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A CONVEXITY THEOREM FOR SEMISIMPLE SYMMETRIC SPACES

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We generalize Kostant's convexity theorem for the Iwasawa decomposition of a real semisimple Lie group G to the following situation. Let τ be an involution of G, and $H = (G^\tau)^0$. Then there exists an Iwasawa decomposition $G = KA_pN$ with certain compatibility properties, e.g. $\tau(K) = K$, $\tau(A_p) = A_p$. Let $\mathfrak{a}_p = \mathrm{Lie}(A_p)$, $\mathfrak{H}\colon G \to \mathfrak{a}_p$ the projection according to the Iwasawa decomposition and E_{pq} the projection of $\mathfrak{a}_p$ onto the -1 eigenspace $\mathfrak{a}_{pq}$ of $d\tau(e)$. Let $X \in \mathfrak{a}_{pq}$. Then the main result of this paper describes the image of the map $H \to \mathfrak{a}_{pq}$, $h \to E_{pq} \circ \mathfrak{H}(\exp(X)\cdot h)$ as the vector sum of a closed convex polyhedral cone and the convex hull of a Weyl group orbit through X. For τ a Cartan involution it gives precisely Kostant's description of $\mathfrak{H}(\exp(X)\cdot K)$.

CONTENTS

0. Introduction. In this paper we prove a generalization of a convexity theorem of Kostant (cf. [**18**]), related to a semisimple symmetric space G/H. Here G is a connected real semisimple Lie group with finite centre, τ an involution of G and H an open subgroup of $G^\tau = \{x \in G;\ \tau(x) = x\}$.

Let K be a τ-stable maximal compact subgroup of G (for its existence, cf. [**6**]) and let θ be the associated Cartan involution. We denote the infinitesimal involutions determined by θ and τ by the same symbols and write $\mathfrak{p}$, $\mathfrak{q}$ for their respective -1 eigenspaces in $\mathfrak{g}$, the Lie algebra of G. The $+1$ eigenspaces of θ and τ in $\mathfrak{g}$ are the respective Lie algebras $\mathfrak{k}$ and $\mathfrak{h}$ of K and H. Since θ and τ commute we have the simultaneous eigenspace decomposition

(0.1) $$\mathfrak{g} = \mathfrak{k} \cap \mathfrak{q} \oplus \mathfrak{k} \cap \mathfrak{h} \oplus \mathfrak{p} \cap \mathfrak{q} \oplus \mathfrak{p} \cap \mathfrak{h}.$$

Fix a maximal abelian subspace $\mathfrak{a}_{pq}$ of $\mathfrak{p} \cap \mathfrak{q}$ and let $\mathfrak{a}_p$ be a τ-stable maximal abelian subspace of $\mathfrak{p}$, containing $\mathfrak{a}_{pq}$. Then

$$\mathfrak{a}_p = \mathfrak{a}_{ph} \oplus \mathfrak{a}_{pq},$$

where $\mathfrak{a}_{ph} = \mathfrak{a}_p \cap \mathfrak{h}$. Let $E_{pq}\colon \mathfrak{a}_p \to \mathfrak{a}_{pq}$ denote the corresponding projection.

The set $\Delta = \Delta(\mathfrak{g}, \mathfrak{a}_{pq})$ of restricted roots of $\mathfrak{a}_{pq}$ in $\mathfrak{g}$ is a (possibly non-reduced) root system (cf. [**24**]). Let Δ^+ be a choice of positive roots for Δ and Δ_p^+ a compatible choice of positive roots for $\Delta_p = \Delta(\mathfrak{g}, \mathfrak{a}_p)$. To the latter choice corresponds an Iwasawa decomposition

$$G = KA_pN, \tag{0.2}$$

where $A_p = \exp \mathfrak{a}_p$. The real analytic map $\mathfrak{H}\colon G \to \mathfrak{a}_p$ determined by

$$x \in K \exp \mathfrak{H}(x) N \qquad (x \in G)$$

is called the corresponding Iwasawa projection.

The main result of this paper is, for any fixed $a \in A_{pq} = \exp(\mathfrak{a}_{pq})$, a description of the image of the map $F_a\colon H \to \mathfrak{a}_{pq}$, defined by

$$F_a(h) = E_{pq} \circ \mathfrak{H}(ah) \tag{0.3}$$

(see Theorem 1.1). Here H is required to be connected (or to satisfy the slightly weaker condition (1.2)). If τ is a Cartan involution, then $\tau = \theta$, $H = K$, $\mathfrak{a}_p = \mathfrak{a}_{pq}$ and the result is precisely the Kostant convexity theorem.

In the present case the image of F_a is a vector sum

$$\operatorname{im}(F_a) = \operatorname{conv}(W_{K \cap H} \cdot \log a) + \Gamma(\Delta_-^+). \tag{0.4}$$

Here $W_{K \cap H}$ is a certain Weyl group, $\Gamma(\Delta_-^+)$ a closed convex polyhedral cone and we have used the notations "conv" for convex hull and "log" for the inverse of exp: $\mathfrak{a}_p \to A_p$. The cone $\Gamma(\Delta_-^+)$ can be entirely described in terms of a set of roots Δ_-^+. In particular it is independent of a and equals $\operatorname{im}(F_e) = E_{pq} \circ \mathfrak{H}(H)$.

We prove the characterization (0.4) by induction over centralizers in G, using ideas of Heckman [**15**]. However, since there seems to exist no infinitesimal version of (0.4), we cannot use his homotopy argument to reduce to an infinitesimal case. Consequently, we need to compute critical points and Hessians of F_a on the group. This is done in §4 and 5, using ideas of [**8**].

Another complication is caused by the non-compactness of H. It is overcome by showing that the map F_a, apart from a right invariance, is proper (Lemma 3.3), and that its image is not the whole of $\mathfrak{a}_{pq}$ (Lemma

3.9). Lemma 3.3 is proved by comparing F_a with another map P_a (Lemma 3.6). For a restricted class of symmetric spaces, the map P_a has been studied by Oshima and Sekiguchi [**23**], who pointed out its importance for the harmonic analysis on G/H. Lemma 3.3 follows from the properness of P_a. For the purpose of proving the latter, we study the holomorphic continuation of a certain decomposition in Appendix B, generalizing results of [**2**] on the Iwasawa decomposition.

In the recent literature, Kostant's theorem for complex groups has been generalized to a Hamiltonian framework by Atiyah [**1**], and by Guillemin and Sternberg [**13**]. Duistermaat [**7**] obtained such a generalization for the real case. At present I do not know whether the result of this paper fits into such a framework or not.

It is a pleasure for me to thank J. J. Duistermaat and G. J. Heckman for some stimulating discussions on the subject of this paper.

