

Maass–Selberg relations for Whittaker functions on a real reductive group

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Introduction

In this paper we give a complete proof of the Maass–Selberg relations for Whittaker integrals on a real reductive Lie group. These relations were announced in 1982 by Harish-Chandra as an important part of the Plancherel formula for Whittaker functions. Because of his untimely death in 1983, no further details appeared until 2018, when V.S. Varadarajan and R. Gangolli published an edited version of Harish-Chandra’s manuscripts, [7, pp. 141-307]. The results of the present paper are based on a result of [7] for a basic case. Harish-Chandra’s proof for this basic case involves an asymptotic analysis of boundary terms reflecting the non-symmetry of the Casimir operator over an expanding K -invariant domain in G/N_0 whose radial part is a simplex. In particular, this involves the application of Gauss’ divergence theorem on a simplex. A detailed account of Harish-Chandra’s arguments is given in Sections 11 and 12 of the present paper.

Starting with the proof for the basic case, there appears to be a complete proof of the Maass-Selberg relations in [7]; however, we have not been able to understand the details. In the present paper we follow a different approach by combining the result for the basic case with ideas from the theory of reductive symmetric spaces, in which the action of the so-called standard intertwining operators plays a central role. We believe the obtained information is of separate interest. As an application of the Maass-Selberg relations we prove that the normalized Fourier and Wave packet transforms are continuous linear maps between appropriate Schwartz spaces.

N. Wallach [13] independently developed another approach to the Whittaker-Plancherel formula, in which the Whittaker Maass-Selberg relations do not seem to play a role.

The results of the present paper will be of key importance in a proof of the Plancherel theorem that I have outlined in several lectures in recent years. The accompanying slides are available on my website. Details will appear in a follow up paper.

We will now describe the results of our paper in more detail. It is assumed that G is a real reductive Lie group of the Harish-Chandra class, that K is a maximal compact subgroup, and that $G = KAN_0$ is an Iwasawa decomposition. Furthermore, χ is a fixed unitary character of N_0 which is regular in the sense that for any simple root α of $\mathfrak{a}$ in $\mathfrak{n}_0$ the restriction of $\chi_* := d\chi(e)$ to the root space $\mathfrak{g}_\alpha$ is non-zero. Here $\mathfrak{a}$ and $\mathfrak{n}_0$ denote the Lie algebras of A and N_0 , respectively, in accordance with the convention to denote Lie groups by roman capitals and their Lie algebras by the corresponding gothic lower cases.

The root system of $\mathfrak{a}$ in $\mathfrak{g}$ is denoted by Σ . Furthermore, Σ^+ denotes the positive system for which $\mathfrak{n}_0$ is the sum of the associated root spaces $\mathfrak{g}_\alpha$, for $\alpha \in \Sigma^+$.

We denote by $\mathcal{P}$ the finite set of parabolic subgroups of G containing A and by $\mathcal{P}_{\text{st}}$ the subset of the standard ones among them, i.e., the $P \in \mathcal{P}$ such that P contains the minimal parabolic subgroup $P_0 := Z_G(\mathfrak{a})N_0$. Every $Q \in \mathcal{P}$ is conjugate to a unique $P \in \mathcal{P}_{\text{st}}$, under the action of $N_K(\mathfrak{a})$. The action of the latter group on $\mathfrak{a}$ induces an isomorphism from $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ onto $W(A)$, the Weyl group of the root system Σ .

Given $Q \in \mathcal{P}$ we denote its Langlands decomposition by $Q = M_Q A_Q N_Q$. By $\widehat{M}_{Q,\text{ds}}$ we denote the set of equivalence classes of representations in the discrete series of M_Q .

For $Q \in \mathcal{P}$, $\sigma \in \widehat{M}_{Q,\text{ds}}$ and $\nu \in \mathfrak{a}_{Q,\mathbb{C}}^*$ we define $C^\infty(G/Q : \sigma : \nu)$ to be the Fréchet space of smooth functions $\varphi : G \rightarrow \widehat{H}_\sigma$ transforming according to the rule

$$\varphi(xman) = a^{-\lambda - \rho_Q} \sigma(m)^{-1} \varphi(x), \quad (x \in G, (m, a, n) \in M_Q \times A_Q \times N_Q).$$

Equipped with the left regular representation this space realizes the space of smooth vectors for the normalized induced representation

$$\text{Ind}_Q^G(\sigma \otimes \nu \otimes 1). \quad (0.1)$$

Let $C^{-\infty}(G/Q : \sigma : \nu)$ denote the continuous conjugate linear dual of the Fréchet space $C^\infty(G/Q : \sigma : -\bar{\nu})$. Via the standard G -equivariant sesquilinear pairing by integration over K , $C^\infty(G/Q : \sigma : \nu)$ is injectively and G -equivariantly mapped into $C^{-\infty}(G/Q : \sigma : \nu)$. Accordingly, the latter is viewed as the space of generalized vectors for (0.1). We write

$$C^{-\infty}(G/Q : \sigma : \nu)_\chi \quad (0.2)$$

for the subspace of $C^{-\infty}(G/Q : \sigma : \nu)$ consisting of φ such that $\varphi(nx) = \chi(n)\varphi(x)$, for $x \in G, n \in N_0$. Its elements are called the generalized Whittaker vectors of type χ .

If P is an opposite standard parabolic subgroup then $N_0 P$ is open in G . In [2, Thm. 8.6] it is shown that every function $\varphi \in C^{-\infty}(G/P : \sigma : \nu)_\chi$ restricts to a continuous function $N_0 P \rightarrow H_\sigma^{-\infty}$, satisfying $\varphi(nm) = \chi(n)\sigma(m)^{-1}\varphi(e)$ for $n \in N_0$ and $m \in M_P$. In particular, $\varphi(e) \in H_\sigma^{-\infty}$ satisfies

$$\sigma(n)\varphi(e) = \chi(n)\varphi(e), \quad (n \in M_P \cap N_0).$$

We define $\chi_P := \chi|_{(M_P \cap N_0)}$ and put

$$H_{\sigma, \chi_P}^{-\infty} = \{\eta \in H_\sigma^{-\infty} \mid \forall n \in M_P \cap N_0 : \sigma(n)\eta = \chi_P(n)\eta\}.$$

In [2, Prop. 8.15] it was proven that the evaluation map $\text{ev}_e : C^{-\infty}(G/P : \sigma : \nu)_\chi \rightarrow H_{\sigma, \chi_P}^{-\infty}$ is a bijective linear map of finite dimensional linear spaces, for every $\nu \in \mathfrak{a}_{P,\mathbb{C}}^*$. The inverse of ev_e is denoted by

$$j(P : \sigma : \nu) : H_{\sigma, \chi_P}^{-\infty} \rightarrow C^{-\infty}(G/P : \sigma : \nu)_\chi. \quad (0.3)$$

Furthermore, according to [2, Prop. 8.14], for every $\eta \in H_{\sigma, \chi_P}^{-\infty}$ the function $\nu \rightarrow j(P, \sigma, \nu)\eta$ is holomorphic as a function on $\mathfrak{a}_{P,\mathbb{C}}^*$ with values in $C^{-\infty}(K/K_P : \sigma_P)$ (in the compact picture).

In [2, Prop. 8.10] the Whittaker integrals for $P \in \mathcal{P}_{\text{st}}$ are essentially defined as finite sums of matrix coefficients of K -spherical matrix coefficients with the generalized Whittaker vectors for $\text{Ind}_{\bar{P}}^G(\sigma \otimes \bar{\nu} \otimes 1)$. The Maass–Selberg relations give information about their asymptotic behavior towards infinity, on the closed positive Weyl chamber $\text{cl}A^+$. In the theory of reductive symmetric spaces these Maass–Selberg relations can be reformulated in terms of the action of standard intertwining operators on the analogues of the generalized vectors $j(\bar{P} : \sigma : \bar{\nu})\eta$. The known product decomposition of these operators then reduce the Maass–Selberg relations to a basic case, where they can be established more directly. For this approach to work in the Whittaker setting, one needs to define the generalized Whittaker vectors for $\text{Ind}_Q^G(\sigma \otimes \nu \otimes 1)$, with $Q \in \mathcal{P}$ not necessarily opposite standard. This is worked out in Section 1, making use of the existence of an element $v \in N_K(\mathfrak{a})$ such that vQv^{-1} is opposite standard. The orbit $N_0v\bar{Q}$ is the unique open N_0 -orbit on G/Q . In this setting, evaluation at v defines a bijective linear map $\text{ev}_v : C^{-\infty}(G/Q : \sigma : \nu)_{\chi} \rightarrow H_{\sigma, \chi_Q}^{-\infty}$, whose inverse $j(Q : \sigma : \nu)$ gives the appropriate generalization of $j(P, \sigma, \nu)$, $P \in \bar{\mathcal{P}}_{\text{st}}$. At the end of Section 1 we give the definition of the corresponding Whittaker integral as a finite sum of matrix coefficients of K -spherical vectors with the Whittaker vectors of the induced representations $\text{Ind}_Q^G(\sigma \otimes -\nu \otimes 1)$.

In Section 2 it is shown that for parabolic subgroups $P, Q \in \mathcal{P}$ with equal split components (i.e., $\mathfrak{a}_P = \mathfrak{a}_Q$) there is a unique meromorphic map $\nu \mapsto B(Q, P, \sigma, \nu)$, $\mathfrak{a}_{P_{\mathbb{C}}}^* \rightarrow \text{Hom}(H_{\sigma, \chi_P}^{-\infty}, H_{\sigma, \chi_Q}^{-\infty})$ such that

$$A(Q, P, \sigma, \nu)j(P, \sigma, \nu) = j(Q, \sigma, \nu)B(Q, P, \sigma, \nu), \quad (\nu \in \mathfrak{a}_{P_{\mathbb{C}}}^*).$$

Each space $H_{\sigma, \chi_Q}^{-\infty}$, for $Q \in \mathcal{P}$, carries a natural structure of Hilbert space. In terms of these structures, the Maass–Selberg relations can be formulated as

$$B(Q, P, \sigma, -\bar{\nu})^* B(Q, P, \sigma, \nu) = \eta(Q, P, \sigma, \nu), \quad (0.4)$$

where $\eta(Q, P, \sigma, \nu)$ is the scalar meromorphic function on $\mathfrak{a}_{P_{\mathbb{C}}}^* = \mathfrak{a}_{Q_{\mathbb{C}}}^*$ determined by

$$A(Q, P, \sigma, -\bar{\nu})^* A(Q, P, \sigma, \nu) = \eta(Q, P, \sigma, \nu).$$

The interaction of these structures with the Weyl group $W(\mathfrak{a})$ is discussed.

In Section 3 we study the operators $B(Q, P, \sigma, \nu)$ in detail. The Maass–Selberg relations (0.4) for B are formulated in Theorem 3.1. By using the well-known product decomposition of the standard intertwining operators in terms of those with adjacent P and Q we reduce the proof of the relations (0.4) to the setting in which P, Q are adjacent. We discuss the well-known technique of choosing a subgroup $G^{(\alpha)}$ of G in which P and Q determine opposite maximal parabolic subgroups $P(\alpha)$ and $Q(\alpha)$. It is then shown that $B(Q, P, \sigma, \nu)$ is essentially equal to the B -matrix for $G^{(\alpha)}, Q^{(\alpha)}, P^{(\alpha)}, \sigma, \nu$, see Lemma 4.12. The proof of that lemma requires comparison of distributions on G with distributions on $G^{(\alpha)}$. This makes it long and technical, see Sections 4 - 8.

In the end, the proof of the Maass-Selberg relations for B is reduced to the basic setting in which G has compact center and P and Q are opposite maximal parabolic subgroups, see Lemma 4.14.

In Section 9 we introduce the C -functions as coefficients in certain constant terms of the Whittaker integrals, see Theorem 9.2. These C -functions can be expressed in terms of the standard intertwining operators and the B -matrices, see Lemma 9.5. We show that the Maass-Selberg relations for B imply similar Maass-Selberg relations for C -functions $C_{Q|P}(s, \nu)$, with Q, P associated and $s \in W(\mathfrak{a}_Q | \mathfrak{a}_P)$. They eventually take the form

$$C_{Q|P}(s, -\bar{\nu})^* C_{Q|P}(s, \nu) = \eta_*(P, \bar{P}, -\nu)$$

see Theorem 12.8.

Conversely, it is not a priori clear that the Maass-Selberg relations for the C -functions imply those for the B -matrices. However, in the basic setting they do, as is explained in Section 10. The completion of the proof of the Maass-Selberg relations thus depends on their validity for the C -functions in the mentioned basic setting. The latter case is addressed in the next two sections, 11 and 12, which are based on Harish-Chandra's work in [7].

Section 11 is preparatory, determining a useful formula for the radial part of the Casimir operator, which leads to a formula given without proof in [7, p. 208]. That formula allows the application of Gauss' divergence theorem for a simplex, which in turn leads to asymptotic information in Section 12, see Thm. 12.7. At the end of Section 12, the obtained asymptotic information turns out to imply the Maass-Selberg relations for the C -functions in the basic setting.

In Section 13 we introduce the normalized Whittaker integrals $\text{Wh}^\circ(P, \psi)$ and the associated normalized C -functions $C_{Q|P}^\circ$, for $P, Q \in \mathcal{P}_{\text{st}}$, following the definitions of Harish-Chandra, [7]. The Maass-Selberg relations imply the following relations for the normalized C -functions, for $s \in W(\mathfrak{a}_Q | \mathfrak{a}_P)$,

$$C_{Q|P}^\circ(s, -\bar{\nu})^* C_{Q|P}^\circ(s, \nu) = I \quad (\nu \in \mathfrak{a}_{P_C}^*).$$

These, combined with the uniformly tempered estimates obtained in [2], allow us to show that the normalized Whittaker integrals are (finite sums of) functions of type Π'_{hol} . In view of results in our paper [3] (to appear in the near future) this allows us, in Section 14, to define for each standard parabolic $P \in \mathcal{P}_{\text{st}}$ a normalized Fourier transform $\mathcal{F}_P^\circ$ which is continuous linear from the Harish-Chandra type Schwartz space $C(\tau : G/N_0 : \chi)$ to the Euclidean Schwartz space $\mathcal{S}(i\mathfrak{a}_P^*, \mathcal{A}_{2,P})$. The conjugate Wave packet transform $\mathcal{J}_P$ is continuous linear between these Schwartz spaces in the converse direction.

In Section 15 we establish the functional equations for the normalized Whittaker integrals as given by Harish-Chandra [7, §17.1] In turn these imply transformation formulas for the normalized C -functions, the normalized Fourier transform, and the Wave packet transform.

1 Definition of the map $j(Q, \sigma, \nu)$

A parabolic subgroup $Q \in \mathcal{P}$ is said to be opposite standard if $\bar{Q}$ is standard. The set of $Q \in \mathcal{P}$ with $\bar{Q} \in \mathcal{P}_{\text{st}}$ is denoted by $\bar{\mathcal{P}}_{\text{st}}$. For this paper it will be necessary to describe the action of standard intertwining operators on Whittaker vectors of parabolically induced representations. To make this possible, we need to extend the definition of the map $j(Q, \sigma, \nu)$ to the setting of all parabolic subgroups Q from $\mathcal{P}$, beyond those from $\bar{\mathcal{P}}_{\text{st}}$. To prepare for this we start with the description of the open orbits $N_0\nu Q$ in G , for $\nu \in N_K(\mathfrak{a})$. We assume that $Q \in \mathcal{P}$.

Lemma 1.1 *G is a finite union of double cosets of the form $N_0\nu Q$, for $\nu \in N_K(\mathfrak{a})$. The coset $N_0\nu Q$ is open in G if and only if $\nu N_Q\nu^{-1} \subset \bar{N}_0$, which in turn is equivalent to the condition that $\nu Q\nu^{-1} \in \bar{\mathcal{P}}_{\text{st}}$.*

Proof. There exists an $s \in N_K(\mathfrak{a})$ such that $P := sQs^{-1}$ is opposite standard. By the Bruhat decomposition, G is the disjoint union of the sets $P_0\nu\bar{P}_0$ for $\nu \in W(\mathfrak{a})$. Since $P_0\nu\bar{P}_0 = N_0\nu\bar{P}_0 \subset N_0\nu sQs^{-1}$, it follows that G is a finite union of sets of the form $N_0\nu Qs^{-1}$, with $\nu \in N_K(\mathfrak{a})$. Hence, $G = Gs$ is a finite union of orbits $N_0\nu Q$, for $\nu \in N_K(\mathfrak{a})$.

Put ${}^*\mathfrak{n}_0 := \mathfrak{n}_0 \cap \mathfrak{m}_Q$. Then $N_0\nu Q$ is open in G if and only if the map $N_0 \times Q \rightarrow G$, $(n, q) \mapsto nvq$ is submersive at (e, e) . This is equivalent to $\mathfrak{n}_0 + \text{Ad}(\nu)(\mathfrak{q}) = \mathfrak{g}$, which in turn is equivalent to $\text{Ad}(\nu)^{-1}\mathfrak{n}_0 + \mathfrak{q} = \mathfrak{g}$. Since $\text{Ad}(\nu)$ maps $\mathfrak{a}$ -root spaces to $\mathfrak{a}$ -root spaces the latter assertion is equivalent to

$$\text{Ad}(\nu)^{-1}\mathfrak{n}_0 + {}^*\mathfrak{n}_0 + {}^*\bar{\mathfrak{n}}_0 + \mathfrak{n}_Q = {}^*\mathfrak{n}_0 + {}^*\bar{\mathfrak{n}}_0 + \bar{\mathfrak{n}}_Q + \mathfrak{n}_Q.$$

This in turn is equivalent to $\text{Ad}(\nu)^{-1}\mathfrak{n}_0 \supset \bar{\mathfrak{n}}_Q$, hence to $\nu\bar{N}_Q\nu^{-1} \subset N_0$ and to $\nu Q\nu^{-1} \supset \bar{P}_0$. $\square$

We denote by $W_Q(\mathfrak{a})$ the centralizer of $\mathfrak{a}_Q$ in $W(\mathfrak{a})$. The following lemma is well known through its formulation in terms of root systems.

Lemma 1.2 *Let $Q \in \mathcal{P}$, $s, t \in W(\mathfrak{a})$.*

- (a) *If $s \in W(\mathfrak{a})$ is such that $sQs^{-1} = Q$ then $s \in W_Q(\mathfrak{a})$.*
- (b) *The group Q is $W(\mathfrak{a})$ -conjugate to a unique $P \in \bar{\mathcal{P}}_{\text{st}}$.*

Proof. We start by proving (b). The existence part of (b) is well-known (and also follows from the previous lemma). So, there exists an $s \in W(\mathfrak{a})$ and a $P \in \bar{\mathcal{P}}_{\text{st}}$ such that $sQs^{-1} = P$. It follows that $s(\mathfrak{a}_Q^+) = \mathfrak{a}_P^+$. If Q is opposite standard, then $\mathfrak{a}_Q^+ \subset -\text{cl}(\mathfrak{a}^+)$. Fix $X \in \mathfrak{a}_Q^+$; then both X and sX belong to $-\text{cl}(\mathfrak{a}^+)$. Since the latter set is a fundamental domain for the action of $W(\mathfrak{a})$ on $\mathfrak{a}$, we conclude that $sX = X$. Now s can be written as a product of simple reflections in roots vanishing on $-X$ hence on $\mathfrak{a}_P^+$. Therefore, $s \in W_P(\mathfrak{a})$. This in turn implies that $\mathfrak{a}_Q^+ = \mathfrak{a}_P^+$ hence $P = Q$ and uniqueness follows.

Assume now that Q is general, and let $t \in W(\mathfrak{a})$ be such that $P' := tQt^{-1}$ is standard. Then P' is $W(\mathfrak{a})$ -conjugate to P and from the argument above it follows that $P = P'$. This establishes uniqueness of P .

For (a) we fix $t \in W(\mathfrak{a})$ and $P \in \bar{\mathcal{P}}_{\text{st}}$ such that $tQt^{-1} = P$. Then $sQs^{-1} = Q$ implies that conjugation by tst^{-1} fixes P . By (b) this implies that $t^{-1}st$ belongs to $W_P(\mathfrak{a})$. This in turn implies that $s \in tW_P(\mathfrak{a})t^{-1} = W_Q(\mathfrak{a})$. $\square$

Let $Q \in \mathcal{P}$. Then by the lemma above there exists a unique $P \in \mathcal{P}_{\text{st}}$ that is Weyl conjugate to Q . We fix $v \in N_K(\mathfrak{a})$ such that $vQv^{-1} = P$. Then $vN_Qv^{-1} = N_P \subset N_0$ and by Lemma 1.1 it follows that $N_0v\bar{Q}$ is open in G . The image of v in $W_P(\mathfrak{a}) \setminus W(\mathfrak{a})$ is independent of the possible choices of v . Likewise, the image of v in $W(\mathfrak{a})/W_Q(\mathfrak{a})$ is also independent of such choices.

Corollary 1.3 *Let $Q \in \mathcal{P}(A)$. Then precisely one of the N_0 -orbits on G/Q is open. This orbit equals N_0vQ for any $v \in N_K(\mathfrak{a})$ such that vQv^{-1} is opposite standard.*

Corollary 1.3 allows us to make the following choice once and for all.

Definition 1.4 For the remainder of this paper we fix a map $Q \mapsto v_Q, \mathcal{P}(A) \rightarrow N_K(\mathfrak{a})$ such that

- (a) for every $Q \in \mathcal{P}(A)$ the double coset N_0v_QQ is open in G ;
- (b) if $Q \in \bar{\mathcal{P}}_{\text{st}}$ then $v_Q = e$.

We define

$$C^{-\infty}(G/Q : \sigma : \nu)_{\chi} \tag{1.1}$$

to be the subspace of $C^{-\infty}(G/Q : \sigma : \nu)$ consisting of φ such that $\varphi(nx) = \chi(n)\varphi(x)$, for $x \in G, n \in N_0$. The elements of (1.1) are called the generalized functions of type χ .

For $w \in N_K(\mathfrak{a})$ we define the representation $w\sigma$ of wM_Qw^{-1} in H_{σ} by $w\sigma := \sigma \circ w^{-1}$. Furthermore, we write $H_{w\sigma}$ for H_{σ} equipped with the representation $w\sigma$. Let

$$R_w : C^{-\infty}(G/Q : \sigma : \nu) \rightarrow C^{-\infty}(G/wQw^{-1} : w\sigma : w\nu)$$

be the unique continuous linear G -intertwining operator which is given by the right regular action by w on the subspace of smooth functions.

It maps functions of type χ for the left regular action by N_0 bijectively onto functions of the same type in the image space.

Suppose $Q \in \mathcal{P}(A)$ and let $P \in \bar{\mathcal{P}}_{\text{st}}$ be the unique opposite standard parabolic subgroup that is $W(\mathfrak{a})$ -conjugate to Q . Let $v \in N_K(\mathfrak{a})$ be such that $vQv^{-1} = P$; this condition is equivalent to $v \in N_{K_P}(\mathfrak{a})v_Q$ and to $v \in v_QN_{K_Q}(\mathfrak{a})$.

Clearly, $vM_Qv^{-1} = M_P$. Since $M_P = K_P(M_P \cap A)(M_P \cap N_0)$ is an Iwasawa decomposition for M_P it follows that

$$M_Q = K_Q(M_Q \cap A)(M_Q \cap v^{-1}N_0v) \tag{1.2}$$

is an Iwasawa decomposition for M_Q . Its N -component is given by $M_Q \cap v^{-1}N_0v = v^{-1}(M_P \cap N_0)v$. The associated character $M_Q \cap v^{-1}N_0v \rightarrow \mathbb{C}$, given by

$$v^{-1}\chi_P : n \mapsto \chi(vnv^{-1}), \quad (1.3)$$

will in general depend on the particular choice of $v \in N_K(\mathfrak{a})$. To avoid any ambiguity we agree to exclusively use the notation χ_Q for the character (1.3) defined with $v = v_Q$, see Definition 1.4.

Lemma 1.5 *Put $v = v_Q$. The character χ_Q of $M_Q \cap v(M_P \cap N_0)v^{-1}$ defined by (1.3) is regular with respect to the Iwasawa decomposition (1.2).*

Proof. This is immediate from the regularity of $\chi|_{M_P \cap N_0}$ relative to the Iwasawa decomposition $M_P = K_P(M_P \cap A)(M_P \cap N_0)$, see [2, (8.12)]. $\square$

In analogy with [2, (8.12)], we define

$$H_{\sigma, \chi_Q}^{-\infty} := \{\eta \in H_{\sigma}^{-\infty} \mid \forall m \in M_Q \cap v_Q^{-1}N_0v_Q : \sigma(m)\eta = \chi_Q(m)\eta\}. \quad (1.4)$$

In general, if L is a closed subgroup of G , π a unitary representation of L and ξ a unitary character of a closed subgroup $N \subset L$, we agree to write

$$H_{\pi, \xi}^{-\infty} := \{\eta \in H_{\pi}^{-\infty} \mid \forall m \in N : \pi^{-\infty}(m)\eta = \xi(m)\eta\}.$$

Given $v \in G$, we denote by $v\pi$ the unitary representation of vLv^{-1} in H_{π} given by $v\pi(y) = \pi(v^{-1}yv)$. Furthermore, $v\xi$ denotes the character of vNv^{-1} given by $v\xi(z) = \xi(v^{-1}zv)$. It is readily verified that

$$H_{\pi, \xi}^{-\infty} = H_{v\pi, v\xi}^{-\infty}. \quad (1.5)$$

Indeed, the space on the left consists of all $\eta \in H_{\pi}^{-\infty}$ such that $\pi^{-\infty}(m)\eta = \xi(m)\eta$. Substituting $m' = vmv^{-1}$ we see that the condition on $m' \in vNv^{-1}$ is $(v\pi)^{-\infty}(m') = v[\pi^{-\infty}](m') = v\xi(m')$ for all $m' \in vNv^{-1}$. This in turn is equivalent to $\eta \in H_{v\pi, v\xi}^{-\infty}$.

Corollary 1.6 *Let $Q \in \mathcal{P}$ and let $P \in \bar{\mathcal{P}}_{\text{st}}$ be $W(\mathfrak{a})$ -conjugate to Q . Let $v = v_Q$. Then for all $\sigma \in \widehat{M}_{Q, \text{ds}}$,*

$$H_{\sigma, \chi_Q}^{-\infty} = H_{v\sigma, \chi_P}^{-\infty}.$$

Proof. From (1.3) we see that $\chi_Q = v^{-1}\chi_P$. Now apply (1.5). $\square$

Lemma 1.7 *Let $Q \in \mathcal{P}(A)$ and put $v = v_Q$ and $P = vQv^{-1}$. Then $P \in \bar{\mathcal{P}}_{\text{st}}$. For every $\varphi \in C^{-\infty}(G/Q : \sigma : \nu)_{\chi}$ the following assertions are valid:*

- (a) *the restriction of φ to the open subset N_0vQ is a continuous function $N_0vQ \rightarrow H_{\sigma}^{-\infty}$;*
- (b) *$\varphi(v) \in H_{\sigma, \chi_Q}^{-\infty}$.*

Proof. This can be proven in the same fashion as [2, Thm. 8.6] where Q is assumed to be opposite standard and where $v = e$. Alternatively, one may apply the mentioned result as follows. The generalized function $R_v\varphi \in C^{-\infty}(G/P : v\sigma : v\nu)_\chi$, when restricted to N_0P , yields a continuous function $N_0P \rightarrow H_{v\sigma}^{-\infty}$. Consequently, φ is continuous on N_0Pv with values in $H_{v\sigma}^{-\infty} = H_\sigma^{-\infty}$. Now $N_0Pv = N_0vQ$ so (a) follows.

For (b) we note that $\varphi(v) = R_\varphi(e) \in H_{v\sigma, \chi_P}^{-\infty}$, by the mentioned result. We now apply Corollary 1.6. $\square$

Corollary 1.8 *Let $Q \in \mathcal{P}(A)$, $v = v_Q$, and $P := vQv^{-1}$. Then for every $\nu \in \mathfrak{a}_{Q\mathbb{C}}^*$ the following is a commutative diagram of linear maps. All appearing maps are linear isomorphisms between finite dimensional spaces.*

$$\begin{array}{ccc} C^{-\infty}(G/Q : \sigma : \nu)_\chi & \xrightarrow{R_v} & C^{-\infty}(G/P : v\sigma : v\nu)_\chi \\ \downarrow \text{ev}_v & \xRightarrow{=} & \downarrow \text{ev}_e \\ H_{\sigma, \chi_Q}^{-\infty} & & H_{v\sigma, \chi_P}^{-\infty} \end{array}$$

Proof. Fix $\nu \in \mathfrak{a}_{Q\mathbb{C}}^*$. It follows from the arguments of the above lemma that for $\varphi \in C^{-\infty}(G/Q : \sigma : \nu)_\chi$ we have $\text{ev}_e R_v \varphi = \varphi(v) = \text{ev}_v \varphi$. Hence the diagram commutes. It follows from [2, Cor. 14.5] that the vertical map on the right is a linear isomorphism of finite dimensional linear spaces. Clearly the horizontal maps are linear isomorphisms. Therefore, the vertical map on the left is a linear isomorphism. Since the linear spaces on the right are finite dimensional, all appearing spaces are. $\square$

We agree to write ev_Q for the evaluation map

$$\text{ev}_{v_Q} : C^{-\infty}(G/Q : \sigma : \nu)_\chi \rightarrow H_{\sigma, \chi_Q}^{-\infty}$$

appearing in the left column of the diagram of Corollary 1.8. The above definition depends on our choice of $v_Q \in N_K(\mathfrak{a})$. Any alternative choice $v \in N_K(\mathfrak{a})$ must satisfy $vQv^{-1} = P$ or, equivalently, $v = v_Q u$ with $u \in N_{K_Q}(\mathfrak{a})$. The following result expresses the dependence of our definition of ev_Q on the choice of v_Q .

Lemma 1.9 *Let $Q \in \mathcal{P}$ and let $v = v_Q u$ with $u \in N_{K_Q}(\mathfrak{a})$. Then for every $\nu \in \mathfrak{a}_{Q\mathbb{C}}^*$ the following diagram commutes*

$$\begin{array}{ccc} C^{-\infty}(G/Q : \sigma : \nu)_\chi & & \\ \text{ev}_Q \swarrow & & \searrow \text{ev}_v \\ H_{\sigma, \chi_Q}^{-\infty} & \xrightarrow{\sigma(u)^{-1}} & H_{u\sigma, \chi_Q}^{-\infty} \end{array}$$

The horizontal map at the bottom is given by $\eta \mapsto \sigma(u)^{-1}\eta$. All maps in the diagram are linear isomorphisms.

Proof. Let $\varphi \in C^{-\infty}(G/Q : \sigma : \nu)_\chi$. Then $\text{ev}_v(\varphi) = \varphi(v) = \varphi(v_Q u) = \sigma(u)^{-1} \varphi(v_Q) = \sigma(u)^{-1} \text{ev}_Q(\varphi)$. Now ev_Q is a linear isomorphism and the map $\tau : \eta \mapsto \sigma(u)^{-1} \eta$ is a linear automorphism of $H_\sigma^{-\infty}$. Hence, it suffices to show that τ maps $H_{\sigma, \chi_Q}^{-\infty}$ onto $H_{u\sigma, \chi_Q}^{-\infty}$. Let $\eta \in H_{\sigma, \chi_Q}^{-\infty}$, and suppose that $n \in M_Q \cap v_Q^{-1} N_0 v_Q$. Then

$$u\sigma(n)[\tau(\eta)] = u\sigma(n)[\sigma(u)^{-1} \eta] = \sigma(u^{-1} n) \eta = \sigma(u^{-1}) \chi_Q(n) \eta = \chi_Q(n) \tau(\eta),$$

so that $\tau(\eta) \in H_{u\sigma, \chi_Q}^{-\infty}$. In a similar fashion it is shown that $\tau^{-1}(H_{u\sigma, \chi_Q}^{-\infty}) \subset H_{\sigma, \chi_Q}^{-\infty}$. $\square$

Definition 1.10 Let $Q \in \mathcal{P}$. For each $\nu \in \mathfrak{a}_{Q\mathbb{C}}^*$ the map

$$j(Q, \sigma, \nu) : H_{\sigma, \chi_Q}^{-\infty} \rightarrow C^{-\infty}(G/Q : \sigma : \nu)_\chi$$

is defined to be the inverse of $\text{ev}_Q := \text{ev}_{v_Q} : C^{-\infty}(G/Q : \sigma : \nu)_\chi \rightarrow H_{\sigma, \chi_Q}^{-\infty}$.

Corollary 1.11 Let $Q \in \mathcal{P}$, and put $v = v_Q$ and $P := vQv^{-1}$. Then $P \in \bar{\mathcal{P}}_{\text{st}}$. For every $\nu \in \mathfrak{a}_{Q\mathbb{C}}^*$, the following is a commutative diagram of linear maps.

$$\begin{array}{ccc} C^{-\infty}(G/Q : \sigma : \nu)_\chi & \xrightarrow{R_v} & C^{-\infty}(G/P : v\sigma : v\nu)_\chi \\ \uparrow j(Q, \sigma, \nu) & & \uparrow j(P, v\sigma, v\nu) \\ H_{\sigma, \chi_Q}^{-\infty} & \xrightarrow{=} & H_{v\sigma, \chi_P}^{-\infty} \end{array}$$

All maps are linear isomorphisms of finite dimensional spaces.

Proof. This is immediate from Cor. 1.8. $\square$

We chose a non-degenerate $\text{Ad}(G)$ -invariant symmetric bilinear form

$$B : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R} \tag{1.6}$$

as in [2, (2.1)] and define an $\text{Ad}(K)$ positive definite inner product on $\mathfrak{g}$ by $\langle X, Y \rangle := -B(X, \theta Y)$. For $Q \in \mathcal{P}$ and $R > 0$ we put

$$\mathfrak{a}^*(Q, R) = \{\nu \in \mathfrak{a}_{Q\mathbb{C}}^* \mid \langle \text{Re } \nu, \alpha \rangle > R \ (\forall \alpha \in \Sigma(\mathfrak{n}_Q, \mathfrak{a}_Q))\}. \tag{1.7}$$

Note that for $w \in N_K(\mathfrak{a})$ we have $w\mathfrak{a}^*(Q, R) = \mathfrak{a}^*(wQw^{-1}, R)$.

Corollary 1.12 Let $Q \in \mathcal{P}$, $\sigma \in \widehat{M}_{Q, \text{ds}}$ and $\eta \in H_{\sigma, \chi_Q}^{-\infty}$. For every $R \in \mathbb{R}$ there exists a positive integer s such that the assignment $\nu \mapsto j(Q, \sigma, \nu)$ is holomorphic as a map $\mathfrak{a}^*(\bar{Q}, R) \rightarrow C^{-s}(K/K_Q : \sigma_Q)$.

Proof. If Q is opposite standard, this follows from [2, Lemma 14.3]. For general $Q \in \mathcal{P}$ we observe that it follows from Cor. 1.11 that for a fixed $\eta \in H_{\sigma, \chi_Q}^{-\infty}$ we have $j(Q, \sigma, \nu)\eta = R_{v^{-1}} \circ j(P, v\sigma, v\nu)$ for all $\nu \in \mathfrak{a}_{Q\mathbb{C}}^*$. Now $R_{v^{-1}}$ restricts to a continuous linear map $C^{-\infty}(K : v\sigma) \rightarrow C^{-\infty}(K : \sigma)$, independent of ν . Therefore, the required result follows from the established case. $\square$

Remark 1.13 If $\Omega \subset \mathfrak{a}_{Q_C}^*$ is a bounded open subset, then it follows from Cor. 1.12 that there exists a positive integer s such that for all η the assignment $\nu \mapsto j(Q, \sigma, \nu)\eta$ is holomorphic as a map $\Omega \rightarrow C^{-s}(K/K_Q : \sigma_Q)$. The smallest s with this property will be called the order of $j(Q, \sigma, \cdot)$ over Ω .

Remark 1.14 The definition of the space $C^{-s}(K/K_Q : \sigma_Q)$ appearing in the preceding statements is explained in [2, §7].

If $P \in \bar{\mathcal{P}}_{\text{st}}$ we fix an arbitrary positive invariant density $d\bar{m}_P$ on $M_P/M_P \cap N_0$. If $Q \in \mathcal{P}$ is conjugate to P , then $v_Q Q v_Q^{-1} = P$ and conjugation by v_Q induces a diffeomorphism from $M_Q/M_Q \cap v_Q^{-1} N_0 v_Q$ onto $M_P/M_P \cap N_0$. The pull-back of $d\bar{m}_P$ under C_{v_Q} is a positive invariant density on $M_Q/M_Q \cap v_Q^{-1} N_0 v_Q$ which we denote by $d\bar{m}_Q$.

Lemma 1.15 Let $Q, Q' \in \mathcal{P}$ be $N_K(\mathfrak{a})$ -conjugate to the same opposite standard parabolic subgroup P . Put $w = v_{Q'}^{-1} v_Q$. Then $Q' = w Q w^{-1}$, $w(M_Q \cap v_Q^{-1} N_0 v_Q) w^{-1} = (M_{Q'} \cap v_{Q'}^{-1} N_0 v_{Q'})$ and

$$C_w^*(d\bar{m}_{Q'}) = d\bar{m}_Q.$$

Proof. The first assertions are evident. By definition of w we have $C_w^* = C_{v_Q}^* C_{v_{Q'}}^{*-1}$. Hence

$$C_w^*(d\bar{m}_{Q'}) = C_{v_Q}^*(d\bar{m}_P) = d\bar{m}_Q.$$

□

Let $Q \in \mathcal{P}$, $\sigma \in \widehat{M}_{Q, \text{ds}}$. According to [2, Lemma 9.2] with M_Q in place of G , we have the matrix coefficient map

$$\mu_\sigma = \mu_{Q, \sigma} : H_\sigma \otimes \overline{H_{\sigma, \chi_Q}^{-\infty}} \rightarrow L^2(M_Q/M_Q \cap v_Q^{-1} N_0 v_Q : \chi_Q) \quad (1.8)$$

given by $\mu_\sigma(z \otimes \eta)(m) = \langle \sigma(m)^{-1} z, \eta \rangle$, for $z \in H_\sigma^\infty$, $\eta \in H_{\sigma, \chi_Q}^{-\infty}$ and $m \in M_Q$. Here the L^2 -norm is defined with respect to the invariant measure $d\bar{m}_Q$ on $M_Q/M_Q \cap v_Q^{-1} N_0 v_Q$.

From [2, Cor. 9.5] it follows that the finite dimensional space $H_{\sigma, \chi_Q}^{-\infty}$ carries a unique inner product such that the map (1.8) is an isometric linear map onto a closed subspace, which we denote by

$$L^2(M_Q/M_Q \cap v_Q^{-1} N_0 v_Q : \chi_Q)_\sigma.$$

Definition 1.16 Let $Q \in \mathcal{P}$ and $\sigma \in \widehat{M}_{Q, \text{ds}}$. From now on we assume that $H_{\sigma, \chi_Q}^{-\infty}$ is equipped with the unique Hermitean inner product that makes $\mu_{Q, \sigma}$ isometric.

Let Q' be conjugate to Q and put $w = v_{Q'}^{-1} v_Q$. Then it is readily verified that conjugation by C_w^{-1} defomes a diffeomorphism from M_Q onto from $M_{Q'}$, which maps $M_Q \cap v_Q^{-1} N_0 v_Q$ onto the similar intersection with everywhere Q replaced by Q' . In turn this implies that pull-back under conjugation $C_{w^{-1}}$ induces an isometric isomorphism

$$A_w : L^2(M_Q/M_Q \cap v_Q^{-1} N_0 v_Q : \chi_Q)_\sigma \rightarrow L^2(M_{Q'}/M_{Q'} \cap v_{Q'}^{-1} N_0 v_{Q'} : \chi_{Q'})_{w\sigma}.$$

Lemma 1.17 *Let $Q, Q' \in \mathcal{P}$ be such that $wQw^{-1} = Q'$ for $w = v_Q^{-1}v_Q$. Let $\sigma \in \widehat{M}_{Q, \text{ds}}$.*

The linear spaces $H_{\sigma, \chi_Q}^{-\infty}$ and $H_{w\sigma, \chi_{Q'}}^{-\infty}$ are equal as subspaces of $H_{\sigma}^{-\infty}$. The Hermitian inner products, as specified in Definition 1.16 are the same.

Proof. Put $v_Q = v$ and $v' = v_{Q'}$. Then $w = v'^{-1}v$. It follows from Corollary 2.6 1.6 that

$$H_{\sigma, \chi_Q}^{-\infty} = H_{v\sigma, \chi_P}^{-\infty} = H_{v'^{-1}v\sigma, \chi_{Q'}}^{-\infty} = H_{w\sigma, \chi_{Q'}}^{-\infty}.$$

In view of our choices of measure, the map A_w given above is an isometry. For $z \in H_{\sigma}$ and $\eta \in H_{\sigma, \chi_Q}^{-\infty}$ we have that

$$A_w \circ [\mu_{Q, \sigma}(z \otimes \eta)](m) = \mu_{Q, \sigma}(z \otimes \eta)(w^{-1}mw) = \mu_{Q', w\sigma}(w\sigma \otimes \eta)(m).$$

The maps $\mu_{Q, \sigma}$ and $\mu_{Q', w\sigma}$ are unitary by definition, and we see that the identity induces a unitary map

$$H_{\sigma} \otimes \overline{H_{\sigma, \chi_Q}^{-\infty}} \rightarrow H_{w\sigma} \otimes \overline{H_{w\sigma, \chi_{Q'}}^{-\infty}}.$$

Since the identity map $H_{\sigma} \rightarrow H_{w\sigma}$ is unitary, we conclude that the identity map $H_{\sigma, \chi_Q}^{-\infty} \rightarrow H_{w\sigma, \chi_{Q'}}^{-\infty}$ is unitary as well. $\square$

Lemma 1.18 *Let $Q \in \mathcal{P}$ and $u \in N_{K_Q}(\mathfrak{a})$. Then the map $\eta \mapsto \sigma(u)^{-1}\eta$ is an isometry from $H_{\sigma, \chi_Q}^{-\infty}$ onto $H_{u\sigma, \chi_Q}^{-\infty}$.*

Proof. From Lemma 1.9 it follows that the linear map $\tau : H_{\sigma}^{-\infty} \rightarrow H_{\sigma}^{-\infty}$, $\xi \mapsto \sigma(u)^{-1}\xi$ maps $H_{\sigma, \chi_Q}^{-\infty}$ onto $H_{u\sigma, \chi_Q}^{-\infty}$. We will finish the proof by showing that τ is isometric on $H_{\sigma, \chi_Q}^{-\infty}$.

