak{a}_0$ of $\mathfrak{p}$. The restricted roots $\Sigma(\mathfrak{g}, \mathfrak{a}_{0q})$ of $\mathfrak{a}_{0q}$ in $\mathfrak{g}$ constitute a (possibly non-reduced) root system (*cf.* [Ro 79]), which we denote by Σ. Let $\Sigma_0=\Sigma(\mathfrak{g}, \mathfrak{a}_0)$. If Σ_0^+ and Σ^+ are compatible systems of positive roots for Σ_0 and Σ respectively, we agree to write Δ_0 and Δ for the associated fundamental systems. The reflection groups of Σ_0 and Σ are denoted by W_0 and W. Notice that $W_0 \simeq N_K(\mathfrak{a}_0)/Z_K(\mathfrak{a}_0)$. Now let $W_{0\sigma}=\{w \in W;\ \sigma w = w \sigma \text{ on } \mathfrak{a}_0\}$. Then clearly $W_{0\sigma} \simeq N_K(\mathfrak{a}_0) \cap N_K(\mathfrak{a}_{0q})/Z_K(\mathfrak{a}_0)$. Put $\Sigma_0^h=\{\alpha \in \Sigma_0; \alpha \mid \mathfrak{a}_{0q}=0\}$ and let $W(\Sigma_0^h)$ denote the associated reflection group.

LEMMA 1.1. — *Restriction to $\mathfrak{a}_{0q}$ induces a natural surjective map $W_{0\sigma} \to W$ with kernel $W(\Sigma_0^h)$.*

Proof. — The first assertion follows from ([Schl 84], Proposition 7.1.7). The assertion on the kernel follows straightforwardly by application of ([Va 74], Lemma 4.15.15). ∎

LEMMA 1.2. — *The map $N_K(\mathfrak{a}_{0q}) \to \operatorname{End}(\mathfrak{a}_{0q})$, $k \to \operatorname{Ad}(k) \mid \mathfrak{a}_{0q}$ induces an isomorphism $N_K(\mathfrak{a}_{0q})/Z_K(\mathfrak{a}_{0q}) \simeq W$.*

Proof. — Let M_1 be the centralizer of $\mathfrak{a}_{0q}$ in G. Then $\mathfrak{a}_0$ is maximal abelian in $\mathfrak{m}_1 \cap \mathfrak{p}$. Moreover if $k_1 \in N_k(\mathfrak{a}_{0q})$, then k_1 normalizes M_1 and $\operatorname{Ad}(k_1)\mathfrak{a}_0$ is maximal abelian in $\mathfrak{m}_1 \cap \mathfrak{p}$. It follows that $\operatorname{Ad}(k_1)\mathfrak{a}_0=\operatorname{Ad}(k_2)\mathfrak{a}_0$ for some $k_2 \in M_1^0 \cap K$. Let $k=k_2^{-1}k_1$. Then $k \in N_K(\mathfrak{a}_0) \cap N_K(\mathfrak{a}_{0q})$ and it follows that $\operatorname{Ad}(k_1) \mid \mathfrak{a}_{0q}=\operatorname{Ad}(k) \mid \mathfrak{a}_{0q} \in W$ (use Lemma 1.1). By Lemma 1.1 the map $N_K(\mathfrak{a}_{0q}) \to W$ is surjective, whence the result.

Let $W_{0\,K\cap H}$ ($W_{K\cap H}$) denote the image of $N_{K\cap H}(\mathfrak{a}_0)$ ($N_{K\cap H}(\mathfrak{a}_{0\,q})$) in W_0(W). Then obviously $W_{0\,K\cap H}\subset W_{0\,\sigma}$ and the restriction $W_{0\,\sigma}\to W$ maps $W_{0\,K\cap H}$ into $W_{K\cap H}$.

LEMMA 1.3. — *The natural map* $W_{0\,\sigma}\to W$ *induces a bijection between the coset spaces* $W(\Sigma_0^h)\, W_{0\,\sigma}\backslash W_{0\,K\cap H}$ *and* $W/W_{K\cap H}$.

Proof. — By Lemma 1.1 we have $W(\Sigma_0^h)\backslash W_{0\,\sigma}\simeq W$ as groups. Hence it suffices to show that $W_{0\,K\cap H}$ maps onto $W_{K\cap H}$. Let $k_1\in N_{K\cap H}(\mathfrak{a}_{0\,q})$. Then k_1 normalizes $\mathfrak{m}_1\cap\mathfrak{h}$ (*cf.* the proof of Lemma 1.2) and $\mathfrak{a}_0\cap\mathfrak{h}$ is maximal abelian in $(\mathfrak{m}_1\cap\mathfrak{h})\cap\mathfrak{p}$. Hence $\mathrm{Ad}(k_1)(\mathfrak{a}_0\cap\mathfrak{h})=\mathrm{Ad}(k_2)(\mathfrak{a}_0\cap\mathfrak{h})$ for some $k_2\in(M_1\cap H\cap K)^0$. Let $k=k_2^{-1}k_1$. Then $k\in N_{K\cap H}(\mathfrak{a}_0)$ and $\mathrm{Ad}(k)\,|\,\mathfrak{a}_0$ has image $\mathrm{Ad}(k_1)\,a_{0\,q}$ under the natural restriction map. ■

Let $\mathfrak{g}_+$ denote the $+1$-eigenspace of $\sigma\theta$ in $\mathfrak{g}$. It admits the Cartan decomposition $\mathfrak{g}_+=(\mathfrak{k}\cap\mathfrak{h})\oplus(\mathfrak{p}\cap\mathfrak{q})$. Let $\Sigma_+=\Sigma(\mathfrak{g}_+,\mathfrak{a}_{0\,q})$. Its reflection group $W(\Sigma_+)$ is contained in $W_{K\cap H}$. The following result is proved in [Ba 87 II].

LEMMA 1.4. — *We have* $H=H^0 Z_{K\cap H}(\mathfrak{a}_{0\,q})$ *iff* $W(\Sigma_+)=W_{K\cap H}$.

The group H is said to be *essentially connected* if it satisfies the equalities of the above lemma.

2. σθ-stable parabolic subgroups

The purpose of this section is to classify $K\cap H^0$-conjugacy classes of minimal σθ-stable parabolic subgroups. We first consider the Langlands decomposition MAN of any σθ-stable parabolic P. Its θ-stable Levi component $M_1=MA=P\cap\theta(P)$ is σ-stable because σ and θ commute. Since $\mathfrak{a}=\text{center}(\mathfrak{m}_1)\cap\mathfrak{p}$ and $\mathfrak{m}=\mathfrak{m}_1\cap\mathfrak{a}^\perp$ it follows that A and M are σ-stable as well (use that $M=(M_1\cap K)\exp(\mathfrak{m}\cap\mathfrak{p})$). Finally $\bar{N}=\theta(N)=\sigma(N)$. The following lemma is now obvious.

LEMMA 2.1. — *Let* P *be a parabolic subgroup with Langlands decomposition* P=MAN. *Then the following conditions are equivalent:*

(i) P *is* σθ-*stable*;

(ii) A *is* σθ-*stable and* $\Sigma(\mathfrak{n},\mathfrak{a})$ *is* σθ-*stable as a subset of the* $\mathfrak{a}$-*weights in* $\mathfrak{g}$.

If P is a σθ-stable parabolic subgroup with Langlands decomposition P=MAN, then A splits as a direct product $A=A_hA_q$, where $A_h=A\cap H$ and $A_q=\{x\in A;\ \sigma(x)=x^{-1}\}$ are closed subgroups of A. Moreover, $M_\sigma=MA_h$ is a reductive group of Harish-Chandra's class, and we have a decomposition

$$P=M_\sigma A_q N,$$

called the σ-Langlands decomposition of P.

LEMMA 2.2. — *Let* P *be a* σθ-*stable parabolic with* σ-*Langlands decomposition* $P=M_\sigma A_q N$. *Then* $M_1=M_\sigma A_q$ *is the centralizer of* A_q *in* G, *and* $\mathfrak{a}_q=\text{center}(\mathfrak{m}_1)\cap\mathfrak{p}\cap\mathfrak{q}$.

Proof. — Put $\mathfrak{a}_h=\mathfrak{a}\cap\mathfrak{h}$. Then $\mathfrak{a}=\mathfrak{a}_h\oplus\mathfrak{a}_q$. If $\alpha\in\Sigma(\mathfrak{n},\mathfrak{a})$, then $\alpha^{\sigma\theta}\in\Sigma(\mathfrak{n},\mathfrak{a})$, hence $\alpha^{\sigma\theta}\neq-\alpha$. It follows that $\alpha\,|\,\mathfrak{a}_q\neq0$. Using the direct sum decomposition $\mathfrak{g}=\bar{\mathfrak{n}}\oplus\mathfrak{m}_1\oplus\mathfrak{n}$

we deduce that $\mathfrak{m}_1$ equals the centralizer of $\mathfrak{a}_q$ in $\mathfrak{g}$. Now let M_{1c} denote the centralizer of $\mathfrak{a}$ in the complex adjoint group G_c. Then M_{1c} is connected and has Lie algebra $\mathfrak{m}_{1c}$. The centralizer of $\mathfrak{a}_q$ in G_c is also connected, hence equal to M_{1c}. Hence $Z_G(\mathfrak{a}_q) = \mathrm{Ad}_G^{-1}(M_{1c}) = Z_G(\mathfrak{a}) = M_1$. ■

We now turn our attention to $K \cap H^0$-conjugacy classes of $\sigma\theta$-stable parabolic subgroups. If P is a parabolic subgroup of G, we denote its Lie algebra by the corresponding German capital $\mathfrak{P}$. Let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ be the decomposition of $\mathfrak{g}$ in the $+1$ and -1 eigenspaces for the involution $\sigma\theta$. Notice that $\mathfrak{g}_+$ is a reductive subalgebra with Cartan decomposition $\mathfrak{g}_+ = (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{q})$. Given a $\sigma\theta$-stable subspace $\mathfrak{b} \subset \mathfrak{g}$ we often write $\mathfrak{b}_+ = \mathfrak{b} \cap \mathfrak{g}_+$ and $\mathfrak{b}_- = \mathfrak{b} \cap \mathfrak{g}_-$.

LEMMA 2.3. — *Let* P *be a* $\sigma\theta$-*stable parabolic subgroup of* G, $\mathfrak{P}$ *its Lie algebra. Then* $\mathfrak{P}_+$ *is a parabolic subalgebra of* $\mathfrak{g}_+$.

Proof. — Since $\bar{\mathfrak{n}}$ and $\mathfrak{P}$ are $\sigma\theta$-stable we have a direct sum decomposition $\mathfrak{g}_+ = \bar{\mathfrak{n}}_+ \oplus \mathfrak{P}_+$. If $\alpha \in \Sigma(\mathfrak{n}, \mathfrak{a})$, then $\alpha | \mathfrak{a}_q \neq 0$ (*cf.* proof of Lemma 2.2). Hence if $Y \in \bar{\mathfrak{n}}_+ \setminus \{0\}$, there exists $X \in \mathfrak{a}_q$ such that $[X, Y] \in \bar{\mathfrak{n}}_+ \setminus \{0\}$. This implies that $\mathfrak{P}_+$ equals its own normalizer in $\mathfrak{g}_+$. ■

COROLLARY 2.4. — *Let* P *be a* $\sigma\theta$-*stable parabolic subgroup of* G. *Then there exists a* $k \in K \cap H^0$ *such that* P^k *contains* A_{0q} (*and is of course still* $\sigma\theta$-*stable*).

Proof. — Since $\mathfrak{a}_{0q}$ is maximal abelian in $\mathfrak{p} \cap \mathfrak{q}$, there exists a $k \in K \cap H^0$ such that $\mathfrak{P}_+^k$ contains $\mathfrak{a}_{0q}$. ■

The description of $K \cap H^0$-conjugacy classes of $\sigma\theta$-stable parabolics will be completed in terms of standard parabolics. For the remainder of this section, let Σ_0^+ and Σ^+ be compatible systems of positive roots (notations as in Section 1).

If $F \subset \Delta_0$, we let $\mathfrak{P}_F$ denote the associated standard parabolic subalgebra of $\mathfrak{g}$, $\mathfrak{P}_F = \mathfrak{m}_F \oplus \mathfrak{a}_F \oplus \mathfrak{n}_F$ its Langlands decomposition, and $P_F = M_F A_F N_F$ its normalizer in G (*cf.* [Va 77]). Moreover, we write P_0 for $P_\varnothing$, etc.

LEMMA 2.5. — *Let* P *be a* $\sigma\theta$-*stable parabolic subgroup of* G, *containing* A_{0q}. *Then* P *contains* A_0. *Moreover, there exists a* $k \in N_K(\mathfrak{a}_0) \cap N_K(\mathfrak{a}_{0q})$ *such that* P^k *is a* ($\sigma\theta$-*stable*) *standard parabolic.*

Proof. — Let $\mathfrak{m}_\sigma \oplus \mathfrak{a}_q \oplus \mathfrak{n}$ be the σ-Langlands decomposition of $\mathfrak{P}$. Then $\mathfrak{m}_1 = \mathfrak{m}_\sigma \oplus \mathfrak{m}_q$ equals $\mathfrak{P} \cap \theta(\mathfrak{P})$, hence contains $\mathfrak{a}_{0q}$. Hence $\mathfrak{a}_{0q}$ is a maximal abelian subspace of $\mathfrak{m}_1 \cap \mathfrak{p} \cap \mathfrak{q}$. It follows that there exists $m \in M_1$ such that $\mathrm{Ad}(m)\,\mathfrak{a}_q \subset \mathfrak{a}_{0q}$. Since M_1 centralizes $\mathfrak{a}_q$, this implies that $\mathfrak{a}_{0q}$ contains $\mathfrak{a}_q$. Hence $\mathfrak{l}_1$, the centralizer of $\mathfrak{a}_{0q}$ in $\mathfrak{g}$, is contained in $\mathfrak{m}_1$ (use Lemma 2.2). In particular $\mathfrak{a}_0 \subset \mathfrak{m}_1$, whence the first assertion.

If $\alpha \in \Sigma_0$, put $\mathfrak{P}^\alpha = \mathfrak{g}^\alpha \cap \mathfrak{P}$. Then

$$\mathfrak{P} = (\mathfrak{P} \cap \mathfrak{m}_0) \oplus \mathfrak{a}_0 \oplus \sum_{\alpha \in \Sigma_0}^{\oplus} \mathfrak{P}^\alpha,$$

where $\mathfrak{m}_0$ denotes the orthocomplement of $\mathfrak{a}_0$ in its centralizer. Moreover, if $\mathfrak{P}^\alpha \neq 0$, then $\mathfrak{P}^\alpha = \mathfrak{g}^\alpha$. Let $T = \{\alpha \in \Sigma_0;\ \mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha} \subset \mathfrak{P}\}$, $S = \{\alpha \in \Sigma_0;\ \mathfrak{P}^\alpha = \mathfrak{g}^\alpha,\ \mathfrak{P}^{-\alpha} = 0\}$. Then

$$\Sigma_0 = T \cup S \cup (-S) \qquad \text{(disjoint union)},$$

and clearly T is the set of roots in $\mathfrak{P} \cap \theta(\mathfrak{P})$. If $\alpha \in S$, then $\mathfrak{g}_\alpha \subset \mathfrak{n}$, hence $\sigma(\mathfrak{g}_\alpha) \cap \mathfrak{P} = 0$, whence $\sigma\alpha \in -S$. It follows that σ leaves T invariant, maps S onto $-S$ and vice versa. Fix a σ-compatible system T^+ of positive roots for the root system T, *i.e.* $\alpha \in T^+$, $\sigma\alpha \in T^+ \Rightarrow \sigma\alpha = \alpha$. Then $T^+ \cup S$ is a σ-compatible system of positive roots for Σ_0. Hence $T^+ \cup S = w^{-1}(\Sigma_0^+)$ for a unique $w \in W_0$ with $w \circ \sigma = \sigma \circ w$ (*cf.* [Schl 84], Proposition 7.1.7). Let $k \in N_K(\mathfrak{a}_0)$ be a representative for w. Then $k \in N_K(\mathfrak{a}_{0q})$ and $Ad(k)\mathfrak{P}$ is a parabolic subalgebra containing $\mathfrak{P}_0$. Using Lemma 2.1 we infer its $\sigma\theta$-stability. ■

In the following we complete the description of $\sigma\theta$-stable parabolic subgroups by classifying the standard ones: they correspond $1-1$ to subsets of Δ.

LEMMA 2.6. — *Let* $F \subset \Delta_0$. *Then the following conditions are equivalent:*

(i) P_F *is* $\sigma\theta$-*stable,*

(ii) F *contains* $\{\alpha \in \Delta_0;\ \sigma(\alpha) = \alpha\}$ *and* $\Sigma_{0F} = \Sigma_0 \cap \mathbb{Z} F$ *is* σ-*stable,*

(iii) *there exists a subset* Ψ *of* Δ *such that*

$$F = \{\alpha \in \Delta_0;\ \alpha|\mathfrak{a}_{0q} \in \{0\} \cup \Psi\}.$$

Proof. — Let $\Delta_0^h = \{\alpha \in \Delta_0,\ \alpha|\mathfrak{a}_{0q} = 0\}$. Then there is a permutation $\alpha \to \alpha'$ of order 2 of the set $\Delta_0 \setminus \Delta_0^h$ such that

$$\sigma\alpha = -\alpha' - \sum_{\beta \in \Delta_0^h} n(\alpha, \beta)\beta$$

for all $\alpha \in \Delta_0 \setminus \Delta_0^h$. Here $n(\alpha, \beta) \in N = \{0, 1, 2, \ldots\}$ (*cf.* [Schl 84], Lemma 7.2.3). Let $l = \# \Delta$, and $l_1 = \# \{\alpha \in \Delta_0\ \Delta_0^h;\ \alpha' \neq \alpha\}$. Then the elements of Δ_0 may be enumerated $\alpha_1, \ldots, \alpha_n$ so that

$$\alpha'_j = \alpha_j \qquad (1 \leqq j \leqq l - l_1),$$
$$\alpha'_j = \alpha_{j+l_1} \qquad (l - l_1 < j \leqq l),$$

and $\Delta_0^h = \{\alpha_{l+l_1+1}, \ldots, \alpha_n\}$. Moreover, $\Delta = \{\alpha_j|\mathfrak{a}_{0q};\ 1 \leqq j \leqq l\}$ ([Sch. 82], Lemma 7.2.4).

"(*i*) $\Rightarrow$ (*ii*)". Since $\sigma(\mathfrak{n}_F) = \theta(\mathfrak{n}_F) = \bar{\mathfrak{n}}_F$, σ maps $\Sigma_0^+ \setminus \mathbb{N}F$ into $-(\Sigma_0^+ \setminus \mathbb{N}F)$. So if $\alpha \in \Delta_0 \setminus F$, then $\sigma\alpha < 0$, hence $\alpha \notin \Delta_0^h$ and we infer that $\Delta_0 \setminus F \subset \Delta_0 \setminus \Delta_0^h$, whence $F \supset \Delta_0^h$. Moreover, if $\alpha \in \Delta_0 \setminus F$, then $\sigma\alpha \in -(\Sigma_0^+ \setminus \mathbb{N}F)$, and by the above description of the action of σ on $\Delta_0 \setminus \Delta_0^h$, we infer that $\alpha' \notin F$. We deduce that $\alpha \in F \Leftrightarrow \alpha' \in F$ $(\alpha \in \Delta_0 \setminus \Delta_0^h)$. Hence Σ_{0F} is σ-stable.

"(*ii*) $\Rightarrow$ (*iii*)". From (*ii*) it follows that $F \supset \Delta_0^h$, and that $\alpha \in F \Leftrightarrow \alpha' \in F\ (\alpha \in \Delta_0 \setminus \Delta_0^h)$. Hence if we define $\Psi = \{\alpha_j|\mathfrak{a}_{0q};\ 1 \leqq j \leqq l,\ \alpha_j \in F\}$, then $F = \{\alpha \in \Delta_0;\ \alpha|\mathfrak{a}_{0q} \in \{0\} \cup \Psi\}$.

"(*iii*) $\Rightarrow$ (*i*)". From the description of the action of σ on $\Delta_0 \setminus \Delta_0^h$ we deduce that σ leaves Σ_{0F} invariant. Hence $\sigma(\mathfrak{m}_F) = \mathfrak{m}_F$ and $\sigma(\mathfrak{a}_F) = \mathfrak{a}_F$. Moreover, since $F \supset \Delta_0^h$, it follows that $\sigma(\Delta_0\setminus F) \subset -\Sigma_0^+$, and hence that σ maps $\Sigma_0^+\setminus\mathbb{N} F$ onto $-(\Sigma_0^+\setminus\mathbb{N} F)$. Therefore $\sigma(\mathfrak{n}_F)=\bar{\mathfrak{n}}_F=\theta(\mathfrak{n}_F)$, and $\mathfrak{P}_F$ is $\sigma\theta$-stable. ■

We conclude this section with a description of the minimal $\sigma\theta$-stable parabolic subgroups of G. First, by Corollary 2.4 and Lemmas 2.5, 6 we have:

COROLLARY 2.7. — *Let* $F=\{\alpha\in\Delta_0;\ \alpha\,|\,\mathfrak{a}_{0q}=0\}$. *Then the standard parabolic subgroup* P_F *is minimal* $\sigma\theta$*-stable. Moreover, if* P *is any minimal* $\sigma\theta$*-stable parabolic subgroup then there exist* $k\in K\cap H^0$, $w\in N_K(\mathfrak{a}_0)\cap N_K(\mathfrak{a}_{0q})$ *such that*

$$(P^k)^w=P_F.$$

Let $\mathscr{P}_\sigma(A_{0q})$ denote the set of $\sigma\theta$-stable parabolic subgroups whose split component contains A_{0q}. If $P\in\mathscr{P}_\sigma(A_{0q})$, then $P\supset A_0$ and P^w is standard for some $w\in N_K(\mathfrak{a}_0)\cap N_K(\mathfrak{a}_{0q})$ (*cf.* Lemma 2.5). The split component of P^w contains A_{0q}, so P^w is the standard minimal $\sigma\theta$-stable parabolic subgroup (Lemma 2.6). It follows that $\mathscr{P}_\sigma(A_{0q})$ consists of all minimal $\sigma\theta$-stable parabolic subgroups containing A_0. The action of $N_K(\mathfrak{a}_{0q})$ by conjugation induces an action of W on $\mathscr{P}_\sigma(A_{0q})$ (use Lemma 1.2). For $P\in\mathscr{P}_\sigma(A_{0q})$, let $\Sigma(\mathfrak{n}_P,\mathfrak{a}_{0q})$ denote the set of $\mathfrak{a}_{0q}$-weights in $\mathfrak{n}_P$, the nilpotent radical of $\mathfrak{P}$. The following is now obvious.

LEMMA 2.8. — *The map* $P\mapsto\Sigma(\mathfrak{n}_P,\mathfrak{a}_{0q})$ *defines a bijection from* $\mathscr{P}_\sigma(A_{0q})$ *onto the set of positive systems for* Σ *and commutes with the action of* W. *In particular,* W *acts simply transitively on* $\mathscr{P}_\sigma(A_{0q})$.

Remark. — In particular, the $K\cap H^0$-conjugacy classes of minimal $\sigma\theta$-stable parabolics are in bijective correspondence with $W/W_{K\cap H}$.

3. The principal series for G/H

If P is a parabolic subgroup with Langlands decomposition $P=MAN$, we define $\rho_P\in\mathfrak{a}^*$ by $\rho_P(X)=1/2\,\mathrm{tr}(\mathrm{ad}\,X\,|\,\mathfrak{n})$. Let ξ be a unitary representation of M in a Hilbert space $\mathscr{H}_\xi$, and $\lambda\in\mathfrak{a}_c^*$. Then by $C^\infty(G:P:\xi:\lambda)$, or more briefly $C^\infty(P:\xi:\lambda)$ or $C^\infty(\xi:\lambda)$ we denote the space of C^∞-functions $G\to\mathscr{H}_\xi$ satisfying

$$f(man\,x)=a^{\lambda+\rho_P}\,\xi(m)f(x), \tag{3.1}$$

for $x\in G$, $(m, a, n)\in M\times A\times N$. The right regular representation of G on this space is denoted by $\mathrm{Ind}_P^G(\xi\otimes e^\lambda\otimes 1)$. We define a pairing $C^\infty(\xi:\lambda)\times C^\infty(\xi:-\bar{\lambda})\to\mathbb{C}$ by

$$\langle f\,|\,g\rangle=\int_K (f(k), g(k))_\xi\,dk, \tag{3.2}$$

where $(.,.)_\xi$ denotes the unitary structure of $\mathscr{H}_\xi$ and the vertical bar in the left hand side of the equation indicates that the pairing is anti-linear in the second variable. It is well known that the pairing (3.2) is G-equivariant. In particular, if λ is purely imaginary, it follows that $\mathrm{Ind}_P^G(\xi \otimes e^{\lambda} \otimes 1)$ is unitary (for these and other standard facts concerning parabolic induction, we refer the reader to [K-S 80]).

Let $C^{\infty}(P:\xi:-\bar{\lambda})$ be endowed with the usual structure of Fréchet space. Its topological anti-linear dual is denoted by

$$\mathscr{D}'(G:P:\xi:\lambda), \tag{3.3}$$

or, more briefly, $\mathscr{D}'(P:\xi:\lambda)$ or $\mathscr{D}'(\xi:\lambda)$. Naturally, the group G acts on this space. Moreover, the pairing (3.2) induces a G-equivariant and complex linear embedding

$$C^{\infty}(P:\xi:\lambda) \hookrightarrow \mathscr{D}'(P:\xi:\lambda). \tag{3.4}$$

To ensure that the space

$$\mathscr{D}'(P:\xi:\lambda)^{H}$$

of H-fixed elements in $\mathscr{D}'(P:\xi:\lambda)$ is sufficiently rich, we assume from now on that P is $\sigma\theta$-stable.

LEMMA 3.1. — *Let* $P=MAN$ *be a* $\sigma\theta$-*stable parabolic subgroup. Then* $\rho_P=0$ *on* $\mathfrak{a}\cap\mathfrak{h}$.

Proof. — Since $\sigma\theta$ stabilizes $\mathfrak{a}$ and $\mathfrak{n}$, we have $\rho_P^{\sigma\theta}=\rho_P$. On the other hand, on $\mathfrak{a}\cap\mathfrak{h}$ we have $\rho_P^{\sigma\theta}=\rho_P^{\theta}=-\rho_P$. ■

We now restrict our attention to induction from parabolics $P\in\mathscr{P}_{\sigma}(A_{0q})$ (for the definition and properties of this finite set, *cf.* the end of Section 2). All elements of $\mathscr{P}_{\sigma}(A_{0q})$ have the same $\sigma-$ and θ-stable Levi component $M_1=MA$. Here

$$\mathfrak{a}=\bigcap\{\ker\alpha;\ \alpha\in\Sigma_0,\ \alpha|\mathfrak{a}_{0q}=0\}, \tag{3.5}$$

$$A=\exp\mathfrak{a}, \tag{3.6}$$

$$MA=Z_G(\mathfrak{a}_{0q}) \tag{3.7}$$

(*cf.* Lemma 2.2). From now on we reserve the notations M and A exclusively for the objects defined by (3.5-7). Thus we have $\mathfrak{a}_q=\mathfrak{a}\cap\mathfrak{q}=\mathfrak{a}_{0q}$, and

$$A_q=A_{0q}.$$

The following lemma is easy to prove.

LEMMA 3.2. — *Let* $v\in N_K(\mathfrak{a}_q)$. *Then* v *normalizes* M_1, M_{σ}, M *and* $A\cap H$.

Thus $N_K(\mathfrak{a}_q)$ acts in a natural fashion on representations of M by the rule $v\xi(m)=\xi(v^{-1}mv)$, for ξ a representation of M, $v\in N_K(\mathfrak{a}_q)$, and $m\in M$. Since $Z_K(\mathfrak{a}_q)\subset K\cap M_1=K\cap M$ this induces an action of W on the unitary dual $\hat{M}_u$ of M.

Let now ξ be an irreducible unitary representation of M on a Hilbert space $\mathscr{H}_{\xi}$ (in the sequel we abbreviate this as $[\xi]\in\hat{M}_u$, or by abuse of notation, as $\xi\in\hat{M}_u$). Let $\mathscr{H}_{\xi}^{\infty}$ be the space of C^{∞}-vectors endowed with the usual structure of Fréchet space and let $\mathscr{H}_{\xi}^{-\infty}$ be its topological anti-linear dual. By unitarity we have equivariant embeddings $\mathscr{H}_{\xi}^{\infty}\subset\mathscr{H}_{\xi}\hookrightarrow\mathscr{H}_{\xi}^{-\infty}$. Let $\mathscr{O}=\mathscr{O}(P)$ denote the union of the open double cosets in

P\G/H. If $\mathscr{W}$ is a set of representatives for $W/W_{K \cap H}$ in $N_K(\mathfrak{a}_q)$, then

$$(3.8) \qquad \mathscr{O}(P) = \bigcup_{w \in \mathscr{W}} PwH$$

(*cf.* Appendix B).

LEMMA 3.3. — *Let* $P \in \mathscr{P}_\sigma(A_q)$, $\xi \in \hat{M}_u$, *and* $\lambda \in \mathfrak{a}_c^*$. *If there exists a non-trivial right* H-*invariant function* $f : \mathscr{O}(P) \to \mathscr{H}_\xi^{-\infty}$ *transforming according to the rule* (3.1), *then* $\lambda | \mathfrak{a} \cap \mathfrak{h} = 0$ *and there exists a* $v \in N_K(\mathfrak{a}_q)$ *such that* $(\mathscr{H}_{v\xi}^{-\infty})^{M \cap H} \neq 0$.

Proof. — Let $\mathscr{W}$ be as in (3.8). If f fulfills the above conditions then $f(w) \neq 0$ for some $w \in \mathscr{W}$. Moreover if $a \in A \cap H$ then $a^{\lambda + \rho_P} f(w) = f(ww^{-1}aw) = f(w)$ (use Lemma 3.2). Hence $\lambda = \lambda + \rho_P = 0$ on $\mathfrak{a} \cap \mathfrak{h}$ (use Lemma 3.1). Finally, if $m \in M \cap H$ then $[w^{-1}\xi(m) f](w) = f(wm) = f(w)$ and the last assertion holds for $v = w^{-1}$. ■

Let $\hat{M}_{ps}$ denote the set of $\pi \in \hat{M}_u$ for which there exists a $w \in W$ such that $\mathscr{H}_{w\pi}^{-\infty}$ contains non-trivial $(M \cap H)$-fixed elements. Then by the above lemma, a representation $\mathrm{Ind}_P^G(\xi \otimes e^\lambda \otimes 1)$ can only be expected to have H-fixed distribution vectors with non-trivial restriction to $\mathscr{O}(P)$ if $\xi \in \hat{M}_{ps}$ and $\lambda | \mathfrak{a} \cap \mathfrak{h} = 0$. This motivates the following definition.