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1. A precise formulation of the result. The group N in the Iwasawa decomposition (0.2) is given by $N = \exp(\mathfrak{n})$, where

$$\mathfrak{n} = \sum_{\alpha \in \Delta_p^+} \mathfrak{g}^{\alpha}.$$

If $\alpha \in \Delta = \Delta(\mathfrak{g}, \mathfrak{a}_{pq})$, we let H_α denote the element of $\mathfrak{a}_{pq}$ given by

$$H_\alpha \perp \ker \alpha, \qquad \alpha(H_\alpha) = 1.$$

Here $\perp$ denotes orthogonality with respect to the Killing form $\langle \cdot , \cdot \rangle$ of $\mathfrak{g}$. Moreover, if T is a subset of Δ^+, we put

$$\Gamma(T) = \sum_{\alpha \in T} \mathbf{R}_+ \cdot H_\alpha,$$

where $\mathbf{R}_+ = [0, \infty)$.

Since θ and τ commute, $\theta \circ \tau$ is an involution. The $+1$ and -1 eigenspaces of $\theta \circ \tau$ are $\mathfrak{g}_+ = \mathfrak{k} \cap \mathfrak{h} \oplus \mathfrak{p} \cap \mathfrak{q}$ and $\mathfrak{g}_- = \mathfrak{k} \cap \mathfrak{q} \oplus \mathfrak{p} \cap \mathfrak{h}$ respectively. Now $\theta \circ \tau$ acts as the identity on $\mathfrak{a}_{pq}$. Therefore, it leaves the root spaces $\mathfrak{g}^\alpha (\alpha \in \Delta)$ invariant. Consequently, writing $\mathfrak{g}_+^\alpha = \mathfrak{g}^\alpha \cap \mathfrak{g}_+$ and $\mathfrak{g}_-^\alpha = \mathfrak{g}^\alpha \cap \mathfrak{g}_-$, we have

$$\mathfrak{g}^\alpha = \mathfrak{g}_+^\alpha \oplus \mathfrak{g}_-^\alpha \qquad (\alpha \in \Delta). \tag{1.1}$$

We define

$$\Delta_- = \{\alpha \in \Delta; \mathfrak{g}_-^\alpha \neq 0\},$$

and put $\Delta_-^+ = \Delta_- \cap \Delta^+$.

The notation $\Gamma(\Delta_-^+)$ in (0.4) has now been explained. In addition, the Weyl group $W_{K\cap H}$ is defined as
$$W_{K\cap H} = N_{K\cap H}(\mathfrak{a}_{pq})/Z_{K\cap H}(\mathfrak{a}_{pq}),$$
the normalizer modulo the centralizer of $\mathfrak{a}_{pq}$ in $K \cap H$.

With the above notations we can formulate our main result. We say that H is essentially connected if

$$H = Z_{K\cap H}(\mathfrak{a}_{pq})H^0, \tag{1.2}$$

where H^0 denotes the identity component of H.

THEOREM 1.1. *Let G be a connected real semisimple Lie group with finite centre, τ an involution of G, and H an essentially connected open subgroup of G^τ. If $a \in A_{pq}$, then*
$$\operatorname{im}(F_a) = \operatorname{conv}(W_{K\cap H} \cdot \log a) + \Gamma(\Delta_-^+).$$

2. Some notes on the induction procedure. In the proof of Theorem 1.1 (see §6), induction via centralizers in G will be used. Therefore, we need Theorem 1.1 to be valid under the somewhat more general assumption that G is a reductive group of the Harish-Chandra class (class $\mathfrak{H}$), τ an involution of G and H an open subgroup of G^τ, satisfying condition (1.2). All definitions of §§0 and 1 make sense in the context of a group of class $\mathfrak{H}$. Instead of the Killing form we use a $\mathrm{Ad}(G)$-invariant non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$, which is positive definite on $\mathfrak{k}$, negative definite on $\mathfrak{p}$, and for which the decomposition (0.1) is orthogonal. For the basic theory of a reductive symmetric space G/H of class $\mathfrak{H}$, we refer the reader to [**5**].

LEMMA 2.1. *Let G be a group of class $\mathfrak{H}$, τ an involution of G, and H an essentially connected open subgroup of G^τ. Then Theorem* 1.1 *holds for G, H if it holds for* $\mathrm{Ad}(G)^0$, $\mathrm{Ad}(H)^0$.

Proof. Let $\mathfrak{v} = \operatorname{centre}(\mathfrak{g}) \cap \mathfrak{p}$. Then $V = \exp(\mathfrak{v})$ is a closed vector subgroup of G, and we have a direct product
$$G = {}^0GV,$$
where ${}^0G = \bigcap\{\ker|\chi|;\ \chi\colon G \to \mathbf{R}^* \text{ a continuous homomorphism}\}$ (cf. e.g. [**27**, p. 196]). Obviously ${}^0G$ and V are τ-invariant, so that
$$H = (H \cap {}^0G)(H \cap V).$$
Now clearly $E_{pq} \circ \mathfrak{H}$ is right $H \cap V$-invariant, and if $a \in {}^0G \cap A_{pq}$, $a' \in V \cap A_{pq}$, then
$$E_{pq} \circ \mathfrak{H}(a'ah) = E_{pq} \circ \mathfrak{H}(ah) + \log a'$$

for all $h \in H$. It thus easily follows that we may reduce the proof to the case that $G = {}^0G$. Moreover, $E_{pq} \circ \mathfrak{H}$ is right $Z_{K \cap H}(\mathfrak{a}_{pq})$-invariant, so that by (1.2) we may reduce the proof to the case that H is connected. But then we may as well assume that G is connected. Finally, the observation that $E_{pq} \circ \mathfrak{H}$ is right centre(G)-invariant completes the proof.

For the remainder of this section, let G be a group of class $\mathfrak{H}$.

Let $W(\Delta_+)$ denote the reflection group of the root system Δ_+ defined by

$$\Delta_+ = \{\alpha \in \Delta;\, \mathfrak{g}_+^\alpha \neq 0\}$$

(cf. (1.1)). Since Δ_+ can also be viewed as the root system of $\mathfrak{a}_{pq}$ in $\mathfrak{g}_+$, it follows from standard semisimple theory, applied to $[\mathfrak{g}_+, \mathfrak{g}_+]$, that

$$W(\Delta_+) \simeq W_{K \cap H^0}. \tag{2.1}$$

PROPOSITION 2.2. *Let H be an open subgroup of G^τ. Then the following conditions are equivalent.*

(i) *H is essentially connected,*

(ii) $W(\Delta_+) \simeq W_{K \cap H}$.