For $v \in H_{\sigma}^{\infty}$ and $\eta \in H_{\sigma, \chi_Q}^{-\infty}$ the matrix coefficient attached to $v \otimes \eta$ is the function in $C^{\infty}(M_Q/M_Q \cap v_Q^{-1}N_0v_Q : \chi_Q)$ defined by

$$\mu_{v \otimes \eta}(m) = \langle \sigma(m)^{-1}v, \eta \rangle, \quad (m \in M_Q).$$

The sesquilinear map $(v, \eta) \mapsto m_{v \otimes \eta}$ induces a linear isometry from the pre-Hilbert space $H_{\sigma}^{\infty} \otimes H_{\sigma, \chi_Q}^{-\infty}$ to $L^2(M_Q/M_Q \cap v_Q^{-1}N_0v_Q : \chi_Q)$. This implies that for all $v \in H_{\sigma}^{\infty}$ and $\eta \in H_{\sigma, \chi_Q}^{-\infty}$,

$$\|v\|_{\sigma}^2 \|\eta\|_{\sigma, \chi_Q}^2 = \int_{M_Q/M_Q \cap v_Q^{-1}N_0v_Q} |\mu_{v \otimes \eta}(m)|^2 d\bar{m}_Q. \quad (1.9)$$

The representation $u\sigma$ of M_Q defined by $u\sigma(m) = \sigma(u^{-1}mu)$ is irreducible unitary and belongs to the discrete series of M_Q again. We write $H_{u\sigma}$ for the Hilbert space H_{σ} equipped with the representation $u\sigma$ and will discuss the induced inner product on $H_{u\sigma, \chi_Q}^{-\infty}$. The identity map is unitary from H_{σ} to $H_{u\sigma}$. If $v \in H_{\sigma}^{\infty}$, then $v \in H_{u\sigma}^{\infty}$. If $\eta' \in H_{u\sigma, \chi_Q}^{-\infty}$ then the associated matrix coefficient $\mu'_{v \otimes \eta'}$ is given by

$$\mu'_{v \otimes \eta'}(m) = \langle [u\sigma](m)^{-1}v, \eta' \rangle = \langle \sigma(u^{-1})\sigma(m^{-1})\sigma(u)v, \eta' \rangle = \mu_{v \otimes \sigma(u)\eta'}(u^{-1}m).$$

Substituting $\eta' = \tau(\eta)$ we find that

$$\mu'_{v \otimes \tau(\eta)}(m) = \mu_{v \otimes \eta}(u^{-1}m).$$

Using the analogue of (1.9) for $u\sigma$ and μ' , combined with the left invariance of the measure $d\bar{m}_Q$, we infer that

$$\|\tau(\eta)\|_{u\sigma, \chi_Q}^2 = \|\eta\|_{\sigma, \chi_Q}^2, \quad (\eta \in H_{\sigma, \chi_Q}^{-\infty}).$$

Hence, τ is isometric as stated. $\square$

In [2] the Whittaker integral $\text{Wh}(P, \cdot)$ for $P \in \mathcal{P}_{\text{st}}$ is expressed in terms of matrix coefficients involving $j(\bar{P}, \cdot)$. Guided by this definition we will now use matrix coefficients involving $j(\bar{Q}, \cdot)$ to define the notion of Whittaker integral $\text{Wh}(Q, \cdot)$, for $Q \in \mathcal{P}$ arbitrary.

As in [2, §9], which in turn relies on Harish-Chandra [6, Lemmas 7.1, 9.1] we aim at defining a linear isomorphism

$$T \mapsto \psi_T, \quad C^\infty(\tau : K/K_Q : \sigma_Q) \otimes \overline{H_{\sigma, \chi_Q}^{-\infty}} \rightarrow L^2(\tau_Q : M_Q/M_Q \cap v_Q^{-1}N_0v_Q : \chi_Q)_\sigma.$$

Here $\sigma \in \widehat{M}_{Q, \text{ds}}$ and σ_Q denotes the restriction of σ to K_Q . Furthermore, (τ, V_τ) is a finite dimensional unitary representation of K and τ_Q denotes the restriction of τ to $K_Q = K \cap M_Q$.

We define the space of spherical functions

$$L^2(\tau_Q : M_Q/M_Q \cap v_Q^{-1}N_0v_Q : \chi_Q) \tag{1.10}$$

to be the subspace of K_Q -fixed elements in $L^2(M_Q/M_Q \cap v_Q^{-1}N_0v_Q : \chi_Q) \otimes V_\tau$. Viewing (1.10) naturally as a space of functions $M_Q \rightarrow V_\tau$ we shall express the spherical behavior of its functions by $f(km) = \tau(k)f(m)$, for $m \in M_Q$ and $k \in K_Q$. The space is equipped with the restriction of the tensor product Hilbert structure, and thus is a Hilbert space of its own right. The space of functions in (1.10) which belong to $L^2(M_Q/M_Q \cap v_Q^{-1}N_0v_Q : \chi_Q)_\sigma \otimes V_\tau$ is indicated by the subscript σ on the right. Since only finitely many representations of the discrete series of M_Q have a K_Q -type in common with τ_Q , it follows that

$$L^2(\tau_Q : M_Q/M_Q \cap v_Q^{-1}N_0v_Q : \chi_Q) = \bigoplus_{\sigma \in \widehat{M}_{Q, \text{ds}}} L^2(\tau_Q : M_Q/M_Q \cap v_Q^{-1}N_0v_Q : \chi_Q)_\sigma$$

is finite dimensional. In particular the (orthogonal) sum over the σ is finite. From this it also follows that

$$L^2(\tau_Q : M_Q/M_Q \cap v_Q^{-1}N_0v_Q : \chi_Q) = C(\tau_Q : M_Q/M_Q \cap v_Q^{-1}N_0v_Q : \chi_Q),$$

where the definition of the space on the right is obvious. By finite dimensionality, the center $\mathfrak{Z}_Q$ of $U(\mathfrak{m}_Q)$ acts finitely on the space on the right. For this reason, that space is also denoted by $\mathcal{A}_{2, \bar{Q}}$.

The subspace $C(M_Q/M_Q \cap v_Q^{-1}N_0v_Q : \chi_Q)_\sigma \otimes V_\tau \cap \mathcal{A}_{2,\bar{Q}}$ is denoted by $\mathcal{A}_{2,\bar{Q},\sigma}$. We have the finite orthogonal direct sum

$$\mathcal{A}_{2,\bar{Q}} = \oplus_{\sigma \in \widehat{M}_{Q,\text{ds}}} \mathcal{A}_{2,\bar{Q},\sigma}.$$

After these preparations we define, for $\varphi \in C^\infty(\tau_Q : K/K_Q : \sigma_Q)$ and $\eta \in H_{\sigma,\chi_Q}^{-\infty}$ the function $\psi_{\varphi \otimes \eta} : M_Q \rightarrow V_\tau$ by

$$\psi_{\varphi \otimes \eta}(m) = \langle (\sigma(m^{-1}) \otimes I)\varphi(e), \eta \rangle_{\sigma,1}, \quad (m \in M_Q),$$

where the sesquilinear pairing $\langle \cdot, \cdot \rangle_{\sigma,1} : (H_\sigma \otimes V_\tau) \times H_{\sigma,\chi_Q}^{-\infty} \rightarrow V_\tau$ is given by $\langle z \otimes v, \eta \rangle_{\sigma,1} = \langle z, \eta \rangle_{\sigma v}$.

Lemma 1.19 *The map $(\varphi, \eta) \mapsto \psi_{\varphi \otimes \eta}$ induces an isometric linear isomorphism $T \mapsto \psi_T$ from $C^\infty(\tau : K/K_Q : \sigma_Q) \otimes \overline{H_{\sigma,\chi_Q}^{-\infty}}$ onto $\mathcal{A}_{2,\bar{Q},\sigma}$.*

Proof. The proof, which relies on an application of Frobenius reciprocity, is identical to the proof for the case that $Q = P \in \mathcal{P}_{\text{st}}$, in [2, Lemma 9.8]. $\square$

Finally, we are prepared to define the Whittaker integral associated with $Q \in \mathcal{P}$.

Definition 1.20 Let $Q \in \mathcal{P}$. The Whittaker integral $\text{Wh}(Q, \psi, \nu)$, for $\psi \in \mathcal{A}_{2,Q}$ and for generic $\nu \in \mathfrak{a}_{Q^\mathbb{C}}^*$ is defined to be the function in $C^\infty(\tau : G/N_0 : \chi)$ determined by the following requirements.

- (a) $\text{Wh}(Q, \psi, \nu)$ depends linearly on $\psi \in \mathcal{A}_{2,Q}$;
- (b) for $\sigma \in \widehat{M}_{Q,\text{ds}}$ and $T = \varphi \otimes \eta \in C^\infty(\tau : K/K_Q : \sigma_Q) \otimes H_{\sigma,\chi_Q}^{-\infty}$, we have

$$\text{Wh}(Q, \psi_T, \nu)(x) = \langle \pi_{\bar{Q},\sigma,-\nu}(x)^{-1}\varphi, j(\bar{Q}, \sigma, \bar{\nu})\eta \rangle, \quad (x \in M_Q).$$

We retain our assumption that $Q \in \mathcal{P}$, $\sigma \in \widehat{M}_{Q,\text{ds}}$ and put $v = v_{\bar{Q}}$. Then $P := vQv^{-1}$ is the unique standard parabolic subgroup which is conjugate to Q . The Whittaker integral $\text{Wh}(Q)$ can now be expressed in terms of $\text{Wh}(P)$.

Lemma 1.21 *There exists a unique isometric linear isomorphism $\mathcal{R}_Q : \mathcal{A}_{2,Q} \rightarrow \mathcal{A}_{2,P}$ such that for all $\sigma \in \widehat{M}_{Q,\text{ds}}$ the following is valid*

- (a) $\mathcal{R}_Q$ maps $\mathcal{A}_{2,Q,\sigma}$ onto $\mathcal{A}_{2,P,v\sigma}$.
- (b) for all $T \in C^\infty(\tau : K/K_Q : \sigma_Q) \otimes H_{\sigma,\chi_Q}^{-\infty}$ we have

$$\mathcal{R}_Q \psi_T = \psi_{(\mathcal{R}_v \otimes I)T}. \quad (1.11)$$

Here $\mathcal{R}_v : C^\infty(\tau : K/K_Q : \sigma_Q) \rightarrow C^\infty(\tau : K/K_P : (v\sigma)_P)$ is the map induced by right translation by v .

Proof. Since $\mathcal{A}_{2,Q}$ decomposes as the orthogonal finite direct sum of the subspaces $\mathcal{A}_{2,Q,\sigma}$ and since for each σ the map $\Psi_\sigma : T \mapsto \psi_T$ is an isometry from $V_\sigma := C^\infty(\tau : K/K_Q : \sigma_Q) \otimes H_{\sigma, \chi_{\bar{Q}}}^{-\infty}$ onto $\mathcal{A}_{2,Q,\sigma}$, uniqueness is obvious.

For each $\sigma \in \widehat{M}_{Q,\text{ds}}$ the map $R_v : C^\infty(\tau : K/K_Q : \sigma_Q) \rightarrow C^\infty(\tau : K/K_P : (v\sigma)_P)$ is induced by right translation by v , which is clearly an isometry. Furthermore, I denotes the identity map from $H_{\sigma, \chi_{\bar{Q}}}^{-\infty}$ to $H_{v\sigma, \chi_{\bar{P}}}^{-\infty}$, which is an isometry by Lemma 1.19. We thus see that ${}_\sigma R_v : T \mapsto (R_v \otimes I)T$ is an isometry from V_σ onto $V_{v\sigma}$. The condition on $\mathcal{R}_Q$ may now be reformulated as $\mathcal{R}_Q \circ \Psi_\sigma = \Psi_{v\sigma} \circ {}_\sigma R_v$ on V_σ . Thus, $\mathcal{R}_Q = \Psi_{v\sigma} \circ {}_\sigma R_v \circ \Psi_\sigma^{-1}$ on $\mathcal{A}_{2,Q,\sigma}$. This establishes the existence. $\square$

Proposition 1.22 *Let $\mathcal{R}_Q$ be as in Lemma 1.21. For all $\sigma \in \widehat{M}_{Q,\text{ds}}$ and all $\psi \in \mathcal{A}_{2,Q,\sigma}$ we have*

$$\text{Wh}(Q, \psi, \nu, x) = \text{Wh}(P, \mathcal{R}_Q \psi, \nu \nu, x),$$

for all $\nu \in \mathfrak{a}_{Q\mathbb{C}}^*$ and $x \in G$.

Proof. Let $\sigma \in \widehat{M}_{Q,\text{ds}}$ and $T = \varphi \otimes \eta \in C^\infty(\tau : K/K_Q : \sigma_Q) \otimes H_{\sigma, \chi_{\bar{Q}}}^{-\infty}$. Then by Definition 1.20 and Corollary 1.11 we have

$$\begin{aligned} \text{Wh}(Q, \psi_T, \nu, x) &= \langle \pi_{\bar{Q}, \sigma, -\nu}(x)^{-1} \varphi, j(\bar{Q}, \sigma, \bar{\nu}, \eta) \rangle \\ &= \langle R_v \pi_{\bar{Q}, \sigma, -\nu}(x)^{-1} \varphi, j(\bar{P}, v\sigma, v\bar{\nu}, \eta) \rangle \\ &= \langle \pi_{\bar{P}, v\sigma, -v\nu}(x)^{-1} R_v \varphi, j(\bar{P}, v\sigma, v\bar{\nu}, \eta) \rangle \\ &= \text{Wh}(P, \psi_{(R_v \otimes I)T}, \nu \nu, x). \end{aligned}$$

The proof is completed by using (1.11). $\square$

Lemma 1.23 *Let $Q \in \mathcal{P}$ and let $v = v_{\bar{Q}}$. Then $P = vQv^{-1}$ is standard. Furthermore, for $\psi \in \mathcal{A}_{2,Q}$ and $\nu \in \mathfrak{a}_{Q\mathbb{C}}^*$ such that $\text{Re} \langle \nu \nu, \alpha \rangle > 0$ for all $\alpha \in \Sigma^+$ we have*

$$\text{Wh}(Q, \psi, \nu)(x) = \int_{N_Q} [v^{-1} \chi_{\bar{Q}}](n)^{-1} \psi_{\bar{Q}, \sigma, -\nu}(x \nu n) \, dn \quad (x \in G).$$

Remark 1.24 For Q standard and $v = e$, we retrieve Harish-Chandra's formula.

Proof. First of all, by Definition 1.4, $v_{\bar{Q}} v^{-1} \in \bar{\mathcal{P}}_{\text{st}}$ and it follows that P is standard. By linearity, it suffices to consider the case that $\psi = \psi_T$, with $T = \varphi \otimes \eta \in C^\infty(\tau : K/K_Q : \sigma_Q) \otimes H_{\sigma, \chi_{\bar{Q}}}^{-\infty}$, where $\sigma \in \widehat{M}_{Q,\text{ds}}$. Write $\psi_{-\nu}$ for the function in $C^\infty(\tau : G/\bar{Q} : \sigma : -\nu)$ given by $\psi_{-\nu}|_K = \psi$. Then

$$\begin{aligned} \text{Wh}(Q, \psi_T, \nu)(x) &= \langle \pi_{\bar{Q}, \sigma, -\nu}(x)^{-1} \varphi, j(\bar{Q}, \sigma, -\nu) \eta \rangle \\ &= \int_{K/K_Q} \langle \varphi_{-\nu}(xk), j(\bar{Q}, \sigma, -\nu, \eta)(k) \rangle \, dk \\ &= \int_{N_Q} \langle \varphi_{-\nu}(x \nu n), j(\bar{Q}, \sigma, -\nu, \eta)(\nu n) \rangle \, dn \\ &= \int_{N_Q} \chi(\nu n \nu^{-1}) \langle \varphi_{-\nu}(x \nu n), \eta \rangle \, dn. \end{aligned}$$

The required representation now follows from the equality

$$\langle \varphi_{-\nu}(y), \eta \rangle = \psi_{\bar{Q}, \sigma, -\nu}(y), \quad (y \in G).$$

By left τ -sphericity and right $\bar{N}_Q$ -invariance, it suffices to prove the latter equality for $y = ma \in M_Q A_Q$. This in turn follows from

$$\langle \varphi_{-\nu}(ma), \eta \rangle = a^{\nu+\rho_Q} \langle \sigma(m)^{-1} \varphi(e), \eta \rangle = a^{\nu+\rho_Q} \psi_T(m) = \psi_{\bar{Q}, \sigma, -\nu}(ma).$$

□

2 Interaction with the Weyl group

We assume that $Q \in \mathcal{P}$ and that $\sigma \in \widehat{M}_{Q, \text{ds}}$.

Lemma 2.1 *Let $w \in N_K(\mathfrak{a})$, and put $Q' = wQw^{-1}$. There exists a unique linear map*

$$\mathcal{R}_{w, Q} : H_{\sigma, \chi_Q}^{-\infty} \rightarrow H_{w\sigma, \chi_{Q'}}^{-\infty}$$

such that for every $\nu \in \mathfrak{a}_{Q\mathbb{C}}^$ the following diagram commutes:*

$$\begin{array}{ccc} C^{-\infty}(G/Q : \sigma : \nu)_{\chi} & \xrightarrow{R_w} & C^{-\infty}(G/Q' : w\sigma : w\nu)_{\chi} \\ \downarrow \text{ev}_Q & & \downarrow \text{ev}_{Q'} \\ H_{\sigma, \chi_Q}^{-\infty} & \xrightarrow{\mathcal{R}_{w, Q}} & H_{w\sigma, \chi_{Q'}}^{-\infty} \end{array} \quad (2.1)$$

The map $\mathcal{R}_{w, Q}$ is a unitary linear isomorphism.

Proof. Uniqueness of the map follows from the fact that the remaining maps in diagram (2.1) are linear isomorphisms. Let P be the unique parabolic subgroup in $\bar{\mathcal{P}}_{\text{st}}$ such that Q is $W(\mathfrak{a})$ -conjugate to P . Let $v = v_Q$ and $v' = v_{Q'}$. Then conjugation by $v'w$ maps Q to P ; hence $v'w = vu$ for a suitable element $u \in N_{K_Q}(\mathfrak{a})$.

We put $w' = wu^{-1}$ so that $w' = (v')^{-1}v$ and observe that R_w equals the composition $R_{w'}R_u$, with $R_u : C^{-\infty}(G/Q : \sigma : \nu)_{\chi} \rightarrow C^{-\infty}(G/Q : u\sigma : u\nu)_{\chi}$ and with $R_{w'} : C^{-\infty}(G/Q : u\sigma : u\nu)_{\chi} \rightarrow C^{-\infty}(G/Q : w\sigma : w\nu)_{\chi}$. If $f \in C^{-\infty}(G/Q : \sigma : \nu)_{\chi}$, then

$$\text{ev}_Q R_u f = f(v_Q u) = \sigma(u)^{-1} \text{ev}_Q(f).$$

From Lemma 1.18 we know that $\tau : \eta \mapsto \sigma(u)^{-1} \eta$ defines an isometric linear isomorphism

$$\tau : H_{\sigma, \chi_Q}^{-\infty} \rightarrow H_{u\sigma, \chi_Q}^{-\infty}.$$

This gives us the first square of maps in the diagram below, for every $\nu \in \mathfrak{a}_{Q\mathbb{C}}^*$.

$$\begin{array}{ccccc}
C^{-\infty}(G/Q : \sigma : \nu)_\chi & \xrightarrow{R_u} & C^{-\infty}(G/Q : u\sigma : \nu)_\chi & \xrightarrow{R_{w'}} & C^{-\infty}(G/Q' : w\sigma : w\nu)_\chi \\
\downarrow \text{ev}_Q & & \downarrow \text{ev}_Q & & \downarrow \text{ev}_{Q'} \\
H_{\sigma, \chi_Q}^{-\infty} & \xrightarrow{\tau} & H_{u\sigma, \chi_Q}^{-\infty} & \xrightarrow{\mathcal{R}_{w', Q}} & H_{w\sigma, \chi_{Q'}}^{-\infty}.
\end{array} \tag{2.2}$$

To understand the second square of maps, let $g \in C^{-\infty}(G/Q : u\sigma : u\nu)_\chi$. Then $\text{ev}_{Q'}(R_{w'}g) = g(v'w') = g(v) = \text{ev}_Q(g)$. Since $w' = v'^{-1}v$, it follows from Lemma 1.17 that there is a unique linear isomorphism $\mathcal{R}_{w', Q}$ from $H_{u\sigma, \chi_Q}^{-\infty}$ onto $H_{w\sigma, \chi_{Q'}}^{-\infty}$, which makes the second square of maps commutative. Furthermore, $\mathcal{R}_{w', Q}$ is isometric.

The map $R_{w, Q} := R_{w', Q} \circ \tau$ makes the diagram (2.1) commutative, and is the composition of two isometries, hence an isometry of its own right. $\square$

Corollary 2.2 *Let $Q \in \mathcal{P}$, $w \in N_K(\mathfrak{a})$ and let $\mathcal{R}_{w, Q} : H_{\sigma, \chi_Q}^{-\infty} \rightarrow H_{w\sigma, \chi_{wQw^{-1}}}^{-\infty}$ be the isometry of Lemma 2.1. The following diagram commutes for every $\nu \in \mathfrak{a}_{Q\mathbb{C}}^*$:*

$$\begin{array}{ccc}
C^{-\infty}(G/Q : \sigma : \nu)_\chi & \xrightarrow{R_w} & C^{-\infty}(G/wQw^{-1} : w\sigma : w\nu)_\chi \\
\uparrow j(Q, \sigma, \nu) & & \uparrow j(wQw^{-1}, w\sigma, w\nu) \\
(H_{\sigma}^{-\infty})_{\chi_Q} & \xrightarrow{\mathcal{R}_{w, Q}} & (H_{w\sigma}^{-\infty})_{\chi_{wQw^{-1}}}
\end{array} \tag{2.3}$$

The map $\mathcal{R}_{w, Q}$ is an isometric linear isomorphism.

Before proceeding we list some properties of the standard intertwining operators for two parabolic subgroups $Q_j \in \mathcal{P}(A)$ ($j = 1, 2$) with equal split components. Let $Q_j = M_j A_j N_j$, be their Langlands decompositions, then $A_1 = A_2$ and $M_1 = M_2$. We denote by $\Sigma(Q_j)$ the set of $\mathfrak{a}_j$ -roots in $\mathfrak{n}_{Q_j}$. For $R \in \mathbb{R}$ we define

$$\mathfrak{a}^*(Q_2|Q_1, R) := \{\nu \in \mathfrak{a}_{Q_1\mathbb{C}}^* \mid \langle \text{Re } \nu, \alpha \rangle > R \ (\forall \alpha \in \Sigma(\bar{Q}_2) \cap \Sigma(Q_1))\}. \tag{2.4}$$

Fix $\sigma \in \widehat{M}_{1\text{ds}}$ and $\nu \in \mathfrak{a}^*(\bar{Q}_2|Q_1, 0)$. Then the standard intertwining operator $A(Q_2, Q_1, \sigma, \nu)$ from $C^\infty(G/Q_1 : \sigma : \nu)$ to $C^\infty(G/Q_2 : \sigma : \nu)$ is given by the usual integral formula

$$A(Q_2, Q_1, \sigma, \nu)f(x) = \int_{N_2 \cap \bar{N}_1} f(x\bar{n}) \, d\bar{n} \tag{2.5}$$

for $f \in C^\infty(G/Q_1 : \sigma : \nu)$.

We agree to equip any nilpotent subalgebra $\mathfrak{n}_*$ of $\mathfrak{g}$ with the Riemannian inner product obtained by restriction of the positive definite inner product $-B(\cdot, \theta(\cdot))$. The associated analytic subgroup N_* of G is equipped with the associated bi-invariant unit Haar measure. As a result, for every $k \in K$ the map $n \mapsto knk^{-1}$ is measure preserving from N_* to kN_*k^{-1} . In particular, all standard intertwining operators will

be normalized in this way. Then for all $w \in N_K(\mathfrak{a})$, all $\nu \in \mathfrak{a}^*(Q_2|Q_1, 0)$ and all $f \in C^\infty(G/Q_1 : \sigma : \nu)$ we have

$$A(Q_2, Q_1, \sigma, \nu)f = R_w^{-1}A(wQ_2w^{-1}, wQ_1w^{-1}, w\sigma, w\nu)R_wf. \quad (2.6)$$

Indeed this follows from the integral formula (2.5) in view of the normalization of measures specified in the above.

We shall now describe the well known meromorphic continuation of the intertwining operator in terms of the compact pictures of the induced representations, where we exploit the topological linear isomorphism

$$C^\infty(G/Q_j : \sigma : \nu) \simeq C^\infty(K/K_{Q_j} : \sigma_{Q_j}),$$

induced by restriction to K . By transfer under this isomorphism, the left regular representation L in the first space becomes a ν -dependent representation $\pi_{Q_j, \sigma, \nu}$ of G in the second space. The operator $A(Q_2, Q_1, \sigma, \nu)$ can now be viewed as a continuous linear operator of Fréchet spaces $C^\infty(K/K_{Q_1} : \sigma) \rightarrow C^\infty(K/K_{Q_2} : \sigma)$. The dependence of the intertwining operator of $\nu \in \mathfrak{a}_{1\mathbb{C}}^*$ is known to be meromorphic, by [11], with singular locus $\mathcal{S}(Q_2, Q_1, \sigma)$ a locally finite union of affine root hyperplanes of the form $\langle \nu, \alpha \rangle = c$, with $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}_j)$ and $c \in \mathbb{R}$. As a result, the equality (2.6) is valid as equality of meromorphic functions on $\mathfrak{a}_{1\mathbb{C}}^*$ with values in $C^\infty(K/K_{Q_1} : \sigma_{Q_1})$.

Via the equivariant sesquilinear pairing $\langle \cdot, \cdot \rangle$ of the space $C^\infty(Q : \sigma : \nu)$ with $C^\infty(Q : \sigma - \bar{\nu})$ by integration over K/K_Q we may embed the first space in the conjugate continuous linear dual of the second, denoted $C^{-\infty}(Q : \sigma : \nu)$. In view of the formula $A(Q_1, Q_2, \sigma : -\bar{\nu})^* = A(Q_2, Q_1, \sigma, \nu)$ one sees that the intertwining operator $A(Q_2, Q_1, \sigma, \nu)$ has a continuous linear extension to a continuous linear intertwining operator $C^{-\infty}(G/Q_1 : \sigma : \nu) \rightarrow C^{-\infty}(G/Q_2 : \sigma : \nu)$ for non-singular values of ν . In [4] this extended operator is shown to depend meromorphically on $\nu \in \mathfrak{a}_{1\mathbb{C}}^*$ in the following way, in terms of the compact picture.

Lemma 2.3 *For every $R \in \mathbb{R}$ there exists a polynomial function $q \in P(\mathfrak{a}_{1\mathbb{C}}^*)$, with zero set contained in $\mathcal{S}(Q_2, Q_1, \sigma)$, and a constant $r \in \mathbb{N}$ such that for every positive integer s the assignment*

$$\nu \mapsto p(\nu)A(Q_2, Q_1, \sigma, \nu)$$

defines a holomorphic function on $\mathfrak{a}^(Q_2|Q_1, R)$ with values in the Banach space $B(C^{-s}, C^{-s-r})$ of bounded linear maps $C^{-s}(K/P_1 : \sigma_{K_{P_1}}) \rightarrow C^{-s-r}(K/P_2 : \sigma_{K_{P_2}})$.*

By continuity and density it now readily follows that (2.6) is valid for all $f \in C^{-\infty}(G/Q_1 : \sigma : \nu)$, provided $\nu \in \mathfrak{a}_{1\mathbb{C}}^* \setminus \mathcal{S}(Q_2, Q_1, \sigma)$.

Corollary 2.4 *Let Ω be a bounded open subset of $\mathfrak{a}_{1\mathbb{C}}^*$. Then there exists an $r \in \mathbb{N}$ such that for every $s \in \mathbb{N}$ the assignment $\nu \mapsto A(Q_2, Q_1, \sigma, \nu)$ defines a meromorphic function on Ω with values in the Banach space $B(C^{-s}, C^{-s-r})$ of bounded linear maps $C^{-s}(K/P_1 : \sigma_{K_{P_1}}) \rightarrow C^{-s-r}(K/P_2 : \sigma_{K_{P_2}})$.*

Definition 2.5 The smallest $r \in \mathbb{N}$ for which the above is valid will be called the order (of smoothness loss) of the family $A(Q_2, Q_1, \sigma, \cdot)$ over Ω .

Lemma 2.6 Let $\Omega \subset \mathfrak{a}_{Q_2}^*$ be open and suppose that $f : \Omega \rightarrow C^{-\infty}(K/K_Q : \sigma_Q)$ is a holomorphic function such that

$$\pi_{Q, \sigma, \nu}(n)f_\nu = \chi(n)f_\nu, \quad (\nu \in \Omega, n \in N_0). \quad (2.7)$$

Then $\nu \mapsto \text{ev}_Q(f_\nu)$ is a holomorphic function $\Omega \rightarrow H_{\sigma, \chi_Q}^{-\infty}$.

Proof. Fix $\nu_0 \in \Omega$. Put $I^{\pm\infty} = C^{\pm\infty}(K/K_Q : \sigma_Q)$. Furthermore, put $V := H_{\sigma, \chi_Q}^{-\infty}$. Then $j(\nu) := j(Q, \sigma, \nu)$ is a linear map $V \rightarrow I^{-\infty}$ which depends holomorphically on ν . From $\text{ev}_Q \circ j(\nu) = \text{id}_V$ it follows that $j(\nu)$ is injective for every $\nu \in \Omega$.

Let $E := j(\nu_0)(V)$ and let $E^\perp$ denote the annihilator of $E \subset I^{-\infty}$ in I^∞ . We fix a linear subspace $W_0 \subset \overline{I^\infty}$ which is complementary to $E^\perp$. Then the restriction map $r : I^{-\infty} \rightarrow W'_0, \xi \mapsto \xi|_{W_0}$ is continuous linear and restricts to a linear isomorphism from E onto W'_0 .

Since r is continuous linear, the function $J : \nu \mapsto r \circ j(\nu)$ is holomorphic with values in $\text{Hom}(V, W'_0)$. As $J(\nu_0)$ is bijective linear $V \rightarrow W'_0$, the same is true for ν in a sufficiently small open neighborhood $\Omega_0 \ni \nu_0$ in Ω . Furthermore, the function $\nu \mapsto J(\nu)^{-1}$ is holomorphic on Ω_0 with values in $\text{Hom}(W'_0, V)$.

Let f satisfy the hypotheses, and put $a(\nu) = \text{ev}_Q f_\nu$. Then $f_\nu = j(\nu)(a(\nu))$ for $\nu \in \Omega$. Furthermore, since $\nu \mapsto f_\nu$ is holomorphic as a map with values in $I^{-\infty}$ it follows that $\nu \mapsto r(f_\nu) = r \circ j(\nu)[a(\nu)]$ is holomorphic in ν with values in W'_0 . It follows that $\nu \mapsto a(\nu) = J(\nu)^{-1}[r(f_\nu)]$ is holomorphic on Ω_0 with values in V . Hence, $\text{ev}_Q \circ f$ is a holomorphic function $\Omega_0 \rightarrow V$. $\square$

Let $Q_1, Q_2 \in \mathcal{P}$ have equal split components: $\mathfrak{a}_{Q_1} = \mathfrak{a}_{Q_2}$. In analogy with the theory of symmetric spaces, we define, for a regular point $\nu \in \mathfrak{a}_{Q_1\mathbb{C}}^*$ of $A(Q_2, Q_1, \sigma, \cdot)$, the linear map $B(Q_2, Q_1, \sigma, \nu) : H_{\sigma, \chi_{Q_1}}^{-\infty} \rightarrow H_{\sigma, \chi_{Q_2}}^{-\infty}$ by

$$B(Q_2, Q_1, \sigma, \nu)\eta := \text{ev}_{Q_2} A(Q_2, Q_1, \sigma, \nu)j(Q_1, \sigma, \nu)\eta, \quad (\eta \in H_{\sigma, \chi_{Q_1}}^{-\infty}). \quad (2.8)$$

Here $\text{ev}_{Q_2} = \text{ev}_{v_2}$, with $v_2 = v_{Q_2}$, so that $v_2 Q_2 v_2^{-1}$ belongs to $\tilde{\mathcal{P}}_{\text{st}}$.

Lemma 2.7 Let $Q_1, Q_2 \in \mathcal{P}$ have the same split component, and let $\sigma \in \widehat{M}_{Q_1, \text{ds}}$.

(a) The function $B(Q_2, Q_1, \sigma, \cdot) : \mathfrak{a}_{1\mathbb{C}}^* \rightarrow \text{Hom}(H_{\sigma, \chi_{Q_1}}^{-\infty}, H_{\sigma, \chi_{Q_2}}^{-\infty})$ is meromorphic.

(b) For every $\eta \in H_{\sigma, \chi_{Q_1}}^{-\infty}$,

$$A(Q_2, Q_1, \sigma, \nu)j(Q_1, \sigma, \nu)\eta = j(Q_2, \sigma, \nu)B(Q_2, Q_1, \sigma, \nu)\eta \quad (2.9)$$

as an identity of meromorphic $C^{-\infty}(K/K_{Q_2} : \sigma_{Q_2})$ -valued functions of $\nu \in \mathfrak{a}_{Q_1\mathbb{C}}^*$.

Proof. For (a), assume that $\nu_0 \in \mathfrak{a}_{Q_1\mathbb{C}}^*$ and let Ω be a sufficiently small bounded neighborhood of ν_0 in $\mathfrak{a}_{Q_1\mathbb{C}}^*$. Then there exists a holomorphic function $\varphi : \mathfrak{a}_{Q_1\mathbb{C}}^* \rightarrow \mathbb{C}$ such that the function

$$\tilde{A} : \nu \mapsto \varphi(\nu)A(Q_2, Q_1, \sigma, \nu)$$

is holomorphic on Ω with values in $\text{End}(C^{-\infty}(K/K_{Q_1} : \sigma_{Q_1}))$ in the sense that there exists an $r > 0$ such that for every $k > 0$ the function $\tilde{A}$ defines a holomorphic function on Ω with values in the Banach space $B(C^{-k}, C^{-k-r})$ of bounded linear operators from $C^{-k}(K/K_{Q_1} : \sigma_{Q_1}) \rightarrow C^{-k-r}(K/K_{Q_2} : \sigma_{Q_2})$. Furthermore, by holomorphy of $j(Q_1, \sigma, \cdot)$ and boundedness of Ω , there exists $s > 0$ such that $j(Q_1, \sigma, \cdot)$ defines a holomorphic function $\Omega \rightarrow B(H_{\sigma, \chi_{Q_1}}^{-\infty}, C^{-s})$. Hence, $f : \nu \mapsto A(\nu)j(Q_1, \sigma, \nu)\eta$ defines a holomorphic function on Ω with values in $C^{-s-r}(K/K_{Q_2} : \sigma_{Q_2})$. By the transformation property under N_0 of $j(Q_1, \sigma, \nu)\eta$ and the equivariance of $A(Q_2, Q_1, \sigma, \nu)$ it follows that f_ν satisfies the transformation property (2.7) with $Q = Q_2$. By application of Lemma 2.4 it now follows that $\text{ev}_2 \circ A(\nu)(Q_1, \sigma, \nu)$ is holomorphic in $\nu \in \mathfrak{a}_{Q_1\mathbb{C}}^*$ with values in $H_{\sigma, \chi_{Q_2}}^{-\infty}$. Hence $\nu \mapsto \varphi(\nu)B(Q_2, Q_1, \sigma, \nu)\eta$ is holomorphic, and assertion (a) follows.

The validity of (b) is checked by applying the evaluation ev_{Q_2} to both sides of the equation (2.9) and using that ev_{Q_2} is a bijection from $C^{-\infty}(G/Q_2 : \sigma : \nu)_\chi$ onto $H_{\sigma, \chi_{Q_2}}^{-\infty}$, for every $\nu \in \mathfrak{a}_{Q_2\mathbb{C}}^*$. $\square$

Corollary 2.8 *Let $w \in N_K(\mathfrak{a})$ and let $Q_1, Q_2 \in \mathcal{P}$ have common split component. Let $\sigma \in \widehat{M}_{Q_1\text{ds}}$. Then*

$$\mathcal{R}_{w, Q_2} \circ B(Q_2, Q_1, \sigma, \nu) = B(wQ_2w^{-1}, wQ_1w^{-1}, w\sigma, w\nu) \circ \mathcal{R}_{w, Q_1}$$

as meromorphic functions of $\nu \in \mathfrak{a}_{1\mathbb{C}}^*$ with values in $\text{Hom}(H_{\sigma, \chi_{Q_1}}^{-\infty}, H_{\sigma, \chi_{Q_2}}^{-\infty})$.

Proof. From (2.6) for generalized functions it follows, for generic $\nu \in \mathfrak{a}_{Q_1\mathbb{C}}^*$ that

$$R_w A(Q_2, Q_1, \sigma, \nu) j(Q_1, \sigma, \nu) = A(wQ_2w^{-1}, wQ_1w^{-1}, w\sigma, w\nu) R_w j(Q_1, \sigma, \nu).$$

Applying $\text{ev}_{wQ_2w^{-1}}$ to both sides of the equation and using Lemma 2.1 and Corollary 2.2 we find the asserted equality. $\square$

Corollary 2.9 *Let $Q_1, Q_2 \in \mathcal{P}$ have the same split component. The following statements are equivalent, for any meromorphic function $\eta : \mathfrak{a}_{Q_1\mathbb{C}}^* \rightarrow \mathbb{C}$ and any $w \in N_K(\mathfrak{a})$.*

- (a) $B(Q_2, Q_1, \sigma, -\bar{\nu})^* B(Q_2, Q_1, \sigma, \nu) = \eta(\nu)I$ for generic $\nu \in \mathfrak{a}_{Q_1\mathbb{C}}^*$,
- (b) $B(wQ_2w^{-1}, wQ_1w^{-1}, w\sigma, -w\bar{\nu})^* B(wQ_2w^{-1}, wQ_1w^{-1}, w\sigma, w\nu) = \eta(\nu)I$ for generic $\nu \in \mathfrak{a}_{Q_1\mathbb{C}}^*$.

Proof. Let $\nu \in \mathfrak{a}_{Q_1\mathbb{C}}^*$ be a regular value for each of the finitely many functions involved. It follows from Lemma 2.1 that $\mathcal{R}_{w, Q} : H_{\sigma, \chi_Q}^{-\infty} \rightarrow H_{w\sigma, \chi_{wQw^{-1}}}^{-\infty}$ is unitary. The result now follows by a simple argument, using Corollary 2.8. $\square$

3 The B-matrices, reduction arguments

Let $P, Q \in \mathcal{P}$ have the same split component. Then $M_Q = M_P$. If $\sigma \in \widehat{M}_{P, \text{ds}}$ then for generic $\nu \in \mathfrak{a}_{P_{\mathbb{C}}}^*$ the composition $A(P, Q, \sigma, \nu)A(Q, P, \sigma, \nu)$ of standard intertwining operators is a self intertwining operator of $\text{Ind}_P^G(\sigma \otimes \nu \otimes 1)$. Since the latter representation is irreducible for generic ν , it follows that

$$A(P, Q, \sigma, \nu)A(Q, P, \sigma, \nu) = \eta(Q, P, \sigma, \nu)\text{id}, \quad (\nu \in \mathfrak{a}_{P_{\mathbb{C}}}^*), \quad (3.1)$$

for a unique meromorphic function $\eta(Q, P, \sigma, \nu)$. It is easy to see that $\eta(Q, P, \sigma, \nu) = \eta(P, Q, \sigma, \nu)$. Furthermore, suppressing id ,

$$A(Q, P, \sigma, -\bar{\nu})^* A(Q, P, \sigma, \nu) = \eta(Q, P, \sigma, \nu), \quad (\nu \in \mathfrak{a}_{P_{\mathbb{C}}}^*).$$

For this and other properties of η , we refer to [9]. In the course of this and the next sections we will prove the following manifestation of the Maass – Selberg relations.

Theorem 3.1 *Let $P, Q \in \mathcal{P}$ and suppose that $\mathfrak{a}_P = \mathfrak{a}_Q$. If $\sigma \in \widehat{M}_{P, \text{ds}}$, we have the following identity of meromorphic functions $\mathfrak{a}_{P_{\mathbb{C}}}^* \rightarrow \text{End}(H_{\sigma, \chi_P}^{-\infty})$:*

$$B(Q, P, \sigma, -\bar{\nu})^* B(Q, P, \sigma, \nu) = \eta(Q, P, \sigma, \nu), \quad (\nu \in \mathfrak{a}_{P_{\mathbb{C}}}^*). \quad (3.2)$$

This result will be proven in the course of the next sections, through reduction to a basic setting where G has compact kernel and P is a maximal standard parabolic subgroup.

Remark 3.2 It immediately follows from (3.1) that $B(P, Q, \sigma, \nu)B(Q, P, \sigma, \nu) = \eta(Q, P, \sigma, \nu)$. Therefore, the identity (3.2) is valid if and only if

$$B(Q, P, \sigma, -\bar{\nu})^* = B(P, Q, \sigma, \nu), \quad (\nu \in \mathfrak{a}_{P_{\mathbb{C}}}^*).$$

Remark 3.3 Note that for a non-cuspidal parabolic subgroup $P \in \mathcal{P}$ the group M_P has no discrete series, so that $\widehat{M}_{P, \text{ds}} = \emptyset$. This means that for trivial reasons the assertion of Theorem 3.1 is automatically fulfilled for $Q \in \mathcal{P}$ with $\mathfrak{a}_Q = \mathfrak{a}_P$.