DEFINITION 3.4. — *Let* $P \in P_\sigma(A_q)$. *We call the series of representations* $\mathrm{Ind}_P^G(\xi \otimes e^\lambda \otimes 1)$, $[\xi] \in \hat{M}_{ps}$, $\lambda \in \mathfrak{a}_{qc}^*$, *the (non-unitary) principal series for* G/H.

Remarks. — Via the form B we view $\mathfrak{a}_{qc}^*$ as a subspace of $\mathfrak{a}_c^*$. Thus $\mathfrak{a}_{qc}^* = \{ \lambda \in \mathfrak{a}_c^* ; \lambda = 0 \text{ on } \mathfrak{a}_q^\perp = \mathfrak{a} \cap \mathfrak{h} \}$.

If $Q \in \mathscr{P}_\sigma(A_q)$, then $Q = P^u$ for some $u \in N_K(\mathfrak{a}_q)$ (Lemma 2.8). The operator $L(u) : C^\infty(P : \xi : \lambda) \to C^\infty(Q : u\xi : u\lambda)$ defined by

$$(3.9) \qquad (L(u)f)(x) = f(u^{-1}x)$$

defines an equivalence between $\mathrm{Ind}_P^G(\xi \otimes e^\lambda \otimes 1)$ and $\mathrm{Ind}_Q^G(u\xi \otimes e^{u\lambda} \otimes 1)$. Thus up to equivalence the above series is independent of the choice of P. Of course a similar argument shows it to be independent of the choice of the maximal abelian subspace $\mathfrak{a}_q$ of $\mathfrak{p} \cap \mathfrak{q}$.

The following lemma gives, among others, a different characterization of the set $\hat{M}_{ps}$. We write $K_M = K \cap M$.

LEMMA 3.5. — (i) *Let* $[\xi] \in \hat{M}_u$. *Then* $[\xi] \in \hat{M}_{ps}$ *if and only if there exists* $w \in W$ *such that* $w[\xi]$ *belongs to the discrete series of* $M/M \cap H$.

(ii) *If* ξ *is a discrete series representation of* $M/M \cap H$, *then* $\dim(\xi) < \infty$, $\xi | (\mathfrak{m} \cap \mathfrak{p}) = 0$ *and* $\xi | K_M$ *is irreducible. Restriction to* K_M *induces a bijection between the discrete series of* $M/M \cap H$ *and those of* $K_M/K_M \cap H$.

Proof. — We first prove (ii). Since $\mathfrak{a}_q$ is maximally abelian in $\mathfrak{p} \cap \mathfrak{q}$ we have $\mathfrak{m} \cap \mathfrak{p} = \mathfrak{m} \cap \mathfrak{p} \cap \mathfrak{h}$. Since $M = K_M \exp(\mathfrak{m} \cap \mathfrak{p})$ it follows that $\exp(\mathfrak{m} \cap \mathfrak{p}) = \exp(\mathfrak{m} \cap \mathfrak{p} \cap \mathfrak{h})$ acts trivially on $L^2(M/M \cap H)$. Moreover, the map $K_M \to M$ induces a bijection $\varphi : K_M/K_M \cap H \to M/M \cap H$ and by pull-back an isometric isomorphism φ^* from $L^2(M/M \cap H)$ onto $L^2(K_M/K_M \cap H)$ (provided we normalize the invariant measures appropriately). The map φ^* is K_M-equivariant and since $\exp(\mathfrak{m} \cap \mathfrak{p})$ acts trivially on $L^2(M/M \cap H)$ it sets up a $1-1$ correspondence between the M-invariant subspaces of $L^2(M/M \cap H)$ and the K_M-invariant subspaces of $L^2(K_M/K_M \cap H)$. All assertions now follow.

It remains to prove (i). For this it suffices to show that $(\mathscr{H}_\xi^{-\infty})^H \neq 0$ if and only if ξ belongs to the discrete series of $M/M \cap H$. If $\eta \in (\mathscr{H}_\xi^{-\infty})^H$, $\eta \neq 0$, then the map $j: \mathscr{H}_\xi^\infty \to C^\infty(M/M \cap H)$ defined by $j(v)(m) = \overline{\langle \eta, \pi(m^{-1}) v \rangle}$ is an equivariant continuous and complex linear embedding. Since $\mathfrak{m} \cap \mathfrak{p}$ acts trivially on $C^\infty(M/M \cap H)$, $\xi \mid K_M$ is irreducible. In particular $\dim(\xi) < \infty$, hence $\mathscr{H}_\xi^\infty = \mathscr{H}_\xi$ and $im(j)$ is a closed subspace of $L^2(M/M \cap H)$. It follows that ξ belongs to the discrete series for $M/M \cap H$. Conversely, if ξ belongs to the discrete series for $M/M \cap H$, then obviously $(\mathscr{H}_\xi^{-\infty})^H = (\mathscr{H}_\xi)^H = (\mathscr{H}_\xi)^{K_M \cap H} \neq 0$. ∎

Remark 3.6. — In particular it follows that $\hat{M}_{ps} \subset \hat{M}_{fu}$, where $\hat{M}_{fu}$ denotes the set of equivalence classes of *finite dimensional* irreducible unitary representations of M.

We conclude this section with a result on the irreducibility of the unitary principal series for G/H.

PROPOSITION 3.7. — *Let* $P \in \mathscr{P}_\sigma(A_q)$, $\xi \in \hat{M}_{ps}$, *and* $\lambda \in i\mathfrak{a}_q^*$. *If* $\langle \lambda, \alpha \rangle \neq 0$ *for all* $\alpha \in \Sigma$, *then the unitary principal series representation* $\mathrm{Ind}_P^G(\xi \otimes e^\lambda \otimes 1)$ *is irreducible.*

Proof. — Since ξ is finite dimensional unitary whereas centre $(\mathfrak{m}) \cap \mathfrak{p} = 0$ we have that $\xi = 1$ on $M \cap A_0$. It follows that the restriction of $\xi \otimes e^\lambda \otimes 1$ to A_0 equals e^λ. If $w \in W$ does not centralize $\mathfrak{a}$, then $w\lambda \neq \lambda$ in view of the hypothesis. The result now follows from ([Br 56], p. 203, Théorème 4). ∎

Remark. — Using a different technique and under the conditions that G is connected semisimple and admits a simply connected complexification and finally that $H = G^\sigma$, [Ol 87] proves the irreducibility of $\mathrm{Ind}_P^G(\xi \otimes e^\lambda \otimes 1)$ ($P \in \mathscr{P}_\sigma(A_q)$ and $\xi \in \hat{M}_{ps}$) for λ in the complement of a countable union of complex algebraic hypersurfaces in $\mathfrak{a}_{qc}^*$. In particular this implies irreducibility for almost all $\lambda \in i\mathfrak{a}_q$.

4. Intertwining operators

Retaining the notations of paragraph 3 recall that any parabolic subgroup $P \in \mathscr{P}_\sigma(A_q)$ has the same Levi component MA. Fix an irreducible unitary representation ξ of M in a finite dimensional Hilbert space $\mathscr{H}_\xi$. Then for generic $\lambda \in \mathfrak{a}_c^*$, the induced representations $\mathrm{Ind}_P^G(\xi \otimes e^\lambda \otimes 1)$ ($P \in \mathscr{P}_\sigma(A_q)$) are related by intertwining operators (*cf.* [K-S 80]). The purpose of this section is to study these operators when λ varies in the (non-generic) subspace $\mathfrak{a}_{qc}^*$ of $\mathfrak{a}_c^*$ (*i.e.* $\lambda \mid \mathfrak{a} \cap \mathfrak{h} = 0$).

Let $P_i = MAN_i$ ($i = 1, 2$) be parabolic subgroups (from their Langlands decomposition we read off that they belong to $P_\sigma(A_q)$; *cf.* Section 2). If $\lambda \in \mathfrak{a}_{qc}^*$, then we formally define an intertwining operator $A(P_2:P_1:\xi:\lambda)$ from $C^\infty(P_1:\xi:\lambda)$ into $C^\infty(P_2:\xi:\lambda)$ by

$$(4.1)\qquad A(P_2:P_1:\xi:\lambda)f(x)=\int_{N_2\cap\bar{N}_1} f(nx)\,dn.$$

Here dn denotes the bi-invariant Haar measure of the nilpotent group $N_2\cap\bar{N}_1$, normalized as in ([K-S 80], Section 4).

We first investigate absolute convergence of the integral (4.1). If $\lambda\in\mathfrak{a}^*_{0c}$, let H_λ denote the element of $\mathfrak{a}_{0c}$ determined by $\langle H_\lambda, H\rangle=\lambda(H)$ for all $H\in\mathfrak{a}_{0c}$. Given a linear subspace $\mathfrak{b}$ of $\mathfrak{a}_0$ we identify $\mathfrak{b}^*_c$ with the subspace $\{\lambda\in\mathfrak{a}^*_{0c}; H_\lambda\in\mathfrak{b}_c\}$ of $\mathfrak{a}^*_{0c}$.

Select a choice of positive roots for $\mathfrak{a}_0\cap\mathfrak{m}$ in $\mathfrak{m}$ and define ρ_M as half the sum of these positive roots, counting multiplicities.

Given $\beta\in\Sigma$, we define

$$(4.2)\qquad C'_\beta=\max\{\rho_M(H_\alpha);\alpha\in\Sigma_0,\alpha|\mathfrak{a}_q=\beta\}.$$

Moreover if $P_1, P_2\in\mathscr{P}_\sigma(A_q)$ we define the open subset $\mathscr{A}(P_2|P_1)$ of $\mathfrak{a}^*_{qc}$ to be the set of all $\lambda\in\mathfrak{a}^*_{qc}$ such that

$$(4.3)\qquad \langle \operatorname{Re}\lambda,\beta\rangle>C'_\beta$$

for all $\beta\in\Sigma$ with $\mathfrak{g}^\beta\subset\bar{\mathfrak{n}}_2\cap\mathfrak{n}_1$.

PROPOSITION 4.1. — *Let P_1, $P_2\in\mathscr{P}_\sigma(A_q)$ have Langlands decompositions $P_i=MAN_i$ ($i=1, 2$). Then for $\lambda\in\mathscr{A}(P_2|P_1)$ the integral in* (4.1) *converges absolutely and defines a continuous linear map $A(P_2:P_1:\xi:\lambda)$ from $C^\infty(P_1:\xi:\lambda)$ into $C^\infty(P_2:\xi:\lambda)$.*

Proof. — For γ an $\mathfrak{a}$-weight in $\mathfrak{g}$, let C_γ be defined as in ([K-S 80], Lemma 6.5), *i.e.* $C_\gamma=\max\{\rho_M(H_\alpha)\}$, where the maximum is taken over all $\alpha\in\Sigma_0$ with $\alpha|\mathfrak{a}=\gamma$. Then obviously $C'_{\gamma|\mathfrak{a}_q}\geq C_\gamma$ for each $\mathfrak{a}$-weight γ in $\mathfrak{g}$. Now let $\lambda\in\mathscr{A}(P_2|P_1)$ and fix an $\mathfrak{a}$-weight γ such that $\mathfrak{g}^\gamma\subset\bar{\mathfrak{n}}_2\cap\mathfrak{n}_1$. Let $\beta=\gamma|\mathfrak{a}_q$. Then $\mathfrak{g}^\beta\cap(\bar{\mathfrak{n}}_2\cap\mathfrak{n}_1)\neq 0$. Since $\bar{\mathfrak{n}}_2\cap\mathfrak{n}_1$ is a direct sum of $\mathfrak{a}_q$-weight spaces, this implies that $\mathfrak{g}^\beta\subset\bar{\mathfrak{n}}_2\cap\mathfrak{n}_1$. It follows that $\langle\operatorname{Re}\lambda,\gamma\rangle=\langle\operatorname{Re}\lambda,\beta\rangle>C'_\beta\geq C_\gamma$. Hence the conditions of ([K-S 80], Theorem 6.6) are fulfilled and the result follows. ■

We define $C^\infty(K:K_M:\xi)$ (or briefly $C^\infty(K:\xi)$) to be the space of all C^∞-functions $f:K\to\mathscr{H}_\xi$, transforming according to the rule

$$(4.4)\qquad f(mk)=\xi(m)f(k)\qquad (k\in K, m\in K_M).$$

Restriction to K induces a topological isomorphism from $C^\infty(P:\xi:\lambda)$ onto $C^\infty(K:\xi)$ and by transportation we obtain a (λ-dependent) representation of G in $C^\infty(K:\xi)$, called the "compact picture" of $\operatorname{Ind}_P^G(\xi\otimes e^\lambda\otimes 1)$.

THEOREM 4.2. — *Let P_1, P_2 be as in Proposition* 4.1, *and let $f\in C^\infty(K:\xi)$. Then as a mapping into $C^\infty(K:\xi)$, the function $\lambda\to A(P_2:P_1:\xi:\lambda)f$, initially defined for $\lambda\in\mathscr{A}(P_2|P_1)$, extends meromorphically to $\mathfrak{a}^*_{qc}$. Moreover for each $\lambda_0\in\mathfrak{a}^*_{qc}$ there is an open neighbourhood $N(\lambda_0)$ of λ_0 in $\mathfrak{a}^*_{qc}$ and a non-zero holomorphic function $\varphi:N(\lambda_0)\to\mathbb{C}$*

such that the map $(\lambda, g) \to \varphi(\lambda) A(P_2 : P_1 : \xi : \lambda) f$ *of* $N(\lambda_0) \times C^\infty(K : \xi)$ *into* $C^\infty(K : \xi)$ *is continuous.*

Remark. — We say that a (densely defined) mapping F from a complex manifold S into a Fréchet space E is meromorphic if for every $a \in S$ there exists an open neighbourhood $U \ni a$ and a non-zero holomorphic function $\varphi : U \to \mathbb{C}$ such that φF is holomorphic as a map $U \to E$.

Before giving the proof of Theorem 4.2 we prove a corollary which will be useful for the application to distributions at the end of this section. If $r \in \mathbb{N}$, let $C^r(K : \xi)$ denote the Banach space of r-times continuously differentiable functions $K \to \mathscr{H}_\xi$ transforming according to (4.4).

COROLLARY 4.3. — *Let the hypotheses of Theorem* 4.2 *be fulfilled and suppose that* $\lambda_0 \in \mathfrak{a}_{qc}^*$. *If* $N(\lambda_0)$ *is chosen sufficiently small, then there exists a* $q \in N$ *such that*

(i) *The map* $(\lambda, g) \to \varphi(\lambda) A(P_2 : P_1 : \xi : \lambda) g$ *extends uniquely to a continuous map* $N(\lambda_0) \times C^q(K : \xi) \to C(K : \xi)$.

(ii) *The above extension is holomorphic in the first variable. For every* $p \in N$ *it maps* $N(\lambda_0) \times C^{p+q}(K : \xi)$ *continuously into* $C^p(K : \xi)$. *The induced map from* $N(\lambda_0)$ *into the Banach space* $B(C^{p+q}, C^p)$ *of bounded linear maps from* C^{p+q} *into* C^p *(endowed with the operator norm) is holomorphic.*

Proof. — (i) Shrinking $N(\lambda_0)$ if necessary, it follows from the above theorem that $(\lambda, g) \to \varphi(\lambda) A(P_2 : P_1 : \xi : \lambda) g(1)$ extends uniquely to a continuous map $N(\lambda_0) \times C^q(K : \xi) \to \mathbb{C}$ for some $q \in \mathbb{N}$. By holomorphic continuation $A(P_2 : P_1 : \xi : \lambda)$ intertwines the G-actions for every $\lambda \in \mathfrak{a}_{qc}^*$ which is not a pole, and so does $\varphi(\lambda) A(P_2 : P_1 : \xi : \lambda)$ for every $\lambda \in N(\lambda_0)$. Since in particular the K-actions are intertwined, we obtain (i).

(ii) Denote the extension $N(\lambda_0) \times C^q(K : \xi) \to C(K : \xi)$ by Ψ. Fix local coordinates $z = (z_1, \ldots, z_n)$ in $\mathfrak{a}_{qc}^*$ such that λ_0 corresponds to 0. Select $\varepsilon > 0$ such that the closure of $D(\varepsilon)^n = \{ z \in \mathbb{C}^n ; |z_j| < \varepsilon, 1 \leqq j \leqq n \}$ is contained in $N(\lambda_0)$. Then for a fixed $f \in C^\infty(K : \xi)$ we have absolutely converging power series

$$\Psi(z, f) = \sum_\alpha c_\alpha(f) z^\alpha, \tag{4.5}$$

for $z \in D(\varepsilon)^n$. Here the summation involves all multi-indices $\alpha \in \mathbb{N}^n$, and we have used the multi-index notation $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$. From the Cauchy integral formulas for the $c_\alpha(f) \in C(K, \mathscr{H}_\xi)$ we deduce that they depend linearly on f and satisfy the Cauchy estimates

$$\| c_\alpha(f) \| \leqq \varepsilon^{-|\alpha|} \sup \{ \| \Psi(z, f) \| ; z \in \overline{D(\varepsilon)^n} \} \tag{4.6}$$

(here $\| . \|$ denotes the sup-norm on $C(K, \mathscr{H}_\xi)$). It now follows from (i) that the c_α extend (uniquely) to continuous linear maps $C^q(K : \xi) \to C(K : \xi)$. Hence (4.5) holds with absolutely converging power series for $f \in C^q(K : \xi)$, $z \in D(\varepsilon)^n$. Shrinking $N(\lambda_0)$ if necessary we thus obtain the first assertion of (ii). In fact, using (4.5,6) and estimating

$|\Psi(z,f)|$ in terms of the C^q-norm of f we even obtain that the map $\tilde{\Psi}: D(\varepsilon)^n \to B(C^q, C)$ defined by $\tilde{\Psi}(z) f = \Psi(z, f)$, is holomorphic in the Banach sense.

Since $\varphi(\lambda) A(P_2 : P_1 : \xi : \lambda)$ intertwines the G-actions, Ψ maps $N(\lambda_0) \times C^{p+q}$ continuously into C^p. Let $f \in C^{p+q}$, and fix an element v in the universal enveloping algebra $U(\mathfrak{k})$ of $\mathfrak{k}$ with $\operatorname{order}(v) \leqq p$. Then $R(v) \Psi(\lambda, f) = \Psi(\lambda, R(v) f)$ depends holomorphically on λ as a function from $N(\lambda_0)$ into $C(K : \xi)$. It follows that $\Psi : N(\lambda_0) \times C^{p+q} \to C^p$ is continuous and in addition holomorphic in the first variable. Finally, applying Cauchy's integral formula in the same manner as above, we see that the latter statement implies that the induced map $N(\lambda_0) \to B(C^{p+q}, C^p)$ is holomorphic. ■

Proof of Theorem 4.2. — First assume that $\sigma = \theta$. Then $A = A_0$ and $M = M_0 = Z_K(a_0)$, *i.e.* $\mathscr{P}_\sigma(A_q)$ consists of the minimal parabolics with split component A_0. If G has split rank one the result follows from ([K-S 71], Theorem 2). This implies the result for groups of higher split rank if P_1 and P_2 are adjacent ([*loc. cit.* Lemma 5.6]). For general P_1 and P_2 the result follows from Schiffmann's product decomposition (*cf.* [S 71], Theorem 1.1) when Corollary 4.3 is applied (see also the remarks in ([K-S 71], p. 563).

The proof in the general case is now essentially contained in ([K-S 80], Theorem 6.6). Following [*loc. cit.*, Section 6], let $P_M = M_0 A_M N_M$ be the standard minimal parabolic subgroup of M with respect to the selected choice of positive roots for $\mathfrak{a}_0 \cap \mathfrak{m}$ in $\mathfrak{m}$. We now embed ξ in a principal series representation of M. Since ξ is finite dimensional and unitary, the embedding is very special.

LEMMA 4.4. — *Let* $\sigma = \xi | M_0$. *Then* $\sigma \in \hat{M}_0$ *and there exists a* M-*equivariant embedding i of* $\mathscr{H}_\xi$ *into* $C^\infty(M : P_M : \sigma : -\rho_M)$.

Proof. — Since ξ is finite dimensional and unitary, whereas $\operatorname{centre}(\mathfrak{m}) \cap \mathfrak{p} = 0$, it follows that ξ is trivial on $\mathfrak{a}_M$ and all its root spaces in $\mathfrak{m}$. Using density of $\bar{N}_M M_0 A_M N_M$ in M we infer that $\sigma = \xi | M_0$ is irreducible. The embedding may be defined by $i(v)(x) = \xi(x) v$, for $v \in \mathscr{H}_\xi$, $x \in M$. ■

We proceed with our proof of Theorem 4.2. Since the *full* representation space $\mathscr{H}_\xi$ is equivariantly embedded onto a closed subspace of $C^\infty(M : P_M : \sigma : -\rho_M)$ we may completely avoid the closure operation occurring in ([K-S 80], Lemma 6.1) and proceed directly. Instead of [*loc. cit.*, Lemma 6.2] we have

LEMMA 4.5. — *For* $j \in \{1, 2\}$ *let* $(P_j)_p$ *denote the minimal parabolic subgroup* $P_M A N_j$ *of* G. *If* $\lambda \in \mathfrak{a}_{qc}^*$ *then the embedding i of* $\mathscr{H}_\xi$ *into* $C^\infty(M : P_M : \sigma : -\rho_M)$ *induces a* G-*equivariant continuous linear embedding* $i^\#$ *of* $C^\infty(G : P_j : \xi : \lambda)$ *onto a closed subspace of* $C^\infty(G : (P_j)_p : \sigma : \lambda - \rho_M)$. *Moreover, as a map* $C^\infty(K : K_M : \xi) \to C^\infty(K : M_0 : \sigma)$ *the embedding* $i^\#$ *is independent of j and* λ.

Proof. — The map $i^\#$ is defined by $(i^\# f)(x) = ev_1 \circ i \circ f(x)$. Here $ev_1 : C^\infty(M : P_M : \sigma : -\rho_M) \to \mathscr{H}_\sigma$ *denotes evaluation at the identity.* We leave it to the reader to verify the statements of the lemma. ■

As in [*loc. cit.*, p. 35] we now have a commutative diagram for $\lambda \in \mathscr{A}(P_2 | P_1)$:

$$\begin{array}{ccc} C^\infty(K:K_M:\xi) & \xrightarrow{A(P_2:P_1:\xi:\lambda)} & C^\infty(K:K_M:\xi) \\ {\scriptstyle i^\#}\downarrow & & \downarrow{\scriptstyle i^\#} \\ C^\infty(K:M_M:\sigma) & \xrightarrow{A((P_2)_p:(P_1)_p:\sigma:\lambda-\rho_M)} & C^\infty(K:M_M:\sigma) \end{array}$$

Now $A((P_2)_p:(P_1)_p:\sigma:\lambda-\rho_M)$ admits a meromorphic continuation, by the first part of our proof. From the fact that $i^\#$ has closed image independent of λ, it follows by holomorphic continuation that $im(i^\#)$ is stable for this meromorphic continuation. The assertions of the theorem now follow from the corresponding results for $A((P_2)_p:(P_1)_p:\sigma:\lambda-\rho_M)$ by an application of the closed graph theorem. ■

The following transformation properties for intertwining operators are straightforward consequences of ([K-S 80], Proposition 7.1 and Corollary 7.7).

PROPOSITION 4.6. — *The analytic continuation of the operators* $A(P_2:P_1:\xi:\lambda)$ *have the following transformation properties as continuous linear operators from* $C^\infty(P_1:\xi:\lambda)$ *into* $C^\infty(P_2:\xi:\lambda)$ $(P_1, P_2 \in \mathscr{P}_\sigma(A_q), \lambda \in \mathfrak{a}_{qc}^*)$:

(i) *If* $P_3 \in \mathscr{P}_\sigma(A_q)$ *is such that* $\mathfrak{n}_2 \cap \bar{\mathfrak{n}}_1 \supseteq \mathfrak{n}_3 \cap \bar{\mathfrak{n}}_1$ *then*

$$A(P_2:P_1:\xi:\lambda) = A(P_2:P_3:\xi:\lambda)\, A(P_3:P_1:\xi:\lambda);$$

(ii) $A(P_2:P_1:\xi:\lambda)^* = A(P_1:P_2:\xi:-\bar{\lambda})$, *where* * *denotes the adjoint with respect to the pairing* (4.2).

Moreover, from ([K-S 80], Proposition 7.3) we immediately obtain:

PROPOSITION 4.7. — *Let* $P_i = MAN_i$ $(i=1, 2)$ *be parabolic subgroups in* $\mathscr{P}_\sigma(A_q)$. *Then there exists a scalar-valued function* $\eta(P_2:P_1:\xi:\lambda)$ *meromorphic in* $\lambda \in \mathfrak{a}_{qc}^*$ *such that*

$$A(P_1:P_2:\xi:\lambda)\, A(P_2:P_1:\xi:\lambda) = \eta(P_2:P_1:\xi:\lambda)\, I.$$

The function η *satisfies* $\eta(P_2:P_1:\xi:\lambda) = \eta(P_1:P_2:\xi:\lambda)$.

PROPOSITION 4.8. — *Let* P_1, $P_2 \in \mathscr{P}_\sigma(A_q)$. *If* $\xi \in \hat{M}_{fu}$, *define* $\eta_\xi(\lambda) = \eta(P_2:P_1:\xi:\lambda)$. *Then*

(*a*) η_ξ *depends only on the class of* ξ,

(*b*) $\eta_\xi(\lambda) = \eta_\xi(-\bar{\lambda})$,

(*c*) $\eta_\xi(\lambda) > 0$ *for all* $\lambda \in i\,\mathfrak{a}_q^*$.

Proof. — Of course η_ξ is the restriction of the corresponding function in ([K-S 80], (7.5)) from $\mathfrak{a}_c^*$ to $\mathfrak{a}_{qc}^*$. The results now follow from ([*loc.cit.*], Theorem 7.6 and Proposition 7.4). For the strict inequality in (*c*) see also the note at the bottom of p. 42 in [K-S 80]. ■

In particular we deduce that $\eta(P_2:P_1:\xi)$ does not vanish identically on $\mathfrak{a}_{qc}^*$. From Proposition 4.7 and ([K-S 80], Theorems 6.6, 7.6 (*v*)) we now obtain:

COROLLARY 4.9. — *There exists a locally finite union* $\mathfrak{s}$ *of complex hyperplanes in* $\mathfrak{a}_{qc}^*$ *such that for* $\lambda_0 \in \mathfrak{a}_{qc}^* \setminus \mathfrak{s}$ *the meromorphic continuations of the intertwining operators* $A(P_2:P_1:\xi:\lambda)$ *and* $A(P_1:P_2:\xi:\lambda)$ *have no pole and are invertible for* $\lambda=\lambda_0$.

For later purposes we also list how intertwining operators behave under conjugation by elements of $N_K(\mathfrak{a}_q)$. Recall that $N_K(\mathfrak{a}_q)$ operates on $\mathscr{P}_\sigma(A_q)$ by conjugation. If $P \in \mathscr{P}_\sigma(A_q)$, $w \in N_K(\mathfrak{a}_q)$, then left translation $L(w)$ by w^{-1} defines an intertwining operator

$$L(w): C^\infty(G:P:\xi:\lambda) \to C^\infty(G:wPw^{-1}:w\xi:w\lambda).$$

The following lemma is easy to verify.

LEMMA 4.10. — *Let* $P_1, P_2 \in \mathscr{P}_\sigma(A_q)$, $\lambda \in \mathfrak{a}_{qc}^*$. *Then*

$$L(w) \circ A(P_2:P_1:\xi:\lambda) = A(wP_2w^{-1}:wP_1w^{-1}:w\xi:w\lambda) \circ L(w),$$

for $w \in N_K(\mathfrak{a}_q)$.

In the final part of this section we extend the definition of intertwining operators to distributions. If M is a C^∞-manifold, $\mathscr{D}'(M)$ denotes the space of distributions on M, *i.e.* the topological linear dual of the locally convex space of C_c^∞-densities on M; here we follow ([Hör 83] § 6.3) to ensure that $C(M)$ is naturally embedded into $\mathscr{D}'(M)$. In view of (3.4), we may identify $\mathscr{D}'(P:\xi:\lambda)$ $(P \in \mathscr{P}_\sigma(A_q))$ with the space of distributions $\varphi \in \mathscr{D}'(G) \otimes_{\mathbf{C}} \mathscr{H}_\xi$ transforming according to the rule

$$(L_{man}^{-1} \otimes 1)\,\varphi = (1 \otimes a^{\lambda+\rho_P}\xi(m))\,\varphi,$$

for $m \in M$, $a \in A$, $n \in N_P$; here L denotes the left regular representation. We define $\mathscr{D}'(K:\xi)$ to be the topological anti-linear dual of $C^\infty(K:\xi)$. Via the pairing (3.2) we may also view it as the space of distributions $f \in \mathscr{D}'(K) \otimes \mathscr{H}_\xi$ transforming according to the rule (4.4). Recall that restriction to K induces a topological isomorphism from $C^\infty(P:\xi:-\bar{\lambda})$ onto $C^\infty(K:\xi)$. By transposition we obtain a linear isomorphism

$$\mathscr{D}'(K:\xi) \xrightarrow{\simeq} \mathscr{D}'(P:\xi:\lambda). \tag{4.7}$$

Its inverse extends the restriction map $C^\infty(P:\xi:\lambda) \to C^\infty(K:\xi)$. We now topologize $\mathscr{D}'(K:\xi)$ as follows. If $q \in \mathbb{N}$, let $C^q(K:\xi)$ denote the space of q-times continuously differentiable functions $\varphi: K \to \mathscr{H}_\xi$ transforming according to (4.4), provided with the usual structure of Banach space. Its (Banach-) anti-linear dual $\mathscr{D}_q'(K:\xi)$ corresponds to the space of distributions in $\mathscr{D}'(K:\xi)$ of order $\leqq q$. We topologize $\mathscr{D}'(K:\xi) = \bigcup_{q=0}^{\infty} \mathscr{D}_q'(K:\xi)$ by taking the inductive limit of locally convex topological vector spaces. The resulting topology actually is the strong dual topology (by a standard application of Ascoli's theorem the embeddings $C^q \to C^{q+1}$ are compact. Now use [Kom 67] (Theorem 11). Similarly we topologize $\mathscr{D}'(P:\xi:\lambda)$ by taking the inductive limit of the Banach spaces $\mathscr{D}'_q(\mathrm{P}:\xi:\lambda)$ in the category of locally convex topological vector spaces. Thus (4.10) becomes an isomorphism of locally convex spaces.