Proof. In view of (2.1) the assertion follows straightforwardly from the fact that

$$H = N_{K \cap H}(\mathfrak{a}_{pq})H^0. \tag{2.2}$$

Now this is seen as follows. H and H^0 are both θ-invariant (cf. [5]), hence admit the Cartan decompositions $H = (K \cap H)\exp(\mathfrak{p} \cap \mathfrak{h})$ and $H^0 = (K \cap H^0)\exp(\mathfrak{p} \cap \mathfrak{h})$. From this we see that $(K \cap H)^0 = K \cap H^0$. Moreover, (2.2) will follow from $K \cap H = N_{K \cap H}(\mathfrak{a}_{pq})(K \cap H)^0$. Thus let $k \in K \cap H$. Then $\mathrm{Ad}(k^{-1})\mathfrak{a}_{pq}$ is maximal abelian in $\mathfrak{p} \cap \mathfrak{q}$. By standard semisimple theory applied to $[\mathfrak{g}_+, \mathfrak{g}_+]$ it follows that there exists a $k_1 \in (K \cap H)^0$ such that $\mathrm{Ad}(k_1^{-1}k^{-1})\mathfrak{a}_{pq} = \mathfrak{a}_{pq}$. Hence $kk_1 \in N_{K \cap H}(\mathfrak{a}_{pq})$ and we are done.

In the proof of Theorem 1.1 we shall use induction via centralizers of elements $Z \in \mathfrak{a}_{pq}$. The following result guarantees that the class of pairs (G, H) under consideration is stable under this induction. If $\mathfrak{b}$ is a subalgebra (or subspace) of $\mathfrak{g}$, we let $\mathfrak{b}_Z$ denote the centralizer of the element $Z \in \mathfrak{a}_{pq}$ in $\mathfrak{b}$. Similarly, if B is a subgroup of G (or a group acting on $\mathfrak{a}_{pq}$), we let B_Z denote the centralizer of Z in B.

PROPOSITION 2.3. *Let $Z \in \mathfrak{a}_{pq}$. Then G_Z is of class $\mathfrak{H}$ and τ-stable. Moreover, if H is essentially connected then the same holds for H_Z.*

Proof. The first assertion is standard (cf. [**27**, p. 286]). The second follows immediately from $\tau(Z) = -Z$.

Clearly, $\Delta_+(Z) = \{\alpha \in \Delta_+; \alpha(Z) = 0\}$ is the root system of $\mathfrak{a}_{pq}$ in $\mathfrak{g}_+ \cap \mathfrak{g}_Z$. In view of (2.1) we have a commutative diagram of natural monomorphisms

$$\begin{array}{ccc} W_{K\cap H_Z} & \xrightarrow{\varphi} & W_{K\cap H} \\ f\uparrow & & g\uparrow \\ W(\Delta_+(Z)) & \xrightarrow{\psi} & W(\Delta_+). \end{array}$$

Here the map g is an isomorphism onto because H is essentially connected (see Proposition 2.2). Obviously φ maps $W_{K\cap H_Z}$ into $(W_{K\cap H})_Z$, and it is well known that $\operatorname{im}(\psi) = W(\Delta_+)_Z$. Since g is compatible with the natural actions of $W_{K\cap H}$ and $W(\Delta_+)$ on $\mathfrak{a}_{pq}$ it follows that $g(W(\Delta_+)_Z) = (W_{K\cap H})_Z$, and we infer that f is surjective. By Proposition 2.2 this implies that H_Z is essentially connected.

From now on we assume again that G is connected and semisimple. In §6 we will prove Theorem 1.1 under the assumption that it has already been established for centralizers G_Z, $Z \in \mathfrak{a}_{pq}$. In view of the results of this section, this induction procedure is legitimate.

3. Some properties of the map F_a. Let L be the centralizer of $\mathfrak{a}_{pq}$ in G, $\mathfrak{l}$ its Lie algebra. The parabolic subgroup $Q = LN$ of G has the Levi decomposition

$$Q = LN_Q,$$

where $N_Q = \exp(\mathfrak{n}_Q)$,

$$\mathfrak{n}_Q = \sum_{\alpha\in\Delta^+} \mathfrak{g}^\alpha.$$

Let $\mathfrak{n}_L = \mathfrak{n} \cap \mathfrak{l}$, $N_L = N \cap L$. Then L normalizes N_Q and we have the semidirect product

$$N = N_L N_Q. \tag{3.1}$$

The $\langle\ \cdot\ ,\cdot\ \rangle$-orthocomplement $\mathfrak{l}_0$ of $\mathfrak{a}_{pq}$ in $\mathfrak{l}$ decomposes as

$$\mathfrak{l}_0 = \mathfrak{l}_{kq} \oplus \mathfrak{l}_{kh} \oplus \mathfrak{l}_{ph},$$

where we have written $\mathfrak{l}_{kq} = \mathfrak{l} \cap \mathfrak{k} \cap \mathfrak{q}$, etc. $\mathfrak{l}_0$ is the Lie algebra of the closed subgroup $L_0 = (K \cap L)\exp(\mathfrak{l}_{ph})$ of L. Moreover, we have a direct product

$$L = L_0 A_{pq}.$$

PROPOSITION 3.1. *Let $x \in G$. Then there exist unique $a \in A_{pq}$, $n \in N_Q$ such that*

$$x \in KL_0 an. \tag{3.2}$$

Moreover, $\log a = E_{pq} \circ \mathfrak{H}(x)$.

Proof. Write $x = ka_1n_1$, where $a_1 \in A_p$, $n_1 \in N$. Then $a_1 = a_0 a$, $n_1 = n_0 n$, with $a_0 \in A_{ph}$, $a \in A_{pq}$, $n_0 \in N_L$, $n \in N_Q$. It follows that $x = ka_0(an_0a^{-1})an$, whence (3.2) and the last assertion is obvious. The uniqueness follows easily from the uniqueness for the decompositions (0.2), (3.1) and $A_p = A_{ph}A_{pq}$.

COROLLARY 3.2. *The map $E_{pq} \circ \mathfrak{H}$ is right L_0-invariant.*

Proof. Use that L_0 normalizes N_Q and centralizes A_{pq}.

In particular, if $a \in A_{pq}$, then F_a: $H \to \mathfrak{a}_{pq}$, defined by (0.3), naturally induces a map $\underline{F}_a$: $H/H \cap L_0 \to \mathfrak{a}_{pq}$.

LEMMA 3.3. *Let $\mathscr{A}$ be a compact subset of A_{pq}. Then the map $\mathscr{A} \times H/H \cap L_0 \to \mathfrak{a}_{pq}, (a, h) \mapsto \underline{F}_a(h)$ is proper.*

We prove this lemma by comparing $\underline{F}_a$ with another map. Using the direct sum decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{l}_q \oplus \mathfrak{n}_Q$ (cf. **[4]**), one easily checks that the map $H \times L \times N_Q \to G$, $(h, l, n) \mapsto hln$ is a submersion onto an open subset Ω of G (see also **[21]**, **[24]**).