We will write $G = M_G A_G$ for the Langlands decomposition of G , viewed as a parabolic subgroup of G . Here $A_G := \exp \mathfrak{a}_G$, where $\mathfrak{a}_G$ is the intersection of the center of $\mathfrak{g}$ with $\mathfrak{p}$. The group G has compact center if and only if $A_G = \{e\}$. The group M_G is also denoted ${}^\circ G$ and equals the intersection of the kernels $\ker \xi$ where ξ runs over the collection $X(G)$ of multiplicative characters $G \rightarrow \mathbb{R}_+$. Write ${}^\circ A = A \cap {}^\circ G$. Then ${}^\circ G$ is of the Harish-Chandra class, with Iwasawa decomposition

$${}^\circ G = K {}^\circ A N_0.$$

Let ${}^\circ \mathcal{P}$ denote the finite set of parabolic subgroups of ${}^\circ G$ containing ${}^\circ A$. Then the map $Q \mapsto {}^\circ Q := Q \cap {}^\circ G$ is a bijection $\mathcal{P} \rightarrow {}^\circ \mathcal{P}$, with inverse ${}^\circ Q \mapsto {}^\circ Q A_G$. Note that every

$v \in N_K(\mathfrak{a})$ leaves ${}^\circ A$ invariant. We define ${}^\circ v : {}^\circ \mathcal{P} \rightarrow N_K({}^\circ A)$ by ${}^\circ v \circ Q = {}^\circ v_Q|_{{}^\circ A}$, for $Q \in \mathcal{P}$.

We note that for $Q \in \mathcal{P}$, ${}^\circ Q$ has the Langlands decomposition ${}^\circ Q = M_Q {}^\circ A_Q N_Q$, where ${}^\circ A_Q = A_Q \cap {}^\circ G$. Thus, if $P, Q \in \mathcal{P}$ then P and Q have the same split components as elements of $\mathcal{P}$ if and only if ${}^\circ P$ and ${}^\circ Q$ have the same split components as elements of ${}^\circ \mathcal{P}$.

Furthermore, if $Q \in \mathcal{P}$, $M_Q \subset {}^\circ G$. Therefore, if $\sigma \in \widehat{M}_{Q, \text{ds}}$, we may identify the space $H_{\sigma, \chi_Q}^{-\infty}$ with the similar space for the pair ${}^\circ G, {}^\circ Q$. The same is true for the inner products on these spaces.

Accordingly, we may define the B matrices for ${}^\circ G$, denoted ${}^\circ B({}^\circ Q, {}^\circ P, \sigma, \mu)$ in the obvious fashion as endomorphisms of $H_{\sigma, \chi_P}^{-\infty}$, for $P, Q \in \mathcal{P}$ with $\mathfrak{a}_P = \mathfrak{a}_Q$ and for $\sigma \in \widehat{M}_{P, \text{ds}}$ and $\mu \in {}^\circ \mathfrak{a}_{P\mathbb{C}}^*$. Then $\mu \mapsto {}^\circ B({}^\circ Q, {}^\circ P, \sigma, \mu)$ is a meromorphic function on ${}^\circ \mathfrak{a}_{P\mathbb{C}}^*$ with values in $\text{End}(H_{\sigma, \chi_Q}^{-\infty})$. We agree to write ${}^\circ v := v|_{{}^\circ \mathfrak{a}_{P\mathbb{C}}}$ for $v \in \mathfrak{a}_{P\mathbb{C}}^*$.

Lemma 3.4 *Let notation be as in the above text. Then for generic $v \in \mathfrak{a}_{P\mathbb{C}}^*$,*

- (a) $\eta(Q, P, \sigma, v) = {}^\circ \eta({}^\circ Q, {}^\circ P, \sigma, {}^\circ v)$;
- (b) $B(Q, P, \sigma, v) = {}^\circ B({}^\circ Q, {}^\circ P, \sigma, {}^\circ v)$.

Proof. The proof is straightforward, but a bit tedious. Details are left to the reader. $\square$

Corollary 3.5 *The assertions of Theorem 3.1 are valid for G if and only if they are valid for ${}^\circ G$.*

Let $P \in \mathcal{P}$. A root of P is defined to be a non-trivial linear functional $\alpha \in \mathfrak{a}_P^*$ such that the space $\mathfrak{g}_\alpha := \cap_{H \in \mathfrak{a}_P} \ker(\text{ad}H - \alpha(H))$ is contained in $\mathfrak{n}_P$. Equivalently this means that α is the restriction of a root $\beta \in \Sigma(\mathfrak{a})$ with $\mathfrak{g}_\beta \subset \mathfrak{n}_P$. The set of P -roots is denoted by $\Sigma(P)$. We note that $\mathfrak{a}_P^+$ is the set of points $H \in \mathfrak{a}_P$ such that $\alpha(H) > 0$ for all $\alpha \in \Sigma(P)$.

A P -root $\alpha \in \Sigma(P)$ is called reduced if the multiples of α in $\Sigma(P)$ are all of the form $c\alpha$, with $c \geq 1$.

For two parabolic subgroups $P, Q \in \mathcal{P}$ with the same split component, a P -root α is said to separate $\mathfrak{a}_P^+$ and $\mathfrak{a}_Q^+$ if the sign of α on $\mathfrak{a}_Q$ is negative, or equivalently, $\alpha \in \Sigma(\bar{Q})$.

The distance $d(P, Q)$ is defined to be the number of reduced P -roots that separate $\mathfrak{a}_P^+$ and $\mathfrak{a}_Q^+$. P and Q are said to be adjacent if $d(P, Q) = 1$. Equivalently this means that all roots in $\Sigma(P) \cap \Sigma(\bar{Q})$ are proportional.

If P, Q have the same split component and are different, then there is a parabolic subgroup $R \in \mathcal{P}$ with split component $\mathfrak{a}_R = \mathfrak{a}_P$ such that P and R are adjacent, and $d(R, Q) < d(P, R)$. It is well known, see e.g. [9, Cor. 7.7], that for $\sigma \in \widehat{M}_{P, \text{ds}}$ one has in this case that

$$A(Q, P, \sigma, v) = A(Q, R, \sigma, v)A(R, P, \sigma, v), \quad (3.3)$$

for generic $v \in \mathfrak{a}_{P\mathbb{C}}^*$.

Lemma 3.6 *In the above setting, $B(Q, P, \sigma, \nu) = B(Q, R, \sigma, \nu)B(R, P, \sigma, \nu)$.*

Proof. This follows from 3.3 by application of Lemma 2.7 and (2.8). $\square$

Lemma 3.7 *If the identity of Thm. 3.1 holds for all adjacent $Q, P \in \mathcal{P}$ then it holds for all $Q, P \in \mathcal{P}$ with $\mathfrak{a}_P = \mathfrak{a}_Q$.*

Proof. We assume that G is fixed and that the identity of Thm. 3.1 holds for all $P, Q \in \mathcal{P}$ with $d(P, Q) = 1$. Arguing by induction on $d(P, Q)$ we will show that the identity holds for all P, Q with the same split component.

Let $k \geq 1$ and suppose that the identity holds for $P, Q \in \mathcal{P}$ with $d(P, Q) \leq k$. Assume now that $d(P, Q) = k + 1$. Then there exists R as in the text leading to (3.3). By induction we know that

$$B(Q, R, \sigma, -\bar{\nu})^* B(Q, R, \sigma, \nu) = \eta(Q, R, \sigma, \nu).$$

Therefore,

$$\begin{aligned} B(Q, P, \sigma, -\bar{\nu})^* B(Q, P, \sigma, \nu) &= \\ &= B(R, P, \sigma, -\bar{\nu})^* B(Q, R, \sigma, -\bar{\nu})^* \circ B(Q, R, \sigma, \nu) B(R, P, \sigma, \nu) \\ &= B(R, P, \sigma, -\bar{\nu})^* \eta(Q, R, \sigma, \nu) B(R, P, \sigma, \nu) \\ &= \eta(Q, R, \sigma, \nu) \cdot B(R, P, \sigma, -\bar{\nu})^* B(R, P, \sigma, \nu). \end{aligned}$$

Since $d(P, R) = 1$, the expression on the last entry of the array equals the product $\eta(Q, R, \sigma, \nu)\eta(R, P, \sigma, \nu)$. In turn, as a consequence of (3.3), this product equals $\eta(Q, P, \sigma, \nu)$. $\square$

Lemma 3.8 *Suppose the identity of Theorem 3.1 holds for G, P, Q, σ and all $\nu \in \mathfrak{a}_{P_C}^*$. If $v \in N_K(\mathfrak{a})$, then the identity of the theorem also holds with $G, vPv^{-1}, vQv^{-1}, v\sigma$ in place of G, P, Q, σ respectively, and all $\nu \in v\mathfrak{a}_{P_C}^*$.*

Proof. Let the hypothesis be fulfilled, then it suffices to prove the following identity for all $\nu \in \mathfrak{a}_{P_C}^*$,

$$B(vQv^{-1}, vPv^{-1}, v\sigma, -v\bar{\nu})^* B(vQv^{-1}, vPv^{-1}, v\sigma, v\nu) = \eta(vQv^{-1}, vPv^{-1}, v\sigma, v\nu). \quad (3.4)$$

Using Cor. 2.8 and the unitarity of $R_{v,Q}$ and $R_{v,P}$ we may rewrite the expression on the left-hand side of 3.4 as

$$\begin{aligned} R_{v,P}^* B(Q, P, \sigma, -\bar{\nu})^* R_{v,Q}^* R_{v,Q} B(Q, P, \sigma, \nu) R_{v,P} &= \\ &= R_{v,P}^* B(Q, P, \sigma, -\bar{\nu})^* B(Q, P, \sigma, \nu) R_{v,P} \\ &= \eta(Q, P, \sigma, \nu). \end{aligned}$$

The result now follows from the observation that $\eta(vQv^{-1}, vPv^{-1}, v\sigma, v\nu) = \eta(Q, P, \sigma, \nu)$, in view of (2.6) and the definition of $\eta(Q, P, \sigma, \nu)$. $\square$

4 Reduction to maximal parabolic subgroups

In this section we will discuss a method that will allow reduction of the proof of the Maass-Selberg relations to those for maximal parabolic subgroups of lower dimensional groups of the Harish-Chandra class.

To prepare for this, we will first discuss well known aspects of the structure of parabolic subgroups. We briefly write $\Sigma(\mathfrak{a}) := \Sigma(\mathfrak{g}, \mathfrak{a})$. If P is a parabolic subgroup of G , we denote its Langlands decomposition by $P = M_P A_P N_P$. The collection of $\mathfrak{a}_P$ -roots in $\mathfrak{n}_P$ is defined as in the text below Cor. 3.5 and denoted by $\Sigma(P)$.

The positive chamber $\mathfrak{a}_P^+$ consists of the points $X \in \mathfrak{a}_P$ such that $\alpha(X) > 0$ for all $\alpha \in \Sigma(P)$.

Given a point $X \in \mathfrak{a}$ we put

$$\Sigma^+(\mathfrak{a}, X) := \{\alpha \in \Sigma(\mathfrak{a}) \mid \alpha(X) > 0\}.$$

We note that for a root $\alpha \in \Sigma(\mathfrak{a})$, one has $\alpha(X) < 0 \iff -\alpha \in \Sigma^+(\mathfrak{a}, X)$ and $\alpha(X) = 0$ iff $\pm\alpha \notin \Sigma^+(\mathfrak{a}, X)$.

We define the equivalence relation $\sim$ on $\mathfrak{a}$ by

$$X \sim Y \iff \Sigma^+(\mathfrak{a}, X) = \Sigma^+(\mathfrak{a}, Y), \quad (X, Y \in \mathfrak{a}). \quad (4.1)$$

For $X \in \mathfrak{a}$ we define the subspace $\mathfrak{p}_X = \mathfrak{m}_X \oplus \mathfrak{a}_X \oplus \mathfrak{n}_X$ of $\mathfrak{g}$ by

$$\mathfrak{p}_X = \mathfrak{m} + \mathfrak{a} + \bigoplus_{\alpha \in \Sigma^+(\mathfrak{a}, X)} \mathfrak{g}_\alpha. \quad (4.2)$$

Then, clearly, $\mathfrak{p}_X = \mathfrak{p}_Y \iff X \sim Y$.

Suppose now that Σ^+ is a positive system for $\Sigma(\mathfrak{a})$ and $\mathfrak{a}^+$ the associated open positive chamber in $\mathfrak{a}$. If $X \in \text{cl}(\mathfrak{a}^+)$ then one readily verifies that $\mathfrak{p}_X = \mathfrak{m}_X \oplus \mathfrak{a}_X \oplus \mathfrak{n}_X$ is the standard parabolic subalgebra $\mathfrak{p}_F$ with F the collection of simple roots in Σ^+ vanishing on X . The indicated Langlands decomposition is determined by

$$\mathfrak{m}_X = {}^\circ \mathfrak{z}_{\mathfrak{g}}(X), \quad \mathfrak{a}_X^+ = [X], \quad \mathfrak{n}_X = \bigoplus_{\alpha \in \Sigma^+(\mathfrak{a}, X)} \mathfrak{g}_\alpha,$$

where $[X]$ denotes the class of X for the equivalence relation $\sim$. Write $\text{spec}(X)$ for the spectrum of $\text{ad}(X) \in \text{End}(\mathfrak{g})$, and $\text{spec}(X)_+$ for its positive part. Then it is readily checked that

$$\mathfrak{p}_X = \ker \text{ad}(X) + \bigoplus_{\lambda \in \text{spec}(X)_+} \ker(\text{ad}(X) - \lambda I).$$

By using $\text{Ad}(K)$ conjugacy, we see that this definition gives a parabolic subalgebra of $\mathfrak{g}$ for any $X \in \mathfrak{g}$ with $\theta(X) = -X$. Furthermore, its Langlands components are given by

$$\mathfrak{m}_X + \mathfrak{a}_X = {}^\circ \mathfrak{z}_{\mathfrak{g}}(X), \quad \mathfrak{n}_X = \bigoplus_{\lambda > 0} \ker(\text{ad}(X) - \lambda I).$$

As usual, the parabolic subgroup with algebra $\mathfrak{p}_X$ is defined by $P_X := N_G(\mathfrak{p}_X)$. Its Langlands components are given by

$$M_X = {}^\circ Z_G(X), \quad A_X = \exp(\mathfrak{a}_X), \quad N_X = \exp(\mathfrak{n}_X).$$

The following lemma will be used repeatedly in the sequel.

Lemma 4.1 *If $\mathfrak{b}$ is an abelian subspace of $\mathfrak{s}$ and $X \in \mathfrak{b}$ then for each $\lambda \in \text{spec}(X)$ the associated eigenspace $\ker(\text{ad}(X) - \lambda)$ in $\mathfrak{g}$ equals the direct sum of the $\text{ad}(\mathfrak{b})$ -weight spaces $\mathfrak{g}_\beta$ for $\beta \in \Sigma(\mathfrak{b}) \cup \{0\}$ with $\beta(X) = \lambda$.*

Proof. For each $X \in \mathfrak{s}$, the endomorphism $\text{ad}X$ of $\mathfrak{g}$ is symmetric with respect to the inner product $\langle \cdot, \cdot \rangle$ hence semisimple with real eigenvalues. The proof is now straightforward. $\square$

Let $P \in \mathcal{P}$ have split component $\mathfrak{a}_P$. The complement of the union of the finitely many hyperplanes $\ker \beta$ with $\beta \in \Sigma(\mathfrak{g}, \mathfrak{a}_P)$ consists of finitely many convex polyhedral components, called the chambers of $\mathfrak{a}_P$. These chambers are readily seen to be equivalence classes for the equivalence relation $\sim$ given by (4.1). For each chamber $[Y]$ the associated parabolic subgroup $P_{[Y]} := P_Y$ has split component $\mathfrak{a}_P$ and positive chamber $\mathfrak{a}_{P_{[Y]}}^+ = [Y]$. Conversely, for each parabolic subgroup Q with split component $\mathfrak{a}_Q = \mathfrak{a}_P$ the positive chamber $\mathfrak{a}_Q^+$ is a chamber in $\mathfrak{a}_P$.

Two parabolic subgroups $P, Q \in \mathcal{P}$ with the same split components are said to be adjacent if their positive chambers are separated by precisely one hyperplane from the collection of hyperplanes $\ker \alpha \subset \mathfrak{a}_P$ for $\alpha \in \Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$. A root $\beta \in \Sigma(\mathfrak{g}, \mathfrak{a}_P)$ is said to be reduced iff all its real multiples in $\Sigma(\mathfrak{g}, \mathfrak{a}_P)$ are of the form $c\beta$ with $|c| \geq 1$. Thus, P and Q are adjacent if there is a unique reduced $\mathfrak{a}_P$ -root $\alpha \in \Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$ such that $\alpha < 0$ on $\mathfrak{a}_Q^+$. Note that $-\alpha$ is the unique reduced root in $\Sigma(\mathfrak{n}_Q, \mathfrak{a}_Q)$ which is negative on $\mathfrak{a}_P$. In this situation it is easy to see that

$$\mathfrak{n}_P \cap \bar{\mathfrak{n}}_Q = \mathfrak{n}_\alpha := \bigoplus_{c \geq 1} \mathfrak{g}_{c\alpha}, \quad (4.3)$$

where the summation is over the real $c \geq 1$ such that $c\alpha$ is a root of $\mathfrak{a}_P$. We note that the reduced root α has the property that $\ker \alpha \cap \text{cl}(\mathfrak{a}_P^+)$ has non-empty interior in $\ker \alpha$. Conversely if such a reduced root α is given, then there is a unique parabolic subgroup Q with split component $\mathfrak{a}_P$ that is adjacent to P . It is determined by the requirement that (4.3).

Let $P \in \mathcal{P}$. We assume that P is *not maximal* and fix a reduced root $\alpha \in \Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$ such that $\ker \alpha \cap \text{cl}(\mathfrak{a}_P^+)$ has non-empty interior as a subset of the hyperplane $\ker \alpha$ in $\mathfrak{a}_P$. A point X in this interior will be called (P, α) -regular if it has the property that for all $\beta \in \Sigma(\mathfrak{g}, \mathfrak{a}_P)$,

$$\beta(X) = 0 \Rightarrow \beta|_{\ker \alpha} = 0.$$

As we explained, the pair (P, α) uniquely determines an adjacent Q . A point X in the interior of $\ker \alpha \cap \text{cl}(\mathfrak{a}_P^+) = \ker \alpha \cap \text{cl}(\mathfrak{a}_Q^+)$ is (P, α) -regular if and only if it is $(Q, -\alpha)$ -regular. We may select a (P, α) -regular point X in the interior of $\ker \alpha \cap \text{cl}(\mathfrak{a}_P^+)$. We fix such an X and define the parabolic subalgebra $\mathfrak{p}_X$ as in (4.2). The corresponding parabolic subgroup $N_G(\mathfrak{p}_X)$ has the Levi decomposition $M_{1X}N_X$. We put $G^{(\alpha)} := M_{1X}$, ignoring the precise dependence on the choice of the (P, α) -generic element X .

Lemma 4.2 *The Lie algebra of $G^{(\alpha)}$ is given by*

$$\mathfrak{g}^{(\alpha)} = \bar{\mathfrak{n}}_\alpha \oplus \mathfrak{m}_{1P} \oplus \mathfrak{n}_\alpha. \quad (4.4)$$

Proof. By Lemma 4.1, $\mathfrak{g}^{(\alpha)} = \mathfrak{m}_{1X} = \oplus_{\beta} \mathfrak{g}_{\beta}$ where the sum is taken over the $\beta \in \Sigma(\mathfrak{a}_P) \cup \{0\}$ for which either (a): $\beta|_{\mathfrak{a}_P} = 0$ or (b): $\beta|_{\mathfrak{a}_P} \neq 0$ and $\beta(X) = 0$. Condition (a) is equivalent to $\mathfrak{g}_{\beta} \subset \mathfrak{m}_{1P}$. Condition (b) is equivalent to $\beta|_{\ker \alpha} = 0$ which in turn is equivalent to $\beta = c\alpha$ for a constant $c \in \mathbb{R}$. The validity of (4.4) follows. $\square$

Lemma 4.3 $\mathfrak{n}_P = \mathfrak{n}_{\alpha} \oplus \mathfrak{n}_X$.

Proof. $\mathfrak{n}_P$ is the direct sum of the weight spaces $\mathfrak{g}_{\beta}$ for $\beta \in \Sigma(\mathfrak{a}_P) \cup \{0\}$ such that $\beta|_{\mathfrak{a}_P^+} > 0$. This collection of β splits into (a) those such that $\beta(X) > 0$ and (b) those such that $\beta(X) = 0$ and $\beta|_{\mathfrak{a}_P^+} > 0$. The terms satisfying (a) are contained in $\mathfrak{n}_X$, those satisfying (b) satisfy $\beta|_{\ker \alpha} = 0$ and $\beta \neq 0$ hence $\beta = c\alpha$ for $c > 0$. It follows that $\mathfrak{n}_P \subset \mathfrak{n}_{\alpha} \oplus \mathfrak{n}_X$. Conversely, $\mathfrak{n}_X$ is the direct sum of the spaces $\mathfrak{g}_{\beta}$ with $\beta(X) > 0$. The latter condition implies $\beta > 0$ on $\mathfrak{a}_P^+$ hence $\mathfrak{g}_{\beta} \in \mathfrak{n}_P$. We see that $\mathfrak{n}_X \subset \mathfrak{n}_P$. Finally, $\mathfrak{n}_{\alpha}$ is the sum of the root spaces $\mathfrak{g}_{c\alpha}$ with $c \geq 1$. Since $\alpha > 0$ on $\mathfrak{a}_P^+$, it follows that $\mathfrak{n}_{\alpha} \subset \mathfrak{n}_P$. This proves the required identity. $\square$

Lemma 4.4 $\mathfrak{p} \subset \mathfrak{p}_X$.

Proof. $\mathfrak{p}$ is the direct sum of the spaces $\mathfrak{g}_{\beta}$ where $\beta \in \Sigma(\mathfrak{a}) \cup \{0\}$ is such that $\beta \geq 0$ on $\mathfrak{a}_P^+$. For all such β one has $\beta(X) \geq 0$ so that $\mathfrak{g}_{\beta} \subset \mathfrak{p}_X$. $\square$

From $\mathfrak{p} \subset \mathfrak{p}_X$ it follows that

$$P^{(\alpha)} := P \cap M_{1X} = P \cap G^{(\alpha)}$$

is a parabolic subgroup of $G^{(\alpha)}$ with Langlands decomposition

$$P^{(\alpha)} = M_P A_P N_{\alpha},$$

The centralizer of $G^{(\alpha)}$ in $\mathfrak{a}_P$ equals $\ker \alpha$, which has codimension 1. From this we see that $P^{(\alpha)}$ is a maximal parabolic subgroup of $G^{(\alpha)}$.

Let now Q be the adjacent parabolic subgroup determined by the pair (P, α) . Then X is $(Q, -\alpha)$ generic, and it follows from an easy adaptation of the proof of Lemma 4.3 that

$$\mathfrak{n}_Q = \bar{\mathfrak{n}}_{\alpha} + \mathfrak{n}_X, \tag{4.5}$$

and that $Q^{(\alpha)} = Q \cap G^{(\alpha)}$ is a parabolic subgroup of $G^{(\alpha)}$, with the Langlands decomposition

$$Q^{(\alpha)} = M_P A_P \bar{N}_{\alpha} = \overline{P^{(\alpha)}}. \tag{4.6}$$

Lemma 4.5 *The group $G^{(\alpha)}$ is of the Harish-Chandra class, and $P^{(\alpha)}$ and $Q^{(\alpha)}$ are maximal parabolic subgroups of $G^{(\alpha)}$. They have the common split component A_P and are adjacent and opposite.*

Proof. Since the group $G^{(\alpha)} = M_{1X}$ is the centralizer of $\mathfrak{a}_X$ in G , it belongs to the Harish-Chandra class. From (4.6) it follows that both $P^{(\alpha)}$ and $Q^{(\alpha)}$ have split component A_P .

The maximality of the parabolic subgroups $P^{(\alpha)}$ and $Q^{(\alpha)}$ was established in the above. These parabolics are opposite. In view of their maximality it follows that they are adjacent as well. $\square$

The following results will turn out to be key for the argument reducing the proof of the Maass–Selberg relations for B to the case of maximal parabolic subgroups. It makes that certain data for G and for $G^{(\alpha)}$ are suitably compatible.

Lemma 4.6 *The group $G^{(\alpha)}$ normalizes $\mathfrak{n}_P \cap \mathfrak{n}_Q$.*

Proof. The intersection $\mathfrak{n}_P \cap \mathfrak{n}_Q$ is the sum of the root spaces $\mathfrak{g}_\beta$, ($\beta \in \Sigma(\mathfrak{a})$) with $\beta > 0$ on both $\mathfrak{a}_P^+$ and $\mathfrak{a}_Q^+$. From the choice of X on $\text{cl}\mathfrak{a}_P^+ \cap \text{cl}\mathfrak{a}_Q^+$ we see that the condition on β is equivalent to $\beta(X) > 0$. The latter condition in turn is equivalent to $\mathfrak{g}_\beta \subset \mathfrak{n}_X$. It follows that $\mathfrak{n}_P \cap \mathfrak{n}_Q = \mathfrak{n}_X$. As $G^{(\alpha)} = M_{1X}$, the result follows. $\square$

Lemma 4.7 *We have the following direct sums as linear spaces:*

$$(a) \quad \mathfrak{p} = \mathfrak{m}_{1P} \oplus \mathfrak{n}_\alpha \oplus \mathfrak{n}_X,$$

$$(b) \quad \mathfrak{q} = \mathfrak{m}_{1P} \oplus \bar{\mathfrak{n}}_\alpha \oplus \mathfrak{n}_X.$$

If P is standard, then

$$(c) \quad \mathfrak{n}_0 = \mathfrak{n}_0^{(\alpha)} \oplus \mathfrak{n}_X.$$

Proof. Since P and Q are adjacent, $\mathfrak{m}_{1P} = \mathfrak{m}_{1Q}$. Consequently, (a) and (b) follow from Lemma 4.3 and (4.5). If P is standard, so is P_X , in view of Lemma 4.4. It follows that $\mathfrak{n}_0 = (\mathfrak{n}_0 \cap \mathfrak{m}_{1X}) \oplus \mathfrak{n}_X = \mathfrak{n}_0^{(\alpha)} \oplus \mathfrak{n}_X$. $\square$

The algebra $\mathfrak{a}$ is maximal abelian in the -1 eigenspace of the Cartan involution $\theta|_{\mathfrak{g}^{(\alpha)}}$. It follows that the normalizer of A in $K^{(\alpha)}$ maps onto the Weyl group $W^{(\alpha)}$ of the root system $\Sigma^{(\alpha)} = \Sigma(\mathfrak{g}^{(\alpha)}, \mathfrak{a})$. It follows from Lemma 4.6 that $W^{(\alpha)}$ preserves $\mathfrak{n}_X = \mathfrak{n}_P \cap \mathfrak{n}_Q$.

Lemma 4.8 *Let $v \in N_{K^{(\alpha)}}(\mathfrak{a})$ be such that $N_0^{(\alpha)} v \bar{Q}^{(\alpha)}$ is open in $G^{(\alpha)}$. If P is standard, then $N_0 v \bar{Q}$ is open in G .*

Proof. The orbit $N_0^{(\alpha)} v \bar{Q}^{(\alpha)}$ is open in $G^{(\alpha)}$ iff $\mathfrak{n}_0^{(\alpha)} + \text{Ad}(v)\bar{\mathfrak{q}}^{(\alpha)} = \mathfrak{g}^{(\alpha)}$. By adding $\mathfrak{n}_X + \bar{\mathfrak{n}}_X$ to the left and the right of the latter expression, we find, using that $\text{Ad}(v)$ normalizes $\bar{\mathfrak{n}}_X$,

$$(\mathfrak{n}_0^{(\alpha)} + \mathfrak{n}_X) + \text{Ad}(v)(\bar{\mathfrak{q}}^{(\alpha)} + \bar{\mathfrak{n}}_X) = \mathfrak{g}. \quad (4.7)$$

From Lemma 4.7 (b), we see that $\mathfrak{q}^{(\alpha)} + \mathfrak{n}_X = \mathfrak{q}$. Using this and (4.7), we find, by taking Lemma 4.7 (c) into account that

$$\mathfrak{n}_0 + \text{Ad}(v)\bar{\mathfrak{q}} = \mathfrak{g}.$$

This implies that $N_0 v \bar{Q}$ is open in G . $\square$

From now on, we will assume that P is standard. Since $\theta X = -X$, the group $G^{(\alpha)} = M_{1X} = Z_G(X)$ is invariant under θ . The restriction $\theta^{(\alpha)} := \theta|_{G^{(\alpha)}}$ is a Cartan involution. Its group of fixed points is the maximal compact subgroup $K^{(\alpha)} = K \cap G^{(\alpha)}$ of $G^{(\alpha)}$.

Furthermore, $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{s}^{(\alpha)} = \mathfrak{s} \cap \mathfrak{g}^{(\alpha)}$. The algebra $\mathfrak{g}^{(\alpha)} = \mathfrak{m}_{1X}$ is the direct sum of the weight spaces $\mathfrak{g}_\beta$ for $\beta \in \Sigma(\mathfrak{a}) \cup \{0\}$ such that $\beta(X) = 0$. It follows that

$$\Sigma(\mathfrak{g}^{(\alpha)}, \mathfrak{a}) = \{\beta \in \Sigma(\mathfrak{a}) \mid \beta(X) = 0\}.$$

Since P is standard, $\text{cl}(\mathfrak{a}_P^+) \subset \text{cl}(\mathfrak{a}^+)$, hence also $X \in \text{cl}(\mathfrak{a}^+)$. Therefore,

$$\Sigma^+(\mathfrak{g}^{(\alpha)}, \mathfrak{a}) := \Sigma(\mathfrak{g}^{(\alpha)}, \mathfrak{a}) \cap \Sigma^+(\mathfrak{g}, \mathfrak{a})$$

is a positive system for $\Sigma(\mathfrak{g}^{(\alpha)}, \mathfrak{a})$. It is well-known that the associated set of simple roots is given by

$$\Delta^{(\alpha)} := \{\beta \in \Delta \mid \beta(X) = 0\}.$$

The standard minimal parabolic subgroup P_0 of G is contained in P_X . Hence, $P_0^{(\alpha)} = P_0 \cap G^{(\alpha)}$ is a minimal parabolic subgroup of $G^{(\alpha)} = M_{1X}$. The nilpotent radical of $P_0^{(\alpha)}$ equals $N_0^{(\alpha)} = N_0 \cap G^{(\alpha)}$. Accordingly, the Iwasawa decompositions

$$G = KAN_0 \quad \text{and} \quad G^{(\alpha)} = K^{(\alpha)}AN_0^{(\alpha)}$$

are compatible. The restriction $\chi^{(\alpha)} := \chi|_{N_0^{(\alpha)}}$ is a unitary character of $N_0^{(\alpha)}$.

Lemma 4.9 *The character $\chi^{(\alpha)}$ is regular with respect to $G^{(\alpha)}, A, N_0^{(\alpha)}$.*

Proof. The $\mathfrak{a}$ -roots in $\mathfrak{n}_0^{(\alpha)}$ form the positive system $\Sigma^+(\mathfrak{g}^{(\alpha)}, \mathfrak{a})$. Let β be a simple root for this positive system. Then $\beta \in \Delta$ and $\beta(X) = 0$. The simple root space $\mathfrak{g}_\beta^{(\alpha)}$ equals $\mathfrak{g}_\beta$. The derivative $d\chi^{(\alpha)}(e)$ is the restriction of $d\chi(e)$ to $\mathfrak{n}_0^{(\alpha)}$. Since χ is regular and β simple,

$$d\chi^{(\alpha)}(e)|_{\mathfrak{g}_\beta} = d\chi(e)|_{\mathfrak{g}_\beta} \neq 0.$$

It follows that $\chi^{(\alpha)}$ is regular. □

The groups $P^{(\alpha)} = M_P A_P N_\alpha$ and $Q^{(\alpha)} = \theta P^{(\alpha)}$ are opposite maximal parabolic subgroups of $G^{(\alpha)}$ with the same split component. They are adjacent and compatible with Q and P . Let σ be a representation of the discrete series of $M_P = M_Q$. We consider the characters

$$\chi_P = \chi|_{M_P \cap N_0} \quad \text{and} \quad \chi_{P^{(\alpha)}}^{(\alpha)} = \chi^{(\alpha)}|_{M_P \cap N_0^{(\alpha)}}.$$

Lemma 4.10

- (a) $N_0 \cap M_P = N_0^{(\alpha)} \cap M_P$;
- (b) $\chi_P = \chi_{P^{(\alpha)}}^{(\alpha)}$;
- (c) $H_{\sigma, \chi_P}^{-\infty} = H_{\sigma, \chi_{P^{(\alpha)}}^{(\alpha)}}^{-\infty}$;
- (d) *the Hermitian inner products on the spaces in (c) are equal.*

Proof. If $\beta \in \Sigma^+(\alpha)$ is such that $\mathfrak{g}_\beta \subset \mathfrak{m}_P$ then $\beta|_{\mathfrak{a}_P} = 0$ so that $\beta(X) = 0$ which in turn implies that $\mathfrak{g}_\beta \subset \mathfrak{n}_0^{(\alpha)}$. This implies (a). Since clearly

$$\chi|_{M_P \cap N_0^{(\alpha)}} = \chi^{(\alpha)}|_{M_P \cap N_0^{(\alpha)}}$$

it follow from (a) that (b). Assertion (c) is now immediate. For (d) we note that the inner product on $H_{\sigma, \chi_P}^{-\infty}$ is determined by the requirement that the matrix coefficient map

$$\mu_\sigma : H_\sigma^\infty \otimes \overline{H_{\sigma, \chi_P}^{-\infty}} \rightarrow L^2(M_P/M_P \cap N_0 : \chi_P)$$

is an isometry. In view of (a), (b) and (c), the matrix coefficient map μ_σ coincides with the matrix coefficient map

$$\mu_\sigma^{(\alpha)} : H_\sigma^\infty \otimes \overline{H_{\sigma, \chi_{P^{(\alpha)}}^{(\alpha)}}^{-\infty}} \rightarrow L^2(M_P/M_P \cap N_0^{(\alpha)} : \chi_P^{(\alpha)}).$$

Remark 4.11 For (d) it is essential that we agree to equip $M_P/M_P \cap N_0^{(\alpha)}$ with the same positive invariant measure as $M_P/M_P \cap N_0$.

We have now introduced all ingredients needed for the definition of $B^{(\alpha)}(Q^{(\alpha)}, P^{(\alpha)}, \sigma, \nu)$ for the group $G^{(\alpha)}$, the adjacent parabolic subgroups $P^{(\alpha)}, Q^{(\alpha)}$ and any $\sigma \in \widehat{M}_{P, \text{ds}}$, as an $\text{End}(H_{\sigma, \chi_{P^{(\alpha)}}^{(\alpha)}}^{-\infty})$ -valued meromorphic function of $\nu \in \mathfrak{a}_{P_{\mathbb{C}}}^*$. The following is a crucial reduction result.

Lemma 4.12 *Let $P, Q \in \mathcal{P}$ be as before, and assume P is standard. Then for all $\sigma \in \widehat{M}_{P, \text{ds}}$ we have*

$$B(\bar{Q}, \bar{P}, \sigma, \nu) = B^{(\alpha)}(P^{(\alpha)}, \bar{P}^{(\alpha)}, \sigma, \nu),$$

as $\text{End}(H_{\sigma, \chi_P}^{-\infty})$ -valued meromorphic functions of $\nu \in \mathfrak{a}_{P_{\mathbb{C}}}^$.*

Remark 4.13 Here it is important that the Haar measures on $N_Q \cap \bar{N}_P$ and on $\bar{N}_\alpha$, used in the definition of the standard intertwining operators for G and for $G^{(\alpha)}$, are equal.

The proof of Lemma 4.12 will be given in the next few sections. In the text below we shall explain its role in completing the proof of Theorem 3.1.

By a Whittaker datum we shall mean a triple (G, KAN_0, χ) with G a group of the Harish-Chandra class, KAN_0 an Iwasawa decomposition of G and χ a regular unitary character of N_0 . An MS setting is a Whittaker datum as above together with a tuple P, Q of parabolic subgroups of G containing A , and with equal split components. We will say that such a setting satisfies the assertions of Thm. 3.1 if for all $\sigma \in \widehat{M}_{P,ds}$ the identity (3.2) is valid. Finally, an MS setting $(G = KAN_0, \chi, P, Q)$ is called basic if P, Q are maximal and adjacent. In particular, in this case P and Q are opposite.

Lemma 4.14 *If the assertions of Thm. 3.1 hold for every basic MS-setting, then they hold in general.*

Proof. Suppose the assertions of Thm. 3.1 are valid for every basic setting, and let $(G = KAN_0, \chi)$ be a Whittaker datum. Let P', Q' determine an associated MS setting. By Corollary 3.5 it suffices to prove the assertions of Theorem 3.1 for the setting $(G = KAN_0, \chi, P', Q')$ under the assumption that G has compact center. By Lemma 3.7 it suffices to prove the assertions under the additional condition that P', Q' are adjacent. By Lemma 3.8 we may further reduce to the case that P' is opposite standard. If P' is maximal, so is Q' and hence (G, P', Q') is basic, and by hypothesis there is nothing left to be proven. Thus, we may in addition assume that P' is not maximal. We write $P' = \bar{P}$, with P standard. Then $Q := \bar{Q}'$ is adjacent to P . It remains to prove the assertion of Theorem 3.1 for the setting $(G = KAN_0, \chi, \bar{P}, \bar{Q})$. We now select a subgroup $G^{(\alpha)}$ of G related to the pair (P, Q) as in the previous section. Then $(G^{(\alpha)} = K^{(\alpha)}AN_0^{(\alpha)}, \chi^{(\alpha)})$ is a Whittaker datum, and $(P^{(\alpha)}, Q^{(\alpha)})$ determines an associated MS-setting. By Lemma 4.12 it suffices to prove the assertions of Thm. 3.1 for the latter setting. By Lemma 3.4 we see that it suffices to verify the assertions of Thm. 3.1 for the setting $(G^{(\alpha)} = K^{(\alpha)\circ}AN_0^{(\alpha)}, \chi^{(\alpha)}, {}^\circ P^{(\alpha)}, {}^\circ Q^{(\alpha)})$. Since the latter setting is basic, the validity of the assertions is guaranteed by the hypothesis. $\square$

5 Smoothness of J

In this section, we assume that $P, Q \in \mathcal{P}$ are adjacent parabolic subgroups of G , containing A . In addition we assume that P is standard and not maximal. Furthermore, $\alpha \in \Sigma(P)$ and $X \in \ker \alpha$ are as in Section 4. We retain the notation introduced in the text following Lemma 4.1.

Fix $\eta \in H_{\sigma, \chi \bar{P}}^{-\infty}$; recall that P is standard. For $\nu \in \mathfrak{a}_{P\bar{C}}^*$ we define $\varepsilon_\nu : N_0 \bar{P} \rightarrow H^{-\infty}$ by

$$\varepsilon_\nu(nma\bar{n}) = \chi(n)a^{-\nu+\rho_P}\sigma^{-1}(m)\eta,$$

for $n \in N_P, m \in M_P, a \in A_P$ and $\bar{n} \in \bar{N}_P$. We view the restriction of ε_ν to $K \cap N_0 \bar{P}$ as an almost everywhere defined function $K \rightarrow H_\sigma^{-\infty}$. This function satisfies $\varepsilon_\nu(km) = \sigma(m)^{-1}\varepsilon_\nu(k)$ for almost all $k \in K$ and $m \in K_P$.

From [2, Prop. 8.12] it follows that for $\operatorname{Re} \nu$ P -dominant the Whittaker vector $j_\nu := j(\bar{P}, \sigma, \nu)\eta$ is represented by ε_ν in the sense that in the compact picture one has, for all $\varphi \in C^\infty(K/K_P : \sigma_P)$,

$$\langle j_\nu, \varphi \rangle = \int_K \langle \varepsilon_\nu(k), \varphi(k) \rangle dk, \quad (5.1)$$

with absolutely convergent integral.

Likewise, the element $j_\nu^{(\alpha)} = j(\bar{P}^{(\alpha)}, \sigma, \nu)\eta$ associated with $G^{(\alpha)}$ is, for $\nu \in \mathfrak{a}_{P_C}^*$ with $\operatorname{Re} \nu(H_\alpha) > 0$, given by the almost everywhere defined function

$$\varepsilon^{(\alpha)}(nma\bar{n}) = \chi(n)a^{-\nu+\rho_{P^{(\alpha)}}}\sigma^{-1}(m)\eta,$$

for $n \in N_P^{(\alpha)}, m \in M_P, a \in A_P$ and $\bar{n} \in \bar{N}_{P^{(\alpha)}}$. Here $\rho_{P^{(\alpha)}}$ is the rho of the standard parabolic subgroup $P^{(\alpha)}$ in $G^{(\alpha)}$. We note that the difference $\rho_P - \rho_{P^{(\alpha)}}$ restricts to zero on $\ker \alpha = {}^\circ\mathfrak{a}^{(\alpha)}$ hence does not appear in the analysis on ${}^\circ G^{(\alpha)}$.

For $\mu \in \mathfrak{a}_{P_C}^*$ we define $\varphi_\mu : G \rightarrow \mathbb{C}$ by

$$\varphi_\mu(kma\bar{n}) = a^{-\mu}, \quad (k \in K, m \in M_P, a \in A_P, \bar{n} \in \bar{N}_P).$$

We note that the operator

$$m_{\nu, \mu} : C^\infty(G/\bar{P} : \sigma : \nu) \rightarrow C^\infty(G/\bar{P} : \sigma : \nu), \quad \psi \mapsto \varphi_\mu \psi$$

is given by the identity of $C^\infty(K/K_P : \sigma_P)$ in the compact picture. In particular, it follows that the operator has a unique continuous linear extension to an operator $C^{-\infty}(G/\bar{P} : \sigma : \nu) \rightarrow C^{-\infty}(G/\bar{P} : \sigma : \nu + \mu)$. We denote this operator by $m_{\nu, \mu}$ again, and write $m_{\nu, \mu} : \psi \mapsto \varphi_\nu \psi$. In the compact picture the extended operator $m_{\nu, \mu}$ is given by the identity of $C^{-\infty}(K/K_P : \sigma_P)$.