Let now P_1, $\mathrm{P}_2\in\mathscr{P}_\sigma(\mathrm{A}_q)$ and assume that $\lambda\in\mathfrak{a}^*_{qc}$ is not a pole for the intertwining operator $\mathrm{A}(\mathrm{P}_1:\mathrm{P}_2:\xi:-\bar\lambda)$. Then this operator maps $\mathrm{C}^\infty(\mathrm{P}_2:\xi:-\bar\lambda)$ continuously into $\mathrm{C}^\infty(\mathrm{P}_1:\xi:-\bar\lambda)$. Its transposed $\mathrm{A}(\mathrm{P}_1:\mathrm{P}_2:\xi:-\bar\lambda)^*$ maps $\mathscr{D}'(\mathrm{P}_1:\xi:\lambda)$ into $\mathscr{D}'(\mathrm{P}_2:\xi:\lambda)$. In view of Proposition 4.6 (ii), the operator $\mathrm{A}(\mathrm{P}_1:\mathrm{P}_2:\xi:-\bar\lambda)$ may be viewed as an extension of the operator $\mathrm{A}(\mathrm{P}_2:\mathrm{P}_1:\xi:\lambda)$ from smooth functions to distributions. Hence we also write $\mathrm{A}(\mathrm{P}_2:\mathrm{P}_1:\xi:\lambda)$ for $\mathrm{A}(\mathrm{P}_1:\mathrm{P}_2:\xi:-\bar\lambda)^*$. From Corollary 4.3 we now easily deduce the following.

PROPOSITION 4.11. — *Let P_1, $\mathrm{P}_2\in\mathscr{P}_\sigma(\mathrm{A}_q)$ have Langlands decompositions $\mathrm{P}_i=\mathrm{MAN}_i$ $(i=1,2)$ and suppose that $\lambda_0\in\mathfrak{a}^*_{qc}$. Then there exists an open neighbourhood $\mathrm{N}(\lambda_0)$ of λ_0 in $\mathfrak{a}^*_{qc}$, a holomorphic function $\varphi:\mathrm{N}(\lambda_0)\to\mathbb{C}$ and a constant $q\in\mathbb{N}$ such that for every $p\in\mathbb{N}$:*

(i) *the mapping $(\lambda, u)\to\varphi(\lambda)\,\mathrm{A}(\mathrm{P}_2:\mathrm{P}_1:\xi:\lambda)\,u$ maps $\mathrm{N}(\lambda_0)\times\mathscr{D}'_p(\mathrm{P}_1:\xi:\lambda)$ continuously into $\mathscr{D}'_{p+q}(\mathrm{P}_2:\xi:\lambda)$,*

(ii) *the induced map from $\mathrm{N}(\lambda_0)$ into the space $\mathrm{B}(\mathscr{D}'_p,\mathscr{D}'_{p+q})$ of bounded linear operators from $\mathscr{D}'_p(\mathrm{K}:\xi)$ into $\mathscr{D}'_{p+q}(\mathrm{K}:\xi)$ is holomorphic in the Banach sense.*

Finally notice that by the above, Propositions 4.6,7 and Lemma 4.10 extend to distributions.

We conclude this section with three results that will be needed at a later stage.

LEMMA 4.12. — *Let $\mathrm{P}_i=\mathrm{MAN}_i$ $(i=1, 2)$ be parabolic subgroups in $\mathscr{P}_\sigma(\mathrm{A}_q)$ and suppose that $\lambda\in\mathfrak{a}^*_{qc}$ is not a pole for $\mathrm{A}(\mathrm{P}_2:\mathrm{P}_1:\xi:\lambda)$. If $\varphi\in\mathscr{D}'(\mathrm{P}_1:\xi:\lambda)$, then*

$$\operatorname{supp}\mathrm{A}(\mathrm{P}_2:\mathrm{P}_1:\xi:\lambda)\,\varphi\subset cl((\mathrm{N}_2\cap\bar{\mathrm{N}}_1)\operatorname{supp}\varphi).$$

Proof. — If $\lambda\in\mathscr{A}(\mathrm{P}_2|\mathrm{P}_1)$ and $\varphi\in\mathrm{C}^\infty(\mathrm{P}_1:\xi:\lambda)$, then the statement easily follows from the absolutely convergent integral representation (4.1). If λ_0 is not a pole and $\varphi\in\mathrm{C}^\infty(\mathrm{P}_1:\xi:\lambda_0)$, then we define a holomorphic family $\varphi_\lambda\in\mathrm{C}^\infty(\mathrm{P}_1:\xi:\lambda)$ by $\varphi_\lambda|\mathrm{K}=\varphi|\mathrm{K}$. Applying holomorphic continuation to $\mathrm{A}(\mathrm{P}_2:\mathrm{P}_1:\xi:\lambda)\,\varphi_\lambda$ we obtain the result for φ. An easy density argument completes the proof. ■

To formulate the next result we need some definitions and notations. If $\Omega\subset\mathrm{G}$ is a left P-invariant open subset $(\mathrm{P}\in\mathscr{P}_\sigma(\mathrm{A}_q))$, then $\mathrm{C}^\infty(\Omega:\mathrm{P}:\xi:\lambda)$ denotes the space of C^∞-functions $\Omega\to\mathscr{H}_\xi$ transforming according to the rule (3.1) for $x\in\Omega$. Similarly, if U is a left $\mathrm{K_M}$-invariant open subset of K, then $\mathrm{C}^\infty(\mathrm{U}:\mathrm{K_M}:\xi)$ denotes the space of C^∞-functions $\mathrm{U}\to\mathscr{H}_\xi$ transforming according to (4.4).

A map Φ from *a* C^∞-(resp. complex) manifold S into $\mathscr{D}'(\mathrm{K}:\xi)$ is called smooth (resp. holomorphic) if for every $a\in\mathrm{S}$ there exists an open neighbourhood $\mathrm{U}\ni a$ and an integer $q\geqq 0$ such that Φ map U smoothly (resp. holomorphically) into the Banach space $\mathscr{D}'_q(\mathrm{K}:\xi)$. A densely defined map Φ from a complex manifold S into $\mathscr{D}'(\mathrm{K}:\xi)$ is called meromorphic if for every $a\in\mathrm{S}$ there exists an open neighbourhood $\mathrm{U}\ni a$ and a holomorphic function $\varphi:\mathrm{U}\to\mathbb{C}$ such that $\varphi.\Phi$ is holomorphic on U. Finally recall the definition of $\mathscr{O}(\mathrm{P})$ above (3.8).

LEMMA 4.13. — *Let* $P \in \mathscr{P}_\sigma(A_q)$, *and suppose that an open set* $\Omega \subset \mathbb{C}^n$ *and a holomorphic map* $z \to \lambda(z)$ *from* Ω *into* $\mathfrak{a}_{qc}^*$ *are given and moreover for each* $z \in \Omega$ *a distribution* $u_z \in \mathscr{D}'(G:P:\xi:\lambda(z))$. *If* $(z,h) \to R_h u_z$, *viewed as a map* $\Omega \times H \to \mathscr{D}'(K:\xi)$ *is smooth and in addition holomorphic in its first variable, then* u_z *is smooth on* $\mathscr{O}(P)$ *for every* $z \in \Omega$. *Moreover,* $z \to u_z | (\mathscr{O}(P) \cap K)$ *is holomorphic as a mapping from* Ω *into the Fréchet space* $C^\infty(\mathscr{O}(P) \cap K : K_M : \xi)$.

Proof. — Fix $z_0 \in \Omega$ and put $\lambda_0 = \lambda(z_0)$. We first reduce to the case that $\lambda(z) = \lambda_0$ for all $z \in \Omega$, as follows. For $z \in \Omega$ we define $U_z \in \mathscr{D}'(G:P:\xi:\lambda_0)$ by $r_{\lambda_0}(U_z) = r_\lambda(u_z)$, where r_λ denotes the inverse of the topological isomorphism (4.7). Define real analytic maps ν_P, h_P, μ_P and κ_p from G into N_P, A, $\exp(\mathfrak{m} \cap \mathfrak{p})$ and K respectively by

$$x = \nu_P(x)\, h_P(x)\, \mu_P(x)\, \kappa_P(x) \qquad (x \in G) \tag{4.8}$$

Then by a straightforward calculation one checks that $r_{\lambda_0}(R_h U_z) = \mu_{h,z}\, r_{\lambda(z)}(R_h u_z)$, where $\mu_{h,z} \in C^\infty(K)$ is given by $\mu_{h,z}(x) = h_P(xh)^{\lambda_0 - \lambda(z)}$.

It now easily follows that U_z fulfills the hypotheses of our lemma, and we are done if we can show that the map $z \to U_z | \mathscr{O}(P)$ from Ω into $C^\infty(\mathscr{O}(P):P:\xi:\lambda_0)$ is holomorphic. Localizing in Ω we may assume that there exist an open neighbourhood Ω_H of e in H and an integer $r \geqq 0$ such that the mapping $(z,h) \to R_h U_z$ maps $\Omega \times \Omega_H$ smoothly into $\mathscr{D}'_r(G:P:\xi:\lambda_0)$ and is holomorphic in the first variable. Now suppose that $y_0 \in \mathscr{O}(P)$ and fix local coordinates $x_1, \ldots, x_n$ on an open neighbourhood $N(y_0)$ of y_0 in $\mathscr{O}(P)$, such that y_0 corresponds to $x_i = 0\ (1 \leqq i \leqq n)$. Fix a relatively compact open neighbourhood B of y_0 in $N(y_0)$. Then the space $\mathscr{D}'_{r,B}$ of distributions $u \in \mathscr{D}'(N(y_0)) \otimes_{\mathbb{C}} \mathscr{H}_\xi$ of order $\leqq r$ and with $\operatorname{supp} u \subset cl(B)$, carries a Banach topology. Fix a function $\psi \in C_c^\infty(B)$ such that $\psi = 1$ on an open neighbourhood of y_0. Then the linear map $u \to \psi u|_{N(y_0)}$ from $\mathscr{D}'_r(P:\xi:\lambda_0)$ into $\mathscr{D}'_{r,B}$ is continuous.

Shrinking Ω_H if necessary we may assume that Ω_H is stable under inversion $h \to h^{-1}$ and that there exists an open neighbourhood Ω_P of e in P, stable under inversion, and such that $\Omega_P(\operatorname{supp}\psi)\Omega_H \subset B$. Then the map $(z,p,h) \to L_p R_h(\psi U_z)$ is smooth from $\Omega \times \Omega_P \times \Omega_H$ into $\mathscr{D}'_{r,B}$, and also holomorphic in its first variable. For every $y \in B$, the map $(p,h) \to pyh$, $\Omega_P \times \Omega_H \to \mathscr{O}(P)$ is submersive at (e,e). Hence differentiating in p and h we obtain the following. If D is any smooth linear differential operator on $N(y_0)$, then the mapping

$$(z,p,h) \to D R_h L_p(\psi U_z) \tag{4.9}$$

maps $\Omega \times \Omega_P \times \Omega_H$ smoothly into $\mathscr{D}'_{r,B}$ and is holomorphic in z. In particular, the order of ψU_z does not increase, whatever smooth differential operator we apply. This implies that $\psi U_z \in C^\infty(N(y_0)) \otimes_{\mathbb{C}} \mathscr{H}_\xi$ for every $z \in \Omega$.

Let now $e(y_0)$ denote evaluation at y_0, viewed as a compactly supported distribution density, *i.e.* a continuous linear functional on $C^\infty(N(y_0))$. Then there exists a C_c^r-density φ and a smooth differential operator D on $N(y_0)$ such that $e(y_0) = {}^t D\varphi$ on B (*cf.* [Schw 50], Theorem XXVI). Applying φ to (4.9) we see that the map

$$(z,p,h) \to \psi U_z(p^{-1} y_0 h)$$

is smooth and in addition holomorphic in z. Using the fact that $(p, h) \to p^{-1} y_0 h$ is submersive at (e, e), together with an argument involving a partition of unity, we deduce that $U_z \in C^\infty(\mathcal{O}(P) : P : \xi : \lambda_0)$ for every $z \in \Omega$ and that the map $(z, x) \to U_z(x)$ from $\Omega \times \mathcal{O}(P)$ into $\mathcal{H}_\xi$ is smooth and in addition holomorphic in the first variable. By a standard application of the Cauchy integral formula (*cf.* also the proof of Corollary 4.3) it follows that $z \to U_z | \mathcal{O}(P)$ is a holomorphic map from Ω into $C^\infty(\mathcal{O}(P) : P : \xi : \lambda_0)$. ■

COROLLARY 4.14. — *In addition to the hypotheses of Lemma* 4.13, *let* Ω *be connected,* $P' \in \mathcal{P}_\sigma(A_q)$, *and assume that* $z \to \lambda(z)$ *does not entirely map into the singular set of* $A(P' : P : \xi : \lambda)$. *Then for all* $z \in \Omega$ *such that* $\lambda(z)$ *is not a pole for* $A(P' : P : \xi : \lambda)$, *the distribution* $v_z = A(P' : P : \xi : \lambda(z)) u_z$ *is smooth on* $\mathcal{O}(P')$. *Moreover, the map* $z \to v_z | \mathcal{O}(P')$ *is meromorphic as a map from* Ω *into* $C^\infty(\mathcal{O}(P') \cap K : K_M : \xi)$.

Proof. — Let $z_0 \in \Omega$, $\lambda_0 = \lambda(z_0)$ and let $N(\lambda_0)$, φ be as in Proposition 4.11. Since $A(P' : P : \xi : \lambda)$ commutes with R_h, for $h \in H$, it follows from Proposition 4.11 that $z \to \varphi(\lambda(z)) v_z$ satisfies all hypotheses of Lemma 4.13. ■

5. H-fixed distribution vectors

In this section we fix a parabolic subgroup $P \in \mathcal{P}_\sigma(A_q)$ and an irreducible unitary representation of M in a finite dimensional Hilbert space $\mathcal{H}_\xi$. Let $\mathcal{D}'(P : \xi : \lambda)^H$ denote the space of H-fixed elements in $\mathcal{D}'(P : \xi : \lambda)$. Our objective is to construct, for generic $\lambda \in \mathfrak{a}_{qc}^*$, a basis of $\mathcal{D}'(P : \xi : \lambda)^H$ which depends meromorphically on λ. For $Re(\lambda)$ in a suitable region this basis can be defined directly: it then consists of continuous functions. To obtain a basis for other, in particular imaginary, values of λ, we shall apply Proposition 4.11 to obtain a meromorphic continuation.

Recall that $\mathcal{O}(P)$ denotes the union of the open H-orbits on $P \backslash G$ (*cf.* also (3.8)), and that $C^\infty(\mathcal{O}(P) : P : \xi : \lambda)$ denotes the space of smooth functions $\mathcal{O}(P) \to \mathcal{H}_\xi$ transforming according to the rule (3.1). Identifying $\mathcal{D}'(G : P : \xi : \lambda)$ with a subspace of $\mathcal{D}'(G) \otimes_{\mathbf{C}} \mathcal{H}_\xi$ and using the H-invariance, we see that restriction to $\mathcal{O}(P)$ induces a linear map $r : \mathcal{D}'(G : P : \xi : \lambda)^H \to C^\infty(\mathcal{O}(P) : P : \xi : \lambda)$ (*cf.* also Lemma 4.13).

THEOREM 5.1. — *Let* $P \in \mathcal{P}_\sigma(A_q)$ *and* $[\xi] \in \hat{M}_{fu}$. *If* k *is a nonnegative integer then for* λ *in the complement* $\mathcal{B}_k$ *of a finite union of complex hyperplanes, the restriction map* r *maps* $\mathcal{D}'_k(G : P : \xi : \lambda)^H$ *injectively into* $C^\infty(\mathcal{O}(P) : P : \xi : \lambda)$.

Proof. — The proof heavily relies on Matsuki's description of all P-orbits on G/H ([Ma 79] and [Ma 82]) and secondly on an idea which goes back to Bruhat ([Br 56]). We use the form it has been given in [KKMOOT 78], *see* Appendix A.

Without loss of generality we may assume that P is the standard minimal $\sigma\theta$-stable parabolic. We may fix finitely many elements $x_1, \ldots, x_I \in K$ such that $\{ P x_i H, 1 \leqq i \leqq I \}$ is the set of all non-open P-orbits on G/H.

LEMMA 5.2. — *The elements* $x_i (1 \leqq i \leqq I)$ *may be chosen such that*

$$\mathrm{Ad}(x_i)^{-1}(\mathfrak{a}_0 \setminus \mathfrak{a}_0 \cap \mathfrak{h}) \cap \mathfrak{h} \neq \emptyset.$$

This lemma is proved in Appendix B. We proceed with our proof of Theorem 5.1. Fix $U_i \in \mathfrak{a}_0$ such that $U_i \notin \mathfrak{a}_0 \cap \mathfrak{h}$ and $\mathrm{Ad}(x_i)^{-1} U_i \in \mathfrak{h}$ $(1 \leqq i \leqq I)$. Consider the action λ of the group $P \times H$ on G given by

$$\lambda_{(p,h)} g = pgh^{-1}.$$

Fix $1 \leqq i \leqq I$ for the moment. The stabilizer S_i of x_i in $P \times H$ equals $\{(p, h); p = x_i h x_i^{-1}\}$. Its Lie algebra $\mathfrak{s}_i$ equals $\{(X, Y) \in \mathfrak{P} \times \mathfrak{h}; X = \mathrm{Ad}(x_i) Y\}$. Thus $(U_i, \mathrm{Ad}(x_i^{-1}) U_i) = Z_i$ belongs to $\mathfrak{s}_i$. Let $\lambda_{i1}(Z_i), \ldots, \lambda_{im_i}(Z_i)$ be the eigenvalues of the natural action of Z_i on $T_{x_i} G / T_{x_i}(x_i H)$, counting multiplicities (for the definition of this action, *cf.* Appendix A). Moreover, put

$$\Gamma_i(k) = \left\{ -\sum_{j=1}^{m_i} (\nu_j + 1) \lambda_{ij}(Z_i); \nu \in \mathbb{N}^{m_i}, |\nu| \leqq k \right\}.$$

Then $\Gamma_i(k)$ is a finite subset of $\mathbb{C}$.

We now claim that the open dense subset

$$\mathscr{B}_k = \{\lambda \in \mathfrak{a}_{qc}^*; (\lambda + \rho_P)(Z_i) \notin \Gamma_i(k), 1 \leqq i \leqq I\}$$

fulfills the requirements. Indeed, let τ be the representation of $P \times H$ on $\mathscr{H}_\xi$ defined by $\tau(man, h) v = a^{\lambda + \rho_P} \xi(m)$. Let $\mathscr{D}'(G : \tau)$ be the space of distributions $\varphi \in \mathscr{D}'(G) \otimes \mathscr{H}_\xi$ transforming according to

$$(\lambda^*_{(man, h)} \otimes 1) \varphi = (1 \otimes \tau(man, h)^{-1}) \varphi,$$

for $m \in M$, $a \in A$, $n \in N_P$ and $h \in H$ (*cf.* Appendix A for unspecified notations). In a natural way we have

$$\mathscr{D}'(G : \tau) \simeq \mathscr{D}'(G : P : \xi : \lambda)^H.$$

Now $\tau(Z_i) = (\lambda + \rho_P)(Z_i) I$ since $\xi | (A_0 \cap M)$ is trivial. Hence if $\lambda \in \mathscr{B}_k$, then the hypotheses of Proposition A.1 are fulfilled and it follows that for each $1 \leqq i \leqq I$ there is no distribution $\varphi \in \mathscr{D}'_k(G : P : \xi : \lambda)^H$ such that $P x_i H \cap \operatorname{supp} \varphi$ is non-empty and open in $\operatorname{supp} \varphi$. Now suppose that $\varphi \in \mathscr{D}'_k(G : P : \xi : \lambda)^H$ and $r(\varphi) = 0$. If $\varphi \neq 0$ then it would follow that $\operatorname{supp} \varphi$ is a non-empty finite union of orbits $P x_i H$. Fix an orbit $P x_i H \subset \operatorname{supp} \varphi$ of maximal dimension. Then $P x_i H$ is open in $\operatorname{supp} \varphi$, contradiction. Hence $\varphi = 0$. ■

If $\varphi \in \mathscr{D}'(\xi : \lambda)^H$, then from the proof of Lemma 3.3 it follows that $r(\varphi)(w) \in \mathscr{H}_\xi^{M \cap wHw^{-1}}$ for every $w \in N_K(\mathfrak{a}_q)$. Given $w \in N_K(\mathfrak{a}_q)$, we let the space

$$\mathscr{V}(\xi, w) = \mathscr{H}_\xi^{M \cap wHw^{-1}}$$

inherit the Hilbert structure from $\mathscr{H}_\xi$. Now fix a set $\mathscr{W}$ of representatives for $W/W_{K \cap H}$ in $N_K(\mathfrak{a}_q)$ and define a (formal) direct sum of Hilbert spaces

$$V(\xi) = \bigoplus_{w \in \mathscr{W}} \mathscr{V}(\xi, w). \tag{5.1}$$

The canonical injection of $\mathscr{V}(\xi, w)$ into $V(\xi)$ is denoted by $i(\xi, w)$, the canonical projection $V(\xi) \to \mathscr{V}(\xi, w)$ by $pr(\xi, w)$. The canonical image of $\mathscr{V}(\xi, w)$ in $V(\xi)$ is denoted by $V(\xi, w)$. Thus the spaces $V(\xi, w)$, $w \in \mathscr{W}$ are mutually orthogonal subspaces of $V(\xi)$, whereas the subspaces $\mathscr{V}(\xi, w)$ may not be orthogonal in $\mathscr{H}_\xi$: *e.g.* if $\xi = 1$, then the spaces $\mathscr{V}(\xi, w)$ are all equal to $\mathscr{H}_\xi \cong \mathbb{C}$.

For $w \in N_K(\mathfrak{a}_q)$ we define the evaluation map ev_w from $\mathscr{D}'(\xi : \lambda)^H$ into $\mathscr{V}(\xi, w)$ by $ev_w(\varphi) = r(\varphi)(w)$. We define ev from $\mathscr{D}'(\xi : \lambda)^H$ into $V(\xi)$ by

$$ev = \bigoplus_{w \in \mathscr{W}} ev_w.$$

The following is now obvious.

COROLLARY 5.3. — *For* λ *in the complement* $\mathscr{B} = \bigcap_{k=0}^{\infty} \mathscr{B}_k$ *of a countable union of complex hyperplanes in* $\mathfrak{a}_{qc}^*$, *the evaluation map* ev *maps* $\mathscr{D}'(G : P : \xi : \lambda)^H$ *injectively into the finite dimensional space* $V(\xi)$.

Remarks. — In particular $\mathscr{B}$ is a Baire subset of $\mathfrak{a}_{qc}^*$. We have not been able to decide whether generally $\mathscr{B}$ is *open* dense or not. Notice that $\mathscr{B} \cap i\mathfrak{a}_q^*$ is a Baire subset of $i\mathfrak{a}_q^*$.

Later on we show that for generic λ the map ev actually is a bijection from $\mathscr{D}'(\xi : \lambda)^H$ onto $V(\xi)$. Thus generically $V(\xi)$ will serve as a (λ-independent) model for $\mathscr{D}'(\xi : \lambda)^H$.

Before proceeding we discuss some easy properties of the spaces $\mathscr{V}(\xi, w)$. First, notice that for u, $v \in N_K(\mathfrak{a}_q)$ we have

$$\mathscr{V}(\xi, v) = \mathscr{V}(u\xi, uv). \tag{5.2}$$

In particular $\mathscr{V}(\xi, v) = \mathscr{V}(v^{-1}\xi, 1)$.

The following observation I owe to G. Olafsson.

LEMMA 5.4. — *Assume that every Cartan subgroup of* G *is abelian. If* $\xi \in \hat{M}_{fu}$, *then* $\dim(\mathscr{H}_\xi^{M \cap H}) \leq 1$.

Proof. — In the proof of Lemma 4.4 we saw that $\xi | M_0$ is irreducible. Since G is of Harish-Chandra's class, Ad(G) satisfies the hypotheses of [Kn 82]. From the proof of [Kn 82], Lemma 1.2 it follows in this particular case that $Ad(M_0) = Ad(M_0^0)\tilde{F}$, where $\tilde{F} = Ad(K) \cap \exp(i\,\mathrm{ad}\,\mathfrak{a}_0)$. Put $F = K \cap Ad^{-1}(\tilde{F})$. Then F centralizes $\mathfrak{m}_0 \oplus \mathfrak{a}_0$, hence is abelian by our hypothesis. Moreover,

$$M_0 = M_0^0 F.$$

Since F is central in M_0, the infinitesimal representation $\xi | \mathfrak{m}_0$ is irreducible. Consider the σ-stable compact semisimple algebra $\mathfrak{u} = [\mathfrak{m}_0, \mathfrak{m}_0]$. Then either $\dim(\xi) = 1$ or $\mathfrak{u} \neq 0$ and $\xi | \mathfrak{u}$ is irreducible. In the latter case it follows from standard semisimple theory that $\dim(\mathscr{H}_\xi^{M \cap H}) \leq \dim(\mathscr{H}_\xi^{\mathfrak{u} \cap \mathfrak{h}}) \leq 1$. ■

COROLLARY 5.5. — *Assume that every Cartan subgroup of* G *is abelian. If* $w \in N_K(\mathfrak{a}_q)$, *then* $\dim \mathscr{V}(\xi, w) \leq 1$.

Proof. — Use (5.2) to reduce to $w=1$. ■

Constructing a meromorphic basis for $\mathscr{D}'(\mathrm{P}:\xi:\lambda)^{\mathrm{H}}$ (for generic λ) comes down to constructing an inverse to *ev* which depends meromorphically on λ. Now there exist real analytic maps a_q, n_{P} and $\bar{m}_{\mathrm{P}}$ of PH into A_q, N_{P} and M_σ: $\mathrm{M}_\sigma/(\mathrm{M}_\sigma \cap \mathrm{H}) \simeq \mathrm{M}/(\mathrm{M}\cap\mathrm{H})$ respectively, such that

$$x \in n_{\mathrm{P}}(x)\, a_{\mathrm{P}}(x)\, \bar{m}_{\mathrm{P}}(x)\,\mathrm{H}, \tag{5.3}$$

for $x \in \mathrm{PH}$. This result easily follows from the corresponding result for Ad(G), Ad(H), which is proved in ([Ba 86], Appendix B). Thus, if $\eta \in \mathscr{V}(\xi, 1)$ we may define a function $\varepsilon_1(\mathrm{P}:\xi:\lambda:\eta)$ from G into $\mathscr{H}_\xi$ by

$$\begin{cases} \varepsilon_1(\mathrm{P}:\xi:\lambda:\eta) \equiv 0 & \text{outside PH,} \\ \varepsilon_1(\mathrm{P}:\xi:\lambda:\eta)(namh) = a^{\lambda+\rho_{\mathrm{P}}}\,\xi(m)\,\eta, & \end{cases} \tag{5.4}$$

for $m \in \mathrm{M}$, $a \in \mathrm{A}_q$, $n \in \mathrm{N}_{\mathrm{P}}$ and $h \in \mathrm{H}$. Moreover, if $w \in \mathrm{N}_{\mathrm{K}}(\mathfrak{a}_q)$ and $\eta \in \mathscr{V}(\xi, w)$, we define the map $\varepsilon_w(\mathrm{P}:\xi:\lambda:\eta)$ from G into $\mathscr{H}_\xi$ by

$$\varepsilon_w(\mathrm{P}:\xi:\lambda:\eta) = \mathrm{L}_w\,\varepsilon_1(w^{-1}\mathrm{P}w : w^{-1}\xi : w^{-1}\lambda : \eta) \tag{5.5}$$

Then $\varepsilon_w(\mathrm{P}:\xi:\lambda:\eta)$ is the unique function $\mathrm{G}\to\mathscr{H}_\xi$ transforming according to (3.1), which vanishes outside $\mathrm{P}w\mathrm{H}$ and on $\mathrm{P}w\mathrm{H}$ is determined by

$$ev_w \circ \varepsilon_w(\mathrm{P}:\xi:\lambda:\eta) = \eta.$$

Recall that $\mathfrak{g}_-$ denotes the -1 eigenspace of $\sigma\theta$ in $\mathfrak{g}$. We define a subset of $\mathfrak{a}_q$-roots by

$$\Sigma_- = \{\alpha \in \Sigma;\ \mathfrak{g}^\alpha \cap \mathfrak{g}_- \neq 0\}.$$

PROPOSITION 5.6. — *Assume that* H *is essentially connected, let* $\mathrm{P} \in \mathscr{P}_\sigma(\mathrm{A}_q)$, $[\xi] \in \hat{\mathrm{M}}_{fu}$ *and fix* $\lambda \in \mathfrak{a}_{qc}^*$. *Moreover, let* $w \in \mathrm{N}_{\mathrm{K}}(\mathfrak{a}_q)$ *and suppose that* $\eta \in \mathscr{V}(\xi, w)\setminus\{0\}$. *Then the* $\mathscr{H}_\xi$*-valued function* $\varepsilon_w(\mathrm{P}:\xi:\lambda:\eta)$ *is continuous if and only if* $\langle \mathrm{Re}\,\lambda + \rho_{\mathrm{P}}, \alpha\rangle < 0$ *for all* $\alpha \in \Sigma(\mathfrak{n}_{\mathrm{P}}, \mathfrak{a}_q) \cap w(\Sigma_-)$.

Remark. — Observe that $w(\Sigma_-)$ only depends on the class of w in $\mathrm{W}/\mathrm{W}_{\mathrm{K}\cap\mathrm{H}}$, *i.e.* on the orbit $\mathrm{P}w\mathrm{H}$.