LEMMA 3.4. *If $x \in \Omega$ then there exist unique $l(x) \in (H \cap L_0)\backslash L_0$, $a_{pq}(x) \in A_{pq}$ and $n_Q(x) \in N_Q$ such that*

$$x \in Hl(x)a_{pq}(x)n_Q(x). \tag{3.3}$$

The corresponding maps l, a_{pq} and n_Q are real analytic. Moreover, if $\{x_n\}$ is any sequence in Ω converging to a boundary point $x \in \partial\Omega$, then $\{a_{pq}(x_n)\}$ is not relatively compact in A_{pq}.

REMARKS. (i) We prove this lemma at the end of Appendix B.

(ii) If the involution τ arises from a signature on Δ_p (cf. **[23]**), then $\mathfrak{a}_p = \mathfrak{a}_{pq}$, $\mathfrak{a}_{pq}$ is maximal abelian in $\mathfrak{q}$, and the above result is contained in **[23]**.

In view of Lemma 3.4 we have $H \cap LN = H \cap L = H \cap L_0$, so that the inclusion $H \hookrightarrow G$ induces an embedding

$$i\colon H/H \cap L \to G/Q$$

of $H/H \cap L$ onto an open subset $\underline{\Omega}$ of G/Q (the underlining indicates that $\underline{\Omega}$ is the canonical image of Ω in G/Q). If $a \in A_{pq}$, we let λ_a or $\lambda(a)$ denote left multiplication by a on G/Q, and put $\underline{\Omega}_a = \lambda(a^{-1})\underline{\Omega}$. Let

$$j\colon K/K \cap L \to G/Q$$

be the natural diffeomorphism, and set

$$\underline{\Omega}_{K,a} = j^{-1}(\underline{\Omega}_a).$$

Then $\underline{\Omega}_{K,a}$ is the canonical image of $K \cap a^{-1}\Omega$ in $K/K \cap L$. Since L_0 centralizes A_{pq} and normalizes N_Q, the map a_{pq} is right L_0-invariant. Moreover, $K \cap L_0 = K \cap L$ by definition of L_0, so that for $a \in A_{pq}$ we can define a map $P_a\colon \underline{\Omega}_{K,a} \to \mathfrak{a}_{pq}$ by

$$P_a(k(K \cap L)) = \log \circ a_{pq}(ak). \tag{3.4}$$

If $\mathscr{A}$ is a subset of A_{pq}, we define the subset $\underline{\Omega}_{K,\mathscr{A}}$ of $\mathscr{A} \times K/K \cap L$ by

$$\underline{\Omega}_{K,\mathscr{A}} = \{(a,k) \in \mathscr{A} \times K/K \cap L;\, k \in \underline{\Omega}_{K,a}\}.$$

LEMMA 3.5. *If $\mathscr{A}$ is a compact subset of A_{pq}, then the map $P\colon \underline{\Omega}_{K,\mathscr{A}} \times \mathfrak{a}_{pq} \to \mathfrak{a}_{pq}$, $(a,k) \mapsto P_a(k)$ is proper.*

Proof. Clearly, it suffices to prove that the map a_{pq} restricted to $\Omega_{K,\mathscr{A}} = \{(a,k) \in A \times K;\, ak \in \Omega\}$ is proper. Let $\mathscr{C}$ be a compact subset of A_{pq}. We claim that the set $T = a_{pq}^{-1}(\mathscr{C}) \cap \Omega_{K,\mathscr{A}}$ is compact. For assume not; then it is not closed in $\mathrm{cl}(\Omega_{K,\mathscr{A}})$. Hence there exists a point $(a,k) \in \mathscr{A} \times K$ such that $ak \in \mathrm{cl}(\Omega) \setminus \Omega = \partial\Omega$, and a sequence $\{(a_n, k_n)\}$ in T such that $a_n k_n \to ak$. By Lemma 3.4 the set $\{a_{pq}(a_n k_n)\}$ is not relatively compact in A_{pq}, contradicting the assumption on $\mathscr{C}$. Hence T is compact.

LEMMA 3.6. *Let $a \in A_{pq}$. Then*

$$\underline{F}_a = -P_{a^{-1}} \circ j^{-1} \circ \lambda_a \circ i.$$

Proof. From the definitions it is evident that $j^{-1} \circ \lambda_a \circ i$ is a diffeomorphism of $H/H \cap L$ onto $\underline{\Omega}_{K,a^{-1}}$. Let $h \in H$, and set $ah = kl \exp Yn$, with $(k, l, Y, n) \in K \times L_0 \times \mathfrak{a}_{pq} \times N_Q$. Then $Y = E_{pq} \circ \mathfrak{H}(ah) = \underline{F}_a(h(H \cap L))$. On the other hand, $a^{-1}k = hl^{-1}\exp(-Y)n'$, where $n' = l \exp Y n^{-1}(l \exp Y)^{-1} \in N_Q$, so that $-Y = \log a_{pq}(a^{-1}k)$. This proves the lemma.

Proof of Lemma 3.3. The map $\mathscr{A} \times H/H \cap L \to \underline{\Omega}_{K,\mathscr{A}^{-1}}$, $(a,h) \mapsto j^{-1} \circ \lambda_a \circ i(h)$ is easily seen to be a diffeomorphism. Thus the assertion follows from Lemmas 3.5 and 3.6.

COROLLARY 3.7. *If $a \in A_{pq}$, then the set $E_{pq} \circ \mathfrak{H}(aH)$ is closed in $\mathfrak{a}_{pq}$.*

Observe that in view of Lemma 3.6 the following is an equivalent formulation of Theorem 1.1.

THEOREM 3.8. *Let H be essentially connected, and let $a \in A_{pq}$. Then*

$$\operatorname{im}(P_a) = \operatorname{conv}(W_{K\cap H} \cdot \log a) + (-\Gamma(\Delta_-^+)).$$

We now come to the second main result of this section. It deals with a first restriction on the location of the set $E_{pq} \circ \mathfrak{H}(aH)$. Put

$$\mathfrak{a}_{pq}^+(\Delta^+) = \{U \in \mathfrak{a}_{pq};\, \alpha(U) > 0 \text{ for } \alpha \in \Delta^+\}.$$

LEMMA 3.9. *Let $a \in A_{pq}$, $X \in \operatorname{cl}(\mathfrak{a}_{pq}^+(\Delta^+))$. Then the function $F_{a,X}\colon H \to \mathbf{R}$ defined by*

$$F_{a,X}(h) = \langle X, F_a(h)\rangle,$$

for $h \in H$, is bounded from below.

REMARKS. (i) If τ arises from a signature on Δ_p (cf. [23], see also Remark (ii) following Lemma 3.4), then Lemma 3.9 is a consequence of [23, Prop. 3.8].

(ii) The proof presented below was suggested to me by H. Schlichtkrull and M. Flensted-Jensen independently. Being closely related to that of [10, Lemma 4.6], it is based on a characterization of $\mathfrak{H}$ in terms of finite dimensional representations as in [14]. Our treatment follows the presentation in [26].