It follows from Cor. 1.12 that $\nu \mapsto j_\nu$ defines a holomorphic function $\mathfrak{a}_{P_C}^* \rightarrow C^{-\infty}(K/K_P : \sigma_P)$. Clearly, $(\nu, \mu) \mapsto \varphi_\mu j_{\nu-\mu}$ is given by $(\nu, \mu) \mapsto j_{\nu-\mu}|_K$ in the compact picture, hence is holomorphic on $\mathfrak{a}_{P_C}^* \times \mathfrak{a}_{P_C}^*$.

It follows from [2, Prop. 8.14-15] that for $\operatorname{Re} \nu$ strictly P -dominant the generalized function $\varphi_\mu j_{\nu-\mu} \in C^{-\infty}(G/\bar{P} : \sigma : \nu)$ is represented by $\varphi_\mu \varepsilon_{\nu-\mu}$ in the sense that for all $\varphi \in C^\infty(K/K_P : \sigma_P)$,

$$\langle \varphi_\mu j_{\nu-\mu}, \varphi \rangle = \int_K \langle \varepsilon_{\nu-\mu}(k), \varphi(k) \rangle dk.$$

Put $A(\nu) = A(\bar{Q}, \bar{P}, \sigma, \nu)$ and let $\mathcal{S}_A \subset \mathfrak{a}_{P_C}^*$ denote the singular locus for $A(\cdot)$. For $(\nu, \mu) \in (\mathfrak{a}_{P_C}^* \setminus \mathcal{S}_A) \times \mathfrak{a}_{P_C}^*$, we define

$$J_{\nu, \mu} := A(\nu)[\varphi_\mu j_{\nu-\mu}] \in C^{-\infty}(G/\bar{Q} : \sigma : \nu). \quad (5.2)$$

The extra parameter μ is introduced to allow the choice of pairs (ν, μ) where $A(\nu)$ and $j_{\nu-\mu}$ are simultaneously representable by a convergent integral and a locally integrable

function, respectively. This idea is inspired by an argument of T. Oshima and J. Sekiguchi [10, text prec. Lemma 4.13], developed further by [1, Lemma 7.4], see also [5, Prop. 6].

Our first main goal is to understand the dependence of $\pi_{\bar{Q},\sigma,\nu}(n)J_{\nu,\mu}$ on $(n, \nu, \mu) \in N_P \times \mathfrak{a}_{P_{\mathbb{C}}}^* \times \mathfrak{a}_{P_{\mathbb{C}}}^*$.

From (1.7) and (2.4) we have, for $R \in \mathbb{R}$,

$$\begin{aligned}\mathfrak{a}^*(P, R) &:= \{\nu \in \mathfrak{a}_{P_{\mathbb{C}}}^* \mid \operatorname{Re} \langle \beta, \nu \rangle > R, \ (\beta \in \Sigma(P))\}, \\ \mathfrak{a}^*(Q|P, R) &:= \{\nu \in \mathfrak{a}_{P_{\mathbb{C}}}^* \mid \operatorname{Re} \langle \beta, \nu \rangle > R, \ (\beta \in \Sigma(\bar{Q}) \cap \Sigma(P))\}.\end{aligned}$$

From [2, Prop 14.8] we know that for every $R \in \mathbb{R}$ there exists a positive integer s such that $\nu \mapsto j(\bar{P}, \sigma, \nu)$ is holomorphic as a map $\mathfrak{a}_{P_{\mathbb{C}}}^*(P, R) \rightarrow C^{-s}(K/K_P : \sigma_P)$.

Furthermore, for $\sigma \in \widehat{M}_{P,\text{ds}}$ we know by Lemma 2.3 that for every $R \in \mathbb{R}$ there exists a polynomial function $q : \mathfrak{a}_{P_{\mathbb{C}}}^* \rightarrow \mathbb{C}$ and a constant $r \in \mathbb{N}$ such that for every $t \in \mathbb{N}$ the assignment

$$\nu \mapsto q(\nu) A(\bar{Q}, \bar{P}, \sigma, \nu)$$

is holomorphic as a function on $\mathfrak{a}^*(\bar{Q}|\bar{P}, R)$ with values in $B(C^{-t}, C^{-t-r})$, the Banach space of bounded linear maps $C^{-t}(K/K_P : \sigma_P) \rightarrow C^{-t-r}(K/K_P : \sigma_P)$.

Lemma 5.1 *For every bounded open subset Ω of $\mathfrak{a}_{P_{\mathbb{C}}}^* \times \mathfrak{a}_{P_{\mathbb{C}}}^*$ there exists a $p \in \mathbb{N}$ and a polynomial function $q \in P(\mathfrak{a}_{P_{\mathbb{C}}}^*)$ such that the map $(n, \nu, \mu) \mapsto q(\nu)\pi_{\bar{Q},\sigma,\nu}(n)J_{\nu,\mu}$ is a smooth map $N_P \times \Omega \rightarrow C^{-p}(K/K_P : \sigma_P)$ which is holomorphic in the variable $(\nu, \mu) \in \Omega$.*

Proof. Without loss of generality we may assume that $\Omega = \Omega_1 \times \Omega_2$ with Ω_j bounded open in $\mathfrak{a}_{P_{\mathbb{C}}}^*$. We will write $C^s(K : \sigma_P) := C^s(K/K_P : \sigma_P)$, and keep in mind that $K_P = K_Q$. For every $s \in \mathbb{Z}$ we will write $\psi \mapsto \psi|_K$ for the isomorphism $C^s(G/\bar{P} : \sigma : \nu) \rightarrow C^s(K : \sigma_P)$ induced by restriction to K . Note that with this notation,

$$\pi_{\bar{P},\sigma,\nu}(n)[\varphi_\mu j(\bar{P}, \sigma, \nu - \mu, \eta)]|_K = \psi(n, \mu)[j(\bar{P}, \sigma, \nu - \mu, \eta)|_K]$$

where

$$\psi(n, \mu)(k) = \chi(n)^{-1} \varphi_{-\mu}(n^{-1}k) = \chi(n)^{-1} a_{\bar{P}}(n^{-1}k)^{-\mu}$$

is a smooth function $N_P \times \mathfrak{a}_{P_{\mathbb{C}}}^* \rightarrow C^\infty(K/K_M)$, which is holomorphic in the second variable. Let r be the order of $j(P, \sigma, \cdot)$ over $\Omega_1 - \Omega_2 = \{\nu - \mu \mid \nu \in \Omega_1, \mu \in \Omega_2\}$. Then $(\nu, \mu) \mapsto j_{\nu-\mu}$ defines a holomorphic map $\Omega \rightarrow C^{-r}(K : \sigma_P)$. It follows that the map

$$(n, \nu, \mu) \mapsto \pi_{\bar{P},\sigma,\nu}(n)[\varphi_\mu j_{\nu-\mu}|_K]$$

is smooth $N_P \times \Omega \rightarrow C^{-r}(K : \sigma_P)$ and in addition holomorphic in the second variable.

Let t be the order of the family $A(\bar{Q}, \bar{P}, \sigma, \cdot)$ over Ω_1 . Then there exists a polynomial function $q : \mathfrak{a}_{P_{\mathbb{C}}}^* \rightarrow \mathbb{C}$ such that for every positive integer s the operator $A(\bar{Q}, \bar{P}, \sigma, \cdot)$

defines a holomorphic function $\Omega_1 \rightarrow B(C^{-s}(K : \sigma_P), C^{-s-t}(K : \sigma_P))$. Since the natural map

$$B(C^{-r}(K : \sigma_P), C^{-r-t}(K : \sigma_P)) \times C^{-r}(K : \sigma_P) \rightarrow C^{-r-t}(K : \sigma_P)$$

is a continuous bilinear map of Banach spaces, it follows from the usual rules for differentiation that

$$\begin{aligned} (n, \nu) &\mapsto q(\nu)\pi_{\bar{Q}, \sigma, \nu}(n)J_{\nu, \mu} \\ &= q(\nu)A(\nu)\pi_{\bar{P}, \sigma, \nu}(n)[\varphi_\mu j_{\nu-\mu}|_K] \end{aligned} \quad (5.3)$$

defines a smooth map from $N_P \times \Omega$ to $C^{-r-t}(K : \sigma_P)$ which is holomorphic in the second variable. This proves the result with $p = r + t$. $\square$

In view of the results of Section 1 for the group $G^{(\alpha)}$ in place of G there exists an element $v \in N_{K^{(\alpha)}}(\mathfrak{a})$ such that $N_0^{(\alpha)}v\bar{Q}^{(\alpha)}$ is open in $G^{(\alpha)}$. We may therefore choose

$$v_{\bar{Q}} = v_{\bar{Q}^{(\alpha)}}^{(\alpha)} := v$$

for our maps $v : \mathcal{P} \rightarrow N_K(\mathfrak{a})$ and $v^{(\alpha)} : \mathcal{P}^{(\alpha)} \rightarrow N_{K^{(\alpha)}}(\mathfrak{a})$ for G and $G^{(\alpha)}$ as discussed in Definition 1.4. Then $\bar{P} = v\bar{Q}v^{-1}$. It follows that $G^\circ := N_P v\bar{Q}$ is a right $\bar{Q}$ -invariant (dense) open subset of G . The action map $N_P \rightarrow G/\bar{Q}, n \mapsto nv\bar{Q}$, induces an open embedding into $G/\bar{Q}$ with image $G^\circ/\bar{Q}$. Composing the defined embedding $N_P \rightarrow G/\bar{Q}$ with the inverse of the diffeomorphism $K/K_Q \rightarrow G/\bar{Q}$ we obtain an embedding $N_P \rightarrow K/K_Q$ with image $K^\circ = [N_P v\bar{Q}] \cap K$. The defined maps form a commutative diagram of diffeomorphisms

$$\begin{array}{ccc} N_P & \longrightarrow & G^\circ/\bar{Q} \\ & \searrow & \uparrow \\ & & K^\circ/K_Q. \end{array}$$

By pull-back we then obtain for every $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ a commutative diagram of topological linear isomorphisms

$$\begin{array}{ccc} C^r(N_P, H_\sigma) & \longleftarrow & C^r(G^\circ/\bar{Q} : \sigma : \nu) \\ & \nwarrow & \uparrow \\ & & C^r(K^\circ/K_Q : \sigma_Q). \end{array}$$

The diagram with arrows representing the inverted maps is still a commutative diagram of topological linear isomorphisms. For a given $u \in H_\sigma^\infty$ we consider the embedding $\iota_u : C^p(N_P) \rightarrow C^p(N_P, H_\sigma), f \mapsto f \otimes u$. Combining this with the inverted diagram, we obtain a diagram of continuous linear maps

$$\begin{array}{ccc} C^r(N_P) & \xrightarrow{\iota_u T_\nu} & C^r(G^\circ/\bar{Q} : \sigma : \nu) \\ \iota_u T_\nu \searrow & & \downarrow r_\nu \\ & & C^r(K^\circ/K_Q : \sigma_Q). \end{array} \quad (5.4)$$

Lemma 5.2 For every $u \in H_\sigma^\infty$ and $v \in \mathfrak{a}_{Q\mathbb{C}}^*$,

$$\pi_{\bar{Q},\sigma,v}(n) \circ {}_uT_v = {}_uT_v \circ L_n, \quad (n \in N_P).$$

Proof. This follows readily from the definitions. $\square$

Given a compact subset $\mathcal{K} \subset N_P$ we will denote the canonical image of $\mathcal{K}v$ in $G/\bar{Q}$ by $\mathcal{K}'$ and the image in K/K_Q by $\mathcal{K}''$. Note that $\mathcal{K}' \subset G^\circ/\bar{Q}$ and $\mathcal{K}'' \subset K^\circ/K_Q$. Let r be a positive integer. Identifying $C_{\mathcal{K}'}^r(G/\bar{Q} : \sigma : v)$ with $C_{\mathcal{K}'}^r(G/\bar{Q})$ in the usual way, through extension by zero, and using the analogous identification for functions on K , we infer that the diagram (5.4) induces a commutative diagram

$$\begin{array}{ccc} C_{\mathcal{K}}^r(N_P) & \xrightarrow{{}_uT_v} & C_{\mathcal{K}'}^r(G/\bar{Q} : \sigma : v) \\ {}_uT_v \searrow & & \downarrow r_v \\ & & C_{\mathcal{K}''}^r(K/K_Q : \sigma_Q) \end{array} \quad (5.5)$$

of bounded linear maps between Banach spaces. If $f \in C^r(N_P)$ we denote by ${}_uf_{\bar{Q},\sigma,v}$ the function $G^\circ \rightarrow H_\sigma^\infty$ given by

$${}_uf_{\bar{Q},\sigma,v}(nvman\bar{n}) = a^{-v+\rho_Q} f(n) \otimes \sigma(m)^{-1}u, \quad (5.6)$$

for $n \in N_P, m \in M_Q, a \in A_Q$ and $\bar{n} \in \bar{N}_Q$. Then ${}_uT_v$ and ${}_uT_v$ are given by

$${}_uT_v(f) = {}_uf_{\bar{Q},\sigma,v}, \quad \text{and} \quad {}_uT_v(f) = {}_uT_v(f)|_{K^\circ}, \quad (f \in C^r(N_P)).$$

If $f \in C_{\mathcal{K}}^r(N_P)$ we view ${}_uT_v(f)$ as an element of $C_{\mathcal{K}'}^r(G/\bar{Q} : \sigma : v)$ as explained above. Then ${}_uT_v(f) = {}_uT_v(f)|_K$.

Lemma 5.3 Let $\mathcal{K} \subset N_P$ be compact, and $v \in \mathfrak{a}_{P\mathbb{C}}^*$. For every $r \in \mathbb{N}$ the map $(u, f) \mapsto {}_uT_v(f)$ is continuous bilinear $H_\sigma^\infty \times C_{\mathcal{K}}^r(N_P) \rightarrow C_{\mathcal{K}''}^r(K/K_Q, \sigma_Q)$.

Proof. Straightforward. See also [2, Lemma 8.8] for a related discussion. $\square$

We will now investigate the family $J_{v,\mu} \in C^{-r}(G/\bar{Q} : \sigma : v)$ in more detail.

First of all, if $J \in C^{-r}(G/\bar{Q} : \sigma : v)$ and $u \in H_\sigma^\infty$ we define the continuous linear functional ${}_uJ : C_c^r(N_P) \rightarrow \mathbb{C}$ by

$${}_uJ(f) = \langle J, {}_uT_{-\bar{v}}(\bar{f}) \rangle, \quad (f \in C^r(N_P)). \quad (5.7)$$

We denote by $a_{\bar{Q}}$ the function $K^\circ \rightarrow A_Q$ (uniquely) determined by

$$x \in N_P v M_Q a_{\bar{Q}}(x) \bar{N}_Q, \quad (x \in K^\circ).$$

Then $a_{\bar{Q}}$ is real analytic $K^\circ/K_Q \rightarrow A_Q$. For $r \in \mathbb{N}$ and $v \in \mathfrak{a}_{P\mathbb{C}}^*$ we define the map $m_v : C^r(K^\circ/K_Q : \sigma_Q) \rightarrow C^r(K^\circ/K_Q : \sigma_Q)$ by

$$m_v(f)(k) = a_{\bar{Q}}(k)^{-v} f(k), \quad (f \in C_c^r(K^\circ/K_Q)).$$

Lemma 5.4 *Let $\mathcal{K}$ be a compact subset of N_P . Then for every $\nu \in \mathfrak{a}_{Q_{\mathbb{C}}}^*$ the map m_ν restricts to a bounded automorphism of the Banach space $C_{\mathcal{K}'}^r(K/K_Q : \sigma_Q)$. The assignment $\nu \mapsto m_\nu$ is holomorphic as a function on $\mathfrak{a}_{P_{\mathbb{C}}}^*$ with values in the space of bounded operators $B(C_{\mathcal{K}'}^r(K/K_Q : \sigma_Q))$.*

Proof. Straightforward. $\square$

Lemma 5.5 *Let $u \in H_\sigma^\infty$. Let $\mathcal{K} \subset N_P$ be a compact subset. For $f \in C_{\mathcal{K}}^r(N_P)$ and $\nu \in \mathfrak{a}_{Q_{\mathbb{C}}}^*$ we have*

- (a) ${}_uT_\nu(f) \in C_{\mathcal{K}'}^r(K/K_Q : \sigma : \nu)$;
- (b) ${}_uT_\nu(f) = m_\nu({}_uT_0(f))$;
- (c) *the map ${}_uT_\nu$ is a bounded linear map between the Banach spaces $C_{\mathcal{K}}^r(N_P)$ and $C_{\mathcal{K}'}^r(K/K_Q : \sigma_Q)$;*
- (d) *the assignment $\nu \mapsto {}_uT_\nu$ is holomorphic as a function on $\mathfrak{a}_{Q_{\mathbb{C}}}^*$, with values in the Banach space B of bounded linear maps from $C_{\mathcal{K}}^r(N_P)$ to $C_{\mathcal{K}'}^r(K/K_Q : \sigma_Q)$, equipped with the operator norm.*

Proof. Assertion (a) is true by definition. For (b), fix $k \in \mathcal{K}'$. Then k has a unique decomposition $k = nvman \in \mathcal{K}v\bar{Q} \simeq \mathcal{K}vM_QA_Q\bar{N}_Q \subset N_Pv\bar{Q}$. It follows that

$$\begin{aligned} {}_uT_\nu(f)(k) &= f(n)a^{-\nu+\rho_Q}\sigma(m)^{-1}u \\ &= a_{\bar{Q}}(k)^{-\nu}{}_uT_0(f)(k) = m_\mu({}_uT_0(f))(k). \end{aligned}$$

From (a) and (b) it follows that ${}_uT_\nu = m_\nu \circ {}_uT_0$. Thus, view of Lemma 5.4 it suffices to show (c) for $\nu = 0$.

Put $X = C_{\mathcal{K}}^p(N_P)$, $Y = C_{\mathcal{K}'}^p(K/K_Q : \sigma_Q)$; these are Banach spaces. We equip $B(X, Y)$ and $B(Y)$ with the operator norms and consider the natural map $\beta : B(X, Y) \times B(Y) \rightarrow B(X, Y)$ $(\tau, \mu) \mapsto \mu \circ \tau$. Then β is bilinear and $\|\beta(\tau, \mu)\|_{\text{op}} = \|\mu \circ \tau\|_{\text{op}} \leq \|\mu\| \| \tau \|$. Thus, β is continuous. If $\nu \mapsto \tau_\nu$ and $\mu \mapsto \mu_\nu$ are holomorphic, it follows readily that $\mu \mapsto \beta(Y_\nu, \mu_\nu)$ is holomorphic with values in $B(X, Y)$. Applying this to $\tau_\nu = {}_uT_0$ and $\mu_\nu = m_\nu$, we find that $\nu \mapsto m_\mu \circ {}_uT_\nu = \beta({}_uT_0, m_\nu)$ is holomorphic $\mathfrak{a}_{Q_{\mathbb{C}}}^* \rightarrow B(X, Y)$. This establishes (d). $\square$

Recall the definition of $J_{\nu, \mu}$ from (5.2), for $(\nu, \mu) \in (\mathfrak{a}_{P_{\mathbb{C}}}^* \setminus \mathcal{S}_A) \times \mathfrak{a}_{P_{\mathbb{C}}}^*$. For $u \in H_\sigma^\infty$ we define ${}_uJ_{\nu, \mu}$ as in (5.7). This is an element of $C_{\mathcal{K}}^\infty(N_P)'$ for every compact $\mathcal{K} \subset N_P$.

Theorem 5.6 *For every $u \in H_\sigma^\infty$ and $(\nu, \mu) \in (\mathfrak{a}_{P_{\mathbb{C}}}^* \setminus \mathcal{S}_A) \times \mathfrak{a}_{P_{\mathbb{C}}}^*$ there exists a unique smooth function ${}_u\tilde{J}_{\nu, \mu} \in C^\infty(N_P)$ such that ${}_uJ_{\nu, \mu}$ is represented by the density ${}_u\tilde{J}_{\nu, \mu}dn_P$ in the sense that*

$${}_uJ_{\nu, \mu}(f) = \int_{N_P} {}_u\tilde{J}_{\nu, \mu}(n_P)f(n_P)dn_P, \quad (f \in C_c^\infty(N_P)).$$

If $\Omega \subset \mathfrak{a}_{P_{\mathbb{C}}}^ \times \mathfrak{a}_{P_{\mathbb{C}}}^*$ is a bounded open subset then there exists a polynomial function $q : \mathfrak{a}_{P_{\mathbb{C}}}^* \rightarrow \mathbb{C}$ such that the map $(\nu, \mu) \mapsto q(\nu){}_u\tilde{J}_{\nu, \mu}$ is holomorphic $\Omega \rightarrow C^\infty(N_P)$.*

Proof. Let $u \in H^\infty_\sigma$ be fixed. The asserted uniqueness is clear. Therefore, it suffices to prove the assertions that arise if we replace $(\mathfrak{a}_{P_\mathbb{C}}^* \setminus \mathcal{S}_A) \times \mathfrak{a}_{P_\mathbb{C}}^*$ by $(\Omega_1 \setminus \mathcal{S}_A) \times \Omega_2$ and Ω by $\Omega_1 \times \Omega_2$ for an arbitrary pair Ω_1, Ω_2 of bounded open subsets of $\mathfrak{a}_{P_\mathbb{C}}^*$. Suppose such a pair is fixed and let $q \in P(\mathfrak{a}_{P_\mathbb{C}}^*)$ and $p \in \mathbb{N}_+$ be associated with $\Omega = \Omega_1 \times \Omega_2$ as in Lemma 5.1.

Let $\mathcal{K} \subset N_P$ be a compact subset. We fix an open neighborhood V of e in N_P whose closure in N_P is compact. Then $\mathcal{K}_e := \text{cl}(V^{-1})\mathcal{K}$ is a compact subset of N_P . We fix $\mathcal{K}'_e$ and $\mathcal{K}''_e$ as in the discussion of the diagram (5.5) with $\mathcal{K}_e$ in place of $\mathcal{K}$.

For $f \in C^p_{\mathcal{K}}(N_P)$ and $n \in V$ we have $L_n^{-1}f \in C^p_{\mathcal{K}_e}(N_P)$. We now note that

$$\begin{aligned} \langle L_n[{}_uJ_{\nu,\mu}], \bar{f} \rangle &= \langle {}_uJ_{\nu,\mu}, L_n^{-1}\bar{f} \rangle \\ &= \langle J_{\nu,\mu}, {}_uT_{\mu,-\bar{\nu}}[L_n^{-1}\bar{f}] \rangle \\ &= \langle J_{\nu,\mu}, \pi_{Q,\sigma,-\bar{\nu}}(n)^{-1}[{}_uT_{\mu,-\bar{\nu}}\bar{f}] \rangle \\ &= \langle \pi_{Q,\sigma,\nu}(n)J_{\nu,\mu}, {}_uT_{-\bar{\nu}}(\bar{f}) \rangle. \end{aligned}$$

We write $X = C^p_{\mathcal{K}}(N_P)$ and $Y = C^p(K/K_Q : \sigma_Q)$. Furthermore, $\bar{Y}$ denotes the conjugate of Y and $B(X, \bar{Y})$ the space of bounded linear maps from X to Y , equipped with the operator norm.

It follows from Lemma 5.5 (d) that $f \mapsto {}_uT_{-\bar{\nu}}(\bar{f})$ is an element of $B(X, \bar{Y})$, depending holomorphically on $\nu \in \mathfrak{a}_{P_\mathbb{C}}^*$. On the other hand, $(n, \nu, \mu) \mapsto q(\nu)\pi_{Q,\sigma,\nu}(n)J_{\nu,\mu}$ is a smooth function $V \times \Omega \rightarrow \bar{Y}' = C^{-p}(K/K_Q : \sigma_Q)$, which is holomorphic in (ν, μ) .

We now consider the natural bilinear map $\beta : B(X, \bar{Y}) \times \bar{Y}' \rightarrow X'$ given by $(t, \eta) \mapsto \eta \circ t$. Note that $\|\beta(t, \eta)\| \leq \|\eta\| \|t\|$; this shows that β is continuous bilinear. We observe that

$$L_n[{}_uJ_{\nu,\mu}] = \beta({}_uT_{-\bar{\nu}}, \pi_{Q,\sigma,\nu}(n)J_{\nu,\mu}).$$

By the usual rules for differentiation it follows that $(n, \mu, \nu) \mapsto q(\nu)L_n[{}_uJ_{\nu,\mu}]$ is a smooth map $N_P \times \Omega_{R_1, R_2} \rightarrow C^p_{\mathcal{K}}(N_P)'$, which is holomorphic in the variable from Ω .

Let $v_1, \dots, v_n$ be a basis of the Lie algebra of N_P and let $v_j = L_{v_j}$ be the associated right invariant vector fields on N_P . Then it follows for all $(\nu, \mu) \in \Omega$ and every multi-index $\alpha \in \mathbb{N}^n$ that the distribution $q(\nu)v^\alpha({}_uJ_{\nu,\mu})$ belongs to $C^p_{\mathcal{K}}(N_P)'$. Furthermore,

$$(\nu, \mu) \mapsto q(\nu)v^\alpha({}_uJ_{\nu,\mu}) : \Omega_{R_1, R_2} \rightarrow C^p_{\mathcal{K}}(N_P)'$$

is holomorphic. By application of the lemma of the appendix we now conclude that $q(\nu){}_uJ_{\nu,\mu}$ is a smooth density of the form asserted, with holomorphic dependence on (ν, μ) . $\square$

6 A useful integral formula

We keep working under the hypothesis of Section 4. Thus, $P \in \mathcal{P}$ is standard non-maximal, Q is adjacent to P and $\alpha \in \Sigma(P) \cap [-\Sigma(Q)]$. The element $X \in \ker \alpha \cap \text{cl}(\mathfrak{a}_P^+)$

is generic. P_X is the unique parabolic subgroup having X in its positive chamber, and $G^{(\alpha)} = M_{1X} = P_X \cap \bar{P}_X$.

We define smooth maps $k_{\bar{P}} : G \rightarrow K$, $m_{\bar{P}} : G \rightarrow M_P \cap \exp \mathfrak{s}$, $a_{\bar{P}} : G \rightarrow A_P$ and $n_{\bar{P}} : G \rightarrow \bar{N}_P$ by

$$x = k_{\bar{P}}(x)m_{\bar{P}}(x)a_{\bar{P}}(x)n_{\bar{P}}(x), \quad (x \in G).$$

The multiplication map $K \times \bar{P} \rightarrow G$ factors through a diffeomorphism $K \times_{K_P} \bar{P} \rightarrow G$. Since $a^{-2\rho_P} dmdad\bar{n}$ defines a right-invariant measure on the group $\bar{P}$, it follows that for a Lebesgue integrable function $f : G \rightarrow \mathbb{C}$ we have

$$\int_G f(x) dx = \int_{K \times M_P \times A_P \times \bar{N}_P} f(kma\bar{n}) dk dmdad\bar{n}.$$

Lemma 6.1 *Let $\varphi : K \rightarrow \mathbb{C}$ be Lebesgue integrable. Then*

$$\int_K \varphi(k) dk = \int_{N_X \times K^{(\alpha)}} \varphi(k_{\bar{P}}(n_X k_{\alpha})) a_{\bar{P}}(n_X k_{\alpha})^{2\rho_P} dn_X dk_{\alpha}.$$

Proof. We fix $\psi \in C_c(\bar{P})$ left K_P -invariant, such that

$$\int_{M_P \times A_P \times \bar{N}_P} \psi(ma\bar{n}) a^{-2\rho_P} dmdad\bar{n} = 1. \quad (6.1)$$

Furthermore, we extend φ to G by the formula

$$\varphi(kma\bar{n}) = \varphi(k)\psi(man),$$

for $k \in K, m \in M_P, a \in A_P$ and $\bar{n} \in \bar{N}_P$. Then φ is Lebesgue integrable on G and

$$\begin{aligned} \int_G \varphi(x) dx &= \int_{K \times M_P \times A_P \times \bar{N}_P} \varphi(k)\psi(man) a^{-2\rho_P} dk dmdad\bar{n} \\ &= \int_K \varphi(k) dk. \end{aligned}$$

Since $K^{(\alpha)}$ normalizes N_X , the density $dn_X dk_{\alpha}$ on $N_X \times K^{(\alpha)}$ is left invariant. Therefore,

$$\int_G \varphi(x) dx = \int_{N_X \times K^{(\alpha)} \times M_P \times A_P \times \bar{N}_P} \varphi(n_X k_{\alpha} ma\bar{n}) a^{-2\rho_P} dn_X dk^{(\alpha)} dmdad\bar{n}, \quad (6.2)$$

provided the measures are suitably normalized. The pull-back of $a^{-2\rho_P} dmdadn$ on $\bar{P}$ under left multiplication by $m_1 a_1 \bar{n}_1$ equals $a_1^{2\rho_P} a^{-2\rho_P} dmdadn$. Therefore, the second integral in (6.2) equals

$$\begin{aligned} &\int_{N_X \times K^{(\alpha)} \times M_P \times A_P \times \bar{N}_P} \varphi(k_{\bar{P}}(n_X k_{\alpha})) a_{\bar{P}}(n_X k_{\alpha})^{2\rho_P} \psi(ma\bar{n}) a_{\bar{P}}^{-2\rho_P} dmdad\bar{n} dn_X dk_{\alpha} \\ &= \int_{N_X \times K^{(\alpha)}} \varphi(k_{\bar{P}}(n_X k_{\alpha})) a(n_X k_{\alpha})^{2\rho_P} dn_X dk_{\alpha}. \end{aligned}$$

□

Lemma 6.2 Let $f : G \rightarrow \mathbb{C}$ be such that $f|_K$ is Lebesgue integrable and

$$f(xma\bar{n}) = a^{2\rho_P} f(x),$$

for all $x \in G, (m, a, \bar{n}) \in M_P \times A_P \times \bar{N}_P$. Then

$$\int_K f(k) dk = \int_{N_X \times K^{(\alpha)}} f(n_X k_\alpha) dn_X dk_\alpha.$$

Proof.

$$\begin{aligned} \int_K f(k) dk &= \int_{N_X \times K^{(\alpha)}} f(k_{\bar{P}}(n_X k_\alpha)) a_{\bar{P}}(n_X k_\alpha)^{2\rho_P} dn_X dk_\alpha. \\ &= \int_{N_X \times K^{(\alpha)}} f(n_X k_\alpha) dn_X dk_\alpha. \end{aligned}$$

□

Remark 6.3 In proof given above we have not used the particular definitions of N_X and $K^{(\alpha)}$. The proof works under the assumptions that $\mathfrak{n}_X, \mathfrak{n}_\alpha$ are sums of $\mathfrak{a}$ -root spaces such that $\mathfrak{n}_X \oplus \mathfrak{n}_\alpha = \mathfrak{n}_P$, $N_X = \exp \mathfrak{n}_X$, $\mathfrak{k}^{(\alpha)} = \mathfrak{k} \cap (\mathfrak{n}_\alpha + \bar{\mathfrak{n}}_\alpha) + \mathfrak{k}_P$ and $K^{(\alpha)}$ is the group generated by $\exp(\mathfrak{k}^{(\alpha)})K_P$. In particular the proof works for the case $\mathfrak{n}_\alpha = 0$, so that in particular $N_X = N_P$ and $K^{(\alpha)} = K_P$. In this setting the above result is well known.

In the sequel we will also need the following result, for $P \in \mathcal{P}_{\text{st}}$, and $Q = v^{-1}Pv$, $v \in N_K(\mathfrak{a})$.

Lemma 6.4 Let $f : G \rightarrow \mathbb{C}$ be right $M_Q \bar{N}_Q$ -invariant, and let $R_a f = a^{2\rho_Q} f$ for all $a \in A_Q$. If $f|_K$ is Lebesgue integrable, then

$$\int_K f(k) dk = \int_{N_P} f(nv) dn.$$

Proof. The function $L_v f : G \rightarrow \mathbb{C}$ has the same $\bar{Q}$ -equivariance on the right at f . In view of Remark 6.3 we obtain

$$\int_K L_v f(k) dk = \int_{N_Q} f(vn_Q v^{-1}v) dn_Q = \int_{N_P} f(nv) dv.$$

Since dk is left invariant, the desired result follows. □

7 Comparison of J with $J^{(\alpha)}$

The preceding discussion applies to any Whittaker datum $(G = KAN_0, \chi)$. In particular it applies to the group $G^{(\alpha)} = K^{(\alpha)}AN_0^{(\alpha)}$, with the character $\chi^{(\alpha)} = \chi|_{N_0^{(\alpha)}}$; see Lemma 4.9 and its adjacent parabolic subgroups $P^{(\alpha)} = P \cap G^{(\alpha)}$ and $Q^{(\alpha)} = Q \cap G^{(\alpha)}$. Their respective nilpotent radicals are N_α and $\bar{N}_\alpha$. Accordingly, $P^{(\alpha)}$ and $Q^{(\alpha)}$ are opposite parabolic subgroups of $G^{(\alpha)}$ with split components A_P . Since they are maximal parabolic subgroups of $G^{(\alpha)}$ they are adjacent. The element v belongs to the normalizer in $K^{(\alpha)}$ of $\mathfrak{a}$. In particular, from $P = vQv^{-1}$ it follows that $P^{(\alpha)} = vQ^{(\alpha)}v^{-1}$. Thus, N_0vQ is open in G and $N_0^{(\alpha)}vQ^{(\alpha)}$ is open in $G^{(\alpha)}$. Moreover, $N_0v\bar{Q} = N_Pv\bar{Q}$ and $N_0^{(\alpha)}v\bar{Q}^{(\alpha)} = N_\alpha v\bar{Q}^{(\alpha)}$.

We recall that $\mathfrak{n}_P = \mathfrak{n}_\alpha \oplus \mathfrak{n}_X$. Since both $\mathfrak{n}_\alpha$ and $\mathfrak{n}_X$ are subalgebras of $\mathfrak{n}_P$ and each of them is a direct sum of root spaces $\mathfrak{g}_\beta$ with $\beta \in \Sigma(\mathfrak{a})$, it follows that the multiplication map $N_X \times N_\alpha \rightarrow N_P$ is a diffeomorphism. Since $G^{(\alpha)}$ normalizes N_X , it follows that N_X is a normal subgroup of N_P and

$$N_P = N_X N_\alpha = N_X \rtimes N_\alpha \quad (\text{semidirect product}).$$

We note that also $N_X K^{(\alpha)}$ is a closed subgroup of G ; clearly

$$N_X K^{(\alpha)} = N_X \rtimes K^{(\alpha)}.$$

For $\nu \in \mathfrak{a}_{P\mathbb{C}}^*$ with $\text{Re } \nu$ strictly $P^{(\alpha)}$ -dominant the Whittaker vector $j^{(\alpha)}(\bar{P}^{(\alpha)}, \sigma, \nu, \eta)$ is represented by the function $\varepsilon^{(\alpha)} : G^{(\alpha)} \rightarrow \mathbb{C}$ defined by

$$\varepsilon_\nu^{(\alpha)}(n_\alpha m a \bar{n}_\alpha) = \chi(n_\alpha) \sigma(m)^{-1} a^{-\nu + \rho_{P^{(\alpha)}}} \eta$$

for $(n_\alpha, m, a, \bar{n}_\alpha) \in N_\alpha \times M_P \times A_P \times \bar{N}_\alpha$ and by zero on the complement of $N_\alpha M_P A_P \bar{N}_\alpha$. For $\mu \in \mathfrak{a}_{P\mathbb{C}}^*$ the function $\varphi_\mu^{(\alpha)} : G^{(\alpha)} \rightarrow \mathbb{C}$ is defined by

$$\varphi_\mu^{(\alpha)}(k_\alpha m a n_\alpha) = a^\mu,$$

for $(k_\alpha, m, a, n_\alpha) \in K^{(\alpha)} \times M_P \times A_P \times N_P$. We define the character $\xi : A_P \rightarrow \mathbb{R}$ by

$$\xi(a) = a^{\rho_P - \rho_{P^{(\alpha)}}}$$

and note that $\xi = 1$ on $\exp \mathbb{R} H_\alpha$. Now $\exp \mathbb{R} H_\alpha \subset M_X = {}^\circ G^{(\alpha)}$ commutes with $A_X = \exp \ker \alpha$. It follows that ξ uniquely extends to a character of $G^{(\alpha)}$ which is 1 on M_X . This extension is also denoted by ξ . The following result is straightforward.

Lemma 7.1 *For all $\nu, \mu \in \mathfrak{a}_{P\mathbb{C}}^*$,*

$$\varepsilon_\nu|_{G^{(\alpha)}} = \xi \varepsilon_\nu^{(\alpha)} \quad \text{and} \quad \varphi_\mu|_{G^{(\alpha)}} = \varphi_\mu^{(\alpha)}.$$

As the character ξ is only non-trivial on the center of $G^{(\alpha)}$ its role is easily understood in the calculations that follow. From the lemma it follows that $\varphi_\mu \varepsilon_{\nu-\mu}$ restricts to $\xi \varphi_\mu^{(\alpha)} \varepsilon_{\nu-\mu}^{(\alpha)}$. This suggests that $J_{\nu,\mu}$ and $J_{\nu,\mu}^{(\alpha)}$ might be related.

In fact, we will show that the following is valid.

Proposition 7.2 *Let $u \in H_\sigma^\infty$. Then, for $(\nu, \mu) \in \mathfrak{a}_{P_\mathbb{C}}^* \times \mathfrak{a}_{P_\mathbb{C}}^*$,*

$${}_u \widetilde{J}_{\nu,\mu}|_{N_\alpha} = {}_u \widetilde{J}_{\nu,\mu}^{(\alpha)}.$$

For the proof we need some preparation. First we will describe a direct relationship between $\varphi_\mu j_{\nu-\mu}$ and $\varphi_\mu^{(\alpha)} j_{\nu-\mu}^{(\alpha)}$.

Lemma 7.3 *Let $\Omega \subset \mathfrak{a}_{P_\mathbb{C}}^* \times \mathfrak{a}_{P_\mathbb{C}}^*$ be a bounded connected open subset which contains a point (μ, ν) such that $\text{Re}(\nu - \mu)$ is $\bar{P}$ -dominant. There exists a positive integer r such that the following assertions are valid.*

- (a) *The map $(\mu, \nu) \mapsto j_{\nu-\mu}$ is holomorphic $\Omega \rightarrow C^{-r}(K/K_P : \sigma_P)$.*
- (b) *For every $f \in C^r(K/K_P : \sigma_P)$ the function $N_X \times \Omega \rightarrow C^r(K^{(\alpha)}/K_P : \sigma_P)$,*

$$(n_X, \nu, \mu) \mapsto L_{n_X}^{-1}(\varphi_{\bar{\mu}}) L_{n_X}^{-1}(f_{P,\sigma,-\bar{\nu}})|_{K^{(\alpha)}}$$

is smooth, and holomorphic in the variable (ν, μ) from Ω .

- (c) *For every $f \in C^r(K/K_P : \sigma_P)$ with support contained in $K \cap N_P \bar{P}$,*

$$\langle \varphi_\mu j_{\nu-\mu}, f \rangle = \int_{N_X} \chi(n_X) \langle j_{\nu-\mu}^{(\alpha)}|_{K^{(\alpha)}}, L_{n_X}^{-1}(\varphi_{\bar{\mu}}) L_{n_X}^{-1}(f_{\bar{P},\sigma,-\bar{\nu}})|_{K^{(\alpha)}} \rangle dn_X$$

for all $(\nu, \mu) \in \Omega$.

Proof. Assertion (a) follows from Corollary 1.12. Assertion (b) is obvious. We address (c). The set Ω_0 of points $(\mu, \nu) \in \Omega$ such the real part $\text{Re}(\nu - \mu)$ is strictly $\bar{P}$ -dominant is non-empty and open. Let $(\nu, \mu) \in \Omega_0$; then it follows that for $f \in C^r(K/K_P : \sigma_P)$ the function $\langle \varphi_\mu j_{\nu-\mu}, f \rangle_\sigma$ is integrable over K/K_P . We now observe that the function $F : x \mapsto \langle \varphi_\mu(x) j_{\nu-\mu}(x), f_{Q,\sigma,-\bar{\nu}}(x) \rangle_\sigma$ on G is right $M_P \bar{N}_P$ -invariant, and satisfies $R_a F = a^{2\rho_P} F$ for all $a \in A_P$. In view of Lemma 6.2 it follows that

$$\begin{aligned} \langle \varphi_\mu j_{\nu-\mu}, f \rangle &= \int_{N_X \times K^{(\alpha)}} \langle \varepsilon_{\nu-\mu}(n_X k_\alpha), (\varphi_{\bar{\mu}} f_{\bar{P},\sigma,-\bar{\nu}})(n_X k_\alpha) \rangle dn_X dk_\alpha = \\ &= \int_{N_X} \chi(n_X) \int_{K^{(\alpha)}} \langle \varepsilon_{\nu-\mu}^{(\alpha)}(k_\alpha), \varphi_{\bar{\mu}}(n_X k_\alpha) f_{\bar{P},\sigma,-\bar{\nu}}(n_X k_\alpha) \rangle dk_\alpha dn_X \\ &= \int_{N_X} \chi(n_X) \langle j_{\nu-\mu}^{(\alpha)}|_{K^{(\alpha)}}, L_{n_X}^{-1}(\varphi_{\bar{\mu}}) L_{n_X}^{-1}(f_{\bar{P},\sigma,-\bar{\nu}})|_{K^{(\alpha)}} \rangle dn_X. \end{aligned}$$

This establishes the identity of (c) for $(\nu, \mu) \in \Omega_0$.