Proof. — Using (5.2), (5.5) and the fact that $\Sigma(\mathfrak{n}_{w^{-1}\mathrm{P}w}, \mathfrak{a}_q) = w^{-1}\Sigma(\mathfrak{n}_{\mathrm{P}}, \mathfrak{a}_q)$ we may restrict ourselves to the case $w=1$.

Write $\Omega = \mathrm{PH}$. Because of the real analyticity of the decomposition (5.3), the function $\varepsilon_1(\mathrm{P}:\xi:\lambda:\eta)$ is real analytic on the complement of $\partial\Omega$. By P-equivariance it follows that $\varepsilon_1(\mathrm{P}:\xi:\lambda:\eta)$ is continuous iff for every sequence $\{x_n\}$ in $\Omega_{\mathrm{K}} = \Omega \cap \mathrm{K}$ converging to a point $x \in \partial\Omega_{\mathrm{K}} = \partial\Omega \cap \mathrm{K}$ we have $\lim_{n\to\infty} \varepsilon_1(\mathrm{P}:\xi:\lambda:\eta)(x_n) = 0$. The restriction $\underline{a}_{\mathrm{P}}$ of a_{P} to Ω_{K} is a proper map $\Omega_{\mathrm{K}} \to \mathrm{A}_q$ (*cf.* [Ba 86], Lemma 3.5). Moreover, (*cf.* [*loc. cit.*, Theorem 3.8]).

$$im(\log \circ \underline{a}_{\mathrm{P}}) = \Gamma(\mathrm{P}), \tag{5.6}$$

where $\Gamma(P)$ is the closed convex cone in $\mathfrak{a}_q$, spanned by the vectors H_α, $\alpha \in \Sigma(\mathfrak{n}_P, \mathfrak{a}_q) \cap \Sigma_-$. Now suppose $\mathrm{Re}(\lambda+\rho_P)(H_\alpha)<0$ for all $\alpha \in \Sigma(\mathfrak{n}_P, \mathfrak{a}_q) \cap \Sigma_-$. If $\{x_n\}$ is a sequence in Ω_K tending to a boundary point $x \in \partial\Omega_K$, then $\mathrm{Re}(\lambda+\rho_P) \log \underline{a}_P(x_n)$ tends to $-\infty$ by the properness of $\underline{a}_P$ and the characterization (5.6) of its image. It follows that $\|\varepsilon_1(\lambda)(x_n)\| = \underline{a}_P(x_n)^{\mathrm{Re}(\lambda+\rho_P)} \|\eta\| \to 0$. This proves the continuity of $\varepsilon_1(\lambda)$.

Conversely if $\varepsilon_1(\lambda)$ is continuous, we must have that $\underline{a}_P(x_n)^{\mathrm{Re}(\lambda+\rho_P)} \to 0$ for every sequence $\{x_n\} \subset \Omega_K$ tending to a point $x \in \partial\Omega_K$. Fix $\alpha \in \Sigma(\mathfrak{n}_P, \mathfrak{a}_q) \cap \Sigma_-$, a sequence $\{C_n\} \subset [0, \infty[$ such that $C_n \to +\infty$ and select $x_n \in \Omega_K$ such that $\log \underline{a}_P(x_n) = C_n H_\alpha$. Passing to a subsequence if necessary we may assume that x_n tends to a boundary point $x \in \partial\Omega_K$ (use that $\underline{a}_P$ is proper). It follows that $C_n \mathrm{Re}(\lambda+\rho_P)(H_\alpha) = \log \underline{a}_P(x_n)^{\mathrm{Re}(\lambda+\rho_P)} \to -\infty$. Consequently $\mathrm{Re}(\lambda+\rho_P)(H_\alpha)<0$ for all $\alpha \in \Sigma(\mathfrak{n}_P, \mathfrak{a}_q) \cap \Sigma_-$. ■

Remark. — More generally, if H is not assumed to be essentially connected, then every open orbit PwH ($w \in N_K(\mathfrak{a}_q)$) is a finite union $\bigcup_i P x_i H^0$, with $x_i \in N_{K \cap H}(\mathfrak{a}_q)$ (use that $H = N_{K \cap H}(\mathfrak{a}_q) H^0$, (*cf.* [Ba 86], (2.2)). Applying the above proposition to the functions ε_i defined by $\varepsilon_i = \varepsilon_w$ on $P x_i H^0$ and $\varepsilon_i = 0$ outside $P x_i H^0$, we deduce that $\varepsilon_w = \varepsilon_w(P:\xi:\lambda:\eta)$ is continuous if $\mathrm{Re}\,\lambda + \rho_P$ is strictly $\bar{P}$-dominant, *i.e.* $\langle \mathrm{Re}\,\lambda + \rho_P, \alpha \rangle < 0$ for all $\alpha \in \Sigma(\mathfrak{n}_P, \mathfrak{a}_q)$.

Retaining the notations introduced before Proposition 5.6, for $\mathrm{Re}(\lambda+\rho_P)$ strictly $\bar{P}$-dominant we define the map $j(P:\xi:\lambda)$ from $V(\xi)$ into $\mathscr{D}'(G:P:\xi:\lambda)^H$ by

$$j(P:\xi:\lambda) = \sum_{w \in \mathscr{W}} \varepsilon_w(P:\xi:\lambda) \circ pr(\xi, w).$$

Here we have used the embedding of $C^0(G:P:\xi:\lambda)$ into $\mathscr{D}'(G:P:\xi:\lambda)$ determined by the pairing (3.2). The following is now obvious.

LEMMA 5.7. — *For every $\eta \in V(\xi)$, the family $\lambda \to j(P:\xi:\lambda:\eta)$ ($\lambda \in \mathfrak{a}_{qc}^*$ and $\mathrm{Re}\,\lambda + \rho_p$ strictly $\bar{P}$-dominant) is holomorphic as a family in $\mathscr{D}'(K:\xi)$. Moreover,*

$$ev \circ j(P:\xi:\lambda) = \mathrm{id} \quad on \quad V(\xi).$$

If in addition $\lambda \in \mathscr{B}$, then $j(P:\xi:\lambda)$ is a bijection.

Remark. — For the exact meaning of holomorphy, *cf.* the definitions preceding Lemma 4.13.

Before turning our attention to the meromorphic continuation of $j(P:\xi)$, we discuss its dependence on the particular choice of $\mathscr{W}$.

LEMMA 5.8. — *Let $\mathscr{W}'$ be a second choice of representatives for $W/W_{K \cap H}$ in $N_K(\mathfrak{a}_q)$ and define the map $j'(P:\xi:\lambda)$ from $V'(\xi) = \bigoplus_{u \in \mathscr{W}'} \mathscr{V}(\xi, u)$ into $\mathscr{D}'(P:\xi:\lambda)^H$ accordingly. Then there exists a unique linear map $R: V(\xi) \to V'(\xi)$ such that*

$$j'(P:\xi:\lambda) \circ R = j(P:\xi:\lambda) \qquad (5.7)$$

for all $\lambda \in \mathfrak{a}_{qc}^*$ *with* $\mathrm{Re}(\lambda+\rho_p)$ *strictly* $\bar{\mathrm{P}}$*-dominant. The map* R *does not depend on* P. *Moreover, if* $w \in \mathscr{W}$ *and* $w' \in \mathscr{W}'$ *represent the same element of* $\mathrm{W}/\mathrm{W}_{\mathrm{K} \cap \mathrm{H}}$, *then* R *maps* $\mathrm{V}(\xi, w)$ *unitarily onto* $\mathrm{V}(\xi, w')$.

Proof. — In view of Lemma 5.7 it suffices to show that for $\lambda \in \mathscr{B}$ the map $\mathrm{R}=ev' \circ j(\mathrm{P}:\xi:\lambda)$ does neither depend on λ, nor on P and maps $\mathrm{V}(\xi, w)$ unitarily onto $\mathrm{V}(\xi, w')$. Now $w'=lwk$, where $l \in \mathrm{M} \cap \mathrm{H}$, and $k \in \mathrm{K} \cap \mathrm{H}$. Fix $\eta \in \mathrm{V}(\xi, w)$. Then $\mathrm{supp}(j(\mathrm{P}:\xi:\lambda:\eta)) \subset \mathrm{P} w \mathrm{H}=\mathrm{P} w' \mathrm{H}$, hence $\mathrm{R}(\eta) \in \mathrm{V}(\xi, w')$. Moreover,

$$\begin{aligned} pr'(\xi, w') \circ \mathrm{R}(\eta) &= ev_{w'} \circ j(\mathrm{P}:\xi:\lambda)\,\eta \\ &= \xi(l) \circ ev_w \circ j(\mathrm{P}:\xi:\lambda)\,\eta \\ &= \xi(l) \circ pr(\xi, w)\,\eta. \end{aligned}$$

Hence R is independent of λ and P, and maps $\mathrm{V}(\xi, w)$ unitarily onto $\mathrm{V}(\xi, w')$ (notice that $\xi(l)\,\mathscr{V}(\xi, w)=\mathscr{V}(\xi, w')$). ■

The remainder of this section is devoted to the meromorphic continuation of $j(\mathrm{P}:\xi)$. The first step is to prove the existence of *some* meromorphic basis for $\mathscr{D}'(\mathrm{P}:\xi:\lambda)$. For a given real number r we define

$$\mathscr{A}(\mathrm{P}, r)=\{\lambda \in \mathfrak{a}_{qc}^*;\ (\mathrm{Re}\,\lambda, \alpha)>r \text{ for } \alpha \in \Sigma(\mathfrak{n}_{\mathrm{P}}, \mathfrak{a}_q)\}.$$

LEMMA 5.9. — *For every* $r \in \mathbb{R}$ *there exists a meromorphic map* J_r *from* $\mathscr{A}(\mathrm{P}, r)$ *into* $\mathrm{Hom}_{\mathbb{C}}(\mathrm{V}(\xi), \mathscr{D}'(\mathrm{K}:\xi))$ *such that for* λ *in the complement of a countable union of (complex) hyperplanes in* $\mathscr{A}(\mathrm{P}, r)$, *the map* $\mathrm{J}_r(\lambda)$ *maps* $\mathrm{V}(\xi)$ *bijectively onto* $\mathscr{D}'(\mathrm{G}:\mathrm{P}:\xi:\lambda)^{\mathrm{H}}$. *The poles of* J_r *are contained in a locally finite union of hyperplanes.*

Proof. — We first prove the assertion for $r=r_1$ sufficiently large. In that case $j(\bar{\mathrm{P}}:\xi)$ may be viewed as a holomorphic map from $\mathscr{A}(\mathrm{P}, r_1)$ into $\mathrm{Hom}_{\mathbb{C}}(\mathrm{V}(\xi), \mathscr{D}'(\mathrm{K}:\xi))$ (use Lemma 5.7). By Proposition 4.8 the map J_{r_1} defined by

$$\mathrm{J}_{r_1}(\lambda)=\mathrm{A}(\mathrm{P}:\bar{\mathrm{P}}:\xi:\lambda) \circ j(\bar{\mathrm{P}}:\xi:\lambda)$$

is meromorphic. Since $\mathrm{A}(\mathrm{P}:\bar{\mathrm{P}}:\xi:\lambda)$ is defined and bijective for λ in the complement of a locally finite union of hyperplanes (*cf.* Corollary 4.9), all assertions now follow from Lemma 5.7.

To obtain the result for other values of r we apply multiplication by the matrix coefficient of a finite dimensional representation. Let $\mu \in \mathfrak{a}_{qc}^*$ be as in Lemma C.1 (Appendix C) and select a non-trivial real analytic function ψ in $\mathrm{C}^\infty(\mathrm{G}:\mathrm{P}:1:\mu-\rho_{\mathrm{P}})^{\mathrm{H}}$. Then the mapping

$$\mathrm{M}_\psi : u \to \psi u$$

maps $\mathscr{D}'(\mathrm{P}:\xi:\lambda)$ continuously into $\mathscr{D}'(\mathrm{P}:\xi:\lambda+\mu)$. Now assume we have found J_r, fulfilling our requirements. Let

$$m=\min\{\langle \mu, \alpha \rangle;\ \alpha \in \Sigma(\bar{\mathfrak{n}}_{\mathrm{P}}, \mathfrak{a}_q)\}.$$

Then $m>0$ (Lemma C.1, (*i*)) and $\mathscr{A}(P, r)+\mu$ contains $\mathscr{A}(P, r-m)$. Hence

$$J_{r-m}(\lambda)=M_{\psi}\circ J_r(\lambda-\mu)$$

defines a meromorphic map from $\mathscr{A}(P, r-m)$ into $\mathrm{Hom}_{\mathbb{C}}(V(\xi), \mathscr{D}'(K:\xi))$ whose set of poles is as required. To prove that it fulfills the other requirements we only have to show that the mapping M_{ψ} maps $\mathscr{D}'(P:\xi:\lambda)^H$ bijectively onto $\mathscr{D}'(P:\xi:\lambda+\mu)^H$ for λ in the complement of a countable union of hyperplanes in $\mathscr{A}(P, r)$. Define a linear mapping $m_{\psi}: V(\xi)\to V(\xi)$ by $m_{\psi}(\eta)=i(\xi, u).\psi(u).pr(\xi, u)\eta$, for $\eta\in V(\xi, u)$, $u\in\mathscr{W}$. By real analyticity and equivariance, $\psi(u)\neq 0$ for all $u\in\mathscr{W}$, hence m_{ψ} is bijective. The following diagram obviously commutes for every $\lambda\in\mathfrak{a}_{qc}^*$:

$$\begin{array}{ccc} \mathscr{D}'(\xi:\lambda)^H & \xrightarrow{M_{\psi}} & \mathscr{D}'(\xi:\lambda+\mu) \\ {\scriptstyle ev}\downarrow & & \downarrow{\scriptstyle ev} \\ V(\xi) & \xrightarrow{m_{\psi}} & V(\xi) \end{array}$$

By Corollary 5.3 the evaluation map at the right is injective for $\lambda\in-\mu+\mathscr{B}$. Moreover, the evaluation map at the left of the diagram is bijective for λ contained in the complement of a countable union of hyperplanes in $\mathscr{A}(P, r)$, by our assumption on J_r (use also Corollary 5.3 and count dimensions). We conclude that M_{ψ} is bijective for λ in the complement of a countable union of hyperplanes in $\mathscr{A}(P, r)$. ■

THEOREM 5.10. — *Viewed as a function with values in* $\mathrm{Hom}_{\mathbb{C}}(V(\xi), \mathscr{D}'(K:\xi))$, *the map* $j(P:\xi)$ *extends meromorphically to* $\mathfrak{a}_{qc}^*$. *Moreover, if* $\lambda\in\mathfrak{a}_{qc}^*$ *is not a pole for this map then* $j(P:\xi:\lambda)$ *maps* $V(\xi)$ *injectively into* $\mathscr{D}'(P:\xi:\lambda)^H$. *Finally, for* λ *in the complement of a countable union of complex analytic hypersurfaces in* $\mathfrak{a}_{qc}^*$, *the mapping* $j(P:\xi:\lambda)$ *is a bijection from* $V(\xi)$ *onto* $\mathscr{D}'(P:\xi:\lambda)^H$ *with inverse ev.*

Proof. — Once we have established the meromorphic continuation, the other assertions follow from first applying holomorphic continuation to Lemma 5.7 and then using Corollary 5.3.

By Lemma 5.7 the function $j(P:\xi)$ is holomorphic on $\mathscr{A}(\bar{P}, 0)-\rho_P$. Fixing the real number r sufficiently close to $-\infty$ we may assume that $[\mathscr{A}(\bar{P}, 0)-\rho_P]\cap\mathscr{A}(P, r)\neq\emptyset$. It suffices to show that then $j(P:\xi)$ extends meromorphically to $\mathscr{A}(P, r)$. Let J_r be a meromorphic map from $\mathscr{A}(P, r)$ into $\mathrm{Hom}_{\mathbb{C}}(V(\xi), \mathscr{D}'(K:\xi))$ as in the above lemma. Then the map $L:\lambda\to ev\circ J_r(\lambda)$ is meromorphic from $\mathscr{A}(P, r)$ into $\mathrm{End}_{\mathbb{C}}(V(\xi))$. Moreover, for λ in the complement $\mathscr{C}$ of a countable union of complex hyperplanes in $\mathscr{A}(P, r)$, the endomorphism $L(\lambda)$ is bijective from $V(\xi)$ onto $V(\xi)$ (use Corollary 5.3 and Lemma 5.7). Since $\mathscr{C}$ is a Baire subset, hence dense in $\mathscr{A}(P, r)$, the determinant $\det L(\lambda)$ does not vanish identically and $L(\lambda)^{-1}$ depends meromorphically on λ. We claim that

$$j(P:\xi:\lambda)=J_r(\lambda)\circ L(\lambda)^{-1} \tag{5.8}$$

defines the meromorphic extension of $j(\mathrm{P}:\lambda)$ to $\mathscr{A}(\mathrm{P}, r)$. Indeed if $\lambda \in \mathscr{A}(\mathrm{P}, r) \cap (\mathscr{A}(\bar{\mathrm{P}}, 0) - \rho_{\mathrm{P}})$ is not a pole $\mathrm{L}(\lambda)^{-1}$, then $ev \circ \mathrm{J}_r(\lambda) \circ \mathrm{L}(\lambda)^{-1} = \mathrm{I}_{\mathrm{V}(\xi)}$, whence (5.8). ■

If $[\xi] \in \hat{\mathrm{M}}_{fu}$, we define the integer $d_\xi = \dim \mathrm{V}(\xi)$. The following result now justifies our definition of principal series once more.

COROLLARY 5.11. — *Let* $\mathrm{P} \in \mathscr{P}_\sigma(\mathrm{A}_q)$, *and* $\xi \in \hat{\mathrm{M}}_{fu}$. *Then*

(i) $d_\xi \neq 0 \Leftrightarrow \xi \in \hat{\mathrm{M}}_{ps}$,

(ii) *for* λ *in the complement of a countable union of complex analytic hypersurfaces in* $\mathfrak{a}_{qc}^*$, $\dim_{\mathbb{C}} \mathscr{D}'(\mathrm{P}:\xi:\lambda)^{\mathrm{H}} = d_\xi$.

Remarks. — (i) Note that $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\xi \otimes \lambda \otimes 1)$ is an admissible representation of finite length (use Lemma 4.5) in a Fréchet space. Hence $\mathscr{D}'(\mathrm{P}:\xi:\lambda)^{\mathrm{H}}$ is known to be finite dimensional for every $\lambda \in \mathfrak{a}_{qc}^*$ (*cf.* [Ba 87 II]).

(ii) If every Cartan subgroup of G is abelian, then by Corollary 5.5 we have:

$$|\mathrm{W}_{\mathrm{K} \cap \mathrm{H}}| . d_\xi = \# \{ w \in \mathrm{W};\, w[\xi] \in (\mathrm{M}/\mathrm{M} \cap \mathrm{H})\hat{\ } \}.$$

6. Elementary properties of the matrix B.

Retaining the notations of the previous section recall that $j(\mathrm{P}:\xi:\lambda)$ maps $\mathrm{V}(\xi)$ injectively into $\mathscr{D}'(\mathrm{G}:\mathrm{P}:\xi)^{\mathrm{H}}$ whenever λ is not a pole for $j(\mathrm{P}:\xi)$. The image $\mathrm{im}(j(\mathrm{P}:\xi:\lambda))$ is preserved by the action of intertwining operators. More precisely we have

PROPOSITION 6.1. — *Let* $[\xi] \in \hat{\mathrm{M}}_{fu}$, P_1, $\mathrm{P}_2 \in \mathscr{P}_\sigma(\mathrm{A}_q)$. *Then there exists a unique meromorphic map* $\mathrm{B}(\mathrm{P}_2:\mathrm{P}_1:\xi)$ *from* $\mathfrak{a}_{qc}^*$ *into* $\mathrm{End}(\mathrm{V}(\xi))$ *such that*

$$\mathrm{A}(\mathrm{P}_2:\mathrm{P}_1:\xi:\lambda) \circ j(\mathrm{P}_1:\xi:\lambda) = j(\mathrm{P}_2:\xi:\lambda) \circ \mathrm{B}(\mathrm{P}_2:\mathrm{P}_1:\xi:\lambda). \tag{6.1}$$

Remark. — If $\xi \notin \hat{\mathrm{M}}_{ps}$, then $\mathrm{V}(\xi) = 0$ by Corollary 5.11 and the proposition is vacuous. We have used the above formulation for reasons of induction that will become apparent later on.

Proof. — Uniqueness is obvious by Theorem 5.10. Define

$$\mathrm{B}(\mathrm{P}_2:\mathrm{P}_1:\xi:\lambda) = ev \circ \mathrm{A}(\mathrm{P}_2:\mathrm{P}_1:\xi:\lambda) \circ j(\mathrm{P}_1:\xi:\lambda).$$

By Theorem 5.10 and Corollary 4.14, this map is meromorphic. The identity (6.1) now holds for all λ in a Baire subset of $\mathfrak{a}_{qc}^*$ (use Theorem 5.10), hence in a meromorphic sense. ■

From the uniqueness statement in Proposition 6.1 we immediately deduce the following transformation properties of B from the corresponding properties of intertwining operators (*cf.* Propositions 5.6 and 4.7 and the remarks below Proposition 4.11).

PROPOSITION 6.2. — *Let* $[\xi] \in \hat{M}_{fu}$ *and let* $P_1, P_2 \in \mathscr{P}_\sigma(A_q)$. *Then for* $\lambda \in \mathfrak{a}_{qc}^*$ *we have*

(i) $B(P_1 : P_2 : \xi : \lambda) \circ B(P_2 : P_1 : \xi : \lambda) = \eta(P_2 : P_1 : \xi : \lambda) I$;

(ii) $B(P_2 : P_3 : \xi : \lambda) \circ B(P_3 : P_1 : \xi : \lambda) = B(P_2 : P_1 : \xi : \lambda)$, *for any* $P_3 \in \mathscr{P}_\sigma(A_q)$ *with* $\mathfrak{n}_3 \cap \bar{\mathfrak{n}}_1 \subset \mathfrak{n}_2 \cap \bar{\mathfrak{n}}_1$.

A priori it is not clear whether the analogue of Proposition 4.6 (ii) holds for the matrix $B(P_2 : P_1 : \xi : \lambda)$. The reason is that the pairing (3.2) does not induce a pairing of $\mathscr{D}'(P : \xi : \lambda)^H$ and $\mathscr{D}'(P : \xi : -\bar{\lambda})^H$. Surprisingly, the analogue does hold.

THEOREM 6.3. — *Assume that every Cartan subgroup of* G *is abelian, and let* $[\xi] \in \hat{M}_{fu}$ *and* $P_1, P_2 \in \mathscr{P}_\sigma(A_q)$. *Then for* $\lambda \in \mathfrak{a}_{qc}^*$ *we have*

$$B(P_2 : P_1 : \xi : \lambda)^* = B(P_1 : P_2 : \xi : -\bar{\lambda}). \tag{6.2}$$

Here * *indicates that the adjoint with respect to the unitary structure of* $V(\xi)$ *defined by* (5.1) *has been taken.*

Remarks. — (1) A consequence of the above is that an analogue of Harish-Chandra's Maass-Selberg relations (*cf.* [HC 76, Theorem 14.1]) holds for Eisenstein integrals related to minimal σθ-stable parabolic subgroups. We prove this in a second paper [Ba 88].

(2) The matrix B has (in slightly different form) for the first time been studied by Oshima and Sekiguchi for the restricted class of semisimple symmetric spaces of K_ε-type. Then $\mathfrak{a}_q$ is maximally abelian in both $\mathfrak{p}$ and $\mathfrak{q}$ and $H \supset Z_K(\mathfrak{a}_q)$ (*cf.* [O-S 80]). In that case $\hat{M}_{ps}$ consists merely of the trivial representation and the matrices B can be explicitly computed (*cf.* [*loc. cit.*, Lemma 4.14]).

The proof of Theorem 6.3 goes by reduction to the σ-split rank one case (*i.e.* $\dim(\mathfrak{a}_q) = 1$). This reduction, based on an idea of [O-S 80] is carried out in the next section. Finally Section 8 is devoted to the proof for the rank one case.

In the remainder of the present section we derive the elementary properties of B which will be needed in Sections 7, 8. First we describe B's dependence on the choice of $\mathscr{W}$. If $\mathscr{W}'$ is a second choice of representatives we may define the endomorphism $B'(P_2 : P_1 : \xi : \lambda)$ of $V'(\xi)$ by formula (6.1) with j replaced by j' (*cf.* Lemma 5.8). Then the following lemma is an immediate consequence of Lemmas 5.8 and 6.1.

LEMMA 6.4. — *Let* $[\xi] \in \hat{M}_{fu}$, *suppose that* $\mathscr{W}'$ *is a second set of representatives for* $W/W_{K \cap H}$ *in* $N_K(\mathfrak{a}_q)$ *and let* $R : V(\xi) \to V'(\xi)$ *be the unitary map of Lemma* 5.8. *Then for all* $P_1, P_2 \in \mathscr{P}_\sigma(A_q)$ *we have*

$$B'(P_2 : P_1 : \xi : \lambda) \circ R = R \circ B(P_2 : P_1 : \xi : \lambda).$$

Remark 6.5. — Since R is unitary and independent of the choice of parabolics, it suffices to prove Theorem 6.3 for a particular choice of $\mathscr{W}$. It is then valid for any choice of $\mathscr{W}$.

Next we study B's dependence of the choice of ξ in its class. Suppose that $[\eta] \in \hat{M}_{fu}$ and that $T : \mathscr{H}_\xi \to \mathscr{H}_\eta$ is a unitary intertwining operator. Then T naturally induces an intertwining operator $\mathrm{Ind}_P^G(T : \lambda)$ from $\mathscr{D}'(P : \xi : \lambda)$ onto $\mathscr{D}'(P : \eta : \lambda)$ by acting on the second component of the tensor product $\mathscr{D}'(G) \otimes \mathscr{H}_\xi$.

LEMMA 6.6. — *Let* $[\xi]$, $[\eta] \in \hat{M}_{fu}$ *and suppose a unitary intertwining operator* $T: \mathcal{H}_\xi \to \mathcal{H}_\eta$ *is given. Then there exists a unique linear map* $b(T, \eta, \xi): V(\xi) \to V(\eta)$ *such that*

$$\mathrm{Ind}_P^G(T:\lambda) \circ j(P:\xi:\lambda) = j(P:\eta:\lambda) \circ b(T, \eta, \xi)$$

for $P \in \mathcal{P}_\sigma(A_q)$ *and* $\lambda \in \mathfrak{a}_{qc}^*$. *The map* $b(T, \eta, \xi)$ *is unitary and bijective, and maps* $V(\xi, w)$ *into* $V(\eta, w)$ $(w \in \mathcal{W})$.

Proof. — Since $\mathrm{Ind}_P^G(T:\lambda)$ maps $\mathcal{D}'(P:\xi:\lambda)^H$ bijectively onto $\mathcal{D}'(P:\eta:\lambda)^H$, uniqueness follows from Theorem 5.10. The mapping T maps $\mathcal{V}(\xi, u)$ unitarily onto $\mathcal{V}(\eta, u)$, for $u \in \mathcal{W}$. Define the unitary map $b(T, \eta, \xi)$ from $V(\xi)$ into $V(\eta)$ by $b(T, \eta, \xi) = i(\xi, u) \circ T \circ pr(\xi, u)$ on $V(\xi, u)$, $u \in \mathcal{W}$. Then obviously $ev \circ \mathrm{Ind}_P^G(T:\lambda) = b(T, \eta, \xi) \circ ev$ on $\mathcal{D}'(\xi:\lambda)^H$. The result now follows from Theorem 5.10. ■

COROLLARY 6.7. — *Under the assumptions of Lemma* 6.6 *we have*

$$B(P_2:P_1:\eta:\lambda) \circ b(T, \eta, \xi) = b(T, \eta, \xi) \circ B(P_2:P_1:\xi:\lambda),$$

for P_1, $P_2 \in \mathcal{P}_\sigma(A_q)$.

Proof. — The result follows from Proposition 6.1, Lemma 6.6 and the corresponding formula for intertwining operators:

$$A(P_2:P_1:\eta:\lambda) \circ \mathrm{Ind}_{P_2}^G(T:\lambda) = \mathrm{Ind}_{P_1}^G(T:\lambda) \circ A(P_2:P_1:\xi:\lambda). \quad ■$$

Later on it will be of crucial importance to have a similar result when $T: \mathcal{H}_\xi \to \mathcal{H}_\eta$ is an *anti-linear* intertwining operator which is *anti-unitary*, *i.e.* $(Tv, Tw)_\eta = \overline{(v, w)_\xi}$ for $v, w \in \mathcal{H}_\xi$ (for instance, ξ and its contragredient are connected by an operator of this type). The operator T now induces an *anti-linear* intertwining operator $\mathrm{Ind}_P^G(T:\lambda)$ from $\mathcal{D}'(P:\xi:\lambda)$ onto $\mathcal{D}'(P:\eta:\bar{\lambda})$ by acting on the second component of $\mathcal{D}'(G) \otimes \mathcal{H}_\xi$.

LEMMA 6.8. — *Let* $[\xi]$, $[\eta] \in \hat{M}_{fu}$, *and suppose* $T: \mathcal{H}_\xi \to \mathcal{H}_\eta$ *is an anti-unitary and anti-linear intertwining operator. Then there exists a unique anti-linear map* $b(T, \eta, \xi): V(\xi) \to V(\eta)$ *such that*

$$\mathrm{Ind}_P^G(T:\lambda) \circ j(P:\xi:\lambda) = j(P:\eta:\bar{\lambda}) \circ b(T, \eta, \xi),$$

for $P \in \mathcal{P}_\sigma(A_q)$ *and* $\lambda \in \mathfrak{a}_q^*$. *The map* $b(T, \eta, \xi)$ *is anti-unitary and bijective and maps* $V(\xi, w)$ *onto* $V(\eta, w)$ $(w \in \mathcal{W})$.

Proof. — Analogous to the proof of Lemma 6.6. The anti-unitarity follows from the anti-unitarity of T and the fact that the subspaces $V(\xi, w)$ of $V(\xi)$ (resp. $V(\eta, w)$ of $V(\eta)$) are orthogonal (by definition). ■

COROLLARY 6.9. — *Under the assumptions of Lemma* 6.8 *we have*

$$B(P_2:P_1:\eta:\bar{\lambda}) \circ b(T, \eta, \xi) = b(T, \eta, \xi) \circ B(P_2:P_1:\xi:\lambda),$$

for P_1, $P_2 \in \mathscr{P}_\sigma(A_q)$.