The proof of the lemma depends on a few propositions. Via the restriction $\langle\,\cdot\,,\cdot\rangle$ of the Killing form to $\mathfrak{a}_p$, we view $\mathfrak{a}_{pq}^*$ as a linear subspace of $\mathfrak{a}_p^*$. We let $\langle\,\cdot\,,\cdot\rangle$ also denote the dual inner product on $\mathfrak{a}_p^*$, and set

$$\mathfrak{a}_p^{*+} = \{\lambda \in \mathfrak{a}_p^*;\, \langle\lambda,\alpha\rangle > 0 \text{ for } \alpha \in \Delta_p^+\},$$

$$^+\mathfrak{a}_{pq} = \{X \in \mathfrak{a}_{pq};\, \lambda(X) \geq 0 \text{ for } \lambda \in \operatorname{cl}(\mathfrak{a}_p^{*+}) \cap \mathfrak{a}_{pq}^*\}.$$

Then from the fact that $\Delta^+ = \{\alpha|\mathfrak{a}_{pq};\, \alpha \in \Delta_p^+, \alpha|\mathfrak{a}_{pq} \neq 0\}$, it easily follows that

$$^+\mathfrak{a}_{pq} = \{X \in \mathfrak{a}_{pq};\, \langle X, Y\rangle \geq 0 \text{ for } Y \in \mathfrak{a}_{pq}^+(\Delta^+)\}. \tag{3.5}$$

PROPOSITION 3.10. *There exist $\eta_1, \ldots, \eta_l \in \mathfrak{a}_{pq}^*$ ($l = \dim \mathfrak{a}_{pq}$) such that*

(i) $^+\mathfrak{a}_{pq} = \{X \in \mathfrak{a}_{pq};\, \eta_j(X) \geq 0$ *for* $1 \leq j \leq l\}$,

(ii) $\langle\eta_j, \alpha\rangle\langle\alpha,\alpha\rangle^{-1} \in \mathbf{N}$ *for all* $1 \leq j \leq l$, $\alpha \in \Delta_p^+$.

Proof. Let Σ, Σ_p denote the respective sets of simple roots in Δ^+, Δ_p^+, and let $\Sigma_{p0} = \{\alpha \in \Sigma_p;\ \alpha | \mathfrak{a}_{pq} = 0\}$. Then according to [**26**, Lemma 7.2.4] there exist a non-negative integer l_1 and an enumeration $\alpha_1, \dots, \alpha_n$ of Σ_p (where $n = \dim \mathfrak{a}_p$), such that

$$\tau\theta\alpha_j \equiv \alpha_j \bmod \mathbf{N} \cdot \Sigma_{p0} \qquad (1 \leq j \leq l - l_1),$$

$$\tau\theta\alpha_j \equiv \alpha_{j+l_1} \bmod \mathbf{N} \cdot \Sigma_{p0} \qquad (l - l_1 < j \leq l).$$

$$\Sigma_{p0} = \{\alpha_j;\ l + l_1 < j \leq n\}.$$

Moreover, $\Sigma = \{\alpha_1|\mathfrak{a}_{pq}, \dots, \alpha_l|\mathfrak{a}_{pq}\}$.

Now let $\lambda_1, \dots, \lambda_n \in \mathfrak{a}_p^*$ be defined by $\langle \lambda_i, \alpha_j \rangle = 2\langle \alpha_j, \alpha_j \rangle \delta_{ij}$ for all $1 \leq i,\ j \leq n$. Then $\langle \lambda_i, \alpha \rangle \langle \alpha, \alpha \rangle^{-1} \in \mathbf{N}$ for all $1 \leq i \leq n$, $\alpha \in \Delta_p^+$. Define

$$\eta_j = \lambda_j \qquad (1 \leq j \leq l - l_1),$$
$$\eta_j = \lambda_j + \lambda_{j+l_1} \qquad (l - l_1 < j \leq l).$$

Then $\eta_j \in \mathfrak{a}_{pq}^*$ $(1 \leq j \leq l)$ and (ii) holds. Moreover, if $\mu \in \mathfrak{a}_{pq}^*$, then

$$\mu = \frac{1}{2} \sum_{1 \leq j \leq l} \frac{\langle \mu, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} \eta_j.$$

From this we infer that $\mathrm{cl}(\mathfrak{a}_p^{*+}) \cap \mathfrak{a}_{pq}^* = \sum_{1 \leq j \leq l} \mathbf{R}_+ \cdot \eta_j$, whence (i).

PROPOSITION 3.11. *If $x \in G$, then*

$$E_{pq} \circ \mathfrak{H}(x) = E_{pq} \circ \mathfrak{H}(\tau\theta x).$$

Proof. Since $\tau\theta = I$ on $\mathfrak{a}_{pq}$, the algebras $\mathfrak{l}$ and $\mathfrak{n}_Q$ are $\tau\theta$-stable. Hence the decomposition $G = KL_0A_{pq}N_Q$ is $\tau\theta$-stable and the assertion follows from the characterisation of $E_{pq} \circ \mathfrak{H}$ in Proposition 3.1.

PROPOSITION 3.12. *If $x \in H$, then $E_{pq} \circ \mathfrak{H}(x) \in {}^+\mathfrak{a}_{pq}$.*

Proof. The map $E_{pq} \circ \mathfrak{H}$ being left K-invariant, it suffices to prove this for $x = \exp X$, $X \in \mathfrak{p} \cap \mathfrak{h}$. Moreover, by Proposition 3.10 it suffices to show that $\eta \circ \mathfrak{H}(x) \geq 0$ for $\eta \in \{\eta_1, \dots, \eta_l\}$.

By [**17**, Theorem V.4.1.], η is the highest weight of a spherical representation π of G in a finite dimensional complex linear space V. We let π also denote the associated infinitesimal representation of $\mathfrak{g}_c = \mathbf{C} \otimes_{\mathbf{R}} \mathfrak{g}$ in V and endow V with an inner product that makes π unitary on $\mathfrak{k} \oplus i\mathfrak{p}$. Let $v \in V$ be a highest weight vector of norm 1, and put

$f(t) = \|\pi(\exp tX)v\|^2 = \exp\{2\eta \circ \mathfrak{H}(\exp tX)\}$ for $t \in \mathbf{R}$. Then we must show that $f(1) \geq 1$.

Following the proof of [**26**, Lemma 7.6.1], we expand v in an orthogonal sum $v = \Sigma_{a \in \mathbf{R}} v_a$ of eigenvectors of $\pi(X)$, where $\pi(X)v_a = av_a$. Then $f(t) = \Sigma_a e^{2at}\|v_a\|^2$.

On the other hand, by Proposition 3.11 the function f is even, whence $f(t) = \Sigma_a \cosh(2at)\|v_a\|^2 \geq 1$.