If f satisfies the mentioned support condition, it follows that there exists a compact set $\mathcal{K} \subset N_X$ such that

$$L_{n_X^{-1}}(f_{\bar{P}, \sigma, -\bar{v}})|_{K(\alpha)} = 0$$

for all $\nu \in \mathfrak{a}_{P_{\mathbb{C}}}^*$ and all $n_X \in N_X \setminus \mathcal{K}$. From this it is readily seen that the expressions on both sides of the equation in (c) are holomorphic functions of $(\nu, \mu) \in \Omega$. The full result now follows by analytic continuation. $\square$

For $\varphi_X \in C_c^\infty(N_X)$ and $\varphi_\alpha \in C_c^\infty(N_\alpha)$, define $\varphi_X \otimes \varphi_\alpha \in C_c^\infty(N_P)$ by

$$\varphi_X \otimes \varphi_\alpha(n_X n_\alpha) = \varphi_X(n_X) \varphi_\alpha(n_\alpha).$$

We note that for $\nu \in \mathfrak{a}_{P_{\mathbb{C}}}^*$. According to the definition,

$${}_uT_v(\varphi_X \otimes \varphi_\alpha) \in C^r(G/\bar{Q} : \sigma : v)$$

is given by

$${}_uT_v(\varphi_X \otimes \varphi_\alpha)(n_X n_\alpha vman) = a^{-\gamma+\rho_Q} \varphi_X(n_X) \varphi_\alpha(n_\alpha) \sigma(m)^{-1} u$$

for $(n_X, n_\alpha, man\bar{n}) \in N_X \times N_\alpha \times M_P A_P \bar{N}_Q$ and by ${}_u T_v(\varphi_X \otimes \varphi_\alpha) = 0$ on $G \setminus N_P v \bar{Q}$. We recall that

$$T_v(\varphi_X \otimes \varphi_\alpha) = \textstyle\bigcup_u T_v(\varphi_X \otimes \varphi_\alpha)|_K \in C^r(K/K_P : \sigma_P).$$

The map ${}_uT_v^{(\alpha)}$ is defined similarly for the group $G^{(\alpha)}$; note that $N_{P^{(\alpha)}} = N_\alpha$. For $\varphi \in C_c^r(N_\alpha)$, $u \in H_\sigma^\infty$ and $v \in \mathfrak{a}_{P_C}^*$ we define ${}_uT_v^{(\alpha)}(\varphi) \in C^r(G^{(\alpha)}/\bar{Q}^{(\alpha)} : \sigma : v)$ by

$${}_uT_v^{(\alpha)}(\varphi)(n_\alpha vma\bar{n}) = a^{-\nu+\rho_{\mathcal{Q}(\alpha)}} \sigma(m)^{-1} \varphi(n_\alpha) u$$

for $n_\alpha v m \bar{n} \in N_\alpha v M_P A_P \bar{N}_{Q^{(\alpha)}}$ and by ${}_u T_\nu^{(\alpha)}(\varphi) = 0$ on $G^{(\alpha)} \setminus N_\alpha v \bar{Q}^{(\alpha)}$. Finally, ${}_u T_\nu^{(\alpha)} : C_c^r(N_\alpha) \rightarrow C^r(K^{(\alpha)}/K_P : \sigma_P)$ is defined by

$${}_uT_v^{(\alpha)}(\varphi) = \lfloor T_v(\varphi) \rfloor_{K^{(\alpha)}}, \quad (\varphi \in C_c^r(N_\alpha)).$$

Suppose now that R is either of the parabolic subgroups P and Q . In this situation, the natural multiplication map $N_X \times K^{(\alpha)} \times_{K_P} \bar{R} \rightarrow G$ is an open embedding. If $\varphi_X \in C_c^r(N_X)$, $\psi_\alpha \in C^r(K^{(\alpha)}/K_P : \sigma_P)$ and $\nu \in \mathfrak{a}_{P_C}^*$, we define the C^r -function $S_{R,\nu}(\varphi_X \otimes \psi_\alpha) : G \rightarrow H_\sigma$ by

$$S_{R,v}(\varphi_X \otimes \psi_\alpha)(n_X k_\alpha m a \bar{n}) = a^{-v+\rho_R} \varphi_X(n_X) \psi_\alpha(k_\alpha) \sigma(m)^{-1} u,$$

for $(n_X, k_\alpha, m\bar{n}) \in N_X \times K_\alpha \times \bar{R}$, and by $S_{R,\nu}(\varphi_X \otimes \psi_\alpha) = 0$ on $G \setminus N_X K_\alpha \bar{R}$. Thus, $S_{R,\nu}(\varphi_X \otimes \psi_\alpha) \in C^r(G/R : \sigma : \nu)$. As before, we define

$$S_{R,v}(\varphi_X \otimes \psi_\alpha) = S_{R,v}(\varphi_X \otimes \psi_\alpha)|_K$$

Lemma 7.4 *Let $u \in H_\sigma^\infty$. Then for $\varphi_X \in C_c^r(N_X)$ and $\varphi_\alpha \in C_c^r(N_\alpha)$,*

$${}_uT_v(\varphi_X \otimes \varphi_\alpha) = S_{Q,v} \left(\varphi_X \otimes {}_uT_v^{(\alpha)}(\varphi_\alpha) \right),$$

for $v \in \mathfrak{a}_{P_C}^*$.

Proof. We will use the notation σ_v for the character of $\bar{Q}$ given by $\sigma_v(man\bar{n}) = a^{v-\rho_Q}\sigma(m)$.

Let $k \in K$ and suppose ${}_uT_v(\varphi_X \otimes \varphi_\alpha)(k) \neq 0$. Then there exist $n_X \in N_X$ and $n_\alpha \in N_\alpha$ such that $k = n_X n_\alpha v \bar{q}$ with $\bar{q} \in \bar{Q}$ and

$${}_uT_v(\varphi_X \otimes \varphi_\alpha)(k) = \sigma_v(\bar{q})^{-1}(\varphi_X(n_X)\varphi_\alpha(n_\alpha)u).$$

In particular, $\varphi_\alpha(n_\alpha) \neq 0$. Write $n_\alpha v = k_\alpha \bar{q}_\alpha$, with $k_\alpha \in K^{(\alpha)}$ and $\bar{q}_\alpha \in \bar{\mathfrak{q}}^{(\alpha)}$. Then $k = n_X k_\alpha \bar{q}_\alpha \bar{q}$. Hence,

$$\begin{aligned} S_{Q,v}(\varphi_X \otimes {}_uT_v^{(\alpha)}\varphi_\alpha)(k) &= \sigma_v(\bar{q}_\alpha \bar{q})^{-1} \varphi_X(n_X) \left({}_uT_v^{(\alpha)}\varphi_\alpha(k_\alpha) \right) \\ &= \sigma_v(\bar{q}_\alpha \bar{q})^{-1} \varphi_X(n_X) \sigma_v(\bar{q}_\alpha) \varphi_\alpha(n_\alpha) u \\ &= \sigma_v(\bar{q})^{-1} \varphi_X(n_X) \varphi_\alpha(n_\alpha) u \\ &= {}_uT_v(\varphi_X \otimes \varphi_\alpha)(k). \end{aligned} \tag{7.1}$$

Conversely, suppose that

$$S_{Q,v}(\varphi_X \otimes {}_uT_v^{(\alpha)}\varphi_\alpha)(k) \neq 0.$$

Then there exist $n_X \in N_X$, $k_\alpha \in K_\alpha$ such that $k = n_X k_\alpha \bar{q}$ with $\bar{q} \in \bar{Q}$. Moreover,

$$S_{Q,v}(\varphi_X \otimes {}_uT_v^{(\alpha)}\varphi_\alpha)(k) = \sigma(\bar{q})^{-1} \varphi_X(n_X) {}_uT_v^{(\alpha)}(\varphi_\alpha)(k_\alpha).$$

In particular, ${}_uT_v^{(\alpha)}(\varphi_\alpha)(k_\alpha) \neq 0$. Hence, there exist $n_\alpha \in N_\alpha$ and $\bar{q}_\alpha \in \bar{Q}^{(\alpha)}$ such that $k_\alpha = n_\alpha v \bar{q}_\alpha$. Now

$${}_uT_v^{(\alpha)}\varphi_\alpha(k_\alpha) = \sigma_v(\bar{q}_\alpha)^{-1} \varphi_\alpha(n_\alpha).$$

We now have $k = n_X k_\alpha \bar{q}$ and $k_\alpha = n_\alpha v \bar{q}_\alpha$ so that $k = n_X n_\alpha v \bar{q}_\alpha \bar{q}$. It follows that

$$\begin{aligned} {}_uT_v(\varphi_X \otimes \varphi_\alpha)(k) &= \sigma_v(\bar{q}_\alpha \bar{q})^{-1} \varphi_X(n_X) \varphi_\alpha(n_\alpha) u \\ &= \sigma_v(\bar{q})^{-1} \varphi_X(n_X) ({}_uT_v^{(\alpha)}\varphi_\alpha)(k_\alpha) \\ &= S_{Q,v}(\varphi_X \otimes {}_uT_v^{(\alpha)}\varphi_\alpha)(k). \end{aligned}$$

Thus, we have shown that for $k \in K$, ${}_uT_v(\varphi_X \otimes \varphi_\alpha)(k)$ is non-zero if and only if $S_{Q,v}(\varphi_X \otimes {}_uT_v^{(\alpha)}\varphi_\alpha)(k)$ is non-zero, and that the desired equality is valid at such k . In the remaining points k , both functions are zero, hence also equal. $\square$

Lemma 7.5 Suppose that Ω is a bounded connected open subset of $\mathfrak{a}_{P_C}^*$, containing a point ν such that $\operatorname{Re} \nu$ is $\bar{Q}$ -dominant. Let t dominate both the order of $A(\bar{P}, \bar{Q}, \sigma, \cdot)$ over Ω and the order of $A^{(\alpha)}(\bar{P}^{(\alpha)}, \bar{Q}^{(\alpha)}, \sigma, \nu)$ over Ω . Then for $\varphi_X \in C_c^{r+t}(N_X)$ and $\psi \in C_c^{r+t}(K_\alpha/K_P : \sigma_P)$,

$$A(\bar{P}, \bar{Q}, \sigma, \nu) S_{Q, \nu}(\varphi_X \otimes \psi_\alpha) = S_{P, \nu}(\varphi_X \otimes A^{(\alpha)}(\bar{P}^{(\alpha)}, \bar{Q}^{(\alpha)}, \sigma, \nu) \psi_\alpha).$$

as meromorphic functions of $\nu \in \Omega$ with values in $C^r(K/K_P : \sigma)$.

Proof. The statements about meromorphy are well-known, and serve here to allow meromorphic continuation of identities. Let Ω_0 be the set of $\nu \in \Omega$ such that $\operatorname{Re} \nu$ is $\bar{Q}$ -dominant. Then Ω_0 is open and non-empty. For $\nu \in \Omega_0$ the intertwining operators are given by the familiar integral formulas. We use the abbreviated notation $A(\nu)$ and $A^{(\alpha)}(\nu)$ for the above mentioned intertwining operators. Then it suffices to show that

$$A(\nu) {}^{\vee} S_{Q, \nu}(\varphi_X \otimes \psi_\alpha) = {}^{\vee} S_{P, \nu}(\varphi_X \otimes A^{(\alpha)}(\nu) \psi_\alpha)$$

at each point $g := n_X k_\alpha \bar{p} \in N_X K^{(\alpha)} \bar{P}$. Since the elements on both sides of the equation belong to $C^r(G/\bar{P} : \sigma : \nu)$, we may as well assume that $g = n_X k_\alpha$. Then

$$A(\nu) {}^{\vee} S_{P, \nu}(\varphi_X \otimes \psi_\alpha)(g) = \int_{\bar{N}_\alpha} {}^{\vee} S_{P, \nu}(\varphi_X \otimes \psi_\alpha)(n_X k_\alpha \bar{n}_\alpha) d\bar{n}_\alpha.$$

Since $\bar{n}_\alpha \in G^{(\alpha)}$ we may write $\bar{n}_\alpha = \kappa(\bar{n}_\alpha) \bar{p}(n_\alpha)$ with $\kappa(\bar{n}_\alpha) \in K^{(\alpha)}$ and $\bar{p}(\bar{n}_\alpha) \in \bar{P}^{(\alpha)}$ smoothly depending on $\bar{n}_\alpha$. Therefore,

$$\begin{aligned} A(\nu) {}^{\vee} S_{P, \nu}(\varphi_X \otimes \psi_\alpha)(g) &= \int_{\bar{N}_\alpha} \sigma_\nu(\bar{p}(\bar{n}_\alpha))^{-1} {}^{\vee} S_{P, \nu}(\varphi_X \otimes \psi_\alpha)(n_X k_\alpha \kappa(\bar{n}_\alpha)) d\bar{n}_\alpha \\ &= \int_{\bar{N}_\alpha} \sigma_\nu(\bar{p}(\bar{n}_\alpha))^{-1} \varphi_X(n_X) \psi_\alpha(k_\alpha \kappa(\bar{n}_\alpha)) d\bar{n}_\alpha \\ &= \varphi_X(n_X) \int_{\bar{N}_\alpha} \sigma_\nu(\bar{p}(\bar{n}_\alpha))^{-1} \psi_\alpha(k_\alpha \kappa(\bar{n}_\alpha)) d\bar{n}_\alpha \\ &= \varphi_X(n_X) \int_{\bar{N}_\alpha} \psi_\alpha(k_\alpha \bar{n}_\alpha) d\bar{n}_\alpha = \varphi_X(n_X) [A^{(\alpha)}(\nu) \psi_\alpha](k_\alpha) \\ &= {}^{\vee} S_{Q, \nu}(\varphi_X \otimes A^{(\alpha)}(\nu) \psi_\alpha)(g). \end{aligned}$$

□

Proof of Prop. 7.2. Suppose that $u \in H_\sigma^\infty$ is fixed. It suffices to prove the identity for $(\nu, \mu) \in \Omega$, where Ω is an open subset of $\mathfrak{a}_{P_C}^* \times \mathfrak{a}_{P_C}^*$ satisfying the conditions of Lemmas 7.3 and 7.5.

Let $\varphi_X \in C_c^\infty(N_X)$ and $\varphi_\alpha \in C_c^\infty(N_\alpha)$; then

$$\begin{aligned}
& \int_{N_X \times N_\alpha} {}_u\tilde{J}_{\nu,\mu}(n_X, n_\alpha) \varphi_X(n_X) \varphi_\alpha(n_\alpha) dn_X dn_\alpha \quad (7.2) \\
&= {}_uJ_{\nu,\mu}(\varphi_X \otimes \varphi_\alpha) \\
&= \langle J_{\nu,\mu}, {}_uT_{-\bar{\nu}}(\bar{\varphi}_X \otimes \bar{\varphi}_\alpha) \rangle \\
&= \langle J_{\nu,\mu}, S_{Q,-\bar{\nu}}(\bar{\varphi}_X \otimes {}_uT_{-\bar{\nu}}^{(\alpha)}(\bar{\varphi}_\alpha)) \rangle \\
&= \langle A(\bar{Q}, \bar{P}, \sigma, \nu)(\varphi_\mu j_{\nu-\mu}), S_{Q,-\bar{\nu}}(\bar{\varphi}_X \otimes {}_uT_{-\bar{\nu}}^{(\alpha)}(\bar{\varphi}_\alpha)) \rangle \\
&= \langle \varphi_\mu j_{\nu-\mu}, A(-\bar{\nu})S_{Q,-\bar{\nu}}(\bar{\varphi}_X \otimes \bar{\psi}_{\alpha,-\bar{\nu}}) \rangle \\
&= \langle \varphi_\mu j_{\nu-\mu}, S_{P,-\bar{\nu}}(\bar{\varphi}_X \otimes A^{(\alpha)}(-\bar{\nu})\bar{\psi}_{\alpha,-\bar{\nu}}) \rangle \quad (7.3)
\end{aligned}$$

where we have written $A(-\bar{\nu}) = A(\bar{P}, \bar{Q}, \sigma, -\bar{\nu})$, $A^{(\alpha)}(-\bar{\nu}) = A^{(\alpha)}(\bar{P}^{(\alpha)}, \bar{Q}^{(\alpha)}, \sigma, -\bar{\nu})$ and $\bar{\psi}_{\alpha,-\bar{\nu}} = T_{-\bar{\nu}}^{(\alpha)}(\bar{\varphi}_\alpha)$. Put

$$F = S_{P,-\bar{\nu}}(\bar{\varphi}_X \otimes A^{(\alpha)}(-\bar{\nu})\bar{\psi}_{\alpha,-\bar{\nu}}).$$

Then $F \in C^r(K/K_P : \sigma_P)$. Applying Lemma 7.3 with F in place of f , we find that (7.3) equals

$$\langle \varphi_\mu j_{\nu-\mu}, F \rangle = \int_{N_X} \chi(n_X) \langle j_{\nu-\mu}^{(\alpha)}|_{K^{(\alpha)}}, L_{n_X}^{-1}(\varphi_{\bar{\mu}}) L_{n_X}^{-1}(F_{P,\sigma,-\bar{\nu}})|_{K^{(\alpha)}} \rangle dn_X. \quad (7.4)$$

Now

$$F_{P,\sigma,-\bar{\nu}}(n_X k_\alpha) = \bar{\varphi}_X(n_X) A^{(\alpha)}(-\bar{\nu}) \bar{\psi}_{\alpha,-\bar{\nu}}(k_\alpha),$$

so that (7.4) equals

$$\int_{N_X} \chi(n_X) \varphi_X(n_X) \langle j_{\nu-\mu}^{(\alpha)}|_{K^{(\alpha)}}, L_{n_X}^{-1}(\varphi_{\bar{\mu}}) A^{(\alpha)}(-\bar{\nu}) \bar{\psi}_{\alpha,-\bar{\nu}}|_{K^{(\alpha)}} \rangle dn_X. \quad (7.5)$$

The equality of (7.2) with (7.5) for arbitrary $\varphi_X \in C_c^\infty(N_X)$ implies that for every $n_X \in N_X$,

$$\int_{N_\alpha} {}_u\tilde{J}_{\nu,\mu}(n_X, n_\alpha) \varphi_\alpha(n_\alpha) dn_\alpha = \chi(n_X) \langle j_{\nu-\mu}^{(\alpha)}|_{K^{(\alpha)}}, L_{n_X}^{-1}(\varphi_{\bar{\mu}}) A^{(\alpha)}(-\bar{\nu}) \bar{\psi}_{\alpha,-\bar{\nu}}|_{K^{(\alpha)}} \rangle.$$

Substituting $n_X = e$, we find that

$$\begin{aligned}
& \int_{N_\alpha} {}_u\tilde{J}_{\nu,\mu}(e, n_\alpha) \varphi_\alpha(n_\alpha) dn_\alpha \\
&= \langle j_{\nu-\mu}^{(\alpha)}|_{K^{(\alpha)}}, (\varphi_{\bar{\mu}}) A^{(\alpha)}(-\bar{\nu}) \bar{\psi}_{\alpha,-\bar{\nu}}|_{K^{(\alpha)}} \rangle \\
&= \langle A^{(\alpha)}(\bar{Q}^{(\alpha)}, \bar{P}^{(\alpha)}, \sigma, \nu) \varphi_\mu^{(\alpha)} j_{\nu-\mu}^{(\alpha)}, {}_uT_{-\bar{\nu}}^{(\alpha)}(\bar{\varphi}_\alpha) \rangle \\
&= {}_uJ_{\nu,\mu}^{(\alpha)}(\varphi_\alpha) = \int_{N_\alpha} {}_u\tilde{J}_{\nu,\mu}(n_\alpha) \varphi_\alpha(n_\alpha) dn_\alpha.
\end{aligned}$$

As this is valid for any $\varphi_\alpha \in C_c^\infty(N_\alpha)$, we conclude that

$${}_u\tilde{J}_{\nu,\mu}(e, n_\alpha) = {}_u\tilde{J}_{\nu,\mu}^{(\alpha)}(n_\alpha),$$

for all $n_\alpha \in N_\alpha$. □

8 Comparison of B with $B^{(\alpha)}$, proof of Lemma 4.12

We retain the notation of the previous section.

Lemma 8.1 *For every $u \in H_\sigma^\infty$ the assignment $\nu \mapsto {}_u\tilde{J}_{\nu,0}$ is meromorphic $\mathfrak{a}_{P_C}^* \rightarrow C^\infty(N_P)$ and given by*

$${}_u\tilde{J}_{\nu,0}(n) = \chi(n) \langle B(\bar{Q}, \bar{P}, \sigma, \nu) \eta, u \rangle_\sigma, \quad (n \in N_P), \quad (8.1)$$

as an identity of meromorphic functions of the variable ν .

It follows from the assertion about the singular set in Theorem 5.6 that $\nu \mapsto {}_u\tilde{J}_{\nu,0}$ is a genuine meromorphic function of ν with values in $C^\infty(N_P)$. On the other hand, by definition, for regular values of $\nu \in \mathfrak{a}_{P_C}^*$ the following identity of elements of $C^{-\infty}(G/\bar{Q} : \sigma : \nu)_\chi$ is valid:

$$J_{\nu,0} = A(\bar{Q}, \bar{P}, \sigma, \nu) \varphi_0 j(\bar{P}, \sigma, \eta) = j(\bar{Q}, \sigma, \nu) B(\nu) \eta,$$

where $B(\nu) = B(\bar{Q}, \bar{P}, \sigma, \nu)$. From this equality combined with Lemma 1.7 and [2, Thm. 8.6] it follows that for regular ν , the generalized function $J_{\nu,0} \in C^{-\infty}(G/\bar{Q} : \sigma : \nu)$ is continuous H_σ^∞ -valued on the open set $N_P v \bar{Q}$ where it is given by

$$J_{\nu,0}|_{N_P v \bar{Q}}(nvma\bar{n}) = \chi(n) a^{-\nu+\rho_Q} \sigma^{-1}(m) B(\nu) \eta, \quad (n \in N_P, ma\bar{n} \in \bar{Q}),$$

in the sense that the identity is valid after testing with any function φ from $C^\infty(G/\bar{Q} : \sigma : -\bar{\nu})$ whose support is contained in $N_P v \bar{Q}$, i.e.,

$$\langle J_{\nu,0}, \varphi \rangle = \int_K \langle J_{\nu,0}|_{N_P v \bar{Q}}, \varphi \rangle_\sigma(k) dk.$$

Let $u \in H_\sigma^\infty$; then for any $f \in C_c^\infty(N_P)$ the function $\varphi = {}_u T_{-\bar{\nu}}(\bar{f})$ is of this type. The function $F := \langle J_{\nu,0}, {}_u T_{-\bar{\nu}}(\bar{f}) \rangle_\sigma$ is a continuous function $G \rightarrow \mathbb{C}$ which is right $M_Q \bar{N}_Q$ -invariant and transforming according to the rule $R_a F = e^{2\rho_Q} F$. Hence, by Lemma 6.4 its integral over K is given by

$$\langle J_{\nu,0}, {}_u T_{-\bar{\nu}}(\bar{f}) \rangle = \int_{N_P} \langle J_{\nu,0}(nv), {}_u T_{-\bar{\nu}}(\bar{f})(nv) \rangle dn \quad (8.2)$$

$$\begin{aligned} &= \int_{N_P} \langle J_{\nu,0}(nv), \bar{f}(n)u \rangle dn \\ &= \int_{N_P} \chi(n) \langle B(\nu) \eta, u \rangle f(n) dn. \end{aligned} \quad (8.3)$$

The expression on the left-hand side of (8.2) is equal to

$$\langle J_{\nu,0}, {}_u T_{-\bar{\nu}}(\bar{f}) \rangle = {}_u J_{\nu,0}(f) = \int_{N_P} f(n) {}_u \tilde{J}_{\nu,0}(n) dn \quad (8.4)$$

see the text preceding Theorem 5.6. It follows that the integral in 8.4 is equal to the integral in (8.3) for all $f \in C_c^\infty(N_P)$. Since ${}_u \tilde{J}_{\nu,0}$ and χ are continuous functions on N_P the desired identity follows. $\square$

End of proof Lemma 4.12. All arguments presented so far in this section are valid for the triple $G^{(\alpha)}, P^{(\alpha)}, Q^{(\alpha)} = \bar{P}^{(\alpha)}$ in place of G, P, Q . In particular, if $u \in H_\sigma^\infty$, then the function ${}_u\tilde{J}_{\nu,0}^{(\alpha)} \in C^\infty(N_\alpha)$ depends meromorphically on $\nu \in \mathfrak{a}_{P_\mathbb{C}}^*$ and is given by

$${}_u\tilde{J}_{\nu,0}^{(\alpha)}(n) = \chi^{(\alpha)}(n) \langle B^{(\alpha)}(\bar{Q}^{(\alpha)}, \bar{P}^{(\alpha)}, \sigma, \nu)\eta, u \rangle \quad (8.5)$$

for $n \in N_{P^{(\alpha)}} = N_\alpha$. From Proposition 7.2 it follows that

$${}_u\tilde{J}_{\nu,0}(e) = {}_u\tilde{J}_{\nu,0}^{(\alpha)}(e)$$

as meromorphic functions of $(\nu, \mu) \in \mathfrak{a}_{P_\mathbb{C}}^* \times \mathfrak{a}_{P_\mathbb{C}}^*$. Combining this with (8.5) and (8.1) we obtain that

$$\langle B(\bar{Q}, \bar{P}, \sigma, \nu)\eta, u \rangle = \langle B^{(\alpha)}(\bar{Q}^{(\alpha)}, \bar{P}^{(\alpha)}, \sigma, \nu)\eta, u \rangle.$$

Since this holds for every $u \in H_\sigma^\infty$ the proof is complete. $\square$

9 The C-functions and the Maass–Selberg relations

From now on we assume that $\mathfrak{t} \subset \mathfrak{m}$ is maximal abelian, so that $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ is a maximally split Cartan subalgebra of $\mathfrak{g}$. Let $R \in \mathcal{P}$. We write ${}^*\mathfrak{h}_R$ for the orthocomplement of $\mathfrak{a}_R$ in $\mathfrak{h}$. This is a maximally split Cartan subalgebra of $\mathfrak{m}_R$, which decomposes as

$${}^*\mathfrak{h}_R = \mathfrak{t} \oplus {}^*\mathfrak{a}_R.$$

We consider the (τ -spherical) Whittaker integral $\text{Wh}(R, \psi, \nu)$, for $\psi \in \mathcal{A}_{2,R} = \mathcal{A}_2(\tau : M_R/M_R \cap v_R^{-1}N_0v_R : \chi_R)$. If R is non-cuspidal, then $\mathcal{A}_{2,R} = 0$ so that the Whittaker integral is trivial. Therefore, we assume R to be cuspidal.

Let $\Lambda \in {}^*\mathfrak{h}_{R_\mathbb{C}}^*$ be the infinitesimal character of a representation of the discrete series of M_R . For $\varepsilon > 0$ we define $\mathfrak{a}_R^*(\varepsilon) = \{\nu \in \mathfrak{a}_{R_\mathbb{C}}^* \mid |\text{Re}(\nu)| < \varepsilon\}$. For $r, \varepsilon > 0$ we consider the set $\Pi_{\text{hol}}(\Lambda, \mathfrak{a}_R, \varepsilon, r, \tau)$ of families of type Π_{hol} as defined in [3, §7]. This set consists of families $(f_\nu)_{\nu \in \mathfrak{a}_R^*(\varepsilon)}$ of functions $f_\nu \in C^\infty(\tau : G/N_0 : \chi)$ such that

- (a) the function $\nu \mapsto f_\nu$ is holomorphic $\mathfrak{a}_R^*(\varepsilon) \rightarrow C^\infty(\tau : G/N_0 : \chi)$;
- (b) $Zf_\nu = \gamma(Z, \Lambda + \nu)f_\nu \quad (\nu \in \mathfrak{a}_R^*(\varepsilon), Z \in \mathfrak{Z})$;
- (c) for every $u \in U(\mathfrak{g})$ there exist $C > 0$ and $N \in \mathbb{N}$ such that

$$|L_u f_\nu(x)| \leq C_N |(x, \nu)|^N e^{-\rho H(x) + r|\text{Re } \nu||H(x)|}, \quad ((\nu, x) \in \mathfrak{a}_R^*(\varepsilon) \times G).$$

Here $|(x, \nu)| := (1 + |H(x)|)(1 + |\nu|)$.

Lemma 9.1 *Let for $\sigma \in \widehat{M}_{R,\text{ds}}$. There exist constants $\varepsilon > 0, r > 0$ such that for every $\psi_{2,R,\sigma} \in \mathcal{A}_{2,R,\sigma}$ the family $\mathfrak{a}_{R_\mathbb{C}}^* \rightarrow C^\infty(\tau : G/N_0 : \chi)$,*

$$\nu \mapsto \text{Wh}(R, \psi, \nu)$$

belongs to $\Pi_{\text{hol}}(\Lambda, \mathfrak{a}_R, \varepsilon, r, \tau)$; here Λ denotes the infinitesimal character of σ .

Proof. See [3]. $\square$

In particular, the function $\text{Wh}(R, \psi, \nu)$ belongs to the space $\mathcal{A}(\tau : G/N_0 : \chi)$ of τ -spherical tempered Whittaker functions. More precisely, it follows from Lemma 9.1 and from the theory developed in [3] that for $\nu \in i\mathfrak{a}^*$, its constant term along a parabolic subgroup $Q \in \mathcal{P}$, defined as in [7, §1.4], is denoted by $\text{Wh}_Q(R, \psi, \nu)$.

It follows from the theory in [3] that for $\varepsilon > 0$ sufficiently close to zero this constant term extends to a holomorphic function $\text{Wh}_Q(R, \psi, \cdot)$ on $\mathfrak{a}_R^*(\varepsilon)$ with values in $C^\infty(\tau : G/N_0 : \chi)$.

For this constant term to be non-zero for any particular value of $\nu \in i\mathfrak{a}_R^*$, the parabolic subgroup Q needs to be standard (see [7] and [3]), and there needs to be a standard parabolic subgroup P contained in Q such that $P \sim R$ (meaning that $\mathfrak{a}_R$ and $\mathfrak{a}_P$ are conjugate under $W(\mathfrak{a})$). In this case, if $Q \not\sim R$ the function $m \mapsto R_a[\text{Wh}_Q(R, \psi, \nu)]$ is perpendicular to $L_{\text{ds}}^2(\tau : M_Q/M_Q \cap N_0 : \chi)$ for all $a \in A_Q$. If $Q \sim R$, then the function $m \mapsto R_a[\text{Wh}_Q(R, \psi, \nu)]$ belongs to $L_{\text{ds}}^2(\tau : M_Q/M_Q \cap N_0 : \chi)$ for all $a \in A_Q$. In this case, the precise form of the constant term is given in the following result.

If $Q, R \in \mathcal{P}$ then $W(\mathfrak{a}_Q|\mathfrak{a}_R)$ denotes the set of $s \in \text{Hom}(\mathfrak{a}_R, \mathfrak{a}_Q)$ for which there exists a $w \in W(\mathfrak{a})$ such that $s = w|_{\mathfrak{a}_R}$.

Theorem 9.2 *Let $R \in \mathcal{P}$. Then for $\varepsilon > 0$ sufficiently small and for every $Q \in \mathcal{P}_{\text{st}}$ with $Q \sim R$ there exist unique meromorphic functions $C_{Q|R}(s, \cdot)$ on $\mathfrak{a}_{R\mathbb{C}}^*(\varepsilon)$ with values in $\text{Hom}(\mathcal{A}_{2,R}, \mathcal{A}_{2,Q})$ such that for all $\nu \in \mathfrak{a}_{R\mathbb{C}}^*(\varepsilon)$ and $\psi \in \mathcal{A}_{2,R}$, we have*

$$\text{Wh}_Q(R, \psi, \nu)(ma) = \sum_{s \in W(\mathfrak{a}_Q|\mathfrak{a}_R)} a^{s\nu} C_{Q|R}(s, \nu)(\psi)(m), \quad (9.1)$$

for $m \in M_Q$ and $a \in A_Q$, as meromorphic functions of ν .

From now on we will assume that $\varepsilon > 0$ is sufficiently small. We proceed to obtain more detailed information on the C -functions from their characterization through Theorem 9.2.

Lemma 9.3 *Let $Q \in \mathcal{P}_{\text{st}}$. Then for each $\sigma \in \widehat{M}_{Q,\text{ds}}$ appearing as an isotype in $\mathcal{A}_2(\tau : M_Q/M_Q \cap N_0 : \chi_Q)$, and for all $T \in C^\infty(\tau : K/K_Q : \sigma_Q) \otimes H_{\sigma, \chi_Q}^{-\infty}$, we have*

$$C_{Q|Q}(1 : \nu)\psi_T = \psi_{[A(Q, \bar{Q}, \sigma, -\nu) \otimes I]T} \quad (9.2)$$

as an identity of meromorphic functions of $\nu \in \mathfrak{a}_Q^*(\varepsilon)$.

Proof. By linearity we may assume that $T = \varphi \otimes \eta$, with $\varphi \in C^\infty(\tau : K/K_Q : \sigma_Q)$ and $\eta \in H_{\sigma, \chi_Q}^{-\infty}$. Then by Definition 1.20 we have, for all $\nu \in \mathfrak{a}_{Q\mathbb{C}}^*$ with $\langle \text{Re } \nu, \alpha \rangle > 0$ for all

$\alpha \in \Sigma(\mathfrak{a}_Q, \bar{\mathfrak{n}}_Q)$, and for all $(m, a) \in M_Q \times A_Q$,

$$\begin{aligned}
\text{Wh}(Q, \psi_T, \nu)(ma) &= \int_{K/K_Q} \langle \varphi_{\bar{Q}, \sigma, -\nu}(mak), j(\bar{Q}, \sigma, \bar{\nu}, \eta)(k) \rangle dk \\
&= \int_{N_Q} \langle \varphi_{\bar{Q}, \sigma, -\nu}(man), j(\bar{Q}, \sigma, \bar{\nu}, \eta)(n) \rangle dn \\
&= a^{\nu-\rho_Q} \int_{N_Q} \langle \varphi_{\bar{Q}, \sigma, -\nu}(mn), j(\bar{Q}, \sigma, \bar{\nu}, \eta)(a^{-1}na) \rangle dn \\
&= a^{\nu-\rho_Q} \int_{N_Q} \chi(a^{-1}n^{-1}a) \langle \sigma(m)^{-1} \varphi_{\bar{Q}, \sigma, -\nu}(n), \eta \rangle dn.
\end{aligned}$$

The integrand of the final integral may be estimated by $\varepsilon(n) := Ce^{(\nu-\rho_Q)H_{\bar{Q}}(n)}$, with $C > 0$ uniform in $n \in N_Q$ and $a \in A_Q$. Since the mentioned function ε is absolutely integrable over N_Q , it follows by dominated convergence that

$$\begin{aligned}
\lim_{\substack{\bar{Q} \\ a \rightarrow \infty}} a^{-(\nu-\rho_Q)} \text{Wh}(Q, \psi_T, \nu)(ma) &= \int_{N_Q} \langle \sigma(m)^{-1} \varphi_{\bar{Q}, \sigma, -\nu}(n), \eta \rangle dn \\
&= \langle \sigma(m)^{-1} A(Q, \bar{Q}, \sigma, -\nu) \varphi, \eta \rangle = \psi_{[A(Q, \bar{Q}, \sigma, -\nu) \varphi \otimes \eta]}(m).
\end{aligned}$$

Here the limit means that $a^\alpha \rightarrow \infty$ for each $\bar{Q}$ -root α . On the other hand, it follows from (9.1) that for $\nu \in \mathfrak{a}_Q^*(\varepsilon)$ with $\text{Re } \nu$ $\bar{Q}$ -dominant the limit is given by $C_{Q|Q}(1 : \nu) \psi_T$. This establishes (4.2) for all ν in a non-empty open subset of $\mathfrak{a}_Q^*(\varepsilon)$. The validity of (4.2) for all generic $\nu \in \mathfrak{a}_Q^*(\varepsilon)$ follows by application of analytic continuation. $\square$

Lemma 9.4 *Let $P, P' \in \mathcal{P}$ have the same split component and suppose $\sigma \in \widehat{M}_{P, \text{ds}}$. If T is an element of $C^\infty(\tau : K/K_P : \sigma_P) \otimes H_{P, \chi_P}^{-\infty}$, then for generic $\nu \in \mathfrak{a}_P^*(\varepsilon)$,*

$$\text{Wh}(P, \psi_T, \nu) = \text{Wh}(P', \psi_{[A(\bar{P}, \bar{P}', \sigma, -\nu)^{-1} \otimes B(\bar{P}', \bar{P}, \sigma, \bar{\nu})]T}, \nu). \quad (9.3)$$

Proof. We may assume that $T = \varphi \otimes \eta$. It follows from Lemma 2.7 that

$$A(\bar{P}', \bar{P}, \sigma, \nu) j(\bar{P}, \sigma, \nu) \eta = j(\bar{P}', \sigma, \nu) B(\bar{P}', \bar{P}, \sigma, \nu) \eta.$$

Using this and the identity $A(\bar{P}, \bar{P}', \sigma, \bar{\nu})^* = A(\bar{P}', \bar{P}, \sigma, -\nu)$, we infer that

$$\begin{aligned}
\text{Wh}(P, \psi_T, \nu)(x) &= \langle \pi_{\bar{P}, \sigma, -\nu}(x)^{-1} \varphi, j(\bar{P}, \sigma, \bar{\nu}) \eta \rangle \\
&= \langle \pi_{\bar{P}, \sigma, -\nu}(x)^{-1} \varphi, A(\bar{P}', \bar{P}, \sigma, \bar{\nu})^{-1} j(\bar{P}', \sigma, \bar{\nu}) B(\bar{P}', \bar{P}, \sigma, \bar{\nu}) \eta \rangle \\
&= \langle \pi_{\bar{P}', \sigma, -\nu}(x)^{-1} A(\bar{P}, \bar{P}', \sigma, -\nu)^{-1} \varphi, j(\bar{P}', \sigma, \bar{\nu}) B(\bar{P}', \bar{P}, \sigma, \bar{\nu}) \eta \rangle.
\end{aligned}$$

The required identity now follows. $\square$

Lemma 9.5 *Let $P \in \mathcal{P}$ and $Q \in \mathcal{P}_{\text{st}}$ have the same split component and suppose $\sigma \in \widehat{M}_{P,\text{ds}}$. If T is an element of $C^\infty(\tau : K/K_P : \sigma_P) \otimes H_{P,\chi_P}^{-\infty}$, then*

$$C_{Q|P}(1, \nu)\psi_T = \psi_{[A(Q, \bar{P}, \sigma, -\nu) \otimes B(\bar{Q}, \bar{P}, \sigma, \bar{\nu})]T}$$

In particular, $\nu \mapsto C_{Q|P}(1, \nu)$ extends to a meromorphic $\text{Hom}(\mathcal{A}_{2,P}, \mathcal{A}_{2,Q})$ -valued function on $\mathfrak{a}_{P\mathbb{C}}^$.*

Proof. For any $P' \in \mathcal{P}$ with the same split component as P we obtain, by taking the constant terms of the Whittaker integrals in (9.3) along Q and comparing coefficients of exponents,

$$C_{Q|P}(1, \nu)\psi_T = C_{Q|P'}(1, \nu)\psi_{[A(\bar{P}, \bar{P}', \sigma, -\nu)^{-1} \otimes B(\bar{P}', \bar{P}, \sigma, \bar{\nu})]T}$$

In particular, substituting $P' = Q$ and using Lemma 9.3 we obtain

$$\begin{aligned} C_{Q|P}(1, \nu)\psi_T &= \psi_{[A(Q, \bar{Q}, \sigma, -\nu)A(\bar{P}, \bar{Q}, \sigma, -\nu)^{-1} \otimes B(\bar{Q}, \bar{P}, \sigma, \bar{\nu})]T} \\ &= \psi_{[A(Q, \bar{P}, \sigma, -\nu) \otimes B(\bar{Q}, \bar{P}, \sigma, \bar{\nu})]T}. \end{aligned}$$

□

Corollary 9.6 *Let $Q \in \mathcal{P}_{\text{st}}$ and suppose $\sigma \in \widehat{M}_{Q,\text{ds}}$. If T is an element of $C^\infty(\tau : K/K_Q : \sigma_Q) \otimes H_{Q,\chi_Q}^{-\infty}$, then*

$$C_{Q|\bar{Q}}(1, \nu)\psi_T = \psi_{[I \otimes B(\bar{Q}, Q, \sigma, \bar{\nu})]T}, \quad (\nu \in \mathfrak{a}_{Q\mathbb{C}}^*).$$

Proof. This follows from the previous lemma by taking $P = \bar{Q}$, □

The next step is to obtain a formula for $C_{Q|P}(s, \nu)$, for $s \in N_K(\mathfrak{a})$. Let $Q \in \mathcal{P}_{\text{st}}$, $P \in \mathcal{P}$ and let $s \in N_K(\mathfrak{a})$ such that $sPs^{-1} = Q$. Then the right regular action of s defines an intertwining operator $R_s : C^\infty(G/P : \sigma : \nu) \rightarrow C^\infty(G/Q : s\sigma : s\nu)$. According to Cor. 2.2 applied with P, s in place of Q, w there exists an isometric isomorphism

$$\mathcal{R}_{s,P} : H_{\chi_P}^{-\infty} \rightarrow H_{s\sigma, \chi_{sPs^{-1}}}^{-\infty}$$

such that $R_s \circ j(P, \sigma, \nu) = j(sPs^{-1}, s\sigma, s\nu) \circ \mathcal{R}_{s,P}$. Furthermore, R_s induces an isometric isomorphism $\mathcal{R}_s : C^\infty(\tau : K/K_P : \sigma_P) \rightarrow C^\infty(\tau : K/K_Q : s\sigma_Q)$. As in Lemma 1.21 it follows that there exists a unique isometric isomorphism $\underline{\mathcal{R}}_s : \mathcal{A}_{2,P} \rightarrow \mathcal{A}_{2,sPs^{-1}}$, such that for every $\sigma \in \widehat{M}_{P,\text{ds}}$ and every $T \in C^\infty(\tau : K/K_P : \sigma_P) \otimes H_{\sigma, \chi_P}^{-\infty}$

$$\underline{\mathcal{R}}_s(\psi_T) = \psi_{(\mathcal{R}_s \otimes \mathcal{R}_{s,P})T} \in \mathcal{A}_{2,sPs^{-1}, s\sigma}.$$

Lemma 9.7 *Let $P \in \mathcal{P}$ and $s \in N_K(\mathfrak{a})$. Then, for $x \in G$,*

$$\text{Wh}(P, \psi, \nu)(x) = \text{Wh}(sPs^{-1}, \underline{\mathcal{R}}_s\psi, s\nu), \quad (\nu \in \mathfrak{a}_{P\mathbb{C}}^*).$$

Proof. This is derived from the intertwining property of R_s as follows. Let $\sigma \in \widehat{M}_{P,\text{ds}}$ be a type appearing in $\mathcal{A}_{2,P}$. By linearity it suffices to check the identity for $\psi \in \mathcal{A}_{2,P,\sigma}$. Then $\psi = \psi_T$ with $T \in C^\infty(\tau : K/K_P : \sigma_P) \otimes H_{\sigma,\chi_P}^{-\infty}$. By linearity we may assume that $T = \varphi \otimes \eta$. Then

$$\begin{aligned} \text{Wh}(P, \psi, \nu)(x) &= \langle \pi_{\bar{P},\sigma,-\nu}(x)^{-1} \varphi, j(\bar{P}, \sigma, \bar{\nu}) \eta \rangle \\ &= \langle R_s \pi_{\bar{P},\sigma,-\nu}(x)^{-1} \varphi, R_s j(\bar{P}, \sigma, \nu) \eta \rangle \\ &= \langle \pi_{s\bar{P}s^{-1},s\sigma,-s\nu}(x)^{-1} \mathcal{R}_s \varphi, j(s\bar{P}s^{-1}, s\sigma, s\bar{\nu}) \mathcal{R}_{s,P} \eta \rangle \\ &= \text{Wh}(sPs^{-1}, \psi_{\mathcal{R}_s \varphi \otimes \mathcal{R}_{s,P} \eta}, s\nu)(x) \\ &= \text{Wh}(sPs^{-1}, \underline{\mathcal{R}}_s \psi_T, s\nu)(x). \end{aligned}$$

□

Corollary 9.8 *Let $s \in N_K(\mathfrak{a})$ be such that $s(\mathfrak{a}_P) = \mathfrak{a}_Q$. Then*

$$C_{Q|P}(s, \nu) \psi_T = C_{Q|sPs^{-1}}(1, s\nu) \underline{\mathcal{R}}_s \psi_T$$

for generic $\nu \in i\mathfrak{a}_P^*$.