Proof. — From formula (4.1) we obtain:

$$A(P_2 : P_1 : \eta : \lambda) \circ \operatorname{Ind}_{P_1}^G(T : \lambda) = \operatorname{Ind}_{P_2}^G(T : \bar{\lambda}) \circ A(P_2 : P_1 : \xi : \bar{\lambda})$$

(use analytic continuation). Now proceed as in the proof of Corollary 6.7. ■

We shall now describe how B transforms under the action of the intertwining operator $L(w)$ from $\mathscr{D}'(P:\xi:\lambda)$ onto $\mathscr{D}'(wPw^{-1}:w\xi:w\lambda)$ (where $w \in N_K(\mathfrak{a}_q)$). We define an action τ of $N_K(\mathfrak{a}_q)$ on $\mathscr{W}$ as follows. If $w \in N_K(\mathfrak{a}_q)$, $u \in \mathscr{W}$, then $\tau(w)u$ is the element of $\mathscr{W}$ which represents the same element in $W/W_{K \cap H}$ as wu.

LEMMA 6.10. — *Let* $[\xi] \in \hat{M}_{fu}$ *and* $w \in N_K(\mathfrak{a}_q)$. *Then there exists a unique linear map* $L(\xi, w) : V(\xi) \to V(w\xi)$ *such that*

$$(6.3) \qquad L(w) \circ j(P:\xi:\lambda) = j(wPw^{-1}:w\xi:w\lambda) \circ L(\xi, w),$$

for $P \in \mathscr{P}_\sigma(A_q)$, *and* $\lambda \in \mathfrak{a}_{qc}^*$. *The map* $L(\xi, w)$ *is unitary and maps* $V(\xi, u)$ *into* $V(w\xi, \tau(w)u)$ $(u \in \mathscr{W})$.

Proof. — Since $L(w)$ maps $\mathscr{D}'(P:\xi:\lambda)^H$ bijectively onto $\mathscr{D}'(wPw^{-1}:w\xi:w\lambda)^H$, uniqueness follows from Theorem 5.10. As for the existence, notice that $\mathscr{W}' = w^{-1}\mathscr{W}$ is a set of representatives for $W/W_{K \cap H}$. Let R be the map $V(\xi) \to V'(\xi)$ of Lemma 5.8. If $u \in \mathscr{W}$, then $R_u = pr(\xi, w^{-1}\tau(w)u) \circ R \circ i(\xi, u)$ maps $\mathscr{V}(\xi, u)$ unitarily onto $\mathscr{V}(\xi, w^{-1}\tau(w)u) = \mathscr{V}(w\xi, \tau(w)u)$. We define the map $L(\xi, w)$ by

$$L(\xi, w) = i(w\xi, \tau(w)u) \circ R_u \circ pr(\xi, u) \qquad \text{on } V(\xi, u).$$

Then it remains to prove (6.3). Fix $u \in \mathscr{W}$. Then

$$ev_{\tau(w)u} \circ j(wPw^{-1}:w\xi:w\lambda) \circ L(\xi, w) = pr(w\xi, \tau(w)u) \circ L(\xi, w) = R_u \circ pr(\xi, u).$$

On the other hand,

$$ev_{\tau(w)u} \circ L(w) \circ j(P:\xi:\lambda) = ev_{w^{-1}\tau(w)u} \circ j'(P:\xi:\lambda) \circ = pr'(\xi, w^{-1}\tau(w)u) \circ R = R_u \circ pr(\xi, u).$$

It follows that the equality (6.3) holds at every $v \in \mathscr{W}$. The proof is completed by using Theorem 5.10. ■

COROLLARY 6.11. — *Let* $[\xi] \in \hat{M}_{fu}$ *and* $w \in N_K(\mathfrak{a}_q)$. *Then for any two parabolic subgroups* P_1, $P_2 \in \mathscr{P}_\sigma(A_q)$ *we have*

$$L(\xi, w) \circ B(P_2 : P_1 : \xi : \lambda) = B(wP_2w^{-1} : wP_1w^{-1} : w\xi : w\lambda) \circ L(\xi, w).$$

Proof. — Use Lemmas 4.10, 6.1 and 6.10. ■

The existence of automorphisms with certain invariance properties is the source of other transformation properties. If φ is an automorphism of G which commutes with σ and θ and leaves A_q invariant, then φ induces a map $P \to \varphi(P)$ from $\mathscr{P}_\sigma(A_q)$ into itself. Moreover, φ leaves M_1, hence $A = \text{centre}(M_1) \cap \exp \mathfrak{p}$ and

$$M = (M_1 \cap K) \exp(p \cap \mathfrak{m}_1 \cap \mathfrak{a}^\perp)$$

invariant. Hence given $\xi \in \hat{M}_{fu}$ we may define $\xi^\varphi \in \hat{M}_{fu}$ by

$$\xi^\varphi(m) = \xi(\varphi^{-1}(m)),$$

for $m \in M$. Also, we define

$$\lambda^\varphi = \lambda \circ \varphi^{-1},$$

for $\lambda \in \mathfrak{a}_{qc}^*$. Finally, if $P \in \mathscr{P}_\sigma(A_q)$, then φ induces a linear map $(\varphi^{-1})^*$ from $\mathscr{D}'(G:P:\xi:\lambda)$ onto $\mathscr{D}'(G:\varphi(P):\xi^\varphi:\xi^\varphi)$, defined by

$$(\varphi^{-1})^* f = f \circ \varphi^{-1}.$$

LEMMA 6.12. — *Let* P_1, $P_2 \in \mathscr{P}_\sigma(A_q)$, $\xi \in \hat{M}_{fu}$. *Then*

$$(\varphi^{-1})^* \circ A(P_2:P_1:\xi:\lambda) = A(\varphi(P_2):\varphi(P_1):\xi^\varphi:\lambda^\varphi) \circ (\varphi^{-1})^*.$$

Proof. — Use formula (4.1) and analytic continuation. Using that φ stabilizes K and A it is straightforward to verify that the normalizations of Haar measures agreed upon in paragraph 4 transform appropriately. ■

The automorphism φ^{-1} stabilizes A_q and K, hence $N_K(\mathfrak{a}_q)$ and $Z_K(\mathfrak{a}_q)$. Therefore it induces an automorphism φ_* of W (*cf.* Lemma 1.2). We know already that $\varphi(H^0) = H^0$, since φ commutes with σ. If we assume in addition that $\varphi(H) = H$, then φ_* stabilizes $W_{K \cap H}$ and induces a bijection φ_* of $W/W_{K \cap H}$ onto itself (use the definition of $W_{K \cap H}$ above Lemma 1.3).

By transportation we obtain a bijection $\varphi_* : \mathscr{W} \to \mathscr{W}$. Though $(\varphi^{-1})^*$ is not an intertwining operator, it does map $\mathscr{D}'(G:P:\xi:\lambda)^H$ onto $\mathscr{D}'(G:\varphi(P):\xi^\varphi:\lambda^\varphi)^H$, since φ stabilizes H.

LEMMA 6.13. — *Let* $\xi \in \hat{M}_{fu}$. *Then there exists a unique linear map* $b(\varphi, \xi)$ *from* $V(\xi)$ *onto* $V(\xi^\varphi)$ *such that*

$$(\varphi^{-1})^* \circ j(P:\xi:\lambda) = j(\varphi(P):\xi^\varphi:\lambda^\varphi) \circ b(\varphi, \xi) \tag{6.4}$$

for $P \in \mathscr{P}_\sigma(A_q)$, $\lambda \in \mathfrak{a}_{qc}^*$. *The map* $b(\varphi, \xi)$ *is unitary and maps* $V(\xi, w)$ *onto* $V(\xi^\varphi, \varphi_* w)$, *for* $w \in \mathscr{W}$.

Proof. — Uniqueness follows from Theorem 5.10. For existence, put $\mathscr{W}' = \varphi^{-1}(\mathscr{W})$ and let R be the map $V(\xi) \to V'(\xi)$ of Lemma 5.8. Then R is unitary and maps $V(\xi, w)$ onto $V(\xi, \varphi^{-1} \circ \varphi_*(w))$, for $w \in \mathscr{W}$. Now $\mathscr{V}(\xi, \varphi^{-1}(w)) = \mathscr{V}(\xi^\varphi, w)$ (use the fact that φ leaves $M \cap H$ invariant). We denote the induced map $V'(\xi) \to V(\xi^\varphi)$ by $\mathscr{I}$, and define $b(\varphi, \xi) = \mathscr{I} \circ R$. Then $b(\varphi, \xi)$ is unitary and maps $V(\xi, w)$ onto $V(\xi^\varphi, \varphi_*(w))$. To see that it is the required map, fix $w \in \mathscr{W}$. Then

$$\begin{aligned} ev_w \circ (\varphi^{-1})^* \circ j(\mathrm{P}:\xi:\lambda) &= ev_{\varphi^{-1}(w)} \circ j(\mathrm{P}:\xi:\lambda) \\ &= ev_{\varphi^{-1}(w)} \circ j'(\mathrm{P}:\xi:\lambda) \circ \mathrm{R} \\ &= pr'(\xi, \varphi^{-1}(w)) \\ &= pr(\xi^{\varphi}, w) \circ \mathscr{I} \circ \mathrm{R} \\ &= ev_w \circ j(\varphi(\mathrm{P}):\xi^{\varphi}:\lambda^{\varphi}) \circ b(\varphi, \xi) \end{aligned}$$

and (6.4) follows. ■

COROLLARY 6.14. — *Let* φ *be an automorphism of* G *which commutes with* θ *and* σ *and leaves* A_q *and* H *invariant. Then for* $\mathrm{P}_1, \mathrm{P}_2 \in \mathscr{P}_\sigma(\mathrm{A}_q)$, $\xi \in \hat{\mathrm{M}}_{fu}$ *and* $\lambda \in \mathfrak{a}_{qc}^*$ *we have*

$$b(\varphi, \xi) \circ \mathrm{B}(\mathrm{P}_2:\mathrm{P}_1:\xi:\lambda) = \mathrm{B}(\varphi(\mathrm{P}_2):\varphi(\mathrm{P}_1):\xi^{\varphi}:\lambda^{\varphi}) \circ b(\varphi, \xi)$$

Proof. — Use Proposition 6.1 and Lemmas 6.12, 13. ■

COROLLARY 6.15. — *Let* $\mathrm{P}_1, \mathrm{P}_2 \in \mathscr{P}_\sigma(\mathrm{A}_q)$ *and* $\xi \in \hat{\mathrm{M}}_{fu}$. *Then*

$$\mathrm{B}(\mathrm{P}_2:\mathrm{P}_1:\xi:\lambda) = \mathrm{B}(\bar{\mathrm{P}}_2:\bar{\mathrm{P}}_1:\xi:-\lambda).$$

Proof. — We apply Corollary 6.14 with $\varphi = \theta$. First of all, using the notations of the proof of Lemma 6.13, we have that $\theta_* = \theta$ on $\mathscr{W}$. Hence R is the identity of $\mathrm{V}(\xi)$. Next, ξ is trivial on $\mathfrak{m} \cap \mathfrak{p}$ (*cf.* the proof of Lemma 4.4), hence $\xi^{\theta} = \xi$. We see that $b(\theta, \xi)$ is the identity of $\mathrm{V}(\xi)$ and the result follows. ■

In the rest of this paper we assume that every Cartan subgroup of G *is abelian.*

Given $\xi \in \hat{\mathrm{M}}_{fu}$, we write $\xi^{\vee}$ for the contragredient representation on the complex linear dual $\mathscr{H}_\xi^*$ of $\mathscr{H}_\xi$. It is unitary for the dual inner product on $\mathscr{H}_\xi^*$.

LEMMA 6.16. — *If* $\xi \in \hat{\mathrm{M}}_{fu}$ *possesses a* $(\mathrm{M} \cap \mathrm{H})$*-fixed vector, then there exists an anti-linear and anti-unitary map* $\mathrm{S}:\mathscr{H}_\xi \to \mathscr{H}_\xi$ *which intertwines* ξ *with* ξ^{σ}, i.e. $\xi^{\sigma}(m) \circ \mathrm{S} = \mathrm{S} \circ \xi(m)$ *for all* $m \in \mathrm{M}$. *Moreover, any such* S *satisfies* $\mathrm{S}^2 = 1$.

Proof. — Consider $\mathscr{F} = \mathrm{L}^2(\mathrm{M}/\mathrm{M} \cap \mathrm{H})$ together with the left regular representation L. Fix $\eta \in \mathscr{H}_\xi^{\mathrm{M} \cap \mathrm{H}} \setminus 0$, and define $j:\mathscr{H}_\xi \to \mathscr{F}$ by $j(v)(m) = \langle v, \xi(m)\eta \rangle$, for $v \in \mathscr{H}_\xi$, $m \in \mathrm{M}$. Then j is an equivariant embedding. By Lemma 5.4, its image equals the space $\mathscr{F}(\xi)$ of functions of type ξ in $\mathscr{F}$. Via the equivariant L^2-inner product we see that the map $\overline{\mathrm{C}:f \mapsto \bar{f}}$ maps $\mathscr{F}(\xi)$ onto the space $\mathscr{F}(\xi^{\vee})$. In particular the function $\varphi : m \mapsto \overline{\langle \eta, \xi(m)\eta \rangle}$ belongs to $\mathscr{F}(\xi^{\vee})$. Clearly

$$\varphi(m) = \langle \eta, \xi(\sigma(m))\eta \rangle \tag{6.5}$$

for $m \in \exp(\mathfrak{m}_0 \cap \mathfrak{q})$. The group F introduced in the proof of Lemma 5.4 is σ-stable and abelian, hence $m \in \mathrm{F} \Rightarrow m\,\sigma(m) \in \mathrm{F} \cap \mathrm{H}$. Since F is central in M_0, we see that (6.5) holds for $m \in \mathrm{F}\exp(\mathfrak{m}_0 \cap \mathfrak{q})(\mathrm{M}_0^0 \cap \mathrm{H})$, hence by real analytic continuation for $m \in \mathrm{FM}_0^0 = \mathrm{M}_0$. From the proof of Lemma 4.4, we recall that ξ is trivial on $\bar{\mathrm{N}}_{\mathrm{M}}$, A_{M}, and N_{M}. By density of $\bar{\mathrm{N}}_{\mathrm{M}}\mathrm{M}_0\mathrm{A}_{\mathrm{M}}\mathrm{N}_{\mathrm{M}}$ in M, we see that (6.5) holds for all $m \in \mathrm{M}$. It follows that $0 \neq \varphi \in \mathscr{F}(\xi^\sigma) \cap \mathscr{F}(\check{\xi})$. Hence $\mathscr{F}(\xi^\sigma) = \mathscr{F}(\check{\xi})$ and $\xi^\sigma \sim \check{\xi}$. The mapping $\sigma^* : f \mapsto f \circ \sigma$ maps $\mathscr{F}(\xi^\sigma)$ into $\mathscr{F}(\xi)$. One readily verifies that

$$S = j^{-1} \circ \sigma^* \circ C \circ j$$

fulfills the requirements.

If S′ is a second anti-linear and anti-unitary operator $\mathscr{H}_\xi \to \mathscr{H}_\xi$ intertwining ξ with ξ^σ, then $S' = \lambda S$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. It follows that $(S')^2 = \lambda\bar{\lambda} S^2 = 1$. ■

LEMMA 6.17. — *Let* $P_1, P_2 \in \mathscr{P}_\sigma(A_q)$ *and assume that* $\xi \in \hat{M}_{fu}$ *possesses a* $(M \cap H)$-*fixed vector. Then*

$$\text{(6.6)} \quad B(P_2^\sigma : P_1^\sigma : \xi : -\bar{\lambda}) \circ b(\sigma, \xi^\sigma) \circ b(S, \xi^\sigma, \xi) = b(\sigma, \xi^\sigma) \circ b(S, \xi^\sigma, \xi) \circ B(P_2 : P_1 : \xi : \lambda).$$

Moreover, $b(\sigma, \xi^\sigma) \circ b(S, \xi^\sigma, \xi)$ *is an anti-unitary, anti-linear map from* $V(\xi)$ *onto itself. It maps* $V(\xi, w)$ *onto* $V(\xi, \sigma_* w)$ $(w \in \mathscr{W})$ *and its square is* 1.

Proof. — Except for the assertion about the square, all assertions follow straightforwardly from Corollaries 6.9, 14. In view of Lemmas 6.8 and 6.13, the map $b(\sigma, \xi^\sigma) \circ b(S, \xi^\sigma, \xi)$ is characterized by

$$\text{(6.7)} \qquad \sigma^* \circ \mathrm{Ind}_P^G(S : \xi : \lambda) \circ j(P : \xi : \lambda) = j(P^\sigma : \xi : -\bar{\lambda}) \circ b(\sigma, \xi^\sigma) \circ b(S, \xi^\sigma, \xi).$$

Now observe that S also intertwines ξ^σ with $\xi^{\sigma^2} = \xi$, so that $\sigma^* \circ \mathrm{Ind}_{\sigma P}^G(S : \xi : -\bar{\lambda}) = \mathrm{Ind}_P^G(S : \xi : \bar{\lambda}) \circ \sigma^*$. Applying the latter map to the left hand side of (6.7), we obtain $\mathrm{Ind}_{\sigma P}^G(S : \xi^\sigma : \bar{\lambda}) \circ \mathrm{Ind}_P^G(S : \xi : \lambda) \circ j(P : \xi : \lambda)$, which equals $j(P : \xi : \lambda)$, since $S^2 = I$. On the other hand, application of the map to the right hand side of (6.7) yields $j(P : \xi : \sigma) \circ (b(\sigma, \xi^\sigma) \circ b(S, \xi^\sigma, \xi))^2$ (use Lemmas 6.8 and 6.13 once more). In view of Theorem 5.10 the obtained equality implies the result. ■

7. Reduction to the σ-split rank one case.

The purpose of this section is to reduce the proof of Theorem 6.3 to the σ-split rank one case, *i.e.* $\dim \mathfrak{a}_q = 1$. As before we fix $[\xi] \in \hat{M}_{fu}$. The first step is the usual decomposition of intertwining operators as a product of operators coming from adjacent parabolic subgroups.

Two parabolics $P, P' \in \mathscr{P}_\sigma(A_q)$ are called $(\sigma-)$ adjacent iff $P \neq P'$ and all $\mathfrak{a}_q$-roots in $\bar{\mathfrak{n}} \cap \mathfrak{n}'$ are proportional. Notice that P′ is adjacent to P iff $P' = P^{s_\alpha}$, for α a simple root in $\Sigma(\mathfrak{n}_P, \mathfrak{a}_q)$ (the latter is a positive system for Σ). A $(\sigma-)$ minimal string of parabolics from P to P′ is defined to be a sequence of parabolics $P_0, \ldots, P_r \in \mathscr{P}_\sigma(A_q)$ of smallest possible length $r \geqq 0$ such that $P_0 = P$, $P_r = P'$ and P_i, P_{i+1} are adjacent for $0 \leqq i < r$. Clearly any two parabolics in $\mathscr{P}_\sigma(A_q)$ are connected by a minimal string. Since W acts simply transitively on $\mathscr{P}_\sigma(A_q)$, minimal strings correspond to reduced expressions of Weyl group elements just as in the case of (ordinary) minimal parabolics. Indeed $P' = P^w$ for a uniquely determined $w \in W$. If $w = s_{\alpha_1} \circ \ldots \circ s_{\alpha_k}$ is a reduced expression for w (in terms of the simple roots in $\Sigma(\mathfrak{n}_P, \mathfrak{a}_q)$) then a minimal string $\{P_j\}$ is defined by

$P_0=P$ and $P_j=s_{\alpha_1}\circ\ldots\circ s_{\alpha_j}P s_{\alpha_j}\circ\ldots\circ s_{\alpha_1}$ for $1\leqq j\leqq k$. Conversely any minimal string $\{P_j\}$ from P to P′ determines a reduced expression for w satisfying the above equalities. From the corresponding result for intertwining operators (a consequence of Proposition 4.6) we immediately obtain the following.

PROPOSITION 7.1. — *Let* $P, P'\in\mathscr{P}_\sigma(A_q)$ *and suppose that* P_i, $0\leqq i\leqq r$ *is a minimal string from* P *to* P′. *Then*

$$B(P':P:\xi:\lambda)=B(P_r:P_{r-1}:\xi:\lambda)\circ\ldots\circ B(P_1:P_0:\xi:\lambda).$$

In view of the above it suffices to prove Theorem 6.3 when P_1 and P_2 are adjacent. So assume this to be the case and let α be the reduced $\mathfrak{a}_q$-root whose root space $\mathfrak{g}^\alpha$ is contained in $\bar{\mathfrak{n}}_2\cap\mathfrak{n}_1$. Then $P_2=P_1^{s_\alpha}$. We agree to write $\mathfrak{n}_\alpha=\Sigma_\gamma\mathfrak{g}^\gamma$, the summation extending over $\mathbb{N}\alpha\cap\Sigma$. Thus, putting $\bar{\mathfrak{n}}_\alpha=\theta\,\mathfrak{n}_{\alpha_1}$, $N_\alpha=\exp(\mathfrak{n}_\alpha)$ and $\bar{N}_\alpha=\theta N_\alpha$ we have

$$N_2\cap\bar{N}_1=\bar{N}_\alpha.$$

The group W acts on $W/W_{K\cap H}$ by left multiplication. We denote the induced action on the set $\mathscr{W}$ of representatives by τ.

LEMMA 7.2. — *Suppose* $P_1, P_2\in\mathscr{P}_\sigma(A_q)$ *are adjacent parabolic subgroups and let* α *be the reduced* $\mathfrak{a}_q$*-root with* $\mathfrak{g}^\alpha\subset\bar{\mathfrak{n}}_2\cap\mathfrak{n}_1$. *If* $w\in\mathscr{W}$, *then* $B(P_2:P_1:\xi:\lambda)$ *leaves the subspace* $V(\xi,w)+V(\xi,\tau(s_\alpha)(w))$ *of* $V(\xi)$ *invariant.*

Proof. — It clearly suffices to prove that $B(P_2:P_1:\xi:\lambda)$ maps $V(\xi,w)$ into $V(\xi,w)+V(\xi,\tau(s_\alpha)w)$, for any $w\in\mathscr{W}$. By Remark 6.5 we may pass to a different set of representatives so that $1\in\mathscr{W}$. The map $L(\xi,w)$ of Lemma 6.10 maps $V(\xi,1)$ unitarily onto $V(w\xi,w)$ and $V(\xi,\tau(s_{w^{-1}\alpha})1)$ unitarily onto $V(w\xi,\tau(s_\alpha)w)$. Now put $P=w^{-1}P_1 w$ and $\beta=w^{-1}\alpha$. Then

$$B(P_2:P_1:\xi:\lambda)=L(\xi,w)\circ B(P^{s_\beta}:P:w^{-1}\xi:w^{-1}\lambda)\circ L(\xi,w)^{-1}\tag{7.1}$$

by Corollary 6.11. We see that it suffices to prove the lemma for $w=1$ and all choices of P_1, P_2, ξ and λ. So fix $\eta\in V(\xi,1)$. Then $\operatorname{supp} j(P_1:\xi:\lambda)\eta\subset cl(P_1H)$ (for generic λ). From Lemma 4.12 it follows that $\operatorname{supp} A(P_2:P_1:\xi:\lambda)j(P_1:\xi:\lambda)\eta$ is contained in $cl(\bar{N}_\alpha P_1 H)$. From Lemma 6.1 and the lemma below we infer that $j(P_2:\xi:\lambda)\circ B(P_2:P_1:\xi:\lambda)\eta$ has support contained in $cl(P_1H\cup P_1 s_\alpha H)$, whence the assertion (use Theorem 5.10). ■

LEMMA 7.3. — *Let* $P\in\mathscr{P}_\sigma(A_q)$ *and* α *a simple root in* $\Sigma(\mathfrak{n}_P,\mathfrak{a}_q)$. *Then* $\bar{N}_\alpha P\subset cl(PH\cup Ps_\alpha H)$.

Proof. — The centralizer $G_1(\alpha)$ of $\ker\alpha$ in G is a group of Harish-Chandra's class satisfying assumption (A) (*cf.* Section 1). Its Lie algebra $\mathfrak{g}_1(\alpha)$ equals $\mathfrak{m}\oplus\mathfrak{a}\oplus\Sigma_{\beta\in\Sigma\cap\mathbb{Z}\alpha}\mathfrak{g}^\beta$. Let $P_1(\alpha)$ be the parabolic subgroup of $G_1(\alpha)$ with Lie algebra

$$\mathfrak{P}_1(\alpha)=\mathfrak{m}\oplus\mathfrak{a}\oplus\Sigma_{\beta\in\Sigma\cap\mathbb{N}\alpha}\mathfrak{g}^\beta.$$

Then $P_1(\alpha)$ is a minimal $\sigma\theta$-stable parabolic subgroup of $G_1(\alpha)$, and $H_1(\alpha)=G_1(\alpha)\cap H$ is an open subgroup of $G_1(\alpha)^\sigma$. Hence if s is a representative of s_α in $N_K(\mathfrak{a}_q)\cap G_1(\alpha)$, then by Appendix B we have

$$(7.2)\qquad G_1(\alpha)=cl(P_1(\alpha)H_1(\alpha)\cup P_1(\alpha)sH_1(\alpha)).$$

Let Q be the parabolic subgroup of G with Lie algebra $\mathfrak{P}+\bar{\mathfrak{n}}_\alpha$. Then $\bar{N}_\alpha P\subset Q$. Moreover, $Q=N_Q G_1(\alpha)$ with N_Q its unipotent radical. Now observe that $P=N_Q P_1(\alpha)$. Hence from (7.2) we obtain that $Q=cl(PH_1(\alpha)\cup PsH_1(\alpha))$ and the proof is complete. ■

Reduction to σ-split rank 1, *first step.* — According to Remark 6.5 it suffices to prove Theorem 6.3 for one fixed choice $\mathcal{W}$ of representatives. We fix $\mathcal{W}$ such that $1\in\mathcal{W}$. In view of Proposition 7.1 it suffices to prove the equation (6.2) for adjacent parabolics P and P^{s_α}, and by Lemma 7.2 we may restrict ourselves to proving it on the subspace $V(\xi, w)+V(\xi, \tau(s_\alpha)w)$. By (7.1) and unitarity of the map $L(\xi, w)$ we may even restrict ourselves to proving the equation on $V(\xi, 1)+V(\xi, \tau(s_\alpha)1)$, for all choices of ξ, P, P^{s_α} (α simple in $\Sigma(\mathfrak{n}_P, \mathfrak{a}_q)$) and generic λ.

For the rest of this section we fix $P=MAN$ in $\mathscr{P}_\sigma(A_q)$ and α a simple root in $\Sigma^+=\Sigma(\mathfrak{n}, \mathfrak{a}_q)$. Let $\mathcal{W}_\alpha=\{1, \tau(s_\alpha)1\}$, put

$$V(\xi, \mathcal{W}_\alpha)=V(\xi, 1)+V(\xi, \tau(s_\alpha)1)$$

and define endomorphisms of $V(\xi, \mathcal{W}_\alpha)$ by

$$B_\alpha(P^{s_\alpha s}: P^s: \xi: \lambda)=B(P^{s_\alpha s}: P^s: \xi: \lambda)\,|\,V(\xi, \mathcal{W}_\alpha)$$

$(s=1, s_\alpha)$, whenever $\lambda\in\mathfrak{a}_{qc}^*$ is not a pole. Then according to the above it suffices to prove that

$$(7.3)\qquad B_\alpha(P^{s_\alpha}: P: \xi: \lambda)^*=B_\alpha(P: P^{s_\alpha}: \xi: -\bar{\lambda}).$$

Reduction to σ-split rank 1, *last step.* — The final step in the reduction consists of comparing B_α with the B for a σ-split rank 1 group G(α). Recall the definitions given in the proof of Lemma 7.3. Since ker α is central in $\mathfrak{g}_1(\alpha)$ and contained in $\mathfrak{g}_1(\alpha)\cap\mathfrak{p}$,

$$G(\alpha)=(G_1(\alpha)\cap K)\exp(\mathfrak{g}_1(\alpha)\cap\mathfrak{p}\cap(\ker\alpha)^\perp)$$

is a closed subgroup of Harish-Chandra's class satisfying assumption (A). Moreover $H(\alpha)=H\cap G(\alpha)$ is an open subgroup of $G(\alpha)^\sigma$. The group G(α) has Lie algebra

$$\mathfrak{g}(\alpha)=\mathfrak{m}\oplus\mathfrak{a}(\alpha)\oplus\Sigma_{\beta\in\Sigma\cap\mathbb{Z}\alpha}\mathfrak{g}^\beta,$$

with $\mathfrak{a}(\alpha)=\mathfrak{a}\cap(\ker\alpha)^\perp$.

Its Langlands decomposition is $P(\alpha)=MA(\alpha)N_\alpha$, where $A(\alpha)=\exp\mathfrak{a}(\alpha)$. Since G(α) centralizes ker α, a representative s for $s_\alpha|\mathfrak{a}_q(\alpha)$ in $N_{K(\alpha)}(\mathfrak{a}_q(\alpha))$ is also a representative for s_α in $N_K(\mathfrak{a}_q)$. Hence we may select a set of representatives $\mathcal{W}$ such that $\mathcal{W}_\alpha\subset G(\alpha)$ (*cf.* Remark 6.5). Let W(α) and $W_{K\cap H}(\alpha)$ be the Weyl groups for G(α) defined as in Section 1. Then W(α) naturally embeds into W. Moreover,

$$W(\alpha)\cap W_{K\cap H}=W_{K\cap H}(\alpha),$$

hence $\mathscr{W}_\alpha$ is a set of representatives for $W(\alpha)/W_{K\cap H}(\alpha)$. The following lemma reduces the proof of Theorem 6.3 to the σ-split rank 1 case.

LEMMA 7.4. — *Assume that* $\mathscr{W}_\alpha \subset G(\alpha)$. *Then for all* $\lambda \in \mathfrak{a}_{qc}^*$ *such that* $\lambda_\alpha = \lambda \mid \mathfrak{a}_q(\alpha)$ *is not a pole for* $B(G(\alpha): P(\alpha)^{s_\alpha}: P(\alpha): \xi)$ *we have*

$$B_\alpha(P^{s_\alpha}: P: \xi: \lambda) = B(G(\alpha): P(\alpha)^{s_\alpha}: P(\alpha): \xi: \lambda_\alpha).$$

Remark. — The regions in $\mathfrak{a}_q(\alpha)_c^*$ where $A(G(\alpha): P(\alpha)^{s_\alpha}: P(\alpha): \xi)$ and $j(G(\alpha): P(\alpha): \xi)$ were defined initially (before meromorphic continuation took place) need not have any overlap. Consequently, there are no integral representations for the endomorphisms in the above equation and this is the main difficulty in the proof. It is overcome by using a trick of ([O-S 80], Lemma 4.13).