Proof of Lemma 3.9. If $x \in G$, then $x \in K \exp \mathfrak{H}(x) N$, whence $F_a(x) = E_{pq} \circ \mathfrak{H}(ax) \in E_{pq} \circ \mathfrak{H}(aK) + E_{pq} \circ \mathfrak{H}(x)$. Since $E_{pq} \circ \mathfrak{H}(aK)$ is a compact subset of $\mathfrak{a}_{pq}$, it therefore suffices to show that the function $F_{e,X} = \langle X, E_{pq} \circ \mathfrak{H} \rangle$ is bounded from below on H. Now this follows from Proposition 3.12 and the characterization (3.5) of ${}^{+}\mathfrak{a}_{pq}$.

4. Critical points of the functions $F_{a,X}$. In this section we let $a \in A_{pq}$ and $X \in \mathfrak{a}_{pq}$ be fixed and determine the critical set of the function $F_{a,X}\colon H \to \mathbf{R}$ defined by

$$F_{a,X}(h) = \langle X, \mathfrak{H}(ah) \rangle,$$

for $h \in H$. Moreover, in the next section we shall compute Hessians of $F_{a,X}$ at points of this set. As it turns out, the computations are highly analogous to those in [**8**], and so are the results. As in [**8**], the critical set is a finite union of smooth submanifolds depending only on the subsets $\{\alpha \in \Delta_+; \alpha(\log a) = 0\}$ and $\{\alpha \in \Delta; \alpha(X) = 0\}$ of Δ. Moreover, the Hessian of $F_{a,X}$ at a critical point is non-degenerate transversally to the critical manifold through it. Though such results hold for general $a \in A_{pq}$, we shall only prove them for $a \in A'_{pq}$, this being sufficient for our purposes. Here $A'_{pq} = \exp(\mathfrak{a}'_{pq})$, with

$$\mathfrak{a}'_{pq} = \{Z \in \mathfrak{a}_{pq}; \alpha(Z) \neq 0 \text{ for } \alpha \in \Delta_+\}.$$

If $u \in U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}_c$, we let R_u or $R(u)$ denote the infinitesimal right regular action of u on smooth vector valued functions on G. If f is such a function, we also write

$$f(x; u) = (R_u f)(x) \qquad (x \in G).$$

In view of the Poincaré-Birkhoff-Witt theorem, the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a}_p \oplus \mathfrak{n}$ gives rise to a direct sum decomposition

$$U(\mathfrak{g}) = (\mathfrak{k}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}) \oplus U(\mathfrak{a}_p).$$

Let $E_{\mathfrak{a}}$ denote the corresponding projection $U(\mathfrak{g}) \to U(\mathfrak{a}_p)$. If $v \in U(\mathfrak{a}_p)$, we denote its homogeneous component of degree m by v_m. This makes sense because $\mathfrak{a}_p$ is abelian, so that $U(\mathfrak{a}_p) \simeq S(\mathfrak{a}_p)$, the symmetric algebra of $\mathfrak{a}_{p,c}$. Writing ν for the N-part in the Iwasawa decomposition, we now have the following result (cf. [**8**, Lemma 5.1]).

LEMMA 4.1. *Let $x \in G$, $u \in U(\mathfrak{g})\mathfrak{g}$. Then*

$$\mathfrak{H}(x; u) = \left(E_{\mathfrak{a}}(u^{t(x)})\right)_1.$$

Here $u^{t(x)}$ denotes the image of u under the adjoint action of $t(x) = \exp \mathfrak{H}(x) \cdot \nu(x)$, the "triangular part" of x, and the suffix 1 indicates that the homogeneous component of degree 1 is taken.

Let $F_X\colon G \to \mathbf{R}$ be defined by

$$F_X(x) = \langle X, \mathfrak{H}(x) \rangle \qquad (x \in G).$$

Then the following corollary holds (cf. [**8**. Corollary 5.2]).

COROLLARY 4.2. *If $x \in G$, $U \in \mathfrak{g}$, then*

$$F_X(x; U) = \left\langle U^{t(x)}, X \right\rangle = \left\langle U, X^{\nu(x)^{-1}} \right\rangle.$$

LEMMA 4.3. *$h \in H$ is a critical point for $F_{a,X}$ if and only if $ah \in KA_pN_X$.*

Proof. Write $ah = kbn$, with $k \in K$, $b \in A_p$, $n \in N$. Then $F_{a,X}(h; U) = \langle X^{n^{-1}}, U \rangle$, so h is a critical point iff

$$\mathrm{Ad}(n^{-1})X \perp U \quad \text{for all } U \in \mathfrak{h}.$$

The last statement is equivalent to $\mathrm{Ad}(n^{-1})X \in \mathfrak{q}$, and since $\mathrm{Ad}(n^{-1})X \equiv X \bmod \mathfrak{n}$, this in turn is equivalent to $\mathrm{Ad}(n^{-1})X - X \in \mathfrak{n} \cap \mathfrak{q}$. Since τ maps $\mathfrak{n}$ onto $\bar{\mathfrak{n}} = \theta(\mathfrak{n})$, we have $\mathfrak{n} \cap \mathfrak{q} = 0$ and the proof is complete.

LEMMA 4.4. *Let H be essentially connected, $a \in A'_{pq}$, $X \in \mathfrak{a}_{pq}$. Then the critical set of $F_{a,X}$ equals the set*

$$\mathscr{C}_X = \bigcup_{w \in W(\Delta_+)} wH_X.$$

REMARK. By Proposition 2.2, $W(\Delta_+) \simeq N_{K \cap H}(\mathfrak{a}_{pq})/Z_{K \cap H}(\mathfrak{a}_{pq})$. Since $Z_{K \cap H}(\mathfrak{a}_{pq}) \subset H_X$, the notation wH_X ($w \in W(\Delta_+)$) makes sense.

Proof of Lemma 4.4. If x_w is a representative of $w \in W(\Delta_+)$ in $N_{K \cap H}(\mathfrak{a}_{pq})$, and $h \in H_X$, then $\nu(ax_wh) = \nu(a^{w^{-1}}h) = \nu(h) \in N_X$. Hence x_wh is a critical point for $F_{a,X}$ (Lemma 4.3).