Corollary 9.9 *Let $s \in N_K(\mathfrak{a})$ be such that $sPs^{-1} = Q$, and let $\sigma \in \widehat{M}_{P,\text{ds}}$. Then for $\psi \in \mathcal{A}_{P,2,\sigma}$,*

$$C_{Q|P}(s, -\bar{\nu})^* C_{Q|P}(s, \nu) \psi = \eta(P, \bar{P}, \sigma, -\nu) \psi. \quad (9.4)$$

Proof. It suffices to prove this for $\psi = \psi_T$ with $T = f \otimes v$. Combining Cor. 9.8 and Lemma 9.3 we find

$$C_{Q|P}(s, \nu) \psi_T = C_{Q|Q}(1, s\nu) \psi_T = \psi_{(A(Q, \bar{Q}, s\sigma, -s\nu) \otimes I)T}.$$

Hence,

$$\begin{aligned} C_{Q|P}(s, -\bar{\nu})^* C_{Q|P}(s, \nu) \psi_T &= \psi_{\eta(Q, \bar{Q}, s\sigma, -s\nu) \otimes I)T} = \eta(s^{-1}Qs, s^{-1}\bar{Q}s, \sigma, -\nu) \psi_T \\ &= \eta(P, \bar{P}, \sigma, -\nu) \psi_T. \end{aligned}$$

□

Let $P \in \mathcal{P}$ be a cuspidal parabolic subgroup. We denote by $[P]_{\text{st}}$ the set of $Q \in \mathcal{P}_{\text{st}}$ that are associated with P .

Definition 9.10 The following relations $\text{MSC}(P)$ will be called Maass-Selberg relations for the C -functions of the Whittaker integral $\text{Wh}(P)$

$\text{MSC}(P)$: for all $Q_1, Q_2 \in [P]_{\text{st}}$ and all $s_j \in W(\mathfrak{a}_{Q_j}, \mathfrak{a}_P)$, ($j = 1, 2$),

$$\|C_{Q_1|P}(s_1, \nu) \psi\| = \|C_{Q_2|P}(s_2, \nu) \psi\|, \quad (\psi \in \mathcal{A}_{2,P}), \quad (9.5)$$

for generic $\nu \in i\mathfrak{a}_P^*$.

Lemma 9.11 *The Maass-Selberg relations $\text{MSC}(P)$ for P as stated above are equivalent to the following.*

$\text{MSC}(\mathbf{P})'$: for all $Q \in [P]_{\text{st}}$, all $s \in W(\mathfrak{a}_Q \mid \mathfrak{a}_P)$, all $\sigma \in \widehat{M}_{P,\text{ds}}$ and all generic $\nu \in i\mathfrak{a}_P^*$,

$$C_{Q|P}(s, \nu)^* C_{Q|P}(s, \nu) = \eta(P, \bar{P}, \sigma, -\nu) \text{ on } \mathcal{A}_{2,P,\sigma}.$$

Proof. Assume that $\text{MSC}(\mathbf{P})$ hold. There exists a unique $Q_1 \in \mathcal{P}_{\text{st}}$ which is W -conjugate to P . It is given by $Q_1 = s_1 P s_1^{-1}$, where $s_1 = \nu P$. In particular, $Q_1 \in [P]_{\text{st}}$, $s_1 \in W(\mathfrak{a}_Q \mid \mathfrak{a}_P)$ and it follows from Corollary 9.9 that (9.4) is valid with Q_1, s_1 in place of Q, s , for all $\psi \in \mathcal{A}_{P,2,\sigma}$. For such ψ we find by application of $\text{MSC}(\mathbf{P})$ that

$$\begin{aligned} \langle C_{Q|P}(s, \nu)^* C_{Q|P}(s, \nu) \psi, \psi \rangle &= \langle C_{Q_1|P}(s_1, \nu)^* C_{Q_1|P}(s_1, \nu) \psi, \psi \rangle \\ &= \eta(P, \bar{P}, \sigma, -\nu) \langle \psi, \psi \rangle. \end{aligned}$$

Now $C_{Q|P}(s, \nu)^* C_{Q|P}(s, \nu)$ is Hermitian, and the only eigenvalue of its restriction to $\mathcal{A}_{P,2,\sigma}$ can be $\eta(P, \bar{P}, \sigma, -\nu)$. It follows that the latter Hermitian map is the scalar $\eta(P, \bar{P}, \sigma, -\nu)$ on $\mathcal{A}_{P,2,\sigma}$. Therefore, $\text{MSC}(\mathbf{P})'$ holds.

The converse implication is straightforward. $\square$

We will now compare the Maass–Selberg relations formulated above with those for the B -matrix. Recall from the text following (3.1) that $\eta(P, Q, \sigma, \nu) = \eta(Q, P, \sigma, \nu)$.

Proposition 9.12 *Let $P \in \mathcal{P}$ and $\sigma \in \widehat{M}_{P,\text{ds}}$. Then the following assertions are equivalent, for each $Q \in [P]_{\text{st}}$ and all $s \in W(\mathfrak{a}_Q \mid \mathfrak{a}_P)$.*

- (a) $C_{Q|P}(s, \nu)^* C_{Q|P}(s, \nu) = \eta(P, \bar{P}, \sigma, -\nu)$ on $\mathcal{A}_{P,2,\sigma}$ for generic $\nu \in i\mathfrak{a}_P^*$;
- (b) $B(s^{-1} \bar{Q}s, \bar{P}, \sigma, -\nu)^* B(s^{-1} \bar{Q}s, \bar{P}, \sigma, -\nu) = \eta(s^{-1} Qs, P, \sigma, -\nu)$ on $H_{\sigma, \chi_P}^{-\infty}$, for generic $\nu \in i\mathfrak{a}_P^*$.

Proof. Let $s \in W(\mathfrak{a}_Q \mid \mathfrak{a}_P)$. For $T \in C^\infty(\tau : K/K_P : \sigma_P) \otimes H_{\sigma, \chi_P}^{-\infty}$ we have, by Corollary 9.8 and Lemma 9.5 that

$$\begin{aligned} C_{Q|P}(s : \nu) \psi_T &= C_{Q|sPs^{-1}}(1, s\nu) \psi_-[(\mathcal{R}_s \otimes \mathcal{R}_{s,P})T] \\ &= \psi_-[(A(Q, s\bar{P}s^{-1}, s\sigma, -s\nu) \otimes B(\bar{Q}, s\bar{P}s^{-1}, s\sigma, -s\nu) R_s)T]. \end{aligned}$$

Since $T \mapsto \psi_T$ is unitary from $C^\infty(\tau : K/K_P : \sigma_P) \otimes H_{\sigma, \chi_P}^{-\infty}$ onto $\mathcal{A}_{P,2,\sigma}$, and unitary from $C^\infty(\tau : K/K_P : s\sigma_P) \otimes H_{s\sigma, \chi_P}^{-\infty}$ onto $\mathcal{A}_{2,P,s\sigma}$, it follows from the above that

$$C_{Q|P}(s : \nu)^* \psi_S = \psi_-[R_{s^{-1}}(A(Q, s\bar{P}s^{-1}, s\sigma, -s\nu)^* \otimes B(\bar{Q}, s\bar{P}s^{-1}, s\sigma, s\nu)^*)S].$$

for $S \in C^\infty(\tau : K/K_P : s\sigma_P) \otimes H_{s\sigma, \chi_P}^{-\infty}$. Combining the above, and using that $A(Q, s\bar{P}s^{-1}, s\sigma, -s\nu)^* A(Q, s\bar{P}s^{-1}, s\sigma, -s\nu) = \eta(Q, s\bar{P}s^{-1}, s\sigma, -s\nu)$, we infer

$$C_{Q|P}(s : \nu)^* C_{Q|P}(s : \nu) \psi_T = \psi_-[\eta(Q, s\bar{P}s^{-1}, s\sigma, -s\nu) \otimes b(\nu)]T \quad (9.6)$$

where

$$\begin{aligned} b(\nu) &= R_{s^{-1}} B(\bar{Q}, s\bar{P}s^{-1}, s\sigma, -s\nu)^* B(\bar{Q}, s\bar{P}s^{-1}, s\sigma, -s\nu) R_s \\ &= B(s^{-1} \bar{Q}s, \bar{P}, \sigma, -\nu)^* B(s^{-1} \bar{Q}s, \bar{P}, \sigma, -\nu) \end{aligned}$$

and where we have used the notation $\psi_{-T} := \psi_T$. Suppose now that (a) is valid. Then it follows from (9.6) that $b(\nu)$ must be a multiple of the identity map by the (non-negative real) factor

$$\begin{aligned} \eta(P, \bar{P}, \sigma, -\nu) \eta(Q, s\bar{P}s^{-1}, s\sigma, -s\nu)^{-1} \\ = \eta(P, \bar{P}, \sigma, -\nu) \eta(s^{-1}Qs, \bar{P}, \sigma, -\nu)^{-1} \\ = \eta(s^{-1}Qs, P, \sigma, -\nu), \end{aligned}$$

and (b) follows.

Conversely, suppose that (b) is valid. Then it follows that $b(\nu)$ is the scalar $\eta(s^{-1}Qs, P, \sigma, -\nu)$. From (9.6) we now see that $C_{Q|P}(s, \nu)^* C_{Q|P}(s, \nu)$ is the scalar

$$\begin{aligned} \eta(Q, s\bar{P}s^{-1}, s\sigma, -s\nu) \eta(s^{-1}Qs, P, \sigma, -\nu) \\ = \eta(s^{-1}Qs, \bar{P}, \sigma, -\nu) \eta(s^{-1}Qs, P, \sigma, -\nu) \\ = \eta(\bar{P}, P, \sigma, -\nu). \end{aligned}$$

□

Definition 9.13 The following relations MSB(P) will be called Maass-Selberg relations for the B -matrices associated with P :

MSB(P): for all $Q \in \mathcal{P}$ with $\mathfrak{a}_Q = \mathfrak{a}_P$ and all $\sigma \in \widehat{M}_{P, \text{ds}}$

$$B(\bar{Q}, \bar{P}, \sigma, \nu)^* B(\bar{Q}, \bar{P}, \sigma, \nu) = \eta(\bar{Q}, \bar{P}, \sigma, \nu)$$

for generic $\nu \in i\mathfrak{a}_P^*$.

From Proposition 9.12 we see, for each $P \in \mathcal{P}$, that the validity of the relations MSB(P) implies the validity of the relations MSC(P)'. The converse is not clear a priori, except in the basic setting, where G has compact center and P is maximal. This will be addressed in the next section.

10 Maass–Selberg relations in the basic setting

We consider the basic setting in which G has compact center, and $P \in \mathcal{P}$ is a maximal parabolic subgroup. In this case there is precisely one $Q \in \mathcal{P}$ which is adjacent to P , namely $\bar{P}$. From Proposition 9.12 it follows that the Maass-Selberg relations MSB(P) for the B -matrix imply the relations MSC(P)' for the C -functions, but the converse is not obvious. In the present section we will show that the converse is obvious for the basic setting.

Lemma 10.1 *The following assertions are equivalent.*

- (a) $|W(\mathfrak{a}_P)| = 1$;

(b) $\bar{P}$ is not $W(\mathfrak{a})$ -conjugate to P ;

(c) $[P]_{\text{st}}$ has two elements.

Proof. First of all, by using the action of $W(\mathfrak{a})$ we see that we may as well assume that P is standard.

Suppose (a). Then $\mathfrak{a}_P^+$ and $-\mathfrak{a}_P^+$ are not $W(\mathfrak{a}_P)$ -conjugate hence not $W(\mathfrak{a})$ -conjugate and (b) follows.

Suppose (b), then $\bar{P}$ is $W(\mathfrak{a})$ -conjugate to precisely one $Q \in \mathcal{P}_{\text{st}}$ which we know cannot be P . It follows that $[P]_{\text{st}}$ has at least 2 elements. If R were a third element of $[P]_{\text{st}}$ then there would be an element $s \in W(\mathfrak{a})$ such that $s(\mathfrak{a}_P) = \mathfrak{a}_R$. Then either $s(\mathfrak{a}_P^+) = \mathfrak{a}_R^+$ or $s(-\mathfrak{a}_P^+) = \mathfrak{a}_R^+$. In the first case it would follow that $\mathfrak{a}_Q^+$ and $\mathfrak{a}_R^+$ are $W(\mathfrak{a})$ -conjugate. But then, since $P, R \in \mathcal{P}_{\text{st}}$, it would follow that $R = P$, contradiction. In the second case, it would follow that $Q, \bar{P}$ and R are conjugate under $W(\mathfrak{a})$ hence $R = Q$, contradiction.

Finally, suppose (c). Then there is a parabolic subgroup $Q \in \mathcal{P}_{\text{st}} \setminus \{P\}$ such that $\mathfrak{a}_Q$ is $W(\mathfrak{a})$ -conjugate to $\mathfrak{a}_P$. Hence, $\mathfrak{a}_Q^+$ is conjugate to either $\mathfrak{a}_P^+$ or $-\mathfrak{a}_P^+$. The first cannot be true since then $P = Q$. Therefore, $\mathfrak{a}_P^+$ is not conjugate to $-\mathfrak{a}_P^+$. From this it follows that $-\mathfrak{a}_P^+$ cannot be conjugate to $\mathfrak{a}_P^+$. It follows that $\mathfrak{a}_P^+$ and $-\mathfrak{a}_P^+$ are not conjugate under $W(\mathfrak{a}_P)$. Hence, (a) follows. $\square$

Remark 10.2 It follows from the proof that in any case $[P]_{\text{st}}$ has at most two elements.

Proposition 10.3 Let $P \in \mathcal{P}$ be a maximal parabolic subgroup of G .

(a) If $|W(\mathfrak{a}_P)| = 1$ then $[P]_{\text{st}}$ consists of two distinct elements, $Q_1, Q_2 \in \mathcal{P}_{\text{st}}$. The constant terms of $\text{Wh}(P, \psi, \nu)$ along Q_j , for $j = 1, 2$, are of the form

$$\text{Wh}(P, \psi, \nu)_{Q_j}(ma) = a^{s_j \nu} C_{Q_j|P}(s_j, \nu) \psi(m),$$

with $W(\mathfrak{a}_{Q_j} | \mathfrak{a}_P) = \{s_j\}$. Furthermore, $s_1|_{\mathfrak{a}_P} = -s_2|_{\mathfrak{a}_P}$. In this case the Maass-Selberg relations $\text{MSC}(P)$ are equivalent to

$$\|C_{Q_1|P}(s_1, \nu) \psi\|^2 = \|C_{Q_2|P}(s_2, \nu) \psi\|^2. \quad (10.1)$$

for all $\psi \in \mathcal{A}_{2,P}$ and a dense set of $\nu \in i\mathfrak{a}_P^*$.

(b) If $|W(\mathfrak{a}_P)| = 2$, then $[P]_{\text{st}}$ consists of a single element Q in $\mathcal{P}_{\text{st}}$ and $|W(\mathfrak{a}_Q | \mathfrak{a}_P)| = 2$. The constant term of $\text{Wh}(P, \psi, \nu)$ along Q is of the form

$$\text{Wh}(P, \psi, \nu)_Q(ma) = a^{s\nu} C_{Q|P}(s, \nu) \psi(m) + a^{-s\nu} C_{Q|P}(-s, \nu) \psi(m).$$

where $W(\mathfrak{a}_Q | \mathfrak{a}_P) = \{s, -s\}$. In this case the Maass-Selberg relations $\text{MSC}(P)$ are equivalent to

$$\|C_{Q|P}(s : \nu) \psi\|^2 = \|C_{Q|P}(-s : \nu) \psi\|^2. \quad (10.2)$$

for all $\psi \in \mathcal{A}_{2,R}$ and all regular values of $\nu \in i\mathfrak{a}_P^*$.

Proof. (a) P is conjugate to a standard parabolic subgroup Q_1 . Clearly $Q_1 \in [P]_{\text{st}}$. The latter set has two distinct elements, hence equals $\{Q_1, Q_2\}$ with Q_2 a second maximal parabolic subgroup associated with P . Since $W(\mathfrak{a}_P)$ has a single element, there exist for each $j = 1, 2$ a single element $s_j \in W(\mathfrak{a}_{Q_j}, \mathfrak{a}_P)$. Since Q_1, Q_2 are standard and not equal, they cannot be $W(\mathfrak{a})$ -conjugate. The element $s = s_2 s_1^{-1}$ of $W(\mathfrak{a})$ maps $\mathfrak{a}_{Q_1}$ to $\mathfrak{a}_{Q_2}$ but not $\mathfrak{a}_{Q_1}^+$ to $\mathfrak{a}_{Q_2}^+$. Therefore, there exist a point $X \in \mathfrak{a}_P^+$ such that $s_2^{-1} s_1(X) \notin \mathfrak{a}_P^+$. Since $\mathfrak{a}_P^+$ is one dimensional and since $s_2^{-1} s_1$ is length preserving, it follows that $s_2^{-1} s_1(X) = -X$. Hence, $s_1|_{\mathfrak{a}_P} = -s_2|_{\mathfrak{a}_P}$.

By application of Proposition 9.12 we see that in this case the Maass-Selberg relations associated with P are completely described by (10.1).

We turn to case (b). By Lemma 10.1 there is a unique $Q \in \mathcal{P}_{\text{st}}$ such that $[P]_{\text{st}} = \{Q\}$. Hence $\mathfrak{a}_P$ and $\mathfrak{a}_Q$ are $W(\mathfrak{a})$ -conjugate and $|W(\mathfrak{a}_Q|\mathfrak{a}_P)| = |W(\mathfrak{a}_P)| = 2$. The constant term of $Wh(P, \psi, \nu)$ along Q is described by

$$Wh(P, \psi, \nu)_Q(ma) = \sum_{s \in W(\mathfrak{a}_Q, \mathfrak{a}_P)} a^{s\nu} C_{Q|P}(s : \nu) \psi(m).$$

Take $s \in W(\mathfrak{a}_Q|\mathfrak{a}_P)$, then $W(\mathfrak{a}_Q|\mathfrak{a}_P) = \{\pm s\}$ and it follows that the description of the Maass-Selberg relations is complete. $\square$

Lemma 10.4 *Let G have compact center and let $P \in \mathcal{P}$ be a maximal parabolic subgroup of G . Assume the relations $\text{MSC}(P)$ are valid. Then for every $\sigma \in \widehat{M}_{P, \text{ds}}$ the Maass-Selberg relations $\text{MSB}(P)$ are valid, i.e.*

$$B(P, \bar{P}, \sigma, -\nu)^* B(P, \bar{P}, \sigma, -\nu) = \eta(\bar{P}, P, \sigma, -\nu) \quad (10.3)$$

for generic $\nu \in i\mathfrak{a}_P^*$.

Proof. First assume that we are in case (a): $|W(\mathfrak{a}_P)| = 1$. Then $[P]_{\text{st}} = \{Q_1, Q_2\}$ and s_1, s_2 are as in Proposition 10.3. Fix $\sigma \in \widehat{M}_{P, \text{ds}}$. Then assertion (a) of Prop. 9.12 is valid for each choice $(Q, s) \in \{(Q_1, s_1), (Q_2, s_2)\}$. It follows that assertion (b) is valid for each choice. For one of the choices one has $s^{-1}Qs = \bar{P}$ hence $s^{-1}\bar{Q}s = P$ and the validity of assertion (b) now implies that (10.3).

Next assume that we are in case (b): $|W(\mathfrak{a}_P)| = 2$. Then $[P]_{\text{st}} = [Q]_{\text{st}}$ and there exists a $s \in W(\mathfrak{a}_Q, \mathfrak{a}_P)$ which maps $\mathfrak{a}_P^+$ to $-\mathfrak{a}_Q^+$. Then $sPs^{-1} = \bar{Q}$. The condition (a) is fulfilled hence also (b). We find (10.3). $\square$

Thus, in order to complete the proof of the MS relations for the B -matrix, it suffices to give a proof of the assertions of $\text{MSC}(P)$ for the basic setting, as listed in Proposition 10.3. We will do this, following a method of Harish-Chandra [7] in Sections 11 - 12.

11 The radial part of the Casimir operator

By the Iwasawa decomposition $G = KAN_0$, the multiplication map $m : K \times A \times N_0 \rightarrow G$ is a diffeomorphism. Accordingly, we may define a topological linear isomorphism $T^\uparrow : C^\infty(A, V_\tau) \rightarrow C^\infty(\tau : G/N_0 : \chi)$ by

$$T^\uparrow f(kan) = \chi(n)^{-1} \tau(k) f(a),$$

for $f \in C^\infty(A)$ and $(k, a, n) \in K \times A \times N_0$. The inverse of this isomorphism is given by the restriction map $T^\downarrow : f \mapsto f|_A$.

For an element $u \in U(\mathfrak{g})^{N_0}$ we consider the differential operator R_u on $C^\infty(G, V_\tau)$ given by $R_u(f)(x) := f(x; u)$. Then R_u restricts to a differential operator r_u on $C^\infty(\tau : G/N_0 : \chi)$. The radial part of the latter, denoted $\Pi(u)$, is defined by

$$\Pi(u) = T^\downarrow \circ r_u \circ T^\uparrow.$$

We will determine the radial part of the Casimir element $\Omega \in \mathfrak{Z}$ associated with the invariant symmetric bilinear form B on $\mathfrak{g}$, see (1.6). For each $\alpha \in \Sigma^+$ we fix a basis $X_{\alpha,i}$, $1 \leq i \leq m_\alpha$, which is orthogonal with respect to the positive definite inner product $X, Y \mapsto -B(X, \theta Y)$. Furthermore, we put $X_{-\alpha,i} = -\theta X_{\alpha,i}$. Let $H_\alpha \in \mathfrak{a}$ be defined by $H_\alpha \perp \ker \alpha$ and $\alpha(H_\alpha) = 1$. Then $\alpha = B(H_\alpha, \cdot)$. It is readily seen that

$$[X_{\alpha,i}, X_{-\alpha,i}] = H_\alpha, \quad (1 \leq i \leq m_\alpha).$$

The Casimir operators of $\mathfrak{m}$ and $\mathfrak{a}$, defined relative to the restrictions of B to these Lie algebra's, are denoted by $\Omega_{\mathfrak{m}}$ and $\Omega_{\mathfrak{a}}$. It is now well known that

$$\Omega = \Omega_{\mathfrak{m}} + \Omega_{\mathfrak{a}} + \sum_{\alpha, i} (X_{\alpha,i} X_{-\alpha,i} + X_{-\alpha,i} X_{\alpha,i});$$

here the summation ranges over $\alpha \in \Sigma^+$ and $1 \leq i \leq m_\alpha$. The radial part of an operator $X \in U(\mathfrak{g})^{N_0}$ may be calculated by from a decomposition of the form

$$X = \sum_j f_j(a) z_j^{a^{-1}} u_j v_j, \quad (a \in A),$$

with $f_j \in C^\infty(A)$, $Z_j \in U(\mathfrak{k})$, $u_j \in U(\mathfrak{a})$, $v_j \in U(\mathfrak{n}_0)$. Here the superscript a^{-1} indicates that the image under $\text{Ad}(a)^{-1}$ is taken. Given a decomposition as above the radial component may be expressed by

$$[\Pi(X)\varphi](a) = \sum_j f_j(a) \tau_*(z_j) \varphi(a; u_j) \chi_*(v_j^\vee).$$

Put $Z_{\alpha,i} := X_{\alpha,i} - X_{-\alpha,i}$. Then $Z_{\alpha,i} \in \mathfrak{k}$. Furthermore, for each $\alpha \in \Sigma^+$ and $1 \leq i \leq m_\alpha$,

$$X_{-\alpha,i} = a^{-\alpha} Z_{\alpha,i}^{a^{-1}} + a^{-2\alpha} X_{\alpha,i},$$

It follows from this that

$$X_{-\alpha,i}X_{\alpha,i} = a^{-\alpha}Z_{\alpha,i}^{a^{-1}}X_{\alpha,i} + a^{-2\alpha}X_{\alpha,i}^2.$$

On the other hand,

$$X_{\alpha,i}X_{-\alpha,i} = X_{-\alpha,i}X_{\alpha,i} + H_\alpha.$$

Hence,

$$X_{\alpha,i}X_{-\alpha,i} + X_{-\alpha,i}X_{\alpha,i} = 2X_{-\alpha,i}X_{\alpha,i} + H_\alpha = 2a^{-\alpha}Z_{\alpha,i}^{a^{-1}}X_{\alpha,i} + H_\alpha + 2a^{-2\alpha}X_{\alpha,i}^2$$

Lemma 11.1

$$\Pi(\Omega) = \tau_*(\Omega_{\mathfrak{m}}) + \Omega_{\mathfrak{a}} + \sum_{\alpha} m_{\alpha}H_{\alpha} + \sum_{\alpha,i} -[2a^{-\alpha}\tau(Z_{\alpha,i})\chi_*(X_{\alpha,i}) + 2a^{-2\alpha}\chi_*(X_{\alpha,i})^2]$$

For two functions $f, g \in C^\infty(\tau : G/N_0 : \chi)$ we define the function $[f, g] : G \rightarrow \mathbb{C}$ by

$$[f, g](x) = \langle f(x; \Omega), g(x) \rangle - \langle f(x), g(x; \Omega) \rangle. \quad (11.1)$$

We define $\omega \in U(\mathfrak{a})$ by

$$\omega = \Omega_{\mathfrak{a}} + \sum_{\alpha \in \Sigma^+} m_{\alpha}H_{\alpha}.$$

Lemma 11.2

$$[f, g](a) = \langle f(a; \omega), g(a) \rangle - \langle f(a), g(a; \omega) \rangle.$$

Proof. Since τ and χ are unitary, the operators $\tau_*(Y)$ for $Y \in \mathfrak{k}$ are anti-Hermitian, while $\chi_*(X) \in i\mathbb{R}$ for $X \in \mathfrak{n}_0$. It follows from this that the operators

$$\tau_*(\Omega), \quad \tau(Z_{\alpha,i}), \quad \tau(Z_{\alpha,i})\chi_*(X_{\alpha,i})$$

from $\text{End}(V_\tau)$ are Hermitian, while $\chi_*(X_{\alpha,i})^2 \in \mathbb{R}$. It follows that

$$S(a) := \tau_*(\Omega_{\mathfrak{m}}) + \sum_{\alpha,i} -[2a^{-\alpha}\tau(Z_{\alpha,i})\chi_*(X_{\alpha,i}) + 2a^{-2\alpha}\chi_*(X_{\alpha,i})^2]$$

is Hermitian for all a . Now $\Omega = \omega + S(a)$. Hence

$$\begin{aligned} [f, g](a) &= \langle f(a; \omega) + S(a)f(a), g \rangle - \langle f, g(a; \omega) + S(a)g(a) \rangle \\ &= \langle f(a; \omega), g(a) \rangle - \langle f, g(a; \omega) \rangle + \langle S(a)f(a), g \rangle - \langle f, S(a)g(a) \rangle \\ &= \langle f(a; \omega), g(a) \rangle - \langle f, g(a; \omega) \rangle. \end{aligned}$$

□

Lemma 11.3 $a^\rho \circ \omega \circ a^{-\rho} = \Omega_{\mathfrak{a}} - \langle \rho, \rho \rangle$.

Proof. Let H_j be an orthonormal basis for $\mathfrak{a}$. The dual inner product on $\mathfrak{a}^*$ makes $B : \mathfrak{a} \rightarrow \mathfrak{a}^*$ orthogonal; in particular, $B(H_j)$, for $1 \leq j \leq \ell$, is an orthonormal basis for $\mathfrak{a}^*$. Accordingly, if $\lambda, \mu \in \mathfrak{a}^*$ then $\langle \lambda, B(H_j) \rangle = \lambda(H_j)$ and it follows that

$$\langle \lambda, \mu \rangle = \sum_{1 \leq j \leq \ell} \lambda(H_j) \mu(H_j).$$

The Casimir operator of $\mathfrak{a}$ is given by $\Omega_{\mathfrak{a}} = \sum_{j=1}^{\ell} H_j^2$. Moreover $u \mapsto a^\rho \circ u \circ a^{-\rho}$ equals the algebra automorphism $T = T_{-\rho}$ of $U(\mathfrak{a})$, determined by $T(H) = H - \rho(H)$. From this it follows that

$$T\Omega_{\mathfrak{a}} = \sum_j [H_j - \rho(H_j)]^2 = \Omega_{\mathfrak{a}} - \sum_j 2\rho(H_j)H_j + \langle \rho, \rho \rangle.$$

On the other hand,

$$\sum_j 2\rho(H_j)H_j = \sum_{\alpha > 0} \sum_j m_\alpha \alpha(H_j)H_j = \sum_{\alpha > 0, j} m_\alpha B(H_\alpha, H_j)H_j = \sum_{\alpha > 0} m_\alpha H_\alpha$$

from which

$$T\Omega_{\mathfrak{a}} = \Omega_{\mathfrak{a}} - \sum_{\alpha} m_\alpha H_\alpha + \langle \rho, \rho \rangle.$$

Hence,

$$\begin{aligned} T(\omega) &= T(\Omega_{\mathfrak{a}} + \sum_{\alpha} m_\alpha H_\alpha) \\ &= \Omega_{\mathfrak{a}} - \sum_{\alpha} m_\alpha \rho(H_\alpha) + \langle \rho, \rho \rangle \\ &= \Omega_{\mathfrak{a}} - \langle \rho, \rho \rangle. \end{aligned}$$

□

Let Δ denote the collection of simple roots for the positive system Σ^+ . We define

$${}^\circ \mathfrak{a} := {}^\circ \mathfrak{g} \cap \mathfrak{a}.$$

This is the orthocomplement of the intersection of root hyperplanes $\ker \alpha$, for $\alpha \in \Delta$. Let $\{H_\alpha^0 \mid \alpha \in \Delta\}$ be the B -dual of the basis Δ in ${}^\circ \mathfrak{g}$. This subset of ${}^\circ \mathfrak{a}$ is determined by

$$B(H_\alpha^0, H_\beta) = \delta_{\alpha\beta}, \quad (\alpha, \beta \in \Delta).$$

Lemma 11.4 Suppose G has compact center, then

$$\begin{aligned} &\langle f(a; \Omega_{\mathfrak{a}}), g(a) \rangle - \langle f(a), g(a; \Omega_{\mathfrak{a}}) \rangle \\ &= \sum_{\alpha \in \Delta} R(H_\alpha) (\langle R(H_\alpha^0) f, g \rangle - \langle f, R(H_\alpha^0) g \rangle)(a) \end{aligned}$$

Proof. Since $\{H_\alpha^0\}$ is B -dual to $\{H_\alpha\}$ we have

$$\Omega_{\mathfrak{a}} = \sum_{\alpha \in \Delta} H_\alpha H_\alpha^0.$$

In the following, we will abbreviate $R(H)f$ by Hf . By substituting this in the left hand side of the above equation, and by application of the Leibniz rule for differentiation, we find that the above equation holds provided we add to the right hand side the expression

$$\mathcal{R}(f, g, a) = \sum_{\alpha \in \Delta} \langle H_\alpha^0 f(a), H_\alpha g(a) \rangle - \langle H_\alpha f(a), H_\alpha^0 g(a) \rangle.$$

We will finish the proof by showing that $\mathcal{R}(f, g, a) = 0$. Substituting $H_\alpha = \sum_i B(H_\alpha, H_i)H_i$ and $H_\alpha^0 = \sum_j B(H_\alpha^0, H_j)H_j$, we find that

$$\mathcal{R}(f, g) = \sum_{\alpha, i, j} [B(H_\alpha^0, H_i)B(H_\alpha, H_j) - B(H_\alpha, H_i)B(H_\alpha^0, H_j)] \langle H_i f, H_j g \rangle.$$

Fix i, j . By duality of $\{H_\alpha\}$ and $\{H_\alpha^0\}$, $H_j = \sum_\alpha B(H_\alpha, H_j)H_\alpha^0$. In turn this implies

$$B(H_i, H_j) = \sum_\alpha B(H_\alpha^0, H_i)B(H_\alpha, H_j).$$

By a similar reasoning this identity holds with i and j interchanged. Therefore,

$$\mathcal{R}(f, g) = \sum_{i, j} [B(H_i, H_j) - B(H_j, H_i)] \langle H_i f, H_j g \rangle = 0.$$

□

Given $f, g \in C^\infty(A, V_\tau)$ and $H \in \mathfrak{a}$, we define the function $(f, g)_H : A \rightarrow \mathbb{C}$ by

$$(f, g)_H(a) = d_0(a)^2 [\langle f(a; H), g(a) \rangle - \langle f(a), g(a; H) \rangle], \quad (11.2)$$

for $a \in A$. The following lemma is given without proof in [7, page 208].

Lemma 11.5 *Let $f, g \in C^\infty(\tau : G/N_0 : \chi)$. Then*

$$d_0(a)^2 [f, g](a) = \sum_{\alpha \in \Delta} H_\alpha (f, g)_{H_\alpha^0}(a), \quad (a \in A).$$

Proof. It follows from Lemma 11.2 that

$$d_0(a)^2 [f, g](a) = d_0(a)^2 \langle \omega f(a), g(a) \rangle - \langle f(a), \omega g(a) \rangle.$$

Using Lemma 11.3 we now find that

$$\begin{aligned}
d_0(a)^2[f, g](a) &= \langle \Omega_{\mathfrak{a}}(d_0f), d_0g \rangle(a) - \langle d_0f, \Omega_{\mathfrak{a}}(D_0g) \rangle(a) \\
&= \sum_{\alpha \in \Delta} H_{\alpha} (\langle H_{\alpha}^0 d_0f, d_0g \rangle - \langle d_0f, H_{\alpha}^0 d_0g \rangle)(a) \\
&= \sum_{\alpha \in \Delta} H_{\alpha} d_0^2 (\langle H_{\alpha}^0 f + \rho(H_{\alpha})f, g \rangle(a) - \langle f, H_{\alpha}^0 g + \rho(H_{\alpha})g \rangle(a)) \\
&= \sum_{\alpha \in \Delta} H_{\alpha} d_0^2 (\langle H_{\alpha}^0 f, g \rangle(a) - \langle f, H_{\alpha}^0 g \rangle(a)) \\
&= \sum_{\alpha \in \Delta} H_{\alpha} (f, g)_{H_{\alpha}^0}(a).
\end{aligned}$$

□

12 A result of Harish-Chandra

We retain the assumption that G has compact center.

Let $\mu \in \mathfrak{a}^*$ be defined by $\langle \mu, \alpha \rangle = 1$ for all $\alpha \in \Delta$. Equivalently, $B^{-1}\mu = \sum_{\alpha \in \Delta} H_{\alpha}^0$. For $t > 0$ we define $\mathfrak{a}[t]$ to be the subset of $\mathfrak{a}$ consisting of the points $H \in \mathfrak{a}$ such that for all $\alpha \in \Delta$,

$$\langle H_{\alpha}^0, H \rangle \leq t \text{ and } \mu(H) \geq -t.$$

Clearly, $\mathfrak{a}[t] = t\mathfrak{a}[1]$. We agree to write $A[t] = \exp \mathfrak{a}[t]$ and $G[t] = KA[t]N_0$.

Lemma 12.1 *The set $\mathfrak{a}[1]$ is a compact neighborhood of 0 in $\mathfrak{a}$.*

Proof. Put $\mu_{\alpha} = BH_{\alpha}^0 = \langle H_{\alpha}^0, \cdot \rangle$; then $\mu = \sum_{\alpha \in \Delta} \mu_{\alpha}$. The set $\mathfrak{a}[1]$ is given by the inequalities $\mu_{\alpha} \leq 1$ and $\mu \geq -1$, hence closed, and a neighborhood of 0. It remains to prove its boundedness. If $H \in \mathfrak{a}[1]$ then

$$\mu_{\alpha}(H) = \mu(H) - \sum_{\beta \neq \alpha} \mu_{\beta}(H) \geq -1 - (|\Delta| - 1) = -|\Delta|.$$

Since $\{\mu_{\alpha} \mid \alpha \in \Delta\}$ form a set of linear coordinates for $\mathfrak{a}$, the boundedness follows. □

Note that the argument in fact demonstrates that $\mathfrak{a}[1]$ is an ℓ -dimensional simplex, with $\ell = \dim \mathfrak{a}$.

In the discussion that follows we will make full use of the Euclidean structure on A obtained by transfer of structure under the exponential map $\exp : \mathfrak{a} \rightarrow A$. Our notation will be in terms of the multiplicative group in order to emphasize the connection with the structure of the group G .

For $f, g \in C^{\infty}(\tau : G/N_0 : \chi)$ it is readily checked that the function $[f, g]$ is left K -invariant, and right N_0 -invariant. Hence, for $t > 0$,

$$\int_{G[t]/N_0} [f, g](x) d\bar{x} = \int_{A[t]} d_0(a)^2 [f, g](a) da.$$

Using Lemma 11.5 we find

$$\int_{G[t]/N_0} [f, g](x) d\bar{x} = \int_{A[t]} \sum_{\alpha \in \Delta} H_\alpha(f, g)_{H_\alpha^0}(a) da. \quad (12.1)$$

The integration over $A[t]$ coincides with the Lebesgue integration over $\mathfrak{a}[t]$ and the differentiation on A induced by the right regular action coincides with the usual directional derivative on $\mathfrak{a}$. This makes Gauss' divergence theorem for the simplex $\mathfrak{a}[t]$ in $\mathfrak{a}$ available, and we obtain:

$$\int_{G[t]/N_0} [f, g](x) dx = \sum_{\alpha \in \Delta} \int_{\partial A[t]} \langle \nu, H_\alpha \rangle (f, g)_{H_\alpha^0} ds(a). \quad (12.2)$$

Here ν corresponds to the outward normal vector to the boundary $\partial \mathfrak{a}[t]$ and ds is the $\ell - 1$ dimensional Euclidean Lebesgue measure on the boundary.

We will now introduce some structure that is necessary for a proper understanding of the integral on the right. The boundary $\partial \mathfrak{a}[1]$ of $\mathfrak{a}[1]$ is the union of ℓ simplices $\mathfrak{s}_\gamma$, for $\gamma \in \Delta \cup \{\nu\}$, of dimension $\ell - 1$, namely $\mathfrak{s}_\beta$ for $\beta \in \Delta$ and a remaining simplex $\mathfrak{s}_\mu$. More precisely, $\mathfrak{s}_\beta$ ($\beta \in \Delta$) is the intersection of $\mathfrak{a}[1]$ with the hyperplane $\sigma_\beta := \{H \in \mathfrak{a} \mid \langle H, H_\beta^0 \rangle = |\beta|\}$ and $\mathfrak{s}_\mu$ is the intersection of $\mathfrak{a}[1]$ with the hyperplane $\sigma_\mu := \{H \in \mathfrak{a} \mid \mu(H) = -1\}$. The outward normals are $\nu_{\mathfrak{s}_\beta} = H_\beta^0$ for $\beta \in \Delta$ and $\nu_{\mathfrak{s}_\mu} = -|\mu|^{-1} B^{-1} \mu$. We note that

$$\langle \nu_{\mathfrak{s}_\beta}, H_\alpha \rangle = \delta_{\alpha\beta} \text{ and } \langle \nu_{\mathfrak{s}_\mu}, H_\alpha \rangle = -|\mu|^{-1}.$$

For $t > 0$ we define the multiplication operator $M_t : A \rightarrow A$ by $M_t(\exp H) = \exp tH$ for $H \in \mathfrak{a}$. Then M_t maps $A[1]$ onto $A[t]$ and $\partial A[1]$ onto $\partial A[t]$. Since ν is the outward unit normal, we find that $\nu(M_t a) = \nu(a)$ for $a \in \partial A[1]$. The pull-back of the surface measure ds by M_t is given by $M_t^* ds = t^{\ell-1} ds$.

Write $\widehat{\Delta} = \Delta \cup \{\mu\}$ and put $c_\gamma = 1$ for $\gamma \in \Delta$ and $c_\gamma = -|\mu|^{-1}$ for $\gamma = \mu$. For $\gamma \in \widehat{\Delta}$ we put $S_\gamma = \exp(\mathfrak{s}_\gamma)$. Then $\cup_{\gamma \in \widehat{\Delta}} S_\gamma = \partial A[1]$ and (12.2) takes the following form.