Proof. — Let pr_α denote the orthogonal projection from $V(\xi)$ onto $V(\xi, \mathscr{W}_\alpha)$ and define $ev_\alpha = pr_\alpha \circ ev$. Then we must show that

$$(7.4)\qquad ev_\alpha \circ A(P^{s_\alpha}: P: \xi: \lambda) \circ j(P: \xi: \lambda: \eta) = ev \circ A(\bar{P}(\alpha): P(\alpha): \xi: \lambda_\alpha) \circ j(P(\alpha): \xi: \lambda_\alpha: \eta),$$

for $\eta \in V(\xi, \mathscr{W}_\alpha)$. Fix $\eta \in V(\xi, \mathscr{W}_\alpha)$. We introduce a new parameter $\nu \in \mathfrak{a}_q(\alpha)_c^*$ and define Λ to be the set of all $(\lambda, \nu) \in \mathfrak{a}_{qc}^* \times \mathfrak{a}_q(\alpha)_c^*$ satisfying

(*a*) $\langle \operatorname{Re}\lambda, \alpha \rangle > C'_\alpha$

(*b*) $\langle \operatorname{Re}(\lambda - \nu) + \rho_P, \beta \rangle < 0$ for $\beta \in \Delta$.

Then clearly Λ is a non-empty open set. Moreover, the subset Λ' consisting of $(\lambda, \nu) \in \Lambda$ such that

(*c*) $\mu = \lambda - \nu$ is neither for $j(G: P: \xi: \mu: \eta)$, nor for $j(G(\alpha): P(\alpha): \xi: \mu_\alpha: \eta)$ a pole,

is open and dense in Λ.

Let h_P be the real analytic map $G \to A$ defined by

$$x \in N\, h_P(x)\, MK,$$

for $x \in G$. Then h_P maps $G(\alpha)$ into $A_q(\alpha)$. We define distributions, meromorphically depending on $(\lambda, \nu) \in \mathfrak{a}_{qc}^* \times \mathfrak{a}_q(\alpha)_c^*$, by

$$(7.5)\qquad J(G: \lambda: \nu) = h_P^\nu j(P: \xi: \lambda - \nu: \eta),$$

$$(7.6)\qquad J(G(\alpha): \lambda: \nu) = [h_P \mid G(\alpha)]^\nu j(G(\alpha): P(\alpha): \xi: \lambda_\alpha - \nu: \eta).$$

Then $J(G: \lambda: \nu) \in \mathscr{D}'(G: P: \xi: \lambda)$, and $J(G(\alpha): \lambda: \nu) \in \mathscr{D}'(G(\alpha): P(\alpha): \xi: \lambda_\alpha)$. Moreover, if $(\lambda, \nu) \in \Lambda'$, then by Proposition 5.6 the generalized functions $J(G: \lambda: \nu)$ and $J(G(\alpha): \lambda: \nu)$ are continuous and

$$J(G: \lambda: \nu) \mid G(\alpha) = J(G(\alpha): \lambda: \nu).$$

Moreover, because of condition (*a*),

$$A(P^{s_\alpha}: P: \xi: \lambda)\, J(G: \lambda: \nu) \qquad \text{and} \qquad A(\bar{P}(\alpha): P(\alpha): \xi: \lambda_\alpha)\, J(G(\alpha): \lambda: \nu)$$

are then continuous functions defined by absolutely convergent integrals (use Proposition 4.1 and the fact that $(a) \Leftrightarrow \lambda \in \mathscr{A}(P^{s_\alpha} | P)$). It is readily checked from the definitions that the first of these functions restricts to the latter on $G(\alpha)$. In view of Appendix B, the union of the open $\bar{P}(\alpha) \times H(\alpha)$-orbits in $G(\alpha)$ is given by $\mathcal{O}(\bar{P}(\alpha)) = \bigcup_{w \in \mathscr{W}_\alpha} \bar{P}(\alpha)\, w\, H(\alpha)$, hence contained in $\mathcal{O}(P^{s_\alpha})$. We obtain

$$(7.7) \quad A(P^{s_\alpha}: P: \xi: \lambda)\, J(G: \lambda: \nu) \,|\, \mathcal{O}(\bar{P}(\alpha)) = A(\bar{P}(\alpha): P(\alpha): \xi: \lambda_\alpha)\, J(G(\alpha): \lambda: \nu) \,|\, \mathcal{O}(\bar{P}(\alpha))$$

still for $(\lambda, \nu) \in \Lambda'$. Now fix $(\lambda_0, \nu_0) \in \mathfrak{a}_{qc}^* \times \mathfrak{a}_q(\alpha)_c^*$ for the moment. Then there exists a holomorphic function φ defined on an open neighbourhood $N(\lambda_0, \nu_0)$ of (λ_0, ν_0) in $\mathfrak{a}_{qc}^* \times \mathfrak{a}_q(\alpha)_c^*$ such that $(\lambda, \nu) \to \varphi(\lambda, \nu)\, j(P: \xi: \lambda - \nu: \eta)$ is holomorphic as a map from $N(\lambda_0, \nu_0)$ into $\mathscr{D}'(K: \xi)$. Since $j(P: \xi: \lambda - \nu: \eta)$ is right H-invariant as an element of $\mathscr{D}'(G) \otimes \mathscr{H}_\xi$, whereas h_P is a real analytic function, it follows that the map $(\lambda, \nu, h) \to \varphi(\lambda, \nu)\, [R_h J(G: \lambda: \nu)]$, from $N(\lambda_0, \nu_0) \times H$ into $\mathscr{D}'(K: \xi)$ is smooth and in addition holomorphic in (λ, ν). By Lemma 4.10 we obtain that $(\lambda, \nu) \to A(P^{s_\alpha}: P: \xi: \lambda)\, J(G: \lambda: \nu) \,|\, \mathcal{O}(P^{s_\alpha})$ is meromorphic as a $C^\infty(\mathcal{O}(P^{s_\alpha}) \cap K: K_M: \xi)$-valued map. By a similar reasoning we infer that

$$(\lambda, \nu) \to A(\bar{P}(\alpha): P(\alpha): \xi: \lambda)\, J(G(\alpha): \lambda: \nu) \,|\, \mathcal{O}(\bar{P}(\alpha))$$

is meromorphic as a $C^\infty(\mathcal{O}(\bar{P}(\alpha)) \cap K(\alpha): K_M: \xi)$-valued map. By analytic continuation it now follows that (7.7) holds as an identity of meromorphic functions on the whole of $\mathfrak{a}_{qc}^* \times \mathfrak{a}_q(\alpha)_c^*$. Now if $\lambda \in \mathfrak{a}_{qc}^*$ is not a pole for any of $j(P: \xi: \lambda: \eta)$, $j(G(\alpha): P(\alpha): \xi: \lambda_\alpha: \eta)$, $A(P^{s_\alpha}: P: \xi: \lambda)$ and $A(\bar{P}(\alpha): P(\alpha): \xi: \lambda_\alpha)$, then $(\lambda, 0)$ is not a pole for left or right hand side of equation (7.7) (use definitions (7.5,6)). Substituting $\nu = 0$ in (7.7) we obtain (7.4) on a Baire subset of $\mathfrak{a}_{qc}^*$, hence in a meromorphic sense. ■

8. Proof of Theorem 6.3 when $\dim A_q = 1$.

As before we fix $P = MAN$ in $\mathscr{P}_\sigma(A_q)$, and $\xi \in \hat{M}_{fu}$. If $\xi \notin \hat{M}_{ps}$ then $V(\xi) = 0$ and there is nothing to prove, so we may assume that $w\xi$ possesses a $(M \cap H)$-fixed vector, for some $w \in N_K(\mathfrak{a}_q)$. Applying Corollary 6.11 if necessary we may as well assume that $w = 1$, *i.e.* ξ possesses a $(M \cap H)$-fixed vector.

Put $\Sigma^+ = \Sigma(\mathfrak{n}, \mathfrak{a}_q)$ and let α be the unique simple root in Σ^+. Then $W = \{1, s_\alpha\}$. Consequently there are two cases: either $W/W_{K \cap H}$ has one, or it has two elements. We treat these cases separately.

8 *A. The case* $|W/W_{K \cap H}| = 1$. — In this case we may fix $\mathscr{W} = \{1\}$ (*cf.* Remark 6.5). By Corollary 5.5, $\dim V(\xi) = 1$, so $B(\bar{P}: P: \xi: \lambda)$ is a scalar. Now ξ possesses a $M \cap H$-fixed vector, so we may apply Lemma 6.17. Fix $\eta \in V(\xi) \setminus \{0\}$. Then there exists $\varepsilon \in \mathbb{C}$ with $|\varepsilon| = 1$ such that $b(\sigma, \xi^\sigma) \circ b(S, \xi^\sigma, \xi)$ is given by $z\eta \to \varepsilon \bar{z} \eta$ $(z \in \mathbb{C})$. It follows that $B(\bar{P}^\sigma: P^\sigma: \xi: -\bar{\lambda}) = \overline{B(\bar{P}: P: \xi: \lambda)}$, and since $P^\sigma = \bar{P}$, this implies the result. ■

8 *B. The case* $|W/W_{K\cap H}|=2$. — In this case we have $|W_{K\cap H}|=1$, hence $W(\Sigma_+)=\{1\}$, and $\Sigma_+=\varnothing$ (*cf.* Section 1). Fix a representative s of s_α in $N_K(\mathfrak{a}_q)$. As a set of representatives for $W/W_{K\cap H}$ we shall use $\mathscr{W}=\{1, s\}$ (*cf.* Remark 6.5).

Before turning to the actual proof of Theorem 6.3 we shall prove a few lemmas which seem to distinguish the present case in an essential way from the other σ-split rank 1 cases.

Since $\Sigma_+=\varnothing$, we have $\sigma X=-\theta X$ for all $X\in\mathfrak{n}\oplus\bar{\mathfrak{n}}$. Hence, if $\mathfrak{h}_n=\mathfrak{h}\cap(\mathfrak{n}\oplus\bar{\mathfrak{n}})$, then

$$\mathfrak{h}_n=\mathfrak{p}\cap(\mathfrak{n}\oplus\bar{\mathfrak{n}}). \tag{8.1}$$

LEMMA 8.1. — *The map* $(m, X)\to m\exp X$ *is a diffeomorphism from* $(M\cap H)\times\mathfrak{h}_n$ *onto* H.

Proof. — From (8.1) we infer that

$$\mathfrak{h}\cap\mathfrak{p}=(\mathfrak{h}\cap\mathfrak{m}\cap\mathfrak{p})\oplus\mathfrak{h}_n.$$

Now $\mathrm{ad}(\mathfrak{h}\cap\mathfrak{m}\cap\mathfrak{p})$ normalizes $\mathfrak{h}_n$ but maps $\mathfrak{p}$ into $\mathfrak{k}$, so it actually centralizes $\mathfrak{h}_n$. It follows that $(X, Y)\to\exp X\exp Y$ maps $(\mathfrak{h}\cap\mathfrak{p}\cap\mathfrak{m})\times\mathfrak{h}_n$ diffeomorphically onto $\exp(\mathfrak{h}\cap\mathfrak{p})$. On the other hand, $\mathfrak{h}_n\cap\mathfrak{k}=0$ by (8.1), hence $\mathfrak{h}\cap\mathfrak{k}=\mathfrak{h}\cap\mathfrak{k}\cap\mathfrak{m}$, and we see that $H^0\cap K=(H\cap K)^0\subset H\cap K\cap M$. From our assumptions it follows that $|W_{K\cap H}|=1$, hence $N_{K\cap H}(\mathfrak{a}_q)\subset H\cap K\cap M$.
Since $H=N_{K\cap H}(\mathfrak{a}_{0q})H^0$ (*cf.* [Ba 86], p. 25, (2.2)) it follows that

$$H\cap K=H\cap K\cap M.$$

The proof is now completed by combining the diffeomorphism

$$(\mathfrak{h}\cap\mathfrak{p}\cap\mathfrak{m}\times\mathfrak{h}_n\to\exp(\mathfrak{h}\cap\mathfrak{p})$$

with the Cartan decompositions

$$H=(H\cap K\cap M)\exp(\mathfrak{h}\cap\mathfrak{p})\text{ and }M\cap H=(H\cap K\cap M)\exp(\mathfrak{h}\cap\mathfrak{m}\cap\mathfrak{p}).\quad\blacksquare$$

LEMMA 8.2. — *There exist unique maps* n_p, a_p, m_p *and* $\mathfrak{H}_P$ *from* $\Omega=PH$ *into* N, A_q, M_σ *and* $\exp(h_n)$ *respectively, such that*

$$x=n_P(x)\,a_P(x)\,m_P(x)\,\mathfrak{H}_P(x), \tag{8.2}$$

for $x\in\Omega$. *These maps are real analytic. Moreover,* n_P *and* a_P *are equal to the corresponding maps defined earlier by* (5.3).

Proof. — Let n_P, a_P and $\bar{m}_P$ be the maps defined by formula (5.3), and define a real analytic map $\bar{\mathfrak{H}}_P$ from Ω into $(H\cap M)\backslash M$ by $\bar{\mathfrak{H}}_P(x)=\bar{m}_P(x)^{-1}\,a_P(x)^{-1}\,n_P(x)^{-1}\,x$. By the previous lemma there exists a real analytic map $s:(H\cap M)\backslash H\to\exp\mathfrak{h}_n$ such that $s((H\cap M)h)\in(H\cap M)h$ for $h\in H$. Define

$$\mathfrak{H}_P=s\circ\bar{\mathfrak{H}}_P,\qquad\text{and}\qquad m_P(x)=a_P(x)^{-1}\,n_P(x)^{-1}\,x\,\mathfrak{H}_P(x)^{-1},\quad\text{for } x\in\Omega.$$

Then n_P, a_P, m_P and $\mathfrak{H}_P$ are real analytic maps satisfying (8.2). Their uniqueness follows from the uniqueness statement in ([Ba 86], Lemma 3.4) and Lemma 8.1. ■

LEMMA 8.3. — *The set* $\Omega = PH$ *is right* K_M*-invariant. Moreover, if* $x \in \Omega$, $l \in K_M = M \cap K$, *then*

(i) $a_P(xl) = a_P(x)$,

(ii) $m_P(xl) = m_P(x)\, l$.

Proof. — From (8.1) it follows that K_M normalizes $\mathfrak{h}_n$. From Lemma 8.2 we infer that $\Omega = P\exp(\mathfrak{h}_n)$, whence the invariance. Moreover, (i) and (ii) easily follow from the above and the uniqueness of the $NA_q\, M_\sigma \exp(\mathfrak{h}_n)$-decomposition. ■

The above lemmas enable us to define a special family of intertwining operators. Since $M_\sigma \simeq M \times (A \cap H)$ (*cf.* Section 2), we may extend ξ to a representation of M_σ which is trivial on $A \cap H$. Let $\rho = \rho_P$, and put

$$\mathscr{U}_1 = \{ \lambda \in \mathfrak{a}_{qc}^*;\ \langle \operatorname{Re}\lambda + \rho, \alpha \rangle < 0 \}.$$

Then for $\lambda \in \mathscr{U}_1$ and *any* $\varphi \in \mathscr{H}_\xi$ we define a $\mathscr{H}_\xi$-valued function on G by

$$(8.3)\qquad \begin{cases} \varepsilon(\xi:\lambda:\varphi)(x) = a_P(x)^{\lambda+\rho}\,\xi(m_P(x))\,\varphi & \text{for } x \in \Omega, \\ \qquad = 0 & \text{outside } \Omega. \end{cases}$$

Then it is straightforward to verify that $\varepsilon(\xi:\lambda:\varphi)$ satisfies the transformation rule (3.1).

LEMMA 8.4. — *If* $\lambda \in \mathscr{U}_1$ *and* $\varphi \in \mathscr{H}_\xi$ *then* $\varepsilon(\xi:\lambda:\varphi) \in C(P:\xi:\lambda)$. *Moreover, as a* $B(\mathscr{H}_\xi, C(K:\xi))$*-valued function,* $\varepsilon(\xi:\lambda)$ *depends holomorphically on* $\lambda \in \mathscr{U}_1$.

Proof. — The continuity follows exactly as in the proof of Proposition 5.6 (notice that H is essentially connected, since $W_{K\cap H} = \{1\}$, *see* Section 1). Holomorphy is obvious. ■

We now define $\mathscr{U}$ to be the set of $(\lambda, \nu) \in \mathfrak{a}_{qc}^* \times \mathfrak{a}_{qc}^*$ such that

$$(8.4)\qquad \lambda \in \mathscr{A}(\bar{P}\,|\,P) \quad \text{and} \quad \lambda - \nu \in \mathscr{U}_1.$$

For $(\lambda, \nu) \in \mathscr{U}$, the operator $A(\bar{P}:P:\xi:\lambda)$ is defined by the absolutely convergent integral (4.1) (*cf.* Proposition 4.1). We define real analytic maps $\nu_P : G \to N$, $h_P : G \to A$, $\mu_P : G \to \exp(\mathfrak{p} \cap \mathfrak{m})$ and $\kappa_P : G \to K$ by

$$(8.5)\qquad x = \nu_P(x)\, h_P(x)\, \mu_P(x)\, \kappa_P(x),$$

for $x \in G$. If $(\lambda, \nu) \in \mathscr{U}$ and $w \in N_K(\mathfrak{a}_q)$, we define the linear endomorphism $L(w:\xi:\lambda:\nu)$ of $\mathscr{H}_\xi$ by

$$(8.6)\qquad L(w:\xi:\lambda:\nu)\,\varphi = ev_w \circ A(\bar{P}:P:\xi:\lambda)\,[h_P^\nu\, \varepsilon(\xi:\lambda-\nu:\varphi)],$$

for $\varphi \in \mathscr{H}_\xi$.

LEMMA 8.5. — *The linear endomorphism* $L(w:\xi:\lambda:\nu) \in \operatorname{End}(\mathscr{H}_\xi)$ *depends holomorphically on* $(\lambda, \nu) \in \mathscr{U}$ *and intertwines* ξ *with* $w^{-1}\xi$.

Proof. — The holomorphy follows from Lemma 8.4, Corollary 4.14 and the fact that $\lambda \in \mathscr{A}(\bar{P}|P)$ is not a pole for $A(\bar{P}:P:\xi:\lambda)$. To see that the intertwining property holds, we use that for $(\lambda, \nu) \in \mathscr{U}$ the endomorphism $L(w:\xi:\lambda:\nu)$ is given by the absolutely convergent integral

$$(8.7) \qquad \int_{\bar{N} \cap \Omega w^{-1}} h_p(\bar{n}w)^\nu a_p(\bar{n}w)^{\lambda-\nu+\rho} \xi(m_P(\bar{n}w))\, d\bar{n}$$

[use (4.1) and (8.3) to rewrite (8.6)]. Now fix $l \in K_M$. Then the automorphism $\bar{n} \to l\bar{n}l^{-1}$ transforms the Haar measure $d\bar{n}$ by multiplication by a positive scalar. The scalar must be 1 by compactness of K_M. Moreover, from Lemma 8.3 we infer that $l(\bar{N} \cap \Omega w^{-1})\, l^{-1} = \bar{N} \cap \Omega w^{-1}$. Hence (8.7) equals the integral

$$(8.8) \qquad \int_{\bar{N} \cap \Omega w^{-1}} h_P(l\bar{n}l^{-1} w)^\nu a_P(l\bar{n}l^{-1} w)^{\lambda-\nu+\rho} \xi(m_P(l\bar{n}l^{-1} w))\, d\bar{n}.$$

Now obviously $h_P(l\bar{n}l^{-1} w) = h_P(\bar{n}) = h_P(\bar{n}w)$ and $a_P(l\bar{n}l^{-1} w) = a_P(\bar{n}w^{-1} wl^{-1} w) = a_P(\bar{n}w)$ by Lemma 8.3. Moreover, $m_P(l\bar{n}l^{-1} w) = l\, m_P(\bar{n}w)\, w^{-1} l^{-1} w$ by the same lemma. Substituting these relations in (8.8) we obtain

$$\xi(l) \circ L(w:\xi:\lambda:\nu) \circ \xi(w^{-1} l^{-1} w) = L(w:\xi:\lambda:\nu),$$

for $l \in K_M$. This equation holds for every $l \in M$, since ξ is trivial on $M \cap \exp \mathfrak{p}$. ■

Before studying the operator $L(w:\xi:\lambda:\nu)$ in more detail, we discuss its relation with $B(\bar{P}:P:\xi:\lambda)$.

PROPOSITION 8.6. — *Let u, $v \in \mathscr{W}$, $\varphi \in \mathscr{V}(\xi, u)$. Then $L(vu^{-1}:\xi:\lambda:\nu)\varphi$, originally defined for $(\lambda, \nu) \in \mathscr{U}$ extends to a meromorphic $\mathscr{H}_\xi$-valued function of $(\lambda, \nu) \in \mathfrak{a}_{qc}^* \times \mathfrak{a}_{qc}^*$. Its set of poles does not entirely contain $\mathfrak{a}_{qc}^* \times \{0\}$. Moreover,*

$$(8.9) \qquad pr(\xi, v) \circ B(\bar{P}:P:\xi:\lambda) \circ i(\xi, u)\varphi = L(vu^{-1}:\xi:\lambda:0)\varphi.$$

Proof. — Recall the definition of $\varepsilon_u(P:\xi:\lambda:\varphi)$ by (5.5). Then for $\lambda \in \mathscr{U}_1$ we have

$$(8.10) \qquad \varepsilon_u(P:\xi:\lambda:\varphi) = R_{u^{-1}}\, \varepsilon(\xi:\lambda:\varphi).$$

This is seen as follows. Since $N_K(\mathfrak{a}_q)$ normalizes $\mathfrak{h}_n$ and M, we have

$$PH\, u = P\exp(\mathfrak{h}_n)\, u = P u \exp(\mathfrak{h}_n) = P u H.$$

Hence $R_{u^{-1}}\, \varepsilon(\xi:\lambda:\varphi) = 0$ outside $PuH = Pu\exp(\mathfrak{h}_n)$. On the other hand, if $n \in N$, $a \in A_q$, $m \in M_\sigma$ and $h \in \exp(\mathfrak{h}_n)$, then

$$R_{u^{-1}}\, \varepsilon(\xi:\lambda:\varphi)(namuh) = \varepsilon(\xi:\lambda:\varphi)(namuhu^{-1}) = a^{\lambda+\rho} \xi(m)\varphi = \varepsilon_u(P:\xi:\lambda:\varphi)(namuh),$$

and (8.10) follows. Hence for $(\lambda, \nu) \in \mathscr{U}_1$ we have

$$(8.11) \qquad \begin{aligned} L(vu^{-1}:\xi:\lambda:\nu)\varphi &= ev_{vu^{-1}} \circ A(\bar{P}:P:\xi:\lambda)[h_P^\vee \varepsilon(\xi:\lambda:\nu:\varphi)] \\ &= ev_v \circ A(\bar{P}:P:\xi:\lambda) \circ R_{u^{-1}}[h_P^\vee \varepsilon(\xi:\lambda:\nu:\varphi)] \\ &= ev_v \circ A(\bar{P}:P:\xi:\lambda)[h_P^\vee \varepsilon_u(P:\xi:\lambda-\nu:\varphi)] \\ &= ev_v \circ A(\bar{P}:P:\xi:\lambda)[h_P^\vee j(P:\xi:\lambda-\nu:i(\xi,u)\varphi)]. \end{aligned}$$

Applying Theorem 5.10 and Corollary 4.14 to the latter expression we deduce that $L(vu^{-1}:\xi:\lambda:\nu)\varphi$ extends to a meromorphic function of $(\lambda,\nu)\in\mathfrak{a}_{qc}^*$. Moreover, substitution of $\nu=0$ in (8.11) yields (8.9) (use the formula for B occurring in the proof of Proposition 6.1 and notice that $ev_v = pr(\xi, v)\circ ev$). ■

We now proceed with our investigation of $L(w:\xi:\lambda:\nu)$. By Schur's lemma and Lemma 8.5 there exists a fixed operator $T(w:\xi)\in \mathrm{End}(\mathscr{H}_\xi)$ intertwining ξ with $w^{-1}\xi$ and a scalar $l(w:\xi:\lambda:\nu)$ depending holomorphically on $(\lambda,\nu)\in\mathscr{U}$, such that $L(w:\xi:\lambda:\nu) = l(w:\xi:\lambda:\nu)\circ T(w:\xi)$. These objects can actually be computed from the integral representation (8.7). The following lemma enables us to reduce to a $SL(2,\mathbb{R})$-computation.

LEMMA 8.7. — $\Sigma = \{\alpha, -\alpha\}$.

Proof. — Suppose $2\alpha\in\Sigma$. Then $\mathfrak{g}^{-\alpha}$, $\mathfrak{g}^{2\alpha}\subset\mathfrak{g}_-$ (since $\Sigma_+=\varnothing$). Hence $[\mathfrak{g}^{-\alpha},\mathfrak{g}^{2\alpha}]\subset\mathfrak{g}^\alpha\cap[\mathfrak{g}_-,\mathfrak{g}_-]\subset\mathfrak{g}_+^\alpha=0$. It follows that $\mathfrak{g}^{2\alpha}$ centralizes $\mathfrak{g}^{-\alpha}$ and $\mathfrak{g}^\alpha$, hence centralizes $\mathfrak{a}_q\subset[\mathfrak{g}^{-\alpha},\mathfrak{g}^\alpha]$, contradiction. ■

Let H_α be the unique element of $\mathfrak{a}_q$ with $\alpha(H_\alpha)=2$. We define a subset of $\bar{\mathfrak{n}}=\mathfrak{g}^{-\alpha}$ by

$$S=\{Y\in\bar{\mathfrak{n}};\ -B(Y,\theta Y)=2\langle\alpha,\alpha\rangle\}$$

(since B restricts to the Killing form on $\mathfrak{g}_1$, the bilinear form $-B(.,\theta(.))$ is positive definite on $\bar{\mathfrak{n}}$). If $Y\in S$ we put $X(Y)=-\theta Y$. Then $H=[X(Y),Y]$ belongs to $\mathfrak{m}_1\cap\mathfrak{p}\cap\mathfrak{q}=\mathfrak{a}_q$ and invariance of the Killing form yields $B(H,H_\alpha)=4\langle\alpha,\alpha\rangle$. Hence $H=H_\alpha$ and we see that H_α, $X(Y)$, Y is a standard $sl(2,\mathbb{R})$-triple. Its linear span is denoted by $\mathfrak{g}(Y)$. On $\mathfrak{g}(Y)$, σ is given by $\sigma H_\alpha=-H_\alpha$, $\sigma Y=X(Y)$, and $\sigma X(Y)=Y$ (use that $\sigma\theta=-I$ on $\mathfrak{g}^{-\alpha}$, since $\Sigma_+=\varnothing$). The connected (closed) analytic subgroup of G with Lie algebra $\mathfrak{g}(Y)$ is denoted by $G(Y)$. Given any closed subgroup (subalgebra) $B(\mathfrak{b})$ of $G(\mathfrak{g})$, we get $B(Y)=B\cap G(Y)$ (respectively $\mathfrak{b}(Y)=\mathfrak{b}\cap\mathfrak{g}(Y)$). Thus $A_q(Y)=A_q=\exp(\mathbb{R}H_\alpha)$, $\bar{N}(Y)=\exp(\mathbb{R}Y)$, and $M(Y)=\mathrm{centre}(G(Y))$. Moreover, $K(Y)=\exp(\mathbb{R}(Y+\theta Y))$ and $H(Y)^0=\exp(\mathbb{R}(Y-\theta Y))$. Finally, we define $s(Y)=\exp(\pi/2)(Y+\theta Y)$. Then $s(Y)$ is a representative of s_α in $N_{K(Y)}(\mathfrak{a}_q)$.

LEMMA 8.8. — *Let* $Y\in S$. *Then*:

(1) $\bar{N}(Y)\cap\Omega=\{\exp(tY);\ |t|<1\}$,

(2) $\bar{N}(Y)\cap\Omega s(Y)=\{\exp(tY);\ |t|>1\}$,

(3) $\log\circ h_P(\exp tY)=-(1/2)\log(1+t^2)H_\alpha\ (t\in\mathbb{R})$,

(4) $\log\circ a_P(\exp tY)=-(1/2)\log(1-t^2)H_\alpha\ (|t|<1)$,

(5) $\log\circ a_P(\exp(tY)s(Y))=-(1/2)\log(t^2-1)H_\alpha\ (|t|>1)$,

(6) $m_p(\exp t\mathrm{Y})=1$ $(|t|<1)$,

(7) $m_{\mathrm{P}}(\exp(t\mathrm{Y})\,s(\mathrm{Y}))=s(\mathrm{Y})^2$ $(t>1)$.