Conversely, let h be a critical point for $F_{a,X}$, and write $ah = kbn$ as in the proof of Lemma 4.3. Then $n \in N_X$ and it follows that $k^{-1}ah = bn \in G_X$. Write $h = h_1h_2$, with $h_1 \in H \cap K$, $h_2 \in \exp(\mathfrak{p} \cap \mathfrak{h})$. Then $k^{-1}ah = k^{-1}h_1(h_1^{-1}ah_1)h_2$, where $k^{-1}h_1 \in K$, $h_1^{-1}ah_1 \in \exp(\mathfrak{p} \cap \mathfrak{q})$, $h_2 \in \exp(\mathfrak{p} \cap \mathfrak{h})$. Using uniqueness properties of the decomposition $G = K\exp(\mathfrak{p} \cap \mathfrak{q})\exp(\mathfrak{p} \cap \mathfrak{h})$ and of the analogous decomposition of G_X (cf. [**9**, Thm. 4.1]), we infer that $k^{-1}h_1 \in K_X$, $h_1^{-1}ah_1 \in \exp(\mathfrak{p} \cap \mathfrak{q}_X)$, $h_2 \in \exp(\mathfrak{p} \cap \mathfrak{h}_X)$. By standard semisimple theory, applied to $[\mathfrak{g}_+ \cap \mathfrak{g}_X, \mathfrak{g}_+ \cap \mathfrak{g}_X]$ it follows that there exists a $l \in K \cap H_X^0$ such that $l^{-1}h_1^{-1}ah_1l \in \exp(\mathfrak{a}_{pq})$. Thus, a being regular for the root system $\Delta_+ = \Delta(\mathfrak{g}_+, \mathfrak{a}_{pq})$, it follows that $\mathrm{Ad}(h_1l)$ normalizes $\mathfrak{a}_{pq}$. Hence $h_1 \in N_{K\cap H}(\mathfrak{a}_{pq})(K \cap H_X^0)$, so that $h \in N_{K\cap H}(\mathfrak{a}_{pq})H_X^0$. In view of Proposition 2.2 and the assumption on H this implies that $h \in \mathscr{C}_X$.

Observe that $\mathscr{C}_X$ is a finite union of disjoint smooth manifolds. Moreover, if $y \in \mathscr{C}_X$, then

$$T_y\mathscr{C}_X = d\lambda_y(e)(\mathfrak{h}_X),$$

where $T_y\mathscr{C}_X$ denotes the tangent space at y, $d\lambda_y(e)$ the derivative of the map $\lambda_y\colon G \to G$, $x \mapsto yx$ at e.

5. Hessians of the functions $F_{a,X}$. As in the previous section, we fix $a \in A_{pq}$ and $X \in \mathfrak{a}_{pq}$. In addition, we assume that H is essentially connected. Following [**8**], we write $E_\mathfrak{k}$, $E_\mathfrak{a}$, $E_\mathfrak{n}$ for the projections $\mathfrak{g} \to \mathfrak{k}$, $\mathfrak{a}_p$, $\mathfrak{n}$ according to the Iwasawa decompostion $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a}_p \oplus \mathfrak{n}$. Observe that this definition of $E_\mathfrak{a}$ is compatible with the definition of the map $E_\mathfrak{a}\colon U(\mathfrak{g}) \to U(\mathfrak{a}_p)$ preceding Lemma 4.1.

LEMMA 5.1. *Let $x \in G$; $U, V \in \mathfrak{g}$. Then*

$$F_X(x; UV) = B_X(U^{t(x)}, V^{t(x)}),$$

where the bilinear form B_X on $\mathfrak{g} \times \mathfrak{g}$ is given by

$$B_X(U,V) = \langle E_\mathfrak{a}(UV)_1, X\rangle = \langle [U, E_\mathfrak{k}(V)], X\rangle.$$

Proof. See [**8**, Lemma 6.1].

Motivated by the above formula we first study the map $V \mapsto E_\mathfrak{k} \circ \mathrm{Ad}(t(x))(V)$, $\mathfrak{h} \to \mathfrak{k}$ in more detail. Given $x \in A_{pq}$, let Θ_x be the map $H \to K$ defined by

$$\Theta_x(h) = \kappa(xh),$$

the K-part in the Iwasawa decomposition of xh. Then, writing

$$\dot{\Theta}_x(h) = d\lambda_{\Theta_x(h)}(e)^{-1} \circ d\Theta_x(h) \circ d\lambda_h(e),$$

we have the following result.

LEMMA 5.2. *Let $x \in A_{pq}$, $h \in H$. Then*

$$\dot{\Theta}_x(h) = E_{\mathfrak{k}} \circ \mathrm{Ad}(t(xh)).$$

Proof. Let $V \in \mathfrak{h}$, and set $xh = kt$, with $k \in K$, $t \in A_pN$. For s sufficiently close to zero, we may write

$$xh \exp(sV) = k \exp K(s) \exp A(s) t \exp N(s), \tag{5.1}$$

with $K(s) \in \mathfrak{k}$, $A(s) \in \mathfrak{a}_p$, $N(s) \in \mathfrak{n}$ smoothly depending on s. Clearly $\dot{K}(0) = \dot{\Theta}_x(h)V$. Multiplying both sides of (5.1) by k^{-1} from the left and by t^{-1} from the right and differentiating at $s = 0$, we infer that $\dot{K}(0) = E_{\mathfrak{k}}(\mathrm{Ad}(t)V)$.

LEMMA 5.3. *The map Θ_a maps every coset $h(H \cap L)$ into a coset $\kappa(ah)(K \cap L)$ and induces a diffeomorphism $\underline{\Theta}_a$ of $H/H \cap L$ onto the open subset $\underline{\Omega}_{K,a}$ of $K/K \cap L$. Moreover, for each $w \in W(\Delta_+)$, it maps the submanifold wH_X into wK_X.*

Proof. Fix $h \in H$, $l \in H \cap L$. We may write $ah = \kappa(ah)l_1a_1n_1$, with $l_1 \in L_0$, $a_1 \in A_{pq}$, $n_1 \in N_Q$. Thus

$$ahl = \kappa(ah)l_1la_1l^{-1}n_1l = \kappa(ah)\kappa(l_1l)\left[\exp \mathfrak{H}(l_1l)\nu(l_1l)a_1l^{-1}nl\right].$$

The expression between brackets is easily checked to be contained in A_pN, so that $\kappa(ahl) = \kappa(ah)\kappa(l_1l)$. Since κ maps L into $K \cap L$ this implies that $\Theta_a(h(H \cap L)) \subset \kappa(ah)(H \cap L)$. The induced map $\underline{\Theta}_a$: $H/H \cap L \to K/K \cap L$ is just $j^{-1} \circ \lambda_a \circ i$ (see §3), hence maps $H/H \cap L$ diffeomorphically onto $\underline{\Omega}_{K,a}$. Finally, the last assertion follows from the fact that κ maps G_X into K_X.

Let $\mathfrak{h}^c$, $\mathfrak{k}^c$ denote the orthocomplements of $\mathfrak{l}$ in $\mathfrak{h}$ and $\mathfrak{k}$ respectively. Then $\mathfrak{h} = \mathfrak{h}^c \oplus (\mathfrak{h} \cap \mathfrak{l})$ and $\mathfrak{k} = \mathfrak{k}^c \oplus (\mathfrak{k} \cap \mathfrak{l})$, and the maps

$$\mathfrak{n}_Q \to \mathfrak{h}^c, \quad U \mapsto U + \tau U, \quad \text{and}$$
$$\mathfrak{n}_Q \to \mathfrak{k}^c, \quad U \mapsto U + \theta U$$

are linear isomorphisms. They map $\mathfrak{n}_Q \cap \mathfrak{g}_X$ onto $\mathfrak{h}^c \cap \mathfrak{g}_X$ and $\mathfrak{k}^c \cap \mathfrak{g}_X$ respectively. We now have the following.