Lemma 12.2 *For $t > 0$ and $f, g \in C^\infty(\tau : G/N_0 : \chi)$,*

$$\int_{G[t]/N_0} [f, g](x) dx = \sum_{\gamma \in \widehat{\Delta}} c_\gamma \int_{M_t S_\gamma} (f, g)_{H_\alpha^0}(a) ds(a).$$

We will investigate the asymptotic behavior of the given integrals over the hypersurfaces $M_t S_\gamma$ as $t \rightarrow \infty$. The dominant asymptotic behavior will come from $\gamma \in \Delta$ and neighborhoods of the point $\exp H_\gamma^0 \in S_\gamma$. The integral for $\gamma = \mu$ will turn out to have exponential decay for $t \rightarrow \infty$. The following lemma suggests the relevance of our discussion for the behavior of the constant terms along maximal parabolic subgroups.

If $f_1, f_2 \in C^\infty(A, V_\tau)$ and $H \in \mathfrak{a}$, we define the function $\langle f_1, f_2 \rangle_H : A \rightarrow \mathbb{C}$ by

$$\langle f_1, f_2 \rangle_H(a) = \langle f_1(a; H), f_2(a) \rangle - \langle f_1(a), f_2(a; H) \rangle, \quad (a \in A). \quad (12.3)$$

The following useful lemma is easy to prove.

Lemma 12.3 *Let $\xi : A \rightarrow]0, \infty$ be a character. Then for $f_1, f_2 \in C^\infty(A, V_\tau)$,*

$$\xi^2 \langle f_1, f_2 \rangle_H = \langle \xi f_1, \xi f_2 \rangle_H.$$

If $f_1, f_2 \in C^\infty(\tau : G/N_0 : \chi)$ and $H \in \mathfrak{a}$ then by using the isomorphism $C^\infty(\tau : G/N_0 : \chi) \simeq C^\infty(A, V_\tau)$ we define the function $\langle f_1, f_2 \rangle_H : A \rightarrow \mathbb{C}$ as above. Note that by (11.2) we have

$$(f_1, f_2)_H = d_0^2 \langle f_1, f_2 \rangle_H = \langle d_0 f_1, d_0 f_2 \rangle_H. \quad (12.4)$$

For $f_1, f_2 \in C^\infty(\tau : M_{1F}/(M_{1F} \cap N_0) : \chi)$ we identify f_1, f_2 with functions in $C^\infty(A, V_\tau)$ and then,

$${}^*d^2 \langle f_1, f_2 \rangle_H = \langle {}^*d f_1, {}^*d f_2 \rangle_H. \quad (12.5)$$

Here ${}^*d(a) = d_0(a)/d_F(a)$. Harish-Chandra [7, p. 211] uses the notation $(f_1, f_2)_H$ for the function in (12.5), which he also used for the different function (12.3). We tried to avoid the confusion that may arise from this.

Let $f_1, f_2 \in \mathcal{A}(\tau : G/N_0 : \chi)$, $\alpha \in \Delta$, $F = F_\alpha = \Delta \setminus \{\alpha\}$. We write f_{jF_α} for f_{jP_F} , for $j = 1, 2$.

Lemma 12.4 *Let U_α be a sufficiently small open neighborhood of $\exp H_\alpha^0$ in S_α . Then there exist $C, \delta > 0$ such that, for $t \geq 0$,*

$$\int_{M_t U_\alpha} \left| (f_1, f_2)_{H_\alpha^0}(a) - \langle {}^*d f_{1F_\alpha}, {}^*d f_{2F_\alpha} \rangle_{H_\alpha^0}(a) \right| ds(a) \leq C e^{-\delta t}.$$

Proof. We write F for $F_\alpha = \Delta \setminus \{\alpha\}$ and define $R_j \in C^\infty(A, V_\tau)$, for $j = 1, 2$, by $R_j(a) = d_F(a) f_j(a) - f_{jF}(a)$. Then

$$(f_1, f_2)_{H_\alpha^0} = \langle d_0 f_1, d_0 f_2 \rangle_{H_\alpha^0} = \langle {}^*d (f_{1F} + R_1), {}^*d (f_{2F} + R_2) \rangle_{H_\alpha^0},$$

hence

$$\begin{aligned} (f_1, f_2)_{H_\alpha^0} - \langle {}^*d f_{1F}, {}^*d f_{2F} \rangle_{H_\alpha^0} &= \\ &= \langle {}^*d f_{1F}, {}^*d R_2 \rangle_{H_\alpha^0} + \langle {}^*d R_1, {}^*d f_{2F} \rangle_{H_\alpha^0} + \langle {}^*d R_1, {}^*d R_2 \rangle_{H_\alpha^0}. \end{aligned} \quad (12.6)$$

By the theory of the constant term, there exists an open neighborhood V of H_α^0 in $\mathfrak{a}$, a constant $\delta_1 > 0$ and for every $X \in U(\mathfrak{a})_1$ a constant $C_1 > 0$ such that for all $H \in V$ and $t \geq 0$, one has, for $j = 1, 2$,

$$|{}^*d(\exp tH) R_j(\exp tH)| \leq C_1 e^{-\delta_1 t}.$$

Replacing V by a smaller open neighborhood if necessary, we may arrange to have in addition an estimate of the form

$$|{}^*d(\exp tH) f_{jF}(\exp tH)| \leq C_2 (1+t)^N, \quad (H \in V, t \geq 0),$$

for $j = 1, 2$. Combining these estimates with (12.6) we find for a fixed $0 < \delta_3 < \delta_1$ that there exists a constant $C_3 > 0$ such that for all $H \in V$ and $t \geq 0$ we have the estimate

$$\left| (f_1, f_2)_{H_\alpha^0}(\exp tH) - \langle {}^*d f_{1F_\alpha}, {}^*d f_{2F_\alpha} \rangle_{H_\alpha^0}(\exp tH) \right| \leq C_3 e^{-\delta_3 t}.$$

Let now $U_\alpha = \exp(V) \cap S_\alpha$, then by pulling back the integration over $M_t U$ by M_t , we find

$$\int_{M_t U_\alpha} \left| (f_1, f_2)_{H_\alpha^0}(a) - \langle {}^*d f_{1F_\alpha}, {}^*d f_{2F_\alpha} \rangle_{H_\alpha^0}(a) \right| ds(a) \leq C_3 e^{-\delta_3 t} (1+t)^{\ell-1} \int_{U_\alpha} ds.$$

The proof is now easily completed. $\square$

Following Harish-Chandra, let $f, g \in \mathcal{A}(\tau : G/N_0 : \chi)$, and suppose that $f_F = 0$ for $F \subset \Delta$ with $|\Delta \setminus F| \geq 2$. For $\alpha \in \Delta$ we define the function $\{f, g\}_\alpha : M_{1F_\alpha} \rightarrow \mathbb{C}$ by

$$\{f, g\}_\alpha(m) := \langle f_{F_\alpha}(m; H_\alpha^0), g_{F_\alpha}(m) \rangle - \langle f_{F_\alpha}(m), g_{F_\alpha}(m; H_\alpha^0) \rangle.$$

For $t \in \mathbb{R}$ we define the function $\{f, g\}_{\alpha, t} : M_{F_\alpha} \rightarrow \mathbb{C}$ by

$$\{f, g\}_{\alpha, t}(m) = \{f, g\}_\alpha(m \exp tH_\alpha^0). \quad (12.7)$$

Furthermore, we consider integral

$$J_\alpha(f, g, t) := \int_{M_{F_\alpha}/(M_{F_\alpha} \cap N_0)} \{f, g\}_{\alpha, t} d\dot{m}. \quad (12.8)$$

Lemma 12.5 *Assume that $f_F = 0$ for $F \subset \Delta$ such that $|\Delta \setminus F| > 1$. Let $\alpha \in \Delta$. Then the integral in (12.8) converges absolutely. If U_α is a sufficiently small neighborhood of $\exp H_\alpha^0$ in S_α then there exist constants $C > 0, \delta > 0$ such that for all $t \geq 0$,*

$$\left| \int_{M_t U_\alpha} (d^*)^2 \langle f_{F_\alpha}, g_{F_\alpha} \rangle_{H_\alpha^0} ds(a) - J_\alpha(f, g, t) \right| \leq C e^{-\delta t}. \quad (12.9)$$

Proof. For $t \in \mathbb{R}$ we define the function $f_{\alpha, t} : M_F \rightarrow V_\tau, m \mapsto f_{F_\alpha}(m \exp tH_\alpha^0)$. The function $g_{F_\alpha, t}$ is defined in a similar way. Both of these functions behave finitely under d/dt , hence can be expressed as

$$f_{F_\alpha, t} = \sum_{\eta \in \mathcal{E}, 0 \leq k \leq n} f_{F_\alpha, \eta, k} t^k e^{t\eta} \quad g_{F_\alpha, t} = \sum_{\eta \in \mathcal{E}, 0 \leq k \leq n} g_{F_\alpha, \eta, k} t^k e^{t\eta}, \quad (12.10)$$

where $\mathcal{E} \subset i\mathbb{R}$ a finite subset and $n \in \mathbb{N}$. Since the exponential polynomial functions $t \mapsto t^k e^{\eta t}$ are linearly independent over $\mathbb{C}$, the expressions in (12.10) are unique. By temperedness of f and g , the functions $f_{F_\alpha, \eta, k}$ and $g_{F_\alpha, \eta, k}$ belong to $\mathcal{A}(\tau : M_F/M_F \cap N_0 : {}^*\chi)$. By transitivity of the constant term, each $f_{F_\alpha, \eta, k}$ has constant term zero along every

proper standard parabolic subgroup of M_F . This implies the existence of constants $\varepsilon > 0$ and $C > 0$ such that for all $\eta \in \mathcal{E}$ and $0 \leq k \leq n$,

$$|f_{F_{\alpha}, \eta, k}(\exp {}^*H) \leq C e^{-\rho({}^*H) - \varepsilon |{}^*H|}, \quad ({}^*H \in {}^*\mathfrak{a}).$$

For g there exist constants $C', N > 0$ such that for all $\eta \in \mathcal{E}$ and $0 \leq k \leq n$ we have the tempered estimates

$$|g_{F_{\alpha}, \eta, k}(\exp {}^*H)| \leq C' (1 + |{}^*H|)^N e^{-\rho({}^*H)}, \quad ({}^*H \in {}^*\mathfrak{a}).$$

It follows from the definitions that, for $m \in M_F$ and $t \in \mathbb{R}$,

$$\{f, g\}_{\alpha, t}(m) = \left\langle \frac{d}{dt} f_{F_{\alpha}, t}(m), g_{F_{\alpha}, t}(m) \right\rangle - \left\langle f_{F_{\alpha}, t}(m), \frac{d}{dt} g_{F_{\alpha}, t}(m) \right\rangle.$$

From the estimates given above, we infer the existence of $C'' > 0$ and $\varepsilon' > 0$ such that for all t ,

$$\{f, g\}_{\alpha, t}(\exp {}^*H) \leq C'' (1 + t)^{2n} e^{-2\rho({}^*H) - \varepsilon' |{}^*H|}, \quad ({}^*H \in {}^*\mathfrak{a}).$$

This implies the estimates

$$|\{f, g\}_{\alpha, t}(m)| \leq C'' (1 + t)^{2n} e^{-2\rho(H(m)) - \varepsilon' |H(m)|}, \quad (m \in M_{F_{\alpha}})$$

so that the integral defining $J_{\alpha}(f, g, t)$, see (12.8), converges absolutely, and

$$|J_{\alpha}(f, g, t)| = O((1 + |t|)^{2n}) \quad (t \in \mathbb{R}).$$

Let now U_{α} be a neighborhood of H_{α}^0 in S_{α} . Then the set $V_{\alpha} := \log U_{\alpha}$ is a neighborhood of H_{α}^0 in $\mathfrak{s}_{\alpha} = \partial \mathfrak{a}[1] \cap (H_{\alpha}^0 + {}^*\mathfrak{a})$. Therefore, $V_{\alpha} = {}^*V_{\alpha} + H_{\alpha}^0$, with ${}^*V_{\alpha} := V_{\alpha} \cap \mathfrak{a}_{F_{\alpha}}^{\perp}$ a neighborhood of 0 in ${}^*\mathfrak{a}$. It follows that $U_{\alpha} = {}^*U_{\alpha} \exp H_{\alpha}^0$ with ${}^*U_{\alpha}$ a neighborhood of e in *A . Hence, $M_t U_{\alpha} = M_t({}^*U_{\alpha}) \exp(tH_{\alpha}^0)$. The Euclidean measure ds on $M_t U_{\alpha}$ is the translate of the Euclidean measure d^*a on $M_t({}^*U_{\alpha})$ by $\exp tH_{\alpha}^0$. Consequently, the integral on the left of (12.9) may be rewritten as

$$\int_{M_t {}^*U_{\alpha}} {}^*d(a)^2 \{f, g\}_{\alpha}({}^*a \exp tH_{\alpha}^0) d^*a = \int_{M_t {}^*U_{\alpha}} {}^*d(a)^2 \{f, g\}_{\alpha, t}({}^*a) d^*a.$$

In view of (12.7) the latter integral may be rewritten as

$$\int_{O_t} \{f, g\}_{\alpha, t}(m) d\bar{m}, \quad (12.11)$$

where O_t is the image of $K_F M_t({}^*U_{\alpha})$ in $M_F/M_F \cap N_0$. The difference of (12.11) with $J_{\alpha}(f, g, t)$ is the integral with O_t replaced by its complement O_t^c in $M_F/M_F \cap N_0$. To finish the proof, it suffices to show that there exists a $\delta > 0$ such that

$$\int_{O_t^c} |\{f, g\}_{\alpha, t}(m)| d\bar{m} = O(e^{-\delta t}).$$

Choose $r > 0$ such that the ball $B_r \subset {}^*\mathfrak{a}$ with center 0 and radius $r > 0$ is contained in $\log {}^*U_\alpha$. Then O_t contains the image of $K_F \exp(tB_r)$ in $M_{1F}/M_{1F} \cap N_0$, hence its complement O_t^c is contained in $K_F \exp(tB_r^c)N_0$, so that

$$\begin{aligned} \int_{O_t^c} |\{f, g\}_{\alpha, t}(m)| d\bar{m} &\leq \int_{\exp tB_r^c} |{}^*d(a)^2 \{f, g\}_{\alpha, t}({}^*a)| d{}^*a \\ &= \int_{{}^*\mathfrak{a} \setminus tB_r} C''(1+t)^{2n} e^{-\varepsilon' |{}^*H|} d{}^*H. \end{aligned}$$

Now fix $0 < \delta < \varepsilon'/r$; then by using polar coordinates one readily checks that there exists a constant $C > 0$ such that the latter integral is bounded by $Ce^{-\delta t}$, for $t \geq 0$. $\square$

Lemma 12.6 *With assumptions as in Lemma 12.5, let H_0 be any point of $\partial\mathfrak{a}[1]$ different from the points H_α^0 , for $\alpha \in \Delta$. Then there exists an open neighborhood U of $\exp H_0$ in $\partial A[1]$ and constants $C, \delta > 0$ such that*

$$\int_{M_t U} \left| (f, g)_{H_\alpha^0}(a) \right| ds(a) \leq Ce^{-\delta t}. \quad (12.12)$$

Proof. We will show that there exists an open neighborhood V of H_0 in $\mathfrak{a}$ and constants $C_1, \delta_1 > 0$ such that for all $H \in V$ and $t \geq 1$ we have the estimate

$$|(f, g)_{H_\alpha^0}(\exp(tH))| \leq C_1 e^{-\delta_1 t}. \quad (12.13)$$

Before proving this estimate we will first show that it implies the required estimate (12.12). Indeed, let $U = V \cap A[1]$. By pulling back under M_t and applying substitution of variables we obtain

$$\begin{aligned} \int_{M_t U} \left| (f, g)_{H_\alpha^0}(a) \right| ds(a) &= \int_U \left| (f, g)_{H_\alpha^0}(M_t a) \right| t^{\ell-1} ds(a) \\ &\leq \int_U C_1 e^{-\delta_1 t} t^{\ell-1} ds(a). \end{aligned}$$

From this the result follows for any $0 < \delta < \delta_1$ and for C suitably chosen.

We now turn to the proof of (12.13). Let $H_0 \in \mathfrak{a}[1]$ and assume that $H_0 \neq H_\alpha^0$ for all $\alpha \in \Delta$. First we assume that $H_0 \notin \text{cl}(\mathfrak{a}^+)$. By the argument of [2, Cor. 2.4] it follows that for any $r > 0$ there exists an open neighborhood V of H_0 and a constant $C' > 0$ such that for φ equal to one of the functions $f, L_{H_\alpha^{0\nu}} f, g$ or $L_{H_\alpha^{0\nu}} g$, we have

$$|\varphi(tH)| \leq C' e^{-rt/2}, \quad (H \in V, t \geq 1).$$

In view of (11.2) the above estimate implies

$$|(f, g)_{H_\alpha^0}| \leq (C')^2 d_0(\exp tH)^{-2} e^{-rt} = (C')^2 e^{-r't}$$

with $r' < r - 2 \sup_{H \in V} \rho(H)$.

Note that if $H_0 \in S_\mu$ then $\mu(H_0) < -1$ so that $H_0 \notin \text{cl}(\mathfrak{a}^+)$ and we are in the setting just discussed.

We now assume that $H_0 \in \text{cl}(\mathfrak{a}^+)$. As just noticed, $H_0 \notin S_\mu$ so that $H_0 \in \cup_{\gamma \in \Delta} S_\gamma$. Let F be the collection of $\alpha \in \Delta$ vanishing on H_0 . Since $H_0 \neq H_\alpha^0$ for every $\alpha \in \Delta$, it follows that $|\Delta \setminus F| > 1$, so that by assumption, $f_F = 0$. It now follows by application of [3, Lemma 3.8] that also $[L_{H_\alpha^{0v}} f]_F = 0$. Thus for φ equal to f or $L_{H_\alpha^{0v}} f$ it follows from the estimation of $d_F \varphi - \varphi_F$ in [3, Lemma 3.5] that for a sufficiently small open neighborhood V of H_0 in $\mathfrak{a}$ there exist constants $C, N, \eta > 0$ such that, for $H \in V$ and $t \geq 0$,

$$|d_F \varphi(tH)| \leq C(1+t)^N e^{-\rho_F(t^*H)} e^{-\eta t}.$$

Hence, for $0 < \eta' < \eta$ there exists $C' > 0$ such that

$$|\varphi(tH)| \leq C' e^{-t\rho(H)-t\eta'}, \quad (H \in V, t' \geq 0).$$

Combining this with the tempered estimates for g and $L_{H_\alpha^{*v}} g$, and using (12.3) we obtain (12.13) with $0 < \delta_1 < \eta'$ and a suitable $C_1 > 0$. $\square$

Theorem 12.7 [Harish-Chandra] *Let $f, g \in \mathcal{A}(\tau : G/N_0 : \chi)$ and assume that $f_F = 0$ for each subset $F \subset \Delta$ with $|\Delta \setminus F| > 1$. Then there exists $\delta > 0$ such that*

$$\int_{G[t]/N_0} [f, g](x) d\bar{x} = \sum_{\alpha \in \Delta} J_\alpha(f, g, t) + O(e^{-\delta t}), \quad (t \rightarrow \infty).$$

Proof. For each $\alpha \in \Delta$ let U_α be an open subset of S_α with the properties of Lemma 12.4. Put $U_\mu = \emptyset$. Then it follows from combining Lemmas 12.2, 12.4 and 12.5, that

$$\begin{aligned} \int_{G[t]/N_0} [f, g](x) d\bar{x} - \sum_{\alpha \in \Delta} J_\alpha(f, g, t) &= \\ &= \sum_{\gamma \in \widehat{\Delta}} c_\gamma \int_{M_t(S_\gamma \setminus U_\gamma)} (f_1, f_2)_{H_\alpha^0}(a) ds(a) + O(e^{-\delta t}). \end{aligned}$$

We now consider the compact set $\mathcal{K} = \partial A[1] \setminus \cup_{\gamma \in \widehat{\Delta}} U_\gamma$. It follows by application of Lemma 12.6 and compactness that there exist constants $C, \delta > 0$ such that the estimate of the lemma is valid with U replaced by $\mathcal{K}$. This implies the existence of $\delta > 0$ such that

$$\begin{aligned} \sum_{\gamma \in \Delta} c_\gamma \int_{M_t(S_\gamma \setminus U_\gamma)} (f_1, f_2)_{H_\alpha^0}(a) ds(a) &= \\ &= \sum_{\gamma \in \Delta} c_\gamma \int_{M_t(S_\gamma \cap \mathcal{K})} (f_1, f_2)_{H_\alpha^0}(a) ds(a) = O(e^{-\delta t}). \end{aligned}$$

We will now show that in the basic setting Theorem 12.7 implies the Maass-Selberg relations, see also [7, p. 206]. In the basic setting, G has compact center. $P \in \mathcal{P}_{\text{st}}$ is a maximal standard parabolic subgroup of G . Suppose $\sigma \in \widehat{M}_{P, \text{ds}}$ and $\psi \in \mathcal{A}_{2, P, \sigma}$. Let $\Lambda \in {}^*\mathfrak{h}_{P_{\mathbb{C}}}^*$ be the infinitesimal character of σ . Then Λ is real and regular in the sense that the inner products with the roots of ${}^*\mathfrak{h}_P$ in $\mathfrak{m}_{P_{\mathbb{C}}}$ are real and non-zero. In addition we fix $\nu \in i\mathfrak{a}_P^*$ such that $\nu \neq 0$, $\Lambda + \nu$ is regular. Put $f = f_{\nu} = \text{Wh}(P, \psi, \nu)$. The infinitesimal character of $\text{Ind}_P^G(\sigma \otimes -\nu)$ is given by $Z \mapsto \gamma(Z, \Lambda - \nu)$. From Definition 1.20 one sees that

$$Zf = R_Z f = \gamma(Z, \Lambda - \nu)f, \quad (Z \in \mathfrak{Z}).$$

It follows that the Casimir Ω acts on f by the real eigenvalue $\langle \Lambda, \Lambda \rangle + \langle \nu, \nu \rangle - \langle \rho_P, \rho_P \rangle$. In turn this implies that

$$[f, f] = 0;$$

see (11.1). Since P is maximal, it follows from the discussion below Lemma 9.1 that the constant term of f_{ν} along a standard parabolic subgroup Q is zero if Q is not maximal. From Theorem 12.7 it now follows that

$$\sum_{\alpha \in \Delta} J_{\alpha}(f, f, t) \rightarrow 0 \quad (t \rightarrow \infty). \quad (12.14)$$

For $\alpha \in \Delta$ let $F_{\alpha} = \Delta \setminus \{\alpha\}$. We write $f_{F_{\alpha}}$ for the constant term of f along the standard parabolic subgroup $P_{F_{\alpha}}$ whose split component is $\mathbb{R}H_{\alpha}^0$.

According to Proposition 10.3 there are two possibilities, (a): $|W(\mathfrak{a}_P)| = 1$ and (b): $|W(\mathfrak{a}_P)| = 2$.

In case (a) there exist precisely two distinct roots $\alpha_1, \alpha_2 \in \Delta$ for which $Q_j := P_{F_{\alpha_j}} \sim P$. Moreover, $W(\mathfrak{a}_{Q_j}|\mathfrak{a}_P) = \{s_j\}$ and $s_2 = -s_1$, so, for $m_j \in M_{F_{\alpha_j}}, t \in \mathbb{R}$,

$$f_{F_{\alpha_1}}(m_1 \exp tH_{\alpha}^0) = e^{i\lambda t} \varphi_1(m_1), \quad f_{F_{\alpha_2}}(m_2 \exp tH_{\alpha}^0) = e^{-i\lambda t} \varphi_2(m_2)$$

with $\lambda = -i\nu(H_{\alpha}^0) \in \mathbb{R} \setminus \{0\}$. It now follows that

$$J_{\alpha_1}(f, f, t) + J_{\alpha_2}(f, f, t) = -\lambda^2 \langle \varphi_1, \varphi_1 \rangle + \lambda^2 \langle \varphi_2, \varphi_2 \rangle \rightarrow 0,$$

from which we conclude that $\|\varphi_1\|^2 = \|\varphi_2\|^2$.

In case (b) we have $|W(\mathfrak{a}_P)| = 2$ and there is precisely one simple root $\alpha \in \Delta$ such that $Q := P_{F_{\alpha}} \sim P$. It follows that $W(\mathfrak{a}_Q|\mathfrak{a}_P)$ consists of two elements, s and $-s$. Moreover, the constant term of f along Q is of the form

$$f_{F_{\alpha}}(m \exp tH_{\alpha}^0) = e^{i\lambda t} \varphi_1(m) + e^{-i\lambda t} \varphi_2(m)$$

for $m \in M_{F_{\alpha}}, t \geq 0$. It follows that

$$\begin{aligned} J_{\alpha}(f, f, t) &= i\lambda \langle e^{i\lambda t} \varphi_1 - e^{-i\lambda t} \varphi_2, e^{i\lambda t} \varphi_1 + e^{-i\lambda t} \varphi_2 \rangle - \\ &\quad - i\lambda \langle e^{i\lambda t} \varphi_1 + e^{-i\lambda t} \varphi_2, e^{i\lambda t} \varphi_1 - e^{-i\lambda t} \varphi_2 \rangle \\ &= 2i\lambda (\|\varphi_1\|^2 - \|\varphi_2\|^2) \end{aligned}$$

From $J_{\alpha}(f, f, t) \rightarrow 0$ it now follows that $\|\varphi_1\|^2 = \|\varphi_2\|^2$. □

Completion of the proof of Theorem 3.1. In view of Proposition 10.3, we have now completed the proof of the Maass-Selberg relations MSC(P) for the basic setting. According to Lemma 10.4 this implies the validity of MSB(P) for the basic setting. By Lemma 4.14 this implies the validity of the Maass-Selberg relations for the B -matrix as formulated in Theorem 3.1. $\square$

Proof of the Maass-Selberg relations MSC(P). By Proposition 9.12 we now conclude the validity of all Maass-Selberg relations for the C -functions as formulated in Lemma 9.11. $\square$

For $P \in \mathcal{P}$ we define the meromorphic function $\eta_*(P, \bar{P}) : \mathfrak{a}_{P_{\mathbb{C}}}^* \rightarrow \text{End}(\mathcal{A}_{2,P})$ by

$$\eta_*(P, \bar{P}, \nu)|_{\mathcal{A}_{2,P,\sigma}} = \eta(P, \bar{P}, \sigma, \nu) \text{id}|_{\mathcal{A}_{2,P,\sigma}}.$$

We may now formulate the validity of the entire collection of Maass-Selberg relations for the C -functions as follows.

Theorem 12.8 *Let $P \in \mathcal{P}$, $Q \in \mathcal{P}_{\text{st}}$ and suppose that $Q \sim P$. Then for each $s \in W(\mathfrak{a}_Q|\mathfrak{a}_P)$,*

$$C_{Q|P}(s, -\bar{\nu})^* C_{Q|P}(s, \nu) = \eta_*(P, \bar{P}, -\nu)$$

as an identity of meromorphic functions in $\nu \in \mathfrak{a}_{P_{\mathbb{C}}}^$.*

Proof. The expressions on both sides of the equation define meromorphic functions of $\nu \in \mathfrak{a}_{P_{\mathbb{C}}}^*$. Hence, it suffices to prove the identity

$$C_{Q|P}(s, \nu)^* C_{Q|P}(s, \nu) = \eta_*(P, \bar{P}, -\nu)$$

for generic $\nu \in i\mathfrak{a}_P^*$. This identity is equivalent to the MSC(P) as formulated in Lemma 9.11, which were proven to be valid in the text preceding the theorem. $\square$

13 The normalized Whittaker integral

For $P \in \mathcal{P}_{\text{st}}$ (standard is mandatory) we consider the meromorphic function $\nu \mapsto C_{P|P}(1, \nu), \mathfrak{a}_{P_{\mathbb{C}}}^* \rightarrow \text{End}(\mathcal{A}_{2,P})$. We recall from Lemma 9.3 that for each $\sigma \in \widehat{M}_{P,\text{ds}}$ and all $T \in C^\infty(\tau : K/K_P : \sigma_P) \otimes H_{\sigma, \chi_P}^{-\infty}$ we have

$$C_{P|P}(1, \nu)\psi_T = \psi_{(A(P, \bar{P}, \sigma, -\nu) \otimes I)T}$$

as meromorphic functions of $\nu \in \mathfrak{a}_{P_{\mathbb{C}}}^*$ with values in $\text{End}(\mathcal{A}_{2,P,\sigma})$. We recall that, for $R > 0$,

$$\mathfrak{a}_P^*(P, R) := \{\nu \in \mathfrak{a}_{P_{\mathbb{C}}}^* \mid \langle \text{Re } \nu, \alpha \rangle > R \ (\forall \alpha \in \Sigma(P))\}.$$

For $Q \in \mathcal{P}$ we define $\Pi_{\Sigma(Q), \mathbb{R}}(\mathfrak{a}_Q^*)$ to be the set of polynomial functions $q \in P(\mathfrak{a}_Q^*)$ which can be written as a product of linear factors of the form $\langle \alpha, \cdot \rangle - c$ with $\alpha \in \Sigma(Q)$ and $c \in \mathbb{R}$.

The following result is due to Harish-Chandra.

Lemma 13.1 *For every $R \in \mathbb{R}$ there exist $C, N > 0$ and a polynomial function $q \in \Pi_{\Sigma(P), \mathbb{R}}(\mathfrak{a}_P^*)$ such that the meromorphic function $v \mapsto q(v)C_{P|P}(1, v)$ is regular on $\mathfrak{a}_P^*(P, R)$ and such that*

$$\|q(v)C_{P|P}(1, v)\|_{\text{op}} \leq C(1 + \|v\|)^N, \quad (v \in \mathfrak{a}_P^*(P, R)).$$

Proof. It suffices to prove this for the restriction of $C_{P|P}(1, v)$ to $\mathcal{A}_{2,P,\sigma}$, for each representation of the finite set of $\sigma \in \widehat{M}_{P,\text{ds}}$ for which $\mathcal{A}_{2,P,\sigma} \neq 0$. Since $T \mapsto \psi_T$ is a linear isomorphism of finite dimensional spaces, it suffices to prove a similar estimate for $A(P, \bar{P}, \sigma, -v)$ restricted to the finite dimensional space $C^\infty(\tau : K/K_P : \sigma_P)$. By equivalence of norms on the latter space, that estimate is a consequence of [4, Cor. 1.4] which in turn is a straightforward consequence of [11], see [12]. $\square$

Lemma 13.2 *Let $\sigma \in \widehat{M}_{P,\text{ds}}$. There exist constants $\varepsilon, C, N > 0$ such that the meromorphic function $v \mapsto \eta(\bar{P}, P, \sigma, v)^{-1}$ is regular on $\mathfrak{a}_P^*(\varepsilon)$ and*

$$|\eta(\bar{O}, P, \sigma, v)^{-1}| \leq C(1 + |v|)^N, \quad (v \in \mathfrak{a}_P^*(\varepsilon)).$$

Proof. This result is due to Harish–Chandra for σ a representation of the discrete series of M_P . His notation for $\eta(\bar{P}, P, \sigma, v)^{-1}$ is $\mu_{P,\sigma}(v)$. Under the weaker assumption that σ is unitary with real infinitesimal character, the same result is proven in [11, p. 235]. $\square$

Recall the definition of $\eta_*(\bar{P}, P, v) \in \text{End}(\mathcal{A}_{2,P})$ in the text preceding Theorem 12.8.

Corollary 13.3 *There exist a polynomial function $q \in \Pi_{\Sigma(P), \mathbb{R}}(\mathfrak{a}_P^*)$ and constants $\varepsilon, C, N > 0$ such that the meromorphic function $v \mapsto q(v)C_{P|P}(1, v)^{-1}$ is regular on $\mathfrak{a}_P^*(\varepsilon)$ and*

$$\|q(v)C_{P|P}(1, v)^{-1}\|_{\text{op}} \leq C(1 + |v|)^N, \quad (v \in \mathfrak{a}_P^*(\varepsilon)).$$

Proof. From Theorem 12.8 it follows that

$$C_{P|P}(1, v)^{-1} = \eta_*(P, \bar{P}, v)^{-1}C_{P|P}(1, -\bar{v})$$

as meromorphic functions of $v \in \mathfrak{a}_{P_{\mathbb{C}}}^*$. The result now follows from Lemmas 13.1 and 13.2. $\square$

For $P \in \mathcal{P}_{\text{st}}$ we define the associated normalized Whittaker integral by

$$\text{Wh}^\circ(P, \psi, v)(x) := \text{Wh}(P, C_{P|P}(1, v)^{-1}\psi, v)(x),$$

for $\psi \in \mathcal{A}_{2,P}$, $v \in \mathfrak{a}_{P_{\mathbb{C}}}^*$, $x \in G$.

Let ε, q as in Corollary 13.3, and let $\sigma \in \widehat{M}_{P, \text{ds}}$. It is readily verified that for suitable $0 < \varepsilon' < \varepsilon, r > 0$ and for $\psi \in \mathcal{A}_{2, P, \sigma}$ the function

$$\nu \mapsto q(\nu) \text{Wh}^\circ(P, \psi, \nu) \in C^\infty(\tau : G/N_0 : \chi) \quad (13.1)$$

belongs to $\Pi_{\text{hol}}(\Lambda_\sigma, \mathfrak{a}_P^*, \varepsilon, r, \tau)$. Here Λ_σ denotes the infinitesimal character of σ .

The constant term of (13.1) along a standard parabolic subgroup $Q \in \mathcal{P}_{\text{st}}$ associated with P is given by

$$q(\nu) \text{Wh}_Q^\circ(P, \psi, \nu, ma) = q(\nu) \sum_{s \in W(\mathfrak{a}_Q | \mathfrak{a}_P)} a^{s\nu} [C_{Q|P}^\circ(s, \nu) \psi](m)(a),$$

for $\nu \in i\mathfrak{a}_P^*, m \in M_Q, a \in A_Q$. Here

$$C_{Q|P}^\circ(s, \nu) := C_{Q|P}(s, \nu) C_{P|P}(1, \nu)^{-1}$$

are meromorphic $\text{Hom}(\mathcal{A}_{2, P}, \mathcal{A}_{2, Q})$ -valued functions of $\nu \in \mathfrak{a}_{P\mathbb{C}}^*$. The following result is an important manifestation of the Maass-Selberg relations.

Lemma 13.4 *Let $P, Q \in \mathcal{P}_{\text{st}}$. Then for all $s \in W(\mathfrak{a}_Q | \mathfrak{a}_P)$,*

$$C_{Q|P}^\circ(s, -\bar{\nu})^* C_{Q|P}^\circ(s, \nu) = \text{id}_{\mathcal{A}_{2, P}}$$

as meromorphic functions of $\nu \in \mathfrak{a}_{P\mathbb{C}}^$.*

Proof. This follows from Theorem 12.8. □

Lemma 13.5

(a) *There exists a constant $\varepsilon > 0$ such that $\nu \mapsto C_{Q|P}^\circ(s, \nu)$ is a holomorphic $\text{Hom}(\mathcal{A}_{2, P}, \mathcal{A}_{2, Q})$ -valued function on $\mathfrak{a}_P^*(\varepsilon)$.*

(b) *The constant $\varepsilon > 0$ can be chosen such that there exist $C, N > 0$ such that*

$$\|C_{Q|P}^\circ(s, \nu)\| \leq C(1 + |\nu|)^N, \quad (\nu \in \mathfrak{a}_P^*(\varepsilon)).$$

(c) *The constant $\varepsilon > 0$ can be chosen such that for all $u \in U(\mathfrak{a}_P^*)$ there exist constants $C_u, N_u > 0$ such that*

$$\|C_{Q|P}^\circ(s, \nu; u)\| \leq C_u(1 + |\nu|)^{N_u}.$$

Proof. From Cor. 13.3 and [3, Lemmas 10.1, 10.2] it follows that there exists a $q \in \Pi_{\Sigma(P)}(\mathfrak{a}_P^*)$ and a constant $\varepsilon > 0$ such that $\Gamma : \nu \mapsto q(\nu) C_{Q|P}^\circ(s, \nu)$ is holomorphic on $\mathfrak{a}_P^*(\varepsilon)$ and satisfies the estimate

$$\|\Gamma(\nu)\|_{\text{op}} \leq C(1 + |\nu|)^N \quad (\nu \in \mathfrak{a}_P^*(\varepsilon)). \quad (13.2)$$

Let $\ell : \nu \mapsto \langle \alpha, \nu \rangle - c$ be a linear factor of q . Then $\ell^{-1}(0)$ consists of $\nu \in \mathfrak{a}_{Q\mathbb{C}}^*$ such that $\text{Im} \langle \alpha, \nu \rangle = \text{Im } c$ and $\text{Re} \langle \nu, \alpha \rangle = \text{Re } c$. If $\text{Re } c \neq 0$ then for $0 < \varepsilon < |\text{Re } c|(1 + |\alpha|)^{-1}$ we have $\ell^{-1}(0) \cap \mathfrak{a}_P^*(\varepsilon) = \emptyset$. Furthermore,

$$|\ell(\nu)|^{-1} \leq (|\text{Re } c| - \varepsilon)^{-1} \quad (\nu \in \mathfrak{a}_P^*(\varepsilon)).$$

We may write $q = q_0 q_1$ with q_0 equal to the product of the linear factors ℓ with $\text{Re } c = 0$ and with q_1 equal to the product of the remaining factors. By choosing $\varepsilon > 0$ sufficiently small we may arrange that $|q_1|^{-1}$ is bounded from above on $\mathfrak{a}_P^*(\varepsilon)$. Then $q_0(\nu)C_{Q|P}^\circ(s, \nu) = q_1(\nu)^{-1}\Gamma(\nu)$ is holomorphic in $\nu \in \mathfrak{a}_P^*(\varepsilon)$ and we have an estimate like (13.2) with $q_1^{-1}\Gamma$ in place of Γ . Thus, we may as well assume that $q = q_0$ from the start.

From Lemma 13.4 it follows that the Hilbert-Schmid norm $\|C_{Q|P}^\circ(s, \nu)\|$ is bounded for $\nu \in i\mathfrak{a}_P^* \setminus q^{-1}(0)$. The latter set is open and dense in $i\mathfrak{a}_P^*$. Let ℓ be a linear factor of q , and let $\nu_0 \in \ell^{-1}(0) \cap i\mathfrak{a}_P^*$. There exists a sequence μ_j in $i\mathfrak{a}_P^* \setminus q^{-1}(0)$ with limit ν_0 . The sequence $\|C_{Q|P}^\circ(s, \mu_j)\|$ is bounded, hence $\Gamma(\mu_j) = q(\mu_j)C_{Q|P}^\circ(s, \mu_j)$ tends to zero for $j \rightarrow \infty$. It follows that $\Gamma(\nu_0) = 0$. Hence $\Gamma = 0$ on $\ell^{-1}(0) \cap i\mathfrak{a}_P^*$. The latter set is a hyperplane in the real linear space $i\mathfrak{a}_P^*$. Furthermore, $\ell^{-1}(0) \cap \mathfrak{a}_P^*(\varepsilon)$ is a connected open part of the complex hyperplane $\ell^{-1}(0)$. By analytic continuation it follows that $\Gamma = 0$ on $\ell^{-1}(0) \cap \mathfrak{a}_P^*(\varepsilon)$. We claim that this implies that $\ell^{-1}\Gamma$ extends to a holomorphic function $\tilde{\Gamma}$ on $\mathfrak{a}_P^*(\varepsilon)$. Indeed, by choosing suitable (affine linear) coordinates z_j on $\mathfrak{a}_{P\mathbb{C}}^*$ we may arrange that $\ell = z_1$. By using local power series expansions we find that z_1 divides Γ .

By a straightforward application of Cauchy's integral formula we infer that $\tilde{\Gamma}$ satisfies an estimate of type (13.2) with $\tilde{\varepsilon} = \varepsilon/2$ in place of ε . Repeating this process we reduce q to a non-zero constant, so that (a) and (b) are valid.

Finally, (c) follows from (b) by an easy application of Cauchy's integral formula.

□

We observe that on account of Lemma 13.5 we have

Lemma 13.6 *Let $P, Q \in \mathcal{P}_{\text{st}}$ be associated and suppose that $s \in W(\mathfrak{a}_Q|\mathfrak{a}_P)$. Then the map*

$$\varphi \mapsto s^{-1*}[C_{Q|P}^\circ(s, \cdot)\varphi]$$

is continuous linear from $\mathcal{S}(i\mathfrak{a}_P^, \mathcal{A}_{2,P})$ to $\mathcal{S}(i\mathfrak{a}_Q^*, \mathcal{A}_{2,Q})$. Here s^{-1*} denotes pull-back under $s^{-1} : i\mathfrak{a}_P^* \rightarrow i\mathfrak{a}_Q^*$.*

Corollary 13.7 *There exist $\varepsilon > 0, r > 0$ such that for each $\sigma \in \widehat{M}_{P,\text{ds}}$ the function $\nu \mapsto Wh^\circ(P, \psi, \nu)$ belongs to $\Pi'_{\text{hol}}(\mathfrak{a}_P, \Lambda_\sigma, \varepsilon, r, \tau)$.*

Proof. This follows from [3, Cor. 11.5], in view of Lemma 13.5. □

14 Fourier transform and Wave packets

Let $P \in \mathcal{P}_{\text{st}}$. Since $\mathcal{A}_{2,P}$ is the finite orthogonal direct sum of the finite dimensional non-zero subspaces $\mathcal{A}_{2,P,\sigma}$ with $\sigma \in \widehat{M}_{P,\text{ds}}$ it follows from Cor. 13.6, that the normalized Whittaker integral $\text{Wh}^\circ(P, \psi)$ satisfies the uniformly tempered estimates of [2, Thm. 16.2].