Proof. — Since

$$\Omega\cap\mathrm{G}^0=\bar{\mathrm{N}}\mathrm{A}_q\mathrm{M}_\sigma\exp(h_n)\cap\mathrm{G}^0=\bar{\mathrm{N}}\mathrm{A}_q(\mathrm{M}_\sigma\cap\mathrm{G}^0)\exp(h_n)=(\mathrm{P}\cap\mathrm{G}^0)(\mathrm{H}\cap\mathrm{G}^0),$$

we may restrict ourselves to the case that G is connected. Put $\Omega(\mathrm{Y})=\mathrm{P}(\mathrm{Y})\,\mathrm{H}(\mathrm{Y})$. The disjoint open subsets $\mathrm{G}(\mathrm{Y})\cap\Omega$ and $\mathrm{G}(\mathrm{Y})\cap\Omega s(\mathrm{Y})$ of $\mathrm{G}(\mathrm{Y})$ are $\mathrm{P}(\mathrm{Y})\times\mathrm{H}(\mathrm{Y})$-invariant. Since $\mathrm{G}(\mathrm{Y})$ has at most two open $\mathrm{P}(\mathrm{Y})\times\mathrm{H}(\mathrm{Y})$-orbits we conclude that

$$\mathrm{G}(\mathrm{Y})\cap\Omega=\Omega(\mathrm{Y})\quad\text{and}\quad\mathrm{G}(\mathrm{Y})\cap\Omega s(\mathrm{Y})=\Omega(\mathrm{Y})\,s(\mathrm{Y}).$$

Hence

$$\bar{\mathrm{N}}(\mathrm{Y})\cap\Omega=\bar{\mathrm{N}}(\mathrm{Y})\cap\Omega(\mathrm{Y})\quad\text{and}\quad\bar{\mathrm{N}}(\mathrm{Y})\cap\Omega s(\mathrm{Y})=\bar{\mathrm{N}}(\mathrm{Y})\cap\Omega(\mathrm{Y})\,s(\mathrm{Y})$$

and it suffices to prove (1) and (2) when $\mathfrak{g}\simeq sl(2,\mathbb{R})$. The decompositions

$$\mathrm{G}=\mathrm{NA}\exp(\mathfrak{m}\cap\mathfrak{p})\,\mathrm{K}\quad\text{and}\quad\mathrm{G}(\mathrm{Y})=\mathrm{N}(\mathrm{Y})\,\mathrm{A}(\mathrm{Y})\exp(\mathfrak{m}(\mathrm{Y})\cap\mathfrak{p})\,\mathrm{K}(\mathrm{Y})$$

are compatible. The same holds for the decompositions

$$\Omega=\mathrm{NA}_q\mathrm{M}_\sigma\exp(\mathfrak{h}_n)\quad\text{and}\quad\Omega(\mathrm{Y})=\mathrm{N}(\mathrm{Y})\,\mathrm{A}_q(\mathrm{Y})\,\mathrm{M}_\sigma(\mathrm{Y})\exp(\mathfrak{h}_n(\mathrm{Y}))$$

(where $\mathfrak{h}_n(\mathrm{Y})=\mathfrak{h}_n\cap\mathfrak{g}(\mathrm{Y})$). Hence

$$\mathfrak{h}_{\mathrm{P}}|\mathrm{G}(\mathrm{Y})=\mathfrak{h}_{\mathrm{P}(\mathrm{Y})},\quad\mathfrak{a}_{\mathrm{P}}|\Omega(\mathrm{Y})=\mathfrak{a}_{\mathrm{P}(\mathrm{Y})}\quad\text{and}\quad\mathfrak{m}_{\mathrm{P}}|\Omega(\mathrm{Y})=\mathfrak{m}_{\mathrm{P}(\mathrm{Y})}$$

and we may restrict ourselves to proving the lemma when $\mathrm{G}=\mathrm{G}^0$, $\mathfrak{g}\simeq sl(2,\mathbb{R})$.

If $\mathrm{G}\simeq\mathrm{SL}(2,\mathbb{R})$ then (1)-(5) follow by a straightforward matrix calculation. This implies (1)-(5) for the adjoint group of $\mathfrak{g}$ and via the adjoint representation the results may be lifted to any G with Lie algebra $\simeq sl(2,\mathbb{R})$. Since $\mathrm{M}_\sigma(\mathrm{Y})=$ centre $(\mathrm{G}(\mathrm{Y}))=$ (centre (G)), (6) and (7) require a different treatment. We first notice that the map $t\to m_{\mathrm{P}}(\exp t\mathrm{Y})$ maps $]-1,1[$ continuously into the discrete subgroup centre (G); it must therefore be constant and (6) follows. Similary, $t\to m_{\mathrm{P}}((\exp t\mathrm{Y})\,s(\mathrm{Y}))$ is identically m_0, for some $m_0\in$ centre (G). Now observe that $\mathrm{G}=\mathrm{NA}_q\,\mathrm{K}$ is an Iwasawa decomposition for G and define $\kappa:\mathrm{G}\to\mathrm{K}$ by $x\in\mathrm{NA}_q\kappa(x)$, for $x\in\mathrm{G}$. Then

$$m_{\mathrm{P}}(\exp(t\mathrm{Y})\,s(\mathrm{Y}))=m_{\mathrm{P}}(\kappa(\exp t\mathrm{Y})\,s(\mathrm{Y})).$$

We claim that

$$\kappa(\exp t\mathrm{Y})=\exp(\arctan(t)(\mathrm{Y}+\theta\mathrm{Y})),\tag{8.12}$$

for all $t\in\mathbb{R}$. By analytic continuation it suffices to prove this for t sufficiently close to 0, so we may pass to the adjoint group $\mathrm{SL}(2,\mathbb{R})/\{\pm\mathrm{I}\}$. There the result follows from a straightforward $\mathrm{SL}(2,\mathbb{R})$-computation. Now (8.12) implies that $\lim_{t\to+\infty}\kappa(\exp t\mathrm{Y})=s(\mathrm{Y})$, hence $m_0=\lim_{t\to+\infty}m_{\mathrm{P}}(\kappa(\exp t\mathrm{Y})\,s(\mathrm{Y}))=s(\mathrm{Y})^2$. ■

The above result enables us to separate variables in the integral for $L(w:\xi:\lambda:\nu)$. The map $(t, Y) \to \exp tY$ maps $]0, \infty[\times S$ diffeomorphically onto $\bar{N}\,\{e\}$. Let $d\sigma(Y)$ be the unique measure on S such that for every $\varphi \in C_c(\bar{N})$ we have

$$(8.13) \qquad \int_{\bar{N}} \varphi(\bar{n})\, d\bar{n} = \int_S \int_0^\infty \varphi(\exp tY)\, t^{m(\alpha)-1}\, dt\, d\sigma(Y).$$

Here $m(\alpha) = \dim(\mathfrak{g}^\alpha)$. Obviously $d\sigma(Y)$ is preserved by $-B(.,\theta(.))$-orthogonal transformations.

PROPOSITION 8.9. — *Let $w \in N_K(\mathfrak{a}_q)$. Then there exists a holomorphic function $l(w): \mathcal{U} \to \mathbb{C}$ and an endomorphism $T(w:\xi) \in \operatorname{End}(\mathcal{H}_\xi)$ which intertwines ξ with $w^{-1}\xi$, such that*

$$L(w:\xi:\lambda:\nu) = l(w:\lambda:\nu)\, T(w:\xi),$$

for $(\lambda, \nu) \in \mathcal{U}$. They are given by the following formulas (where we have written $\mu_0 = (1/2)\mu(H_\alpha) = \langle \alpha, \alpha \rangle^{-1} \langle \mu, \alpha \rangle$ for $\mu \in \mathfrak{a}_{qc}^*$).

(i) *If $w \in K_M$, then*

$$l(w:\lambda:\nu) = \int_0^1 (1+t^2)^{-\nu_0} (1-t^2)^{\nu_0 - \lambda_0 - \rho_0}\, t^{m(\alpha)-1}\, dt,$$

$$T(w:\xi) = \left[\int_S d\sigma(Y)\right] \xi(w).$$

(ii) *If $w \in sK_M$, then*

$$l(w:\lambda:\nu) = \int_1^\infty (1+t^2)^{-\nu_0} (t^2-1)^{\nu_0 - \lambda_0 - \rho_0}\, t^{m(\alpha)-1}\, dt,$$

$$T(w:\xi) = \int_S \xi(s(Y)\, w)\, d\sigma(Y).$$

Proof. — We first prove (ii), so assume $w \in sK_M$. In view of Lemma 8.8 (2) we may use the substitution of variables $\bar{n} = \exp tY$, $t > 1$, $Y \in S$ in (8.7). From Lemma 8.8 (3) and (5), we obtain

$$\log \circ h_P(\bar{n}w) = \log \circ h_P(\exp tY) = -\frac{1}{2} \log(1+t^2)\, H_\alpha$$

and

$$\log \circ a_P(\bar{n}w) = \log \circ a_P(\exp(tY)\, w) = \log \circ a_P(\exp(tY)\, s(Y)) = -\frac{1}{2} \log(t^2-1)\, H_\alpha$$

[the second equality follows from Lemma 8.3 (i)]. Next from Lemma 8.3 (ii) and Lemma 8.8 (7) we obtain $m_P(\bar{n}w) = m_P(\exp(tY)\, s(Y))\ s(Y)^{-1} w = s(Y)\, w$. Using the above relations, (8.13) and Fubini's theorem we obtain (ii). The proof of (i) is similar and involves Lemma 8.8 (1), (3), (4) and (6). ■

Remarks. — Notice that $l(w:\lambda:\nu)$ is independent of ξ. From its integral representation we read off that $l(w:\lambda:\nu)$ extends to a meromorphic function of $(\lambda, \nu) \in \mathfrak{a}_{qc}^* \times \mathfrak{a}_{qc}^*$. Hence $L(w:\xi:\lambda:\nu)$ has a $\mathrm{End}(\mathscr{H}_\xi)$-valued meromorphic continuation, the set of poles being independent of ξ. From the integral representations we also deduce that

$$l(w:\lambda:0)=\frac{1}{2}B\left(-\lambda_0-\frac{1}{2}m(\alpha)+1, \frac{1}{2}m(\alpha)\right) \qquad (w \in K_M),$$

$$l(w:\lambda:0)=\frac{1}{2}B\left(\lambda_0, -\lambda_0-\frac{1}{2}m(\alpha)+1\right) \qquad (w \in s K_M),$$

where B denotes Euler's Beta function.

COROLLARY 8.10. — *If* $w \in N_K(\mathfrak{a}_q)$, *then*

$$L(w:\xi:\lambda:\nu)^* = L(w^{-1}:\xi:\bar{\lambda}:\bar{\nu}).$$

Proof. — Since $\overline{l(w:\lambda:\nu)} = l(w:\bar{\lambda}:\bar{\nu})$, it suffices to show that $T(w:\xi)^* = T(w^{-1}:\xi)$. If $w \in K_M$, then this is obvious. If $w \notin K_M$, then

$$\begin{aligned} T(w:\xi)^* &= \int_S \xi(w^{-1} s(-Y))\, d\sigma(Y) \\ &= \int_S \xi(s(-Ad(w^{-1})\theta Y) w^{-1})\, d\sigma(Y). \end{aligned}$$

The endomorphism $Y \to -Ad(w^{-1})\theta Y$ is $-B(.,\theta(.))$-orthogonal hence leaves S invariant and preserves $d\sigma(Y)$. It follows that

$$T(w:\xi)^* = \int_S \xi(s(Y) w^{-1})\, d\sigma(Y) = T(w^{-1}:\xi). \quad ■$$

Completion of the proof of Theorem 6.3. — Let u, $v \in \mathscr{W}$, $\eta_u \in \mathscr{V}(\xi, u)$, $\eta_v \in \mathscr{V}(\xi, v)$. Then by Corollary 8.10 we have

$$(L(vu^{-1}:\xi:\lambda:0)\eta_u, \eta_v)_\xi = (\eta_u, L(uv^{-1}:\xi:\bar{\lambda}:0)\eta_v)_\xi.$$

Using (8.9) we infer that

$$(B(\bar{P}:P:\xi:\lambda)\, i(\xi, u)\eta_u, i(\xi, v)\eta_v) = (i(\xi, u)\eta_u, B(\bar{P}:P:\xi:\bar{\lambda})\, i(\xi, v)\eta_v).$$

Hence

$$B(\bar{P}:P:\xi:\lambda)^* = B(\bar{P}:P:\xi:\bar{\lambda}).$$

The proof is completed by using Corollary 6.15. ■

PROPOSITION 8.11. — *If* ξ *is not equivalent to* $s\xi$ *then* $B(\bar{P}:P:\xi:\lambda)$ *preserves the decomposition* $V(\xi)=V(\xi,1)\oplus V(\xi,s)$.

Proof. — More generally the assumption implies that $\xi\nsim w^{-1}\xi$ when $w\in sK_M$. Since $L(w:\xi:\lambda:\nu)$ intertwines ξ with $w^{-1}\xi$ it follows that $L(w:\xi:\lambda:\nu)=0$. Now use Proposition 8.6. ■

9. Normalized operators, final remarks

Retaining the notations of Sections 3-7, let $\xi\in\hat{M}_{fu}$. Then ξ may be embedded in a non-unitary principal series representation with parameters (σ,Λ_M), where Λ_M is *real* (in fact, $\Lambda_M=-\rho_M$, *cf.* Lemma 4.4). It is this knowledge of Λ_M (and not the stronger condition that ξ'*s* infinitesimal character be a real linear combination of roots), which is used in [K-S 80] to normalize the intertwining operators $A(P_2:P_1:\xi:\lambda)$, for P_1, P_2 (not necessarily $\sigma\theta$-stable) parabolic subgroups with θ-stable Levi component MA, and for $\lambda\in\mathfrak{a}_c^*$. Given such parabolics P_1, P_2 we define a meromorphic function $\gamma(P_2:P_1:\xi):\mathfrak{a}_c^*\to\mathbb{C}$ as in ([K-S 80], p. 50).

LEMMA 9.1. — *If* P_1, $P_2\in P_\sigma(A_q)$, *then* $\gamma(P_2:P_1:\xi)$ *restricts to a non-trivial meromorphic function on* $\mathfrak{a}_{qc}^*$.

Proof. — By definition the function $\gamma=\gamma(P_2:P_1:\xi)$ may be written as a product

$$\gamma(\lambda)=\prod_{\substack{\alpha\in\Sigma(\bar{\mathfrak{n}}_2\cap\mathfrak{n}_1,\mathfrak{a})\\ \alpha\text{ reduced}}}\gamma_\alpha(\langle\lambda,\alpha\rangle),\tag{9.1}$$

where each γ_α is a meromorphic function on $\mathbb{C}$ which does not vanish identically (*cf.* [K-S 80], p. 50). Thus it suffices to show that no $\alpha\in\Sigma(\bar{\mathfrak{n}}_2\cap\mathfrak{n}_1,\mathfrak{a})$ vanishes identically on $\mathfrak{a}_q$. Now this follows from Lemma 2.1. ■

Following [K-S 80] we define normalized intertwining operators

$$\mathscr{A}(P_2:P_1:\xi:\lambda)=\gamma(P_2:P_1:\xi:\lambda)^{-1}A(P_2:P_1:\xi:\lambda)$$

$(P_1,P_2\in\mathscr{P}_\sigma(A_q))$. They depend meromorphically on $\lambda\in\mathfrak{a}_{qc}^*$ in the sense of Corollary 4.3 and Proposition 4.11. Moreover, we define normalized endomorphisms of $V(\xi)$ by

$$\mathscr{B}(P_2:P_1:\xi:\lambda)=\gamma(P_2:P_1:\xi:\lambda)^{-1}B(P_2:P_1:\xi:\lambda).$$

In view of Proposition 6.1 we have

$$\mathscr{A}(P_2:P_1:\xi:\lambda)\circ j(P_1:\xi:\lambda)=j(P_2:\xi:\lambda)\circ\mathscr{B}(P_2:P_1:\xi:\lambda).\tag{9.2}$$

THEOREM 9.2. — *Let* $P_i\in\mathscr{P}_\sigma(A_q)$ $(i=1,2,3)$. *Then*

(i) $\mathscr{B}(P_3:P_2:\xi:\lambda)\circ\mathscr{B}(P_2:P_1:\xi:\lambda)=\mathscr{B}(P_3:P_1:\xi:\lambda)$,

(ii) $\mathscr{B}(P_1:P_2:\xi:\lambda)\circ\mathscr{B}(P_2:P_1:\xi:\lambda)=I_{V(\xi)}$,

(iii) $\mathscr{B}(P_2:P_1:\xi:\lambda)^*=\mathscr{B}(P_1:P_2:\xi:-\bar{\lambda})$.

Proof. — (i) and (ii) follow immediately from ([K-S 80], Lemma 8.3 (i) and Theorem 8.4), (9.2) and Theorem 5.10. In view of Theorem 6.3, (iii) follows from $\overline{\gamma(P_2:P_1:\xi:\lambda)}=\gamma(P_1:P_2:\xi:-\bar{\lambda})$ which is proved in [*loc. cit.*, proof of Proposition 8.5 (iv)]. ■

COROLLARY 9.3. — *Let* P_1, $P_2 \in \mathscr{P}_\sigma(A_q)$. *Then* $\mathscr{B}(P_2:P_1:\xi:\lambda)$ *extends to a holomorphic function of* λ *for* $\lambda \in i\mathfrak{a}_q^*$. *Moreover, for* $\lambda \in i\mathfrak{a}_q^*$, *the endomorphism* $\mathscr{B}(P_2:P_1:\xi:\lambda)$ *is unitary.*

Proof. — We copy the argument of ([K-S 80], Proposition 8.5 (v)). Once we have proved the holomorphy, the unitarity will follow from Theorem 9.2 (ii), (iii) and a density argument. Using Theorem 9.2 (i) we may reduce the proof to the case that P_2 and P_1 are (σ-) adjacent. Let then α be the reduced $\mathfrak{a}_q$-root in $\bar{\mathfrak{n}}_2 \cap \mathfrak{n}_1$. Then every $\mathfrak{a}$-root β in $\bar{\mathfrak{n}}_2 \cap \mathfrak{n}_1$ restricts to α or 2α on $\mathfrak{a}_q$. Hence from (9.1) and Lemma 9.4 below we infer that $\mathscr{B}(P_2:P_1:\xi:\lambda)=\mathscr{B}(\langle\lambda,\alpha\rangle)$, where $\mathscr{B}(z)\in \operatorname{End} V(\xi)$ depends meromorphically on $z\in\mathbb{C}$. Assume that $\mathscr{B}(z)$ has a pole at $z_0 \in i\mathbb{R}$. By meromorphy it has a pole of finite order k, which may be characterized as the smallest integer $k\geqq 0$ for which $(z-z_0)^k\mathscr{B}(z)$ remains bounded as $z\to z_0$, $z\in i\mathbb{R}\{z_0\}$. Since $\mathscr{B}(z)$ is unitary for $z\in i\mathbb{R}\{z_0\}$ sufficiently close to z_0, we see that $k=0$, hence z_0 is removable. ■

LEMMA 9.4. — *Let* P_1, P_2 *be* σ*-adjacent parabolics in* $\mathscr{P}_\sigma(A_q)$, *and let* α *be the reduced* $\mathfrak{a}_q$*-root in* $\bar{\mathfrak{n}}_2\cap\mathfrak{n}_1$. *Then* $B(P_2:P_1:\xi:\lambda)$ *is a meromorphic function of* $\langle\lambda,\alpha\rangle$.

Proof. — Let $w\in\mathscr{W}$. In view of Lemma 7.2 it suffices to show that the restriction of $B(P_2:P_1:\xi:\lambda)$ to $V(\xi,w)+V(\xi,\tau(s_\alpha)w)$ depends only on $\langle\lambda,\alpha\rangle$. Using (7.1) and observing that $\langle w^{-1}\lambda,\beta\rangle=\langle w^{-1}\lambda,w^{-1}\alpha\rangle=\langle\lambda,\alpha\rangle$ we may restrict ourselves to proving this for $w=1$. By Lemma 7.4 the restriction $B_\alpha(P_2:P_1:\xi:\lambda)$ of $B(P_2:P_1:\xi:\lambda)$ to $V(\xi,1)+V(\xi,\tau(s_\alpha)1)$ only depends on $\lambda_\alpha=\lambda|\mathfrak{a}_q(\alpha)$, which in turn only depends on $\langle\lambda,\alpha\rangle$. ■

We conclude this section with a slight improvement on Theorem 5.10.

LEMMA 9.5. — *Let* $P\in\mathscr{P}_\sigma(A_q)$. *Then the singular set of* $j(P:\xi)$ *is a locally finite union of hyperplanes of the form* $\langle\lambda,\alpha\rangle=z_\alpha$ $(\alpha\in\Sigma, z_\alpha\in\mathbb{C})$. *For* λ *in the complement of a countable union of complex hyperplanes in* $\mathfrak{a}_{qc}^*$, $j(P:\xi:\lambda)$ *maps* $V(\xi)$ *bijectively onto* $\mathscr{D}'(P:\xi:\lambda)^H$.

Proof. — Since $ev\circ j(P:\xi:\lambda)=id_{V(\xi)}$, the second assertion will follow once we have established the first one (use Corollary 5.3). We prove the first assertion locally at a point $\lambda_0\in\mathfrak{a}_{qc}^*$. Using the notations of the proofs of Lemma 5.9 and Theorem 5.10, fix $r_1>0$ such that $j(\bar{P}:\xi:\lambda)$ is holomorphic in $\lambda\in\mathscr{A}(P,r_1)$ (use Lemma 5.7). Fix $N\in\mathbb{N}$ such that $\lambda_0-N\mu\in\mathscr{A}(P,r_1)$. Then

$$J(\lambda)=M_\psi^N\circ A(P:\bar{P}:\xi:\lambda-N\mu)\circ j(\bar{P}:\xi:\lambda-N\mu)$$

is defined as a meromorphic function for λ in a neighbourhood $N(\lambda_0)$ of λ_0. Its singular set is contained in the singular set of $A(P:\bar{P}:\xi:\lambda-N\mu)$ which is a locally finite union of hyperplanes of the form $\langle\lambda,\alpha\rangle=z_\alpha$ $(\alpha\in\Sigma, z_\alpha\in\mathbb{C})$, in view of ([K-S 80], Theorem 6.6) (use that $\mathfrak{a}$-roots in $\bar{\mathfrak{n}}$ restrict to non-trivial $\mathfrak{a}_q$-roots in $\bar{\mathfrak{n}}$, *see* also the proof of

Lemma 9.1). Moreover, since $ev \circ M_\psi^N = m_\psi^N \circ ev$, we infer from Proposition 6.1 that

$$ev \circ J(\lambda) = m_\psi^N \circ B(P:\bar{P}:\xi:\lambda - N\mu),$$

so that

$$j(P:\xi:\lambda) = J(\lambda) \circ B(P:\bar{P}:\xi:\lambda - N\mu)^{-1} \circ (m_\psi^N)^{-1}.$$

By the above observation on the singular set of J, it now suffices to show that the poles of $B(P:\bar{P}:\xi:\lambda-N\mu)^{-1}$ in $N(\lambda_0)$ are contained in a locally finite union of hyperplanes of the form $\langle \lambda, \alpha \rangle = z_\alpha$. In view of Proposition 6.2 (ii) we may restrict ourselves to proving a similar statement for $B(P_2:P_1:\xi:\lambda-N\mu)^{-1}$, when P_2, $P_1 \in \mathscr{P}_\sigma(A_q)$ are σ-adjacent. But then the result follows from Lemma 9.4. ■

APPENDIX A
Equivariant distributions

In this appendix we prove a version of results of [Br 56], [KKMOOT 78], which is most suited for our purposes. Let G be a Lie group acting smoothly (*i.e.* C^∞) on a smooth n-dimensional manifold M. We denote the action of $g \in G$ by λ_g. The induced action of G on $C^\infty(G)$ is denoted by λ^*. Thus $\lambda_g^* \varphi(x) = \varphi(\lambda_g^{-1} x)$ for $\varphi \in C^\infty(M)$, $g \in G$, $x \in M$. We extend this action continuously to the space $\mathscr{D}'(M)$ of distributions on M (following [Hör 83], Section 6.3 the latter is defined as the topological linear dual of the space of C_c^∞-densities on M; thus $C^\infty(M)$ is naturally embedded in $\mathscr{D}'(M)$).

Suppose now that a finite dimensional representation τ of G in a complex vector space E of dimension d is given. Let $\mathscr{D}'(M, \tau)$ denote the space of distributions $u \in \mathscr{D}'(M) \otimes_{\mathbb{C}} E$, transforming according to the rule

$$(\lambda_g^* \otimes 1) u = (1 \otimes \tau(g^{-1})) u.$$

Given an orbit V of G on M, we denote the linear space of distributions $u \in \mathscr{D}'(M, \tau)$ satisfying

$$\text{(A.1)} \qquad V \cap \operatorname{supp} u \text{ is open in } \operatorname{supp} u$$

by $\mathscr{D}'(M, \tau, V)$. Moreover, if $k \in \mathbb{N}$ we define $\mathscr{D}'_k(M, \tau, V)$ to be the space of $u \in \mathscr{D}'(M, \tau, V)$ whose order is at most k at any point of V, and we define

$$\mathscr{D}'_k(V, \tau) = \mathscr{D}'_k(M, \tau, V)/\{u \in \mathscr{D}'(M, \tau, V); \operatorname{supp} u \cap V = \varnothing\}.$$

If $x \in V$, then the stabilizer G_x of x in G naturally acts on $T_x M$, the tangent space of M at x, by the rule

$$g . X = d(\lambda_g)(x) X \qquad (g \in G_x, X \in T_x M).$$

This action naturally induces an action on $T_x M/T_x V$. Differentiating once more we obtain a representation of $\mathfrak{g}_x$, the Lie algebra of G_x, in $T_x M/T_x V$. If $H \in \mathfrak{g}_x$, we let

$$\lambda_1(H), \ldots, \lambda_m(H)$$

denote the eigenvalues of the action of H on $T_x M/T_x V$ (so V has codimension m in M). Let $k \in \mathbb{N}$. We say that an element $H \in \mathfrak{g}_x$ satisfies condition C(I, k) iff

$$\Sigma_{i=1}^{m} (\nu_i + 1) \lambda_i(H) + \mu \neq 0 \tag{A.2}$$

for every eigenvalue μ of $\tau(H)$ and every $\nu \in \mathbb{N}^m$ with $|\nu| \leqq k$ (here $|\nu| = \nu_1 + \ldots + \nu_m$). We say that $H \in \mathfrak{g}_x$ satisfies condition C(II, k) iff (A.2) holds for every eigenvalue μ of $\tau(H)$ and every $\nu \in \mathbb{N}^m$ with $1 \leqq |\nu| \leqq k$.

PROPOSITION A.1. — *Let* $k \in \mathbb{N}$.

(i) *If there exists a* $H \in \mathfrak{g}_x$ *satisfying* C(I, k), *then* $\mathcal{D}'_k(V, \tau) = 0$;

(ii) *If there exists a* $H \in \mathfrak{g}_x$ *satisfying* C(II, k), *then* $\mathcal{D}'_k(V, \tau) \subset \mathcal{D}'_0(V, \tau)$ *and the dimension of* $\mathcal{D}'_0(V, \tau)$ *is at most the multiplicity of* $\lambda_1(H) + \ldots + \lambda_m(H)$ *as eigenvalue of* $-\tau(H)$.

The next two lemmas are needed as a preparation for the proof of Proposition A.1. The first of them is straightforward to verify.

LEMMA A.2. — *Let* E, F *be finite dimensional complex vector spaces. Let* $A \in \mathrm{End}(E)$, $B \in \mathrm{End}(F)$ *and let* L *respectively* M *be their sequences of weights, counting multiplicities. Then the weights of* $A \otimes I + I \otimes B$ *are* $\lambda + \mu$ $(\lambda \in L, \mu \in M)$, *counting multiplicities.*

Let now $m \in \mathbb{N} \{0\}$. For $1 \leqq j \leqq m$, let $\partial/\partial s_j$ denote differentiation in the j-th coordinate of $\mathbb{R}^m$, and let $S(\mathbb{R}^m)$ denote the algebra of constant coefficient differential operators on $\mathbb{R}^m$. Then as an algebra over $\mathbb{C}$, $S(\mathbb{R}^m)$ is generated by the vector-fields $\partial/\partial s_1, \ldots, \partial/\partial s_m$. For $n \in \mathbb{N}$, the subspace of homogeneous elements in $S(\mathbb{R}^m)$ of degree n is denoted by $S^n(\mathbb{R}^m)$, and we write $S_n(\mathbb{R}^m) = \Sigma_{k \leqq n} S^k(\mathbb{R}^m)$.

LEMMA A.3. — *Let* $A = (a_{ij})$ *be a complex* $m \times m$*-matrix with eigenvalues* $\lambda_1, \ldots, \lambda_m$, *counting multiplicities. Let the vector field* $\xi(s) = \Sigma_{1 \leqq i, j \leqq m} a_{ij} s_j \partial/\partial s_i$ *act on* $S(\mathbb{R}^m)$ *by the rule*

$$\xi . P = -\mathrm{ad}(\xi) P = [P, \xi].$$

Then the weights of this action on $S_n(\mathbb{R}^m)$ $(n \geqq 1)$ *are contained in the sequence* $(\alpha_1 \lambda_1 + \ldots + \alpha_m \lambda_m)$, $\alpha \in \mathbb{N}^m$, $|\alpha| \leqq n$, *counting multiplicities.*

Proof. — On $\mathbb{C}$ the action of ξ is multiplication by zero. If $1 \leqq k \leqq m$, then

$$\xi . \frac{\partial}{\partial s_k} = \Sigma_{j=1}^{m} a_{jk} \frac{\partial}{\partial s_j},$$

so ξ leaves $S^1(\mathbb{R}^n)$ invariant and the matrix of ξ on $S^1(\mathbb{R}^n)$ is A. Counting multiplicities the eigenvalues are $\lambda_1, \ldots, \lambda_m$. If $P = P_1 P_2$, $P_1, P_2 \in S(\mathbb{R}^n)$, then $\xi . P = (\xi . P_1) P_2 + P_1 (\xi . P_2)$. Hence by induction it follows that ξ preserves degrees and homogeneity. Let Λ_k denote the sequence of weights of ξ's action on $S^k(\mathbb{R}^m)$, and consider the natural surjective map $\psi_k: S^1 \otimes S^k \to S^{k+1}$. Then ψ_k intertwines the actions of ξ if we let ξ act on $S^1 \otimes S^k$ according to the tensor product, *i.e.* as $(\xi . \otimes 1) + (1 \otimes \xi .)$. Applying Lemma A.2 we obtain that $\Lambda_{k+1} \subset (\lambda_i + \mu;\ 1 \leqq i \leqq m,\ \mu \in \Lambda_k)$. The result now follows by induction.