In analogy with the definition of the Fourier transform $\mathcal{F}_P$ in [2, §16] we define the normalized Fourier transform $\mathcal{F}_P^\circ : C(\tau : G/N_0 : \chi) \rightarrow C^0(i\mathfrak{a}_P^*, \mathcal{A}_{2,P})$ by

$$\langle \mathcal{F}_P^\circ(f)(\nu), \psi \rangle = \langle f, \text{Wh}^\circ(P, \psi, \nu) \rangle_2 := \int_{G/N_0} \langle f(x), \text{Wh}^\circ(P, \psi, \nu)(x) \rangle_\tau d\dot{x},$$

for $\psi \in \mathcal{A}_{2,P}$, $\nu \in i\mathfrak{a}_P^*$.

Theorem 14.1 *The normalized Fourier transform $\mathcal{F}^\circ$ defines a continuous linear map.*

$$\mathcal{F}_P^\circ : C(\tau : G/N_0 : \chi) \rightarrow \mathcal{S}(i\mathfrak{a}_P^*, \mathcal{A}_{2,P}). \quad (14.1)$$

Proof. This result is the analogue of [2, Thm. 16.6]. The proof is identical, provided one uses the uniformly tempered estimates for the normalized Whittaker integral. $\square$

Later on it will be convenient to employ a characterization of the normalized Fourier transform in terms of an integral kernel. For this point of view it is convenient to view the normalized Whittaker integral $\text{Wh}^\circ(P, \cdot, \nu)$, for $P \in \mathcal{P}_{\text{st}}$ and $\nu \in \mathfrak{a}_{P_{\mathbb{C}}}^*$, as a function $G \rightarrow \text{Hom}(\mathcal{A}_{2,P}, V_\tau)$. Accordingly we write

$$\text{Wh}^\circ(P, \nu)(x)\psi := \text{Wh}^\circ(P, \psi, \nu, x),$$

for $x \in G$ and $\psi \in \mathcal{A}_{2,P}$. Note that $\nu \mapsto \text{Wh}^\circ(P, \nu)$ may thus be viewed as a meromorphic function with values in the Fréchet space $C^\infty(G, \text{Hom}(\mathcal{A}_{2,P}, \tau))$.

We adopt the similar point of view for the unnormalized Whittaker integral $\text{Wh}(P, \nu, x)$ and note that the two are related by

$$\text{Wh}^\circ(P, \nu, x) = \text{Wh}(P, \nu, x)C_{P|P}(1, \nu)^{-1}$$

We proceed to the promised characterization of $\mathcal{F}_P^\circ$ with an integral kernel. For $A \in \text{Hom}(\mathcal{A}_{2,P}, V_\tau)$ we denote by A^* the Hermitian adjoint in $\text{Hom}(V_\tau, \mathcal{A}_{2,P})$ with respect to the given Hilbert structures on $\mathcal{A}_{2,P}$ and V_τ . Next, we define the dual Whittaker integral $\text{Wh}^*(P, \nu)$ by

$$\text{Wh}^*(P, \nu)(x) := \text{Wh}^\circ(P, -\bar{\nu}, x)^*, \quad (\nu \in \mathfrak{a}_{P_{\mathbb{C}}}^*, x \in G).$$

We note that $\nu \mapsto \text{Wh}^*(P, \nu)$ is a meromorphic function $\mathfrak{a}_{P_{\mathbb{C}}}^* \rightarrow C^\infty(G, \text{Hom}(V_\tau, \mathcal{A}_{2,P}))$ satisfying the transformation laws

$$\text{Wh}^*(P, \nu, kxn) = \chi(n)\text{Wh}^*(P, \nu, x)\tau(k)^{-1},$$

for $x \in G, k \in K$ and $n \in N_0$. If $f \in C(\tau : G/N_0 : \chi)$ we use the notation $\text{Wh}^*(P, \nu)f$ for the function $G/N_0 \rightarrow \mathcal{A}_{2,P}$ defined by

$$\text{Wh}^*(P, \nu)f : x \mapsto \text{Wh}^*(P, \nu)(x)f(x).$$

It is now readily checked that the normalized Fourier transform (14.1) is given by

$$\mathcal{F}_P^\circ f(\nu) = \int_{G/N_0} \text{Wh}^*(P, \nu, x)f(x) dx, \quad (\nu \in i\mathfrak{a}_P^*). \quad (14.2)$$

We retain the assumption that $P \in \mathcal{P}_{\text{st}}$ and denote by $d\nu$ the Lebesgue measure on the real linear space $i\mathfrak{a}_P^*$, normalized in such a way that the usual Euclidean Fourier transform $\mathcal{F}_e : \mathcal{S}(A_P) \rightarrow \mathcal{S}(i\mathfrak{a}_P^*)$ given by $\mathcal{F}_e(f)(\nu) = \int_A f(a)a^{-\nu} da$ is an isometry for the obvious L^2 -inner products on $\mathcal{S}(A_P)$ and $\mathcal{S}(i\mathfrak{a}_P^*)$.

Definition 14.2 The inverse transform $\mathcal{J}_P : \mathcal{S}(i\mathfrak{a}_P^*, \mathcal{A}_{2,P}) \rightarrow C^\infty(\tau : G/N_0 : \chi)$, also called Wave packet transform, is defined by the formula

$$\mathcal{J}_P \varphi(x) = \int_{i\mathfrak{a}_P^*} \text{Wh}^\circ(P, \nu, x)\varphi(\nu) d\nu, \quad (14.3)$$

for $\varphi \in \mathcal{A}_{2,P}, x \in G$.

We note that by the integral (14.3) is absolutely convergent and defines a smooth function of x in view of the uniformly tempered estimates for the normalized Whittaker integral.

Theorem 14.3 *The normalized Wave packet transform $\mathcal{J}_P$, for $P \in \mathcal{P}$, defines a continuous linear map*

$$\mathcal{J}_P : \mathcal{S}(i\mathfrak{a}_P^*, \mathcal{A}_{2,P}) \rightarrow C(\tau : G/N_0 : \chi).$$

Proof. We fix $\sigma \in \widehat{M}_{P,\text{ds}}$ such that $\mathcal{A}_{2,P,\sigma} \neq 0$. Let ψ be an element of the latter space. Then by linearity and finite dimensionality of $\mathcal{A}_{2,P}$ it suffices to show that the map

$$\varphi \mapsto \int_{i\mathfrak{a}_P^*} \varphi(\nu) \text{Wh}^\circ(P, \psi, \nu, x) d\nu$$

is continuous linear $\mathcal{S}(i\mathfrak{a}_P^*) \rightarrow C(\tau : G/N_0 : \chi)$. Since $\text{Wh}^\circ(P, \psi)$ is a family in $\Pi'_{\text{hol}}(\mathfrak{a}_P^*, \Lambda_\sigma, \varepsilon, r, \tau)$, see Cor. 13.7, this follows from [2, Thm. 12.1]. $\square$

Lemma 14.4 *Let $P \in \mathcal{P}_{\text{st}}$. The transforms $\mathcal{F}_P^\circ$ and $\mathcal{J}_P$ are conjugate in the sense that*

$$\langle \mathcal{F}_P^\circ f, \varphi \rangle_2 = \langle f, \mathcal{J}_P \varphi \rangle_2,$$

for $f \in C(\tau : G/N_0 : \chi)$ and $\varphi \in \mathcal{S}(i\mathfrak{a}_P^*) \otimes \mathcal{A}_{2,P}$.

Proof. The brackets on the left indicate the L^2 -type inner product on $L^2(\mathfrak{ia}_P^*, \mathcal{A}_{2,P})$ and the brackets on the right indicated the inner product on $L^2(\tau : G/N_0 : \chi)$. The inclusions $\mathcal{S}(\mathfrak{ia}_P^*, \mathcal{A}_{2,P}) \rightarrow L^2(\mathfrak{ia}_P^*, \mathcal{A}_{2,P})$ and $C(\tau : G/N_0 : \chi) \rightarrow L^2(\tau : G/N_0 : \chi)$ are continuous linear. It follows that the pairings $(f, \varphi) \mapsto \langle \mathcal{F}_P^\circ f, \varphi \rangle$ and $(f, \varphi) \mapsto \langle f, \mathcal{I}_P \varphi \rangle$ are continuous sesquilinear $C(\tau : G/N_0 : \chi) \times \mathcal{S}(\mathfrak{ia}_P^*, \mathcal{A}_{2,P}) \rightarrow \mathbb{C}$. By density of $C_c^\infty(\tau : G/N_0 : \chi)$ in $C(\tau : G/N_0 : \chi)$ and of $C_c^\infty(\mathfrak{ia}_P^*, \mathcal{A}_{2,P})$ in $\mathcal{S}(\mathfrak{ia}_P^*, \mathcal{A}_{2,P})$ it suffices to prove the identity for $f \in C_c^\infty(\tau : G/N_0 : \chi)$ and $\varphi \in C_c^\infty(\mathfrak{ia}_P^*, \mathcal{A}_{2,P})$. For such f and φ we have

$$\begin{aligned} \langle \mathcal{F}_P^\circ f, \varphi \rangle_2 &= \int_{\mathfrak{ia}_P^*} \left\langle \int_{G/N_0} \text{Wh}^*(P, \nu, x) f(x) dx, \varphi(\nu) \right\rangle d\nu \\ &= \int_{\mathfrak{ia}_P^*} \int_{G/N_0} \langle \text{Wh}^*(P, \nu, x) f(x), \varphi(\nu) \rangle dx d\nu \\ &= \int_{G/N_0} \int_{\mathfrak{ia}_P^*} \langle f(x), \text{Wh}^\circ(P, \nu, x) \varphi(\nu) \rangle d\nu dx \\ &= \langle f, \mathcal{I}_P \varphi \rangle_2 \end{aligned}$$

□

15 The functional equations

Based on the results of the previous section, we shall now derive Harish-Chandra's functional equation, see [7, §1.7].

Lemma 15.1 *Let $P, Q \in \mathcal{P}_{\text{st}}$ be associated and let $s \in W(\mathfrak{a}_Q \mid \mathfrak{a}_P)$. Then*

$$\text{Wh}^\circ(P, \nu) = \text{Wh}^\circ(Q, s\nu) C_{Q|P}^\circ(s, \nu) \quad (15.1)$$

as an identity of meromorphic functions of the variable $\nu \in \mathfrak{a}_{P_C}^$.*

Proof. We will first establish the existence of a meromorphic function $F : \mathfrak{a}_{P_C}^* \rightarrow \text{Hom}(\mathcal{A}_{2,P}, \mathcal{A}_{2,Q})$ such that

$$\text{Wh}^\circ(P, \nu) = \text{Wh}^\circ(Q, s\nu) F(\nu), \quad (\nu \in \mathfrak{a}_{P_C}^*). \quad (15.2)$$

Indeed, by Lemma 9.7 there exists an element $F_s \in \text{Hom}(\mathcal{A}_{2,P}, \mathcal{A}_{2,Q})$ such that

$$\text{Wh}(P, \nu) = \text{Wh}(sPs^{-1}, s\nu) F_s, \quad (\nu \in \mathfrak{a}_{P_C}^*).$$

Since sPs^{-1} and Q have the same split component, it follows from Lemma 9.4 that there exists a meromorphic $G : \mathfrak{a}_{Q_C}^* \rightarrow \text{End}(\mathcal{A}_{2,Q})$ such that

$$\text{Wh}(s^{-1}Ps, s\nu) = \text{Wh}(Q, s\nu) G(\nu).$$

Combining these assertions it follows that (15.2) is valid with everywhere Wh in place of Wh° and with $G(\nu)F_s$ in place of $F(\nu)$. If we combine this observation with the definition of the normalized Whittaker integrals, we find that (15.2) is valid with

$$F(\nu) := C_{P|P}^\circ(1, \nu)^{-1} \circ G(\nu) \circ F_s \circ C_{Q|Q}^\circ(1, s\nu), \quad (\nu \in \mathfrak{a}_{P\mathbb{C}}^*).$$

Let $\Omega \subset i\mathfrak{a}_Q^*$ be the set of all $\nu \in i\mathfrak{a}_Q^*$ which are a regular point of all meromorphic functions in the above expression for $\tilde{F}$, and for which the elements $v\nu$, for $v \in W(\mathfrak{a}_Q)$, are mutually distinct. Then Ω is open dense in $i\mathfrak{a}_Q^*$. For $\nu \in \Omega$, the functions on the left and right of (15.2) are tempered and belong to $\mathcal{A}(\tau : G/N_0 : \chi)$. By taking the constant terms of these functions along Q and comparing the appearing exponential functions with exponent $s\nu$, we find that $F(\nu) = C_{Q|P}^\circ(s, \nu)$ for all $\nu \in \Omega$. By analytic continuation this identity is valid as an identity of meromorphic functions of $\nu \in \mathfrak{a}_{Q\mathbb{C}}^*$, and (15.1) follows. $\square$

Corollary 15.2 *Let $P, Q, R \in \mathcal{P}_{\text{st}}$ all be associated to each other. For all $s \in W(\mathfrak{a}_Q | \mathfrak{a}_P)$ and $t \in W(\mathfrak{a}_R | \mathfrak{a}_Q)$,*

$$C_{R|P}^\circ(ts, \nu) = C_{R|Q}^\circ(t, s\nu) C_{Q|P}^\circ(s, \nu) \quad (15.3)$$

as an identity of meromorphic functions of $\nu \in \mathfrak{a}_{P\mathbb{C}}^$.*

Proof. Let $\nu \in i\mathfrak{a}_P^*$ be a regular point for each of the three meromorphic functions appearing in equation (15.1). Then (15.3) follows by taking the constant terms along R of the tempered functions on both sides of (15.1). The proof is finished by application of analytic continuation. $\square$

The functional equations have important consequences for the Fourier and Wave packet transforms.

Corollary 15.3 *Let $P, Q \in \mathcal{P}_{\text{st}}$ be associated and suppose that $s \in W(\mathfrak{a}_Q | \mathfrak{a}_P)$. Then for all $f \in C(\tau : G/N_0 : \chi)$,*

$$C_{Q|P}^\circ(s, \nu) \mathcal{F}_P^\circ f(\nu) = \mathcal{F}_Q^\circ f(s\nu), \quad (\nu \in i\mathfrak{a}_P^*) \quad (15.4)$$

Proof. From Lemma 15.1 we deduce that, for $\nu \in i\mathfrak{a}_P^*$,

$$\text{Wh}^*(P, \nu) = C_{Q|P}^\circ(s, -\bar{\nu})^* \text{Wh}^*(Q, s\nu) = C_{Q|P}^\circ(s, \nu)^{-1} \text{Wh}^*(Q, s\nu).$$

This implies that

$$C_{Q|P}^\circ(s, \nu) \text{Wh}^*(P, \nu) = \text{Wh}^*(Q, s\nu).$$

The identity (15.4) now follows in view of the characterization of the normalized Fourier transform in (14.2). $\square$

Corollary 15.4 *Let $P, Q \in \mathcal{P}_{\text{st}}$ be associated and suppose that $s \in W(\mathfrak{a}_Q | \mathfrak{a}_P)$. Then for all $\varphi \in S(\mathfrak{ia}_P^*, \mathcal{A}_{2,P})$,*

$$\mathcal{J}_P \varphi = \mathcal{J}_Q s^{-1*} [C_{Q|P}^\circ(s : \cdot) \varphi].$$

Here s^{-1*} denotes pull-back by $s^{-1} : \mathfrak{ia}_Q^* \rightarrow \mathfrak{ia}_P^*$.

Proof. By using the definition of $\mathcal{J}_P$ and Lemma 15.1, taking into account Lemma 13.6 we find

$$\begin{aligned} \mathcal{J}_P \varphi &= \int_{\mathfrak{ia}_P^*} \text{Wh}^\circ(Q, s\nu) C_{Q|P}^\circ(s, \nu) \varphi(\nu) \, d\nu \\ &= \int_{\mathfrak{ia}_Q^*} \text{Wh}^\circ(Q, \nu) s^{-1*} [C_{Q|P}^\circ(s, \cdot) \varphi](\nu) \, d\nu. \end{aligned}$$

Now use the definition of $\mathcal{J}_Q$. □

Corollary 15.5 *Let $P, Q \in \mathcal{P}_{\text{st}}$. Then*

$$\mathcal{J}_P \mathcal{F}_P^\circ f = \mathcal{J}_Q \mathcal{F}_Q^\circ f, \quad (f \in C(\tau : G/N_0 : \chi)).$$

Proof. By Corollaries 15.4 and 15.3 we have

$$\mathcal{J}_P \mathcal{F}_P^\circ f = \mathcal{J}_Q s^{-1*} [C_{Q|P}^\circ(s, \cdot) \mathcal{F}_P^\circ f] = \mathcal{J}_Q s^{-1*} [s^*(\mathcal{F}_Q^\circ f)] = \mathcal{J}_Q \mathcal{F}_Q^\circ f. \quad \square$$

16 Appendix: criterion of smoothness for distributions

In this appendix, we will prove a result needed in Section 5. Let U be a smooth manifold of dimension n . For $\mathcal{K} \subset U$ a compact subset and $u \in \mathcal{D}'(U) = C_c^\infty(U)'$ we denote by $u_{\mathcal{K}}$ the restriction of u to $C_{\mathcal{K}}^\infty(\mathbb{R}^n)$. We will say that u has order r on $\mathcal{K}$ if $u_{\mathcal{K}}$ extends to a continuous linear functional on $C_{\mathcal{K}}^r(U)$.

We will write $C_{\mathcal{K}}^\infty(U)$ for the Fréchet space of smooth functions $\varphi \in C^\infty(U)$ with support contained in $\mathcal{K}$. For $p \in \mathbb{N}$ we have $C_{\mathcal{K}}^\infty(U) \subset C_{\mathcal{K}}^p(U)$. The latter space has a natural Banach topology, for which the inclusion is continuous with dense image. Transposition induces a natural inclusion of the dual spaces equipped with the strong dual topologies,

$$C_{\mathcal{K}}^p(U)' \subset C_{\mathcal{K}}^\infty(U)'.$$

The topology on the first of these spaces is the dual Banach topology.

Let Ω be an open subset of $\mathbb{R}^q$ and suppose a map $T : \Omega \rightarrow C_c^\infty(U)'$, $y \mapsto T_y = T(y)$ is given. For a compact subset $\mathcal{K} \subset U$ we define

$$T_{\mathcal{K}} : \Omega \rightarrow C_{\mathcal{K}}^\infty(U)' \quad (16.1)$$

by

$$T_{\mathcal{K}}(y) := T_y|_{C_{\mathcal{K}}^\infty(U)}, \quad (x \in U).$$

We say that $T_{\mathcal{K}}$ maps to the Banach space $C_{\mathcal{K}}^p(U)'$ if $T_{\mathcal{K}}(y) \in C_{\mathcal{K}}^p(U)'$ for all $y \in \Omega$.

Let $v_1, \dots, v_q$ be a collection of smooth vector fields on U , such that for each $x \in U$ the vectors $v_j(x)$ span the tangent space $T_x U$. If $\alpha \in \mathbb{N}^q$, we denote by v^α the differential operator $v_1^{\alpha_1} \cdots v_q^{\alpha_q}$ on U .

Theorem 16.1 *Let $\Omega \subset \mathbb{R}^n$ be open and let $T : \Omega \rightarrow C_c^\infty(U)'$ be a map.*

If for each $\alpha \in \mathbb{N}^q$ and every compact set $\mathcal{K} \subset U$ the map

$$y \mapsto [v^\alpha(T_y)]_{\mathcal{K}} \quad (16.2)$$

is continuous from Ω to the Banach space $C_{\mathcal{K}}^p(U)'$, then

- (a) *for every $y \in \Omega$ the density T_y is of the form $\tau_y dx$ with $\tau_y \in C^\infty(U)$ and dx a smooth positive density on U ;*
- (b) *for each $\alpha \in \mathbb{N}^n$ the function $y \mapsto \partial^\alpha(\tau_y)$, $\Omega \rightarrow C^\infty(U)$, is continuous.*

If $\Omega \subset \mathbb{R}^k \times \mathbb{C}^\ell$ and for every compact $\mathcal{K} \subset U$ the map (16.2) is continuous from Ω to the Banach space $C_{\mathcal{K}}^p(U)'$ and in addition holomorphic in the variable from $\mathbb{C}^\ell$, then for each $\alpha \in \mathbb{N}^q$ the function $y \mapsto v^\alpha(\tau_y)$, $\Omega \rightarrow C^\infty(U)$, is continuous and in addition holomorphic in the variable from $\mathbb{C}^\ell$.

Proof. If $\chi \in C_c^\infty(U)$ then by a straightforward application of the Leibniz rule for differentiation, it follows that the map $\tilde{T} : y \mapsto \chi T_y$ fulfills the hypotheses with $\tilde{T}$ in place of T . Clearly, it suffices to prove the conclusions (a), (b) and the final assertion with $\tilde{T}$ in place of T , for any choice of χ .

Therefore, we may as well assume that there exists a compact set $\mathcal{K}_0$ contained in the interior of a compact subset $\mathcal{K} \subset U$ such that $\text{supp } T_y \subset \mathcal{K}_0$ for all $y \in \Omega$. Then the hypothesis that for every $\alpha \in \mathbb{N}^n$ the function (8.2) is continuous is fulfilled for this particular $\mathcal{K}$. We will keep the sets $\mathcal{K}_0$ and $\mathcal{K}$ fixed from now on and fix a cut off function $\chi \in C_{\mathcal{K}}^\infty(\mathbb{R}^n)$ which is 1 on an open neighborhood of $\mathcal{K}_0$. Then $\chi T_o = T_o$ for every distribution $T_o \in C^\infty(U)'$ with support contained in $\mathcal{K}_0$. In particular this is true for $T_o = T_y$ ($y \in \Omega$).

We denote by $\mathcal{F}$ the Euclidean Fourier transform which on a function f from the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is given by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\xi \cdot x} dx, \quad (\xi \in \mathbb{R}^n).$$

This operator is a topological linear isomorphism from $\mathcal{S}(\mathbb{R})$ onto itself. The inverse transform is given by $\mathcal{S}(\mathbb{R}^n) \ni g \mapsto S\mathcal{F}(g)$, where $Sg(x) = g(-x)$. (No constant is appearing provided Lebesgue measure is replaced by a suitable positive multiple.)

On the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions, the Fourier transform is given by transposition of the transform on Schwartz functions, hence a topological linear isomorphism from the space $\mathcal{S}'(\mathbb{R}^n)$ onto itself; it is also denoted $\mathcal{F}$. This is compatible with the notation for Schwartz functions if we embed $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$ by $f \mapsto f dx$, where dx denotes the standard smooth density on $\mathbb{R}^n$. On the space of tempered distributions, the inverse Fourier transform is given by the transpose of $S\mathcal{F}$.

For every $y \in \Omega$ the distribution T_y is compactly supported, with support in $\mathcal{K}_0$. Let $T_o \in C_c^\infty(U)'$ be any distribution with compact support contained in $\mathcal{K}_0$ and such that $(\partial^\alpha T_o)_\mathcal{K}$ belongs to $C_\mathcal{K}^p(U)'$, for every $\alpha \in \mathbb{N}^n$. In particular, such a T_o is a tempered distribution and the associated Euclidean Fourier transform is the tempered distribution given by the analytic function $\mathbb{R}^n \rightarrow \mathbb{C}$ defined by

$$\mathcal{F}(T_o) : \xi \mapsto T_o(e^{-i\xi}), \quad (\xi \in \mathbb{R}^n).$$

Here $e^{-i\xi}$ denotes the exponential function $x \mapsto e^{-i\xi \cdot x}$, $\mathbb{R}^n \rightarrow \mathbb{C}$. By the inversion formula,

$$T_o = \mathcal{F} \circ S(\mathcal{F}T_o). \quad (16.3)$$

We fix a norm $\|\cdot\|_{\mathcal{K},r}$ which gives the Banach topology on $C_\mathcal{K}^p(U)$,

$$\|\varphi\|_{\mathcal{K},p} := \max_{|\alpha| \leq p} \sup_{\mathcal{K}} |\partial^\alpha \varphi|,$$

and we denote the dual norm on $C_\mathcal{K}^p(U)'$ by $\|\cdot\|^*$. One readily verifies that there exists a constant $c > 0$ such that for every $\xi \in \mathbb{R}^n$,

$$\|\chi e^{-i\xi}\|_{\mathcal{K},p} \leq c(1 + \|\xi\|)^p, \quad (\xi \in \mathbb{R}^n).$$

Suppose now that for every $\alpha \in \mathbb{N}^n$ the distribution $\partial^\alpha(T_o)$ satisfies $(\partial^\alpha T_o)_\mathcal{K} \in C_\mathcal{K}^p(U)'$. Put $\Delta := \sum_{j=1}^n \partial_j^2$. Then for every $N \in \mathbb{N}$ and all $\xi \in \mathbb{R}^n$,

$$\begin{aligned} |(1 + \|\xi\|^2)^N \mathcal{F}(T_o)(\xi)| &= |\mathcal{F}((1 - \Delta)^N T_o)(\xi)| \\ &= |\langle (1 - \Delta)^N T_o, \chi e^{-i\xi} \rangle| \\ &\leq \|(1 - \Delta)^N T_o\|^* \|\chi e^{-i\xi}\|_{\mathcal{K},p} \leq C_N(T_o)(1 + \|\xi\|)^p, \end{aligned}$$

where

$$C_N(T_o) := c\|(1 - \Delta)^N T_o\|^*.$$

This leads to the estimate

$$|\mathcal{F}(T_o)(\xi)| \leq C_N(T_o)(1 + \|\xi\|^2)^{p/2-N}, \quad (\xi \in \mathbb{R}^n) \quad (16.4)$$

for every $N \in \mathbb{N}$. It follows from this estimate that the inverse Fourier transform of $\mathcal{F}(T)$ is the continuous density given by $T_\circ = \tau_\circ dx$ where dx is the standard smooth density on $\mathbb{R}^n$, and where

$$\tau_\circ(x) = S\mathcal{F}\mathcal{F}(T_\circ)(x) = \int_{\mathbb{R}^n} e^{i\xi \cdot x} \mathcal{F}(T_\circ)(\xi) d\xi. \quad (16.5)$$

Using the estimate (16.4) – with N sufficiently large – for domination under the integral sign, we infer that $\tau_\circ$ is a smooth function and for each $\beta \in \mathbb{N}^n$, the derivative $\partial^\beta \tau_\circ$ is given by differentiation under the integral sign. Substituting T_y for $T_\circ$ in the resulting expression, and writing $T_y = \tau_y dx$, we obtain

$$\partial^\beta \tau_y(x) = \int_{\mathbb{R}^n} e^{i\xi \cdot x} (i\xi)^\beta \mathcal{F}(T_y)(\xi) d\xi, \quad (y \in \Omega). \quad (16.6)$$

We note that $y \mapsto (T_y)_\mathcal{K}$ is continuous $\Omega \rightarrow C_\mathcal{K}^p(U)'$ by assumption. On the other hand, it is straightforward that $\xi \mapsto \chi e^{-i\xi}$ is continuous $\mathbb{R}^n \rightarrow C_\mathcal{K}^p(U)$. Since the natural pairing $C_\mathcal{K}^p(U)' \times C_\mathcal{K}^p(U) \rightarrow \mathbb{C}$ is continuous bilinear it follows that the map

$$\Omega \times \mathbb{R}^n \rightarrow \mathbb{C}, (y, \xi) \mapsto \langle T_y, \chi e^{-i\xi} \rangle = \mathcal{F}(T_y)(\xi)$$

is continuous. It now easily follows that the integrand $I(y, x, \xi)$ of (16.6) is a continuous function of $(y, x, \xi) \in \Omega \times U \times \mathbb{R}^n$. On the other hand, the integrand is dominated by

$$C_N(T_y)(1 + \|\xi\|)^{|\beta|} (1 + \|\xi\|)^2)^{p/2-N} \quad (16.7)$$

while $C_N(T_y)$ is locally bounded in y . Since this is valid for N arbitrarily high, we conclude that the function $(y, x) \mapsto \partial^\beta \tau_y(x)$ belongs to $C(\Omega \times U)$. This in turn is equivalent to the assertion that $y \mapsto \partial^\beta \tau_y$ is continuous from Ω to $C(U)$. As $\beta \in \mathbb{N}^n$ was arbitrary we conclude that $y \mapsto \tau_y$ is continuous $\Omega \rightarrow C^\infty(U)$.

We now turn to the statement about holomorphy. Write $y = (z, \lambda)$ according to the decomposition $\mathbb{R}^q \simeq \mathbb{R}^k \times \mathbb{C}^\ell$. It remains to prove the final assertion about holomorphy in the variable λ . For this we first investigate the holomorphy of $\mathcal{F}(T_\lambda)$.

By assumption the map $y \mapsto T_y$ is continuous $\Omega \rightarrow C_\mathcal{K}^p(U)'$ and holomorphic in λ . This implies that $(y, \xi) \mapsto \langle T_y, e^{-i\xi} \rangle$ is continuous $\Omega \times \mathbb{R}^n \rightarrow \mathbb{C}$, with holomorphy in the variable λ . The integrand $I(y, x, \xi)$ introduced above is continuous, and holomorphic in λ , while it is dominated by (16.7). It is a well-known result that this implies that $(x, y) \mapsto \partial^\beta (\tau_y)(x)$ is holomorphic in the variable λ . It follows from this that the map $y \rightarrow \tau_y$ is continuous from Ω to $C^\infty(U)$ and holomorphic in the variable λ . $\square$

17 Appendix: divergence for a convex polyhedron

This appendix gives a rigorous proof of Gauss' divergence theorem for a compact convex polyhedral set in Euclidean space. For a more systematic treatment of Stokes' theorem on manifolds with singularities, we refer the reader to [8].

By an affine hyperplane σ in $\mathbb{R}^n$ we mean a translate of a linear subspace of codimension 1. Its complement $\mathbb{R}^n \setminus \sigma$ is a disjoint union of two open half-spaces. The closures of these are called the closed half-spaces associated with σ . The latter can be retrieved as the intersection of its closed half-spaces.

Any affine hyperplane can be described by a formula of the form $\sigma = \{x \in \mathbb{R}^n \mid \xi(x) = c\}$, where $\xi \in \mathbb{R}^{n*} \setminus \{0\}$ and $c \in \mathbb{R}$. The associated closed half-spaces are then described by $\xi \leq c$ and by $\xi \geq c$.

In this section we assume that C is a compact convex polyhedral subset of $\mathbb{R}^n$, i.e., a compact finite intersection of closed half-spaces.

Assume that C has non-empty interior. Then by a hyperplane facet of C we mean an affine hyperplane σ such that C is contained in precisely one of the two closed half-spaces determined by σ and such that $C_\sigma := C \cap \sigma$ has non-empty interior (denoted by C_σ°) as a subset of σ . The sets C_σ and C_σ° will be called the closed and open facets associated with σ . Note that σ is the affine span of the open facet C_σ° . The collection $\Sigma = \Sigma(C)$ of affine facets of C is finite. Furthermore, it is readily verified that

$$\partial C = \bigcup_{\sigma \in \Sigma} C_\sigma$$

If σ_1 and σ_2 are distinct affine facets, then $C_{\sigma_1}^\circ \cap C_{\sigma_2}^\circ = \emptyset$. If $\sigma \in \Sigma$ then by ν_σ we denote the unit vector in $\sigma^\perp$ which points away from the closed half-space associated with σ that contains C . We write $\nu : \partial C \rightarrow \mathbb{R}^n$ for the partially defined function determined by $\nu(s) = \nu_\sigma$, for $s \in C_\sigma^\circ$. This function is called the outward unit normal to ∂C .

If $f : U \rightarrow \mathbb{R}$ is a C^1 -function on an open subset $U \subset \mathbb{R}^n$ and $H \in \mathbb{R}^n$ then we define the directional derivative $\partial_H f$ by $\partial_H f(x) = d/dt f(x + tH)|_{t=0}$, ($x \in U$).

Lemma 17.1 *Let $f : C \rightarrow \mathbb{R}$ be a continuous function which is partially differentiable on $\text{int}(C)$ with partial derivatives that extend continuously to C . Then for every $H \in \mathbb{R}^n$,*

$$\int_C \partial_H f(x) dx = \int_{\partial C} \langle \nu(s), H \rangle f(s) ds.$$

Remark 17.2 Let Σ denote the set of affine facets of ∂C . Then

$$\int_{\partial C} \langle \nu(s), H \rangle f(s) ds = \sum_{\sigma \in \Sigma} \langle \nu_\sigma, H \rangle \int_{C \cap \sigma} f(s) ds.$$

The lemma will be proven in the rest of this section. We start with investigating partial differentiation for integration over $\mathbb{R}_+^n$, where $\mathbb{R}_+ =]0, \infty[$.

Lemma 17.3 *Let $f : \mathbb{R}_+^n \rightarrow \mathbb{R}$ be a C^1 -function, with partial derivatives up to order 1 extending continuously to $[0, \infty[^n$ and with support bounded in $\mathbb{R}^n$. Then for every $H \in \mathbb{R}^n$,*

$$\int_{\mathbb{R}_+^n} \partial_H f(x) dx = \sum_{j=1}^n \int_{\mathbb{R}_+^{j-1} \times \mathbb{R}_+^{n-j}} -H_j f(x, 0, y) dx dy.$$

Proof. This is elementary from Fubini, and the formula $-f(x, 0, y) = \int_0^\infty \partial_t f(x, t, y) dt$.
 $\square$

As a next step, we consider the cone Γ in $\mathbb{R}^n$ generated by an n -tuple of distinct vectors $\gamma_1, \dots, \gamma_n$. We denote by $\partial_j \Gamma$ the cone generated by the points $\gamma_k, k \neq j$. Then $\partial \Gamma$ is the union of the cones $\partial_j \Gamma$. The outward unit normal to $\partial_j \Gamma$ is denoted by ν_j .

Lemma 17.4 *Let $f : \text{int}(\Gamma) \rightarrow \mathbb{R}$ be a C^1 function with partial derivatives up to order 1 that extend continuously to Γ . In addition it is assumed that f has bounded support. Then*

$$\int_{\Gamma} \partial_H f(x) dx = \sum_{j=1}^n \langle \nu_j, H \rangle \int_{\partial_j \Gamma} f(s) ds.$$

Proof. The previous lemma is a special case. To obtain the more general result from the present lemma, let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the linear map determined by $Te_j = \gamma_j$, for $1 \leq j \leq n$. Then $T(\mathbb{R}_+^n) = \Gamma$. The boundary part $\partial_j \Gamma$ is the image under T of the boundary part $\partial_j := \partial_j \mathbb{R}_+^n$. The outward normal vector n_j at points of the interior of ∂_j relative to the affine span of ∂_j is related to ν_j as follows.

A half-line $x + \mathbb{R}_+ H$ emanates from $\partial_j \Gamma$ in the outward direction if and only if $T^{-1}x + \mathbb{R}_+ T^{-1}H$ emanates from ∂_j in the outward direction. From this it follows that for all $H \in \mathbb{R}^n \setminus \{0\}$, we have that $\langle \nu_j, H \rangle > 0$ if and only if $\langle n_j, T^{-1}H \rangle > 0$. This implies that there exists a constant $c_j > 0$ such that

$$\nu_j = c_j T^{-1*}(n_j). \quad (17.1)$$

Since ν_j has unit length it follows that $c_j = \|T^{-1*}n_j\|^{-1}$.

By linear substitution of variables and application of the previous lemma we obtain

$$\begin{aligned} \int_{\Gamma} \partial_H f(x) dx &= \int_{\mathbb{R}_+^n} \partial_H f(T(z)) dz \\ &= |\det T| \int_{\mathbb{R}_+^n} \partial_{T^{-1}H} [T^* f](z) dz \\ &= |\det T| \sum_{j=1}^n \int_{\mathbb{R}_+^{j-1} \times \mathbb{R}_+^{n-j}} -(T^{-1}H)_j (T^* f)(x, 0, y) dx dy, \end{aligned} \quad (17.2)$$

where $\hat{x}^j = (x_1 \dots x_{j-1} x_{j+1} \dots x_n)$. We consider the j -th term, and compare with the surface integral over $\partial_j \Gamma$. Let $U_j := \mathbb{R}_+^{j-1} \times \{0\} \times \mathbb{R}_+^{n-j}$. Then a regular parametrisation of $\partial_j \Gamma$ is given by the map $T_j : U_j \rightarrow \partial_j \Gamma, (x, y) \mapsto T(x, 0, y)$. Now

$$\int_{\partial_j \Gamma} \langle \nu_j, H \rangle f(s) ds = \int_{U_j} \langle \nu_j, H \rangle f(T(x, 0, y)) |J_j(T)| dx dy, \quad (17.3)$$

where $J_j(T) = Te_1 \times \cdots \times \widehat{Te_j} \times \cdots \times Te_n$. By the definition of the exterior product, we have, for all $v \in \mathbb{R}^n$, that

$$\begin{aligned} \langle J_j(T), v \rangle &= \det(v, Te_1, \dots, \widehat{Te_j}, \dots, Te_n) \\ &= \det T \cdot \det(T^{-1}v, e_1, \dots, \widehat{e_j}, \dots, e_n) \\ &= (-1)^{j-1} \det T \cdot (T^{-1}v)_j = (-1)^j \cdot \det T \cdot \langle n_j, T^{-1}v \rangle, \end{aligned}$$

from which we infer that $J_j(T) = (-1)^j \det T \cdot T^{-1*}n_j$, so that $c_j|J_j(T)| = |\det T|$, see the line below (17.1). Using (17.1) it now follows that

$$\langle v_j, H \rangle |J_j(T)| = c_j \langle n_j, T^{-1}H \rangle |J_j(T)| = \langle n_j, T^{-1}H \rangle |\det T|.$$

Hence the integrals on the right of (17.2) and of (17.3) are identical. $\square$

Completion of the proof of Lemma 17.1. By an easy translation argument, Lemma 17.1 is seen to be valid with domain $a + \Gamma$ ($a \in \mathbb{R}^n$) in place of Γ .

We will now give the proof for C an n -dimensional simplex in $\mathbb{R}^n$, i.e., the convex hull $\text{co}(a_0, \dots, a_n)$ of $n+1$ points in $\mathbb{R}^n$ whose affine span is $\mathbb{R}^n$. For $0 \leq j \leq n$ we denote by Γ_j the cone in $\mathbb{R}^n$ spanned by $a_i - a_j$, for $i \neq j$. Then $C \subset a_j + \Gamma_j$.

By a simple argument there exists a cover of C by bounded open subsets $O_j \subset \mathbb{R}^n$, for $0 \leq j \leq n$, such that $a_j \in O_j$ and $\overline{O_j} \cap \text{co}(a_0, \dots, \widehat{a_j}, \dots, a_n) = \emptyset$. We fix a C^1 -partition of unity $\{\psi_j \mid 0 \leq j \leq n\}$ over C subordinate to the given cover. Thus, $\psi_j \in C_c^1(O_j)$, $0 \leq \psi_j \leq 1$, and $\sum_{j=0}^n \psi_j = 1$ on an open neighborhood of C . Let f satisfy the hypothesis of Lemma 17.1 and define, for each j , $f_j = \psi_j f$. Then $f = \sum_{j=0}^n f_j$ and by linearity it suffices to prove the assertion of the lemma for each f_j ; fix j .

The function $f_j : \text{int}(C) \rightarrow \mathbb{R}$ is continuous and its partial derivatives up to order 1 extend continuously to C . Moreover, since $f_j = 0$ on an open neighborhood of $\text{co}(a_0, \dots, \widehat{a_j}, \dots, a_n)$ it follows that the extension of f_j to $a_j + \Gamma_j$, by requiring it to be zero outside C has bounded support and its partial derivatives of order at most 1 extend continuously to $a_j + \Gamma_j$. By the first part of this proof, it follows that

$$\int_{a_j + \Gamma_j} \partial_H f_j(x) dx = \int_{a_j + \partial\Gamma_j} \langle v_{\Gamma_j}, H \rangle f(s) ds.$$

The integrands of both integrals are zero on an open neighborhood of $(a_j + \Gamma \setminus C)$. Therefore,

$$\int_C \partial_H f_j(x) dx = \int_{C \cap (a_j + \partial\Gamma_j)} \langle v_{\Gamma_j}, H \rangle f(s) ds.$$

The domain of the latter integration equals $\partial C \setminus \text{co}(a_0, \dots, \widehat{a_j}, \dots, a_n)$. Since f_j vanishes on $\text{co}(a_0, \dots, \widehat{a_j}, \dots, a_n)$ it follows that the value of the latter integral remains unchanged if the domain is replaced by ∂C . This completes the proof for C a simplex.

Let now $C \subset \mathbb{R}^n$ be a compact convex polyhedral set with non-empty interior and fix a simplicial decomposition Σ of C . Let Σ_n denote the (finite) set of n -dimensional

simplices $S \in \Sigma$. Then $C = \cup \Sigma_n$ while all points of overlap are contained in the union of the simplices from Σ_{n-1} . The integral of $\partial_H f$ over C equals

$$\begin{aligned} \sum_{S \in \Sigma_n} \int_S \partial_H f(x) dx &= \sum_{S \in \Sigma_n} \int_{\partial S} \langle v_S, H \rangle f(s) ds \\ &= \sum_{S \in \Sigma_n} \sum_{\sigma \in S_{n-1}(\partial S)} \int_{\sigma} \langle v_{\sigma}, H \rangle f(s) ds. \end{aligned} \quad (17.4)$$

where $S_{n-1}[\partial S]$ denotes the set of $\sigma \in S_{n-1}$ which are contained in ∂S (the appearing unit normal v_{σ} points out of S). The double sum presenting the last integral can be rewritten as a sum of integrals over $\sigma \in S_{n-1}$. The elements of $S_{n-1}[\partial C]$ cover ∂C with overlap contained in the negligible set $\cup S_{n-2}$. The remaining elements, from $S_{n-1} \setminus S_{n-1}[\partial C]$, can be grouped in pairs of simplices in $\Sigma_{n-1}[S \cap S']$, ($S, S' \in S_n$) equipped with opposite unit normals. As the contributions of these pairs cancel each other, the final sum in (17.4) can be rewritten as

$$\sum_{\sigma \in S_{n-1}[\partial C]} \int_{\sigma} \langle v_{\sigma}, H \rangle f(s) ds = \int_{\partial C} \langle v_{\sigma}, H \rangle f(s) ds.$$

□

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