Proof of Proposition A.1. — Following ([KKMOOT 78], Appendix I), we fix a coordinate system $(s_1, \ldots, s_m, t_1, \ldots, t_{n-m})$ for a neighbourhood Ω of x, such that x corresponds to the origin and (locally at x) V is defined by $s_1 = \ldots = s_m = 0$. Now let $u \in \mathscr{D}'(M, \tau, V)$. Shrinking Ω if necessary we may assume that supp $u \cap \Omega \subset V$. By equivariance, $u | \Omega$ is smooth in the variables $t_1, \ldots, t_{n-m}$. Now every distribution $\varphi \in \mathscr{D}'(\Omega)$ with supp $\varphi \subset V$, of finite order and smooth in $t_1, \ldots, t_{n-m}$ admits a unique expression

$$\varphi = P(\partial/\partial s, t)\, \delta_m \otimes 1_{n-m},$$

where δ_m is the Dirac measure in $\mathbb{R}^m$ supported at the origin, 1_{n-m} is the constant function 1 on $\mathbb{R}^{n-m}$ and $P(\partial/\partial s, t)$ is a smooth linear differential operator with coefficients depending only on t and of order zero in $\partial/\partial t_i$, $1 \leqq i \leqq n-m$. For $\mu \in \mathbb{R}^{n-m}$ sufficiently close to 0, put $N_\mu = \{ (s, t) \in \Omega;\ t = \mu \}$. Using $s_1, \ldots, s_m$ as coordinates on N_μ, we define the restriction $\varphi | N_\mu$ of φ to N_μ by

$$\varphi | N_\mu = P(\partial/\partial s, \mu)\, \delta_m.$$

One easily checks that if $Q(s, \partial/\partial s, t)$ is a smooth linear differential operator of order 0 in $\partial/\partial t_i$, $1 \leqq i \leqq n-m$, then

$$Q(s, \partial/\partial s, t)\, \varphi | N_\mu = Q(s, \partial/\partial s, \mu)(\varphi | N_\mu).$$

Retaining our previous notations, let v_μ be the restriction of u to N_μ. Then supp $v_\mu \subset \{0\}$.

We identify every $X \in \mathfrak{g}$ with the vector field with flow $(y, t) \mapsto \partial/\partial t\ (\lambda_{\exp(-tX)} y)$ on M. Thus, the distribution u satisfies the equation

$$(X \otimes 1)\, u = -(1 \otimes \tau(X))\, u \qquad (X \in \mathfrak{g}).$$

Let $X_1, \ldots, X_{n-m} \in \mathfrak{g}$ be such that $\mathfrak{g} = \mathfrak{g}_x + \Sigma_{k=1}^{n-m} \mathbb{R} X_k$, and fix $H \in \mathfrak{g}_x$. Then there exist smooth functions $c_1(s, t), \ldots, c_{n-m}(s, t)$, vanishing at the origin, such that

$$(H + \Sigma_{j=1}^{n-m} c_j(s, t) X_j)\, t_k = 0 \quad (1 \leqq k \leqq n-m)$$

(*cf.* [KKMOOT 78], Appendix I). Therefore the vectorfield $H + \Sigma_{j=1}^{n-m} c_j X_j$ is of order zero in $\partial/\partial t_i$, $1 \leqq i \leqq n-m$, and can be restricted to a vector field $\bar{H} = \Sigma_{i=1}^m a_i(s)\, \partial/\partial s_i$ on N_0. Moreover by what we said earlier, the distribution $H + \Sigma_{j=1}^{n-m} c_j X_j)\, u$ restricts to $\bar{H} v_0$ on N_0. Hence

$$(\bar{H} \otimes 1 + 1 \otimes B(s))\, v_0 = 0,$$

where $B(s)=\tau(H)+\Sigma_{j=1}^{n-m} c_j(s,0)\tau(X_j)\in \mathrm{End}(E)$. Moreover, $a_i(0)=0$ $(1\leqq i\leqq m)$ and as in [*loc. cit.*], the eigenvalues of $(\partial a_i/\partial s_j(0))$ are $-\lambda_1(H), \ldots, -\lambda_m(H)$.

Let $\mathscr{D}'$ denote the space of $w\in\mathscr{D}'(\mathbb{R}^m)\otimes E$ with $\mathrm{supp}\, w\subset\{0\}$. Then $v_0\in\mathscr{D}'$. The linear map $\Phi: S(\mathbb{R}^m)\otimes E\to\mathscr{D}'$, defined by

$$\Phi(P\otimes e)=P\delta_m\otimes e \qquad (P\in S^m(\mathbb{R}^m), e\in E),$$

is a bijection. Moreover, order $(\Phi W)=\mathrm{degree}(W)$ for $W\in S(\mathbb{R}^m)\otimes E$ and we may view $\mathscr{D}'$ as a graded vector space. Since $\bar{H}$ vanishes at 0 it follows that $\mathrm{order}[(\bar{H}\otimes 1)w]\leqq\mathrm{order}(w)$ for $w\in\mathscr{D}'$. Let $a_{ij}=\partial a_i/\partial s_j(0)$ $(1\leqq i,j\leqq m)$, $A=(a_{ij})$ and define ξ as in Lemma A.3.

LEMMA A.4. — *Let*

$$L=(\mathrm{ad}\,\xi-\mathrm{tr}(A))\otimes 1+1\otimes\tau(H), \qquad \textit{and} \qquad L'=\mathrm{gr}(\bar{H}\otimes 1+1\otimes B(s)).$$

Then the following diagram commutes

$$\begin{array}{ccc} S(\mathbb{R}^m)\otimes E & \xrightarrow{L} & S(\mathbb{R}^m)\otimes E \\ \Phi\downarrow & & \downarrow\Phi \\ \mathscr{D}' & \xrightarrow{L'} & \mathscr{D}' \end{array}$$

Proof. — If $w\in\mathscr{D}'$, then order $[(1\otimes B(s)-1\otimes B(0))w]\leqq\mathrm{order}(w)$. Hence $gr(1\otimes B(s))\circ\Phi=\Phi\circ(1\otimes\tau(H))$. Moreover, by application of Taylor's formula to the a_i, it follows that

$$\mathrm{order}\left(\Sigma_i\, a_i(s)\frac{\partial}{\partial s_i}w-\xi w\right)<\mathrm{order}(w).$$

Hence $gr(\bar{H}\otimes 1)=gr(\xi\otimes 1)$. If $P\in S^n(\mathbb{R}^m)$, then

$$s_j\frac{\partial}{\partial s_i}P\delta_m=\left[s_j\frac{\partial}{\partial s_i}, P\right]\delta_m+Ps_j\frac{\partial}{\partial s_i}\delta_m.$$

Now $(s_i\,\partial/\partial s_j)\,\delta_m=0$ if $i\neq j$, $=-\delta_m$ if $i=j$. It follows that

$$\xi P\delta_m=[\xi, P]\delta_m-\mathrm{tr}(A)P\delta_m.$$

Since $\mathrm{ad}\,\xi$ preserves degree and homogeneity (proof of Lemma A.3), it follows that $\mathrm{gr}(\xi\otimes 1)\circ\Phi=\Phi\circ(\mathrm{ad}\,\xi-\mathrm{tr}(A))$, whence the commutativety. ■

We proceed with the proof of Proposition A.1. Because of Lemmas A.2-4, the eigenvalues of $\bar{H}\otimes 1+1\otimes B(s)$ on $\mathscr{D}'_k$ are contained in the sequence $\Sigma_{i=1}^m(\nu_i+1)\lambda_i(H)+\mu$, with $\nu\in\mathbb{N}^m$, $|\nu|\leqq k$ and μ an eigenvalue of $\tau(H)$. Thus, if $u\in\mathscr{D}'(M,\tau,v)$ has order $\leqq k$ along V and H satisfies $C(I,k)$, then the restriction v_0 of u to N_0 is zero. Since the condition of Proposition A.1 (i) is independent of the choice of $x\in V$ (use the G-action), it follows by the same reasoning that $v_\mu=u|N_\mu=0$ for all μ

near 0 in $\mathbb{R}^{n-m}$. This implies that u vanishes in a neighbourhood of x, so that $\operatorname{supp} u \cap V = \varnothing$ by equivariance.

Finally suppose that condition C(II, k) holds for $H \in \mathfrak{g}_x$, $u \in \mathscr{D}'_k(M, \tau, V)$. Then the eigenvalues of $\bar{H} \otimes 1 + 1 \otimes B(s)$ on $\mathscr{D}'_k/\mathscr{D}'_0$ are the same as those of $(ad(\xi) - tr(A)) \otimes 1 - 1 \otimes \tau(H)$ on $[\bigoplus_{1 \leq i \leq k} S^i(\mathbb{R}^m)] \otimes E$, hence contained in the sequence $\sum_{i=1}^{m} (\nu_i + 1)\lambda_i(H) + \mu$, with $\nu \in \mathbb{N}^m$, $1 \leq |\nu| \leq k$ and μ an eigenvalue for $\tau(H)$ (*see* the proof of Lemma A.2). Hence for all μ near 0 in $\mathbb{R}^{n-m}$ it follows that order $(v_\mu) = 0$. This implies that u has order 0 along V. The linear map $u \mapsto u | N_0$ from $\mathscr{D}'_0(M, \tau, V)$ into $\mathscr{D}'_0$ has kernel equal to the space of $u \in \mathscr{D}'_0(M, \tau, V)$ with $\operatorname{supp} u \cap V = \varnothing$, and maps into the zero eigenspace of the restriction of $\bar{H} \otimes 1 + 1 \otimes B(s)$ to $\mathscr{D}'_0$. Hence $\dim \mathscr{D}'_0(V, \tau)$ is at most the multiplicity of zero as eigenvalue of $(\operatorname{ad} \xi - \operatorname{tr}(A)) \otimes 1 - 1 \otimes \tau(H)$ restricted to $S^0(\mathbb{R}^m) \otimes E$. Since ξ acts trivially on $S^0(\mathbb{R}^m)$, the proof is completed by using Lemma A.2. ∎

APPENDIX B
H-orbits on P\G

The purpose of this appendix is to derive some properties of the structure of H-orbits on P\G, for P a minimal $\sigma\theta$-stable parabolic subgroup of G containing $A_0 = \exp(\mathfrak{a}_0)$ (notations are as in Sections 1, 2). We do this by comparison with Matsuki's description of the H-orbits on the quotient of a connected semisimple group and a minimal parabolic subgroup (*cf.* [Ma 79]). Thus, the main problem is that P need not be a minimal parabolic subgroup.

PROPOSITION B.1. — *Let* $\mathscr{W}$ *be a set of representatives for* $W/W_{K \cap H}$ *in* $N_K(\mathfrak{a}_{0q})$. *Then the open* H*-orbits on* P\G *are* $1-1$ *parametrized by* PwH, $w \in \mathscr{W}$.

Proof. — Select a minimal parabolic subgroup P_0 of G with $A_0 \subset P_0 \subset P$. The map $P_0 x \mapsto \operatorname{Ad}(x^{-1}) \mathfrak{P}_0$ defines a bijection between $P_0\backslash G$ and the set P_0 of all minimal parabolic subalgebras of $\mathfrak{g}$. Since all elements of P_0 are G^0-conjugate, $P_0\backslash G \simeq (P_0 \cap G^0)\backslash G^0 \simeq \operatorname{Ad}_G(P_0 \cap G^0)\backslash \operatorname{Ad}_G(G^0)$ and we obtain from ([Ma 79], Proposition 3.1) that the open H^0-orbits on $P_0\backslash G$ correspond $1-1$ to $W_{0\sigma}/W_{0K \cap H^0}$ (here the notations are as in Section 1; use that $K \cap H^0 \subseteq K \cap G^0 \cap H^0 = K^0 \cap H^0$). If $\bar{w} \in W_{0\sigma}/W_{0K \cap H^0}$, then the corresponding open orbit is $P_0 w H^0$, where w is any representative of $\bar{w}$ in $N_K(\mathfrak{a}_0)$ (of course the orbit does not depend on the choice of representative). Now fix $\bar{u} \in W_{0\sigma}/W_{0K \cap H^0}$, and select a representative u of $\bar{u}$ in $N_K(\mathfrak{a}_0) \cap N_K(\mathfrak{a}_{0q})$. Then PuH^0 contains the open set $P_0 u H^0$, hence is an open orbit. Let $\mathscr{V}$ be a set of representatives for $W(\Sigma_0^h)$ in $M_1 \cap N_K(\mathfrak{a}_0)$, where M_1 denotes the centralizer of $\mathfrak{a}_{0q}$ in G (*cf.* Section 1). Then $PuH^0 \supset P_0 vuH^0$ for every $v \in V$. We claim that

$$PuH^0 \subset cl\left(\bigcup_{v \in V} P_0 vuH^0\right). \tag{B.1}$$

Indeed, using that $W(\Sigma_0^h)$ is a normal subgroup of $W_{0\sigma}$ (Lemma 1.1), we may reverse the order of v and u in (B.1). Replacing P and P_0 by uPu^{-1} and uP_0u^{-1} if necessary, we may as well assume that $u=1$. Applying Matsuki's orbit description to the minimal parabolic subgroup $P_0 \cap M_1$ of M_1 we obtain that

$$M_1 \subset cl \bigcup_{v \in V} (P_0 \cap M_1) v (H \cap M_1)^0,$$

hence $PH^0 = P_0 M_1 H^0$ is contained in the right hand side of (B.1) (we assumed that $u=1$). This proves the claim. From (B.1) and Lemma 1.3 we infer the validity of the proposition for connected H. The proof is completed by using that $H = N_{K \cap H}(\mathfrak{a}_{0q}) H^0$ (*cf.* [Ba 86], p. 25, (2.2)). ■

LEMMA B.2. — *Let* PxH *be a non-open orbit in* $P\backslash G/H$. *Then there exists a* $x_1 \in K$ *such that* $Px_1H = PxH$ *and*

$$\text{(B.2)} \qquad \mathrm{Ad}(x_1)^{-1}[\mathfrak{a}_0 \setminus \mathfrak{a}_0 \cap \mathfrak{h})] \cap \mathfrak{h} \neq \varnothing.$$

Proof. — Let P_0 be as in the proof of Proposition B.1. Then $PxH \supset P_0 x H^0$, so the latter double coset cannot be open. It suffices to prove the existence of $x_1 \in K$ with $P_0 x_1 H^0 = P_0 x H^0$ and (B.2). For this we may pass to the connected component of the adjoint group of G. Hence we may as well assume the G is connected and semi-simple and that $H = H^0$, so that Matsuki's description of $P_0\backslash G/H$ applies.

Let $\mathfrak{a}_0$, $\mathfrak{a}_i$ $(1 \leqq i \leqq r)$ be a set of representatives of $K \cap H^0$-conjugacy classes of σ-stable maximal abelian subspaces of $\mathfrak{p}$. Then the orbit $P_0 x H$ equals $P_0 x_1 H$, where $x_1 \in K$ is such that $\mathrm{Ad}(x_1^{-1})$ maps $\mathfrak{a}_0$ onto some $\mathfrak{a}_j$, $0 \leqq j \leqq r$. Here j is unique, and x_1 is unique up to left multiplication by $Z_K(\mathfrak{a}_0)$ (or equivalently right multiplication by $Z_K(\mathfrak{a}_j)$). This is an immediate consequence of ([Ma 79], Corollary 1.1). Moreover, the fact that $P_0 x H^0$ is not open implies that one of the following conditions holds (*cf.* [*loc. cit.*], Proposition 3.1):

(*a*) $j=0$ and $\mathrm{Ad}(x_1^{-1})(\mathfrak{a}_0 \cap \mathfrak{h}) \neq \mathfrak{a}_0 \cap \mathfrak{h}$,

(*b*) $j>0$.

In the first case (B.2) follows at once. In the second we have that $\mathfrak{a}_j$ is not $\mathfrak{q}$-maximal (*cf.* [Schl. 84], Lemma 7.1.5), hence $\dim(\mathrm{Ad}(x_1^{-1})\mathfrak{a}_{0q} > \dim(\mathfrak{a}_j \cap \mathfrak{q})$ and it even follows that $\mathrm{Ad}(x_1^{-1})\mathfrak{a}_{0q} \cap (\mathfrak{a}_j \cap \mathfrak{h}) \neq 0$. ■

APPENDIX C

The purpose of this appendix is to prove the following lemma.

LEMMA C.1. — *Let* $P \in \mathscr{P}_\sigma(A_q)$ *have Langlands decomposition* $P = MAN$. *Then there exist a* $\mu \in \mathfrak{a}_q^*$ *and a non-trivial real analytic function* $\psi : G \to \mathbb{C}$ *such that*

(i) $\langle \mu, \alpha \rangle < 0$ *for all* $\alpha \in \Sigma(\mathfrak{n}, \mathfrak{a}_q)$,

(ii) ψ *is a right* G^σ*-invariant element of* $C^\infty(G : P : 1 : \mu - \rho_P)$.

Proof. — Clearly it suffices to prove the result for the case that G is the adjoint group of a semi-simple algebra $\mathfrak{g}$. The function ψ will be obtained from a matrix coefficient of a finite dimensional representation by passing to a suitable dual real form.

Let G_c be the complex adjoint group of $\mathfrak{g}_c$, $\mathfrak{g}^d$ the dual real form $\mathfrak{g}_+ \oplus i\mathfrak{g}_-$ in $\mathfrak{g}_c$, and G^d its adjoint group sitting in G_c. Put $\mathfrak{k}^d = \mathfrak{h}_c \cap \mathfrak{g}^d$, $\mathfrak{p}^d = \mathfrak{q}_c \cap \mathfrak{g}^d$. Then $\mathfrak{g}^d = \mathfrak{k}^d \oplus \mathfrak{p}^d$ is a Cartan decomposition. The corresponding Cartan involution θ^d is the restriction of (the complex linear extension of) σ to $\mathfrak{g}^d$. The involution $\sigma^d = \theta \,|\, \mathfrak{g}^d$ commutes with θ^d. Let $\mathfrak{g}^d = \mathfrak{h}^d \oplus \mathfrak{q}^d$ be the corresponding eigenspace decomposition of $\mathfrak{g}^d$. Then $\mathfrak{a}^d_{0\,q} = \mathfrak{a}_{0\,q}$ is maximal abelian in $\mathfrak{p}^d \cap \mathfrak{q}^d = \mathfrak{p} \cap \mathfrak{q}$. Extend $\mathfrak{a}^d_{0\,q}$ to a maximal abelian subspace $\mathfrak{a}^d_0$ of $\mathfrak{p}^d$ and define Σ^d_0, Σ^d as in Section 1. Then $\Sigma^d = \Sigma$, so $\Sigma^{d+} = \Sigma(\mathfrak{n}, \mathfrak{a}_{0\,q})$ is a choice of positive roots for Σ^d. Select a compatible choice Σ^{d+}_0 of positive roots for Σ^d_0, and define corresponding fundamental systems Δ^d_0 and Δ^d as in Section 1. Then $\Delta^d = \Delta$. Let $\alpha_1, \ldots, \alpha_n$ be a numbering of the elements of Δ^d_0 as in ([Ba 86], Proposition 3.10). Then $\Delta = \{\alpha_j \,|\, \mathfrak{a}^d_{0\,q};\ 1 \leqq j \leqq l\}$, where $l = \dim \mathfrak{a}^d_{0\,q}$. Select $\nu \in (\mathfrak{a}^d_{0\,q})^*$ such that $1/2 \langle \nu, \alpha_j \rangle \langle \alpha_j, \alpha_j \rangle^{-1}$ is a positive integer for every $1 \leqq j \leqq l$. Then (i) holds for $\mu = -2m\nu$, for any positive integer m. Moreover, $\langle \nu, \alpha \rangle \langle \alpha, \alpha \rangle^{-1} \in \mathbb{N}$ for every $\alpha \in \Delta^d_0$ (*cf.* [loc. cit.], proof of Proposition 3.10), hence by ([He 84], Theorem V.4.1) there exists a finite dimensional irreducible spherical representation π of G^d with highest $\mathfrak{a}^d_0$-weight ν. Let V be a representation space for π, e_ν a non-zero highest weight vector for π, and $\varepsilon \in V^*$ a $\pi^\vee(K^d)$-fixed vector (where $K^d = G^\sigma_c \cap G^d$), such that $\langle \varepsilon, e_\nu \rangle = 1$. Let m be the order of the centre of the universal covering group $\tilde{G}_c$ of G_c. Then the real analytic function Ψ on G^d, defined by

$$\Psi(x) = \langle \varepsilon, \pi(x)^{-1} e_\nu \rangle^{2m}$$

extends holomorphically to G_c (the factor 2 in the power will be needed later on). We claim that $\psi = \Psi \,|\, G$ fulfills our requirements. Let $P^d_0 = M^d_0 A^d_0 N^d_0$ be the standard minimal parabolic subgroup of G^d. Writing $\mu = -2m\nu$ as above, we have

$$\text{(C.1)} \qquad \Psi(manxk) = a^\mu \Psi(x),$$

for $m \in M^d_0$, $a \in A^d_0$, $n \in N^d_0$, $k \in K^d$ and $x \in G^d$. The group G^σ_c is a complexification of K^d. Moreover, if F is the (finite) group of quadratic elements in $A^d_{0\,c} = \exp(\mathfrak{a}^d_{0\,c})$, then $G^\sigma_c = F(G^\sigma_c)^0$ (*cf.* [Ko-Ra 71], Proposition 1). Also, if $M^d_{0\,c}$ denotes the centralizer of $\mathfrak{a}^d_{0\,c}$ in G^σ_c, then $M^d_{0\,c} = F(M^d_{0\,c})^0$ (use [War 72], Lemma 1.1.3.8). Now $a \mapsto a^{-m\nu}$ extends holomorphically to $A^d_{0\,c}$ and we see that $a^\mu = (a^2)^{-m\nu} = 1$ for $a \in F$. It follows by holomorphic continuation that (C.1) holds for $m \in M^d_{0\,c}$, $a \in A^d_{0\,c}$, $n \in N^d_{0\,c}$, $k \in G^\sigma_c$ and $x \in P^d_{0\,c} G^\sigma_c$. By density of $P^d_{0\,c} G^\sigma_c$ in G_c the latter condition may be relaxed to $x \in G_c$. Finally, let $M_{1\,c} N_c$ be the θ-stable (hence also σ-stable) Levi decomposition for the normalizer P_c of $\mathfrak{P}_c$ in G_c. Then $M_{1\,c} = M_{\sigma\,c} A_{qc}$, where $M_{\sigma\,c}$ is the algebraic subgroup consisting of all $m \in M_{1\,c}$ with $tr(\mathrm{Ad}(m) \,|\, \mathfrak{n}_c) = 0$ (*cf.* [Ba 86], Lemma B.3.1). Now $(P^d_{0\,c} \cap M_{\sigma\,c})(K^d_c \cap M_{\sigma\,c})$ is a dense subset of $M_{\sigma\,c}$. Moreover,

$$M_{\sigma\,c} \cap P^d_{0\,c} = M^d_{0\,c} \exp(\mathfrak{a}^d_0 \cap \mathfrak{h}^d)(M_{1\,c} \cap N^d_c)$$

and it follows that $\psi \equiv 1$ on $M_{\sigma c}$. Therefore

$$\psi(\operatorname{man} x h) = a^{\mu} \psi(x)$$

for $m \in M_{\sigma c}$, $a \in A_{qc}$, $n \in N_c^d$, $h \in G_c^\sigma$ and $x \in P_c G_c^\sigma$. By density of $P_c K_c^d$ in G_c the latter condition can be relaxed to $x \in G_c$ and (ii) follows for $\psi = \Psi | G$. ■

REFERENCES

[Ba 86] E. P. van den BAN, *A Convexity Theorem for Semisimple Symmetric Spaces* (*Pac. J. Math.*, Vol. **124**, 1986, pp. 21-55).

[Ba 87 I] E. P. van den BAN, *Invariant Differential Operators on a Semisimple Symmetric Space and Finite Multiplicities in a Plancherel Formula* (*Ark. för Mat.*, Vol. **25**, 1987, pp. 175-187).

[Ba 87 II] E. P. van den BAN, *Asymptotic Behaviour of Matrix Coefficients Related to Reductive Symmetric Spaces* (*Proc. Kon. Ned. Akad. Wet.*, Ser. A, Vol. **90**, 1987, pp. 225-249).

[Ba 88] E. P. van den BAN, *The Principal Series for a Reductive Symmetric Space II. Eisenstein Integrals*, in preparation.

[Be 57] M. BERGER, *Les espaces symétriques non-compacts* (*Ann. scient. Ec. Norm. Sup.*, Vol. **74**, 1957, pp. 85-117).

[B 72] I. N. BERNSTEIN, *The analytic continuation of generalized functions with respect to a parameter* (*Funct. Anal. and its Applic.*, Vol. **6**, 1972, pp. 273-285).

[B-G 69] I. N. BERNSTEIN and I. S. GELFAND, *Meromorphic Property of the Function* P^λ (*Funct. Anal. and its Applic.*, Vol. **3**, 1969, pp. 68-69).

[Br 56] F. BRUHAT, *Sur les représentations induites des groupes de Lie* (*Bull. Soc. Math. France*, Vol. **84**, 1956, pp. 97-205).

[D 85] P. DELORME, *Injection de modules sphériques pour les espaces symétriques réductifs dans certaines représentations induites* (*Proceedings Marseille-Luminy* 1985, *LNM* 1243), Springer-Verlag, 1987, pp. 108-134.

[DP 86] G. van DIJK and M. POEL, *The Plancherel Formula for the Pseudo-Riemannian Space* $SL(n, \mathbb{R})/Gl(n-1, \mathbb{R})$ (*Comp. Math.*, Vol. **58**, 1986, pp. 371-397).

[Fa 79] J. FARAUT, *Distributions sphériques sur les espaces hyperboliques* (*J. Math. Pures Appl.*, Vol. **58**, 1979, pp. 369-444).

[FJ 86] M. FLENSTED-JENSEN, *Analysis on non-Riemannian Symmetric Spaces* (*A.M.S. Regional Conference Series* **61**, 1986).

[HC 58 I, II] HARISH-CHANDRA, *Spherical Functions on a Semisimple Lie Group I, II* (*Amer. J. of Math.*, Vol. **80**, 1958, pp. 241-310 and pp. 553-613).

[HC 75] HARISH-CHANDRA, *Harmonic Analysis on Real Reductive Groups I. The theory of the constant term* (*J. Funct. Anal.*, Vol. **19**, 1975, pp. 103-204).

[HC 76] HARISH-CHANDRA, *Harmonic Analysis on Real Reductive Groups II. Wave Packets in the Schwartz Space* (*Invent. Math.*, Vol. **36**, 1979, pp. 1-55).

[HC 76 II] HARISH-CHANDRA, *Harmonic Analysis on Real Reductive Groups III. The Maass-Selberg Relations and the Plancherel Formula* (*Ann. of Math.*, Vol. **104**, 1976, pp. 117-201).

[He 84] S. HELGASON, *Groups and Geometric Analysis*, Academic Press, 1984.

[Hör 83] L. HÖRMANDER, *The Analysis of Linear Partial Differential Operators I*, Springer-Verlag, 1983.

[KKMOOT 78] M. KASHIWARA, A. KOWATA, K. MINEMURA, K. OKAMOTO, T. OSHIMA and M. TANAKA, *Eigenfunctions of Invariant Differential Operators on a Symmetric Space.* (*Ann. of Math.*, Vol. **107**, 1978, pp. 1-39).

[K-S 80] A. W. KNAPP and E. M. STEIN, *Intertwining Operators for Semisimple Groups* (*Ann. of Math.*, Vol. **93**, 1971, pp. 489-578).

[K-S 80] A. W. KNAPP and E. M. STEIN, *Intertwining Operators for Semisimple Groups, II* (*Invent. Math.*, Vol. **60**, 1980, pp. 9-84).

[Kn 82] A. W. KNAPP, *Commutativity of Intertwining Operators for Semisimple Groups* (*Comp. Math.*, Vol. **46**, 1982, pp. 33-84).

[Kom 67] H. KOMATSU, *Projective and Injective Limits of Weakly Compact Sequences of Locally Convex Spaces* (*J. Math. Soc. Japan*, Vol. **19**, 1967, pp. 366-383).

[Ko-Ra 71] B. KOSTANT and S. RALLIS, *Orbits and Representations Associated with Symmetric Spaces* (*Amer. J. Math.*, Vol. **93**, 1971, pp. 753-809).

[Ma 79] T. MATSUKI, *The Orbits of Affine Symmetric Spaces under the Action of Minimal Parabolic Subgroups* (*J. Math. Soc. Japan*, Vol. **31**, 1979, pp. 331-357).

[Ma 82] T. MATSUKI, *Oribts on Affine Symmetric Spaces under the Action of Parabolic Subgroups* (*Hiroshima Math. J.* Vol. **12**, 1982, pp. 307-320).

[O1 87] G. OLAFSSON, *Fourier and Poisson Transformation Associated to a Semisimple Symmetric Space* (*Invent. Math.*, Vol. **90**, 1987, pp. 605-629).

[Os 79] T. OSHIMA, *Poisson Transformations on Affine Symmetric Spaces* (*Proc. Japan Acad.*, Vol. **55**, Série A, 1979, pp. 323-327).

[O-S 80] T. OSHIMA and J. SEKIGUCHI, *Eigenspaces of Invariant Differential Operators on an Affine Symmetric Space* (*Invent. Math.*, Vol. **57**, 1980, pp. 1-81).

[Pe 75] R. PENNEY, *Abstract Plancherel Theorems and a Frobenius Reciprocity Theorem* (*J. of Funct. Anal.*, Vol. **18**, 1975, pp. 177-190).

[Ro 78] W. ROSSMANN, *Analysis on Real Hyperbolic Spaces* (*J. Funct. Anal.*, Vol. **30**, 1978, pp. 448-477).

[Ro 79] W. ROSSMANN, *The Structure of Semisimple Symmetric Spaces* (*Can. J. Math.*, Vol. **31**, 1979, pp. 157-180).

[S 71] G. SCHIFFMANN, *Intégrales d'entrelacement et fonctions de Whittaker* (*Bull. Soc. Math., France*, Vol. **99**, 1971, pp. 3-72).

[Schl 84] H. SCHLICHTKRULL, *Hyperfunctions and Harmonic Analysis on Symmetric Spaces*, Birkhäuser, 1984.

[Schw 50] L. SCHWARTZ, *Théorie des distributions*, Tome I, Hermann, 1950.

[Str 73] R. S. STRICHARTZ, *Harmonic analysis on hyperboloids* (*J. Funct. Anal.*, Vol. **12**, 1973, pp. 341-383).

[Va 74] V. S. VARADARAJAN, *Lie Groups, Lie Algebras, and their Representations*, Prentice Hall, 1974.

[Va 77] V. S. VARADARAJAN, *Harmonic Analysis on Real Reductive Groups* (*Lecture Notes* 576), Springer-Verlag, 1977.

[Vo 81] D. A. VOGAN, *Representations of Real Reductive Lie Groups*, (*P.M.* 15), Birkhäuser, Boston, 1981.

[War 72] G. WARNER, *Harmonic Analysis on Semi-Simple Lie Groups I*, Springer-Verlag, 1972.

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