LEMMA 5.4. *If $h \in H$, then the map $\dot{\Theta}_a(h)$: $\mathfrak{h} \to \mathfrak{k}$ maps $\mathfrak{l} \cap \mathfrak{h}$ into $\mathfrak{l} \cap \mathfrak{k}$ and $\mathfrak{h}_X$ into $\mathfrak{k}_X$. Moreover, the induced maps $\mathfrak{h}/\mathfrak{h} \cap \mathfrak{l} \to \mathfrak{k}/\mathfrak{k} \cap \mathfrak{l}$ and $\mathfrak{h}_Z/\mathfrak{h} \cap \mathfrak{l} \to \mathfrak{k}_X/\mathfrak{k} \cap \mathfrak{l}$ are bijective.*

Proof. The first two assertions follow immediately from Lemma 5.3 and the fact that

$$d\lambda_h(e)^{-1}T_h(H \cap L) = \mathfrak{h} \cap \mathfrak{l},\ d\lambda_{\Theta_a(h)}(e)^{-1}T_{\Theta_a(h)}(K \cap L) = \mathfrak{k} \cap \mathfrak{l},$$

etc. Moreover, by the same lemma the induced map $\underline{\dot{\Theta}}_a(h)$: $\mathfrak{h}/\mathfrak{h} \cap \mathfrak{l} \to \mathfrak{k}/\mathfrak{k} \cap \mathfrak{l}$ must be a linear isomorphism. It maps the canonical image of $\mathfrak{h}_X$ into that of $\mathfrak{k}_X$. In view of the remarks above Lemma 5.4, the last assertion now follows for dimensional reasons.

We now return to the Hessian of $F_{a,X}$.

LEMMA 5.5. *Let $a \in A_{pq}$, $X \in \mathfrak{a}_{pq}$. Then for any $h \in H$; U, $V \in \mathfrak{h}$, we have*:

$$F_{a,X}(h; UV) = \langle U, L_{a,X,h}(V)\rangle,$$

where $L_{a,X,h}$ is the linear map $\mathfrak{h} \to \mathfrak{h}$ given by

$$L_{a,X,h} = -\mathrm{Ad}(h^{-1}) \circ \pi_{\mathfrak{h}} \circ \mathrm{Ad}(a^{-1}) \circ \mathrm{Ad}\,\Theta_a(h) \circ \mathrm{ad}\, X \circ \dot{\Theta}_a(h).$$

Here $\pi_{\mathfrak{h}}$ denotes the projection $\mathfrak{g} \to \mathfrak{h}$ according to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$.

Proof. By Lemma 5.1 we have

$$F_{a,X}(h; UV) = -\left\langle U, \pi_{\mathfrak{h}} \circ \mathrm{Ad}\left(t(ah)^{-1}\right) \circ \mathrm{ad}\, X \circ E_{\mathfrak{k}}(\mathrm{Ad}(t(ah))V)\right\rangle.$$

Now $ah = \Theta_a(h)t(ah)$, so that $t(ah)^{-1} = h^{-1}a^{-1}\Theta_a(h)$, and the assertion follows from Lemma 5.2 and the observation that $\mathrm{Ad}(h^{-1})$ commutes with $\pi_{\mathfrak{h}}$.

LEMMA 5.6. *For each $a \in A'_{pq}$, $X \in \mathfrak{a}_{pq}$ the Hessian of $F_{a,X}$ at any critical point is transversally non-degenerate to the critical set of $F_{a,X}$.*

Proof. Let $h = x_w h'$ be a critical point for $F_{a,X}$. Here x_w is a representative of $w \in W(\Delta_+)$ in $N_{K \cap H}(\mathfrak{a}_{pq})$, and $h' \in H_X$. It is obvious that $d\lambda_h(e)^{-1}T_h(hH_X) = \mathfrak{h}_X$. The bilinear form $\beta(U, V) = F_{a,X}(h; UV)$ on $\mathfrak{h}$ is symmetric. Since $F_{a,X}$ is locally constant on hH_X, we therefore have that $\beta = 0$ on $\mathfrak{h} \times \mathfrak{h}_X$ and $\mathfrak{h}_X \times \mathfrak{h}$. We must show that the induced bilinear form on $\mathfrak{h}/\mathfrak{h}_X$ is non-degenerate. The Killing form being non-degenerate on $\mathfrak{h}$, this comes down to showing that the map $L_{a,X,h}$ of Lemma 5.5 has kernel $\mathfrak{h}_X$. Now $L_{a,X,h} = L_{a',X,h'}$, where $a' = a^{w^{-1}}$ still belongs to A'_{pq}. Therefore we may restrict ourselves to the case that $h \in H_X$. But then $\Theta_a(h) \in K_X$ (Lemma 5.3), so $\mathrm{Ad}(\Theta_a(h))$ and ad X commute. Hence an element $V \in \mathfrak{h}$ belongs to $\ker(L_{a,X,h})$ iff

$$(5.2) \qquad \mathrm{Ad}(a^{-1}) \circ \mathrm{ad}\, X \circ \mathrm{Ad}(\Theta_a(h)) \circ \dot{\Theta}_a(h) V \in \mathfrak{q}.$$

Now $\mathrm{ad}(X) \circ \mathrm{Ad}(\Theta_a(h)) \circ \dot{\Theta}_a(h)$ maps $\mathfrak{h}$ into $\mathfrak{p}$ and if $U \in \mathfrak{p}$, then $\mathrm{Ad}(a^{-1})U \in \mathfrak{q}$ iff $U \in \mathfrak{a}_{pq}$ (see the lemma below). So $V \in \ker(L_{a,X,h})$ iff $\mathrm{ad}(X) \circ \mathrm{Ad}(\Theta_a(h)) \circ \dot{\Theta}_a(h) V \in \mathfrak{a}_{pq}$. Now $\mathrm{Ad}(\Theta_a(h)) \circ \dot{\Theta}_a(h)$ maps $\mathfrak{h}$ into $\mathfrak{k}$, and an easy root space calculation shows that (5.2) is equivalent to

$$\mathrm{Ad}(\Theta_a(h)) \circ \dot{\Theta}_a(h) V \in \mathfrak{k}_X.$$

Since $\Theta_a(h) \in K_X$, $\mathrm{Ad}(\Theta_a(h))$ maps $\mathfrak{k}_X$ bijectively onto itself. Moreover, by Lemma 5.4, $\dot{\Theta}_a(h)$ induces an isomorphism $\mathfrak{h}/\mathfrak{h}_X \to \mathfrak{k}/\mathfrak{k}_X$, and we conclude that (5.2) is equivalent to $V \in \mathfrak{h}_X$.

LEMMA 5.7. *If* $a \in A'_{pq}$, $U \in \mathfrak{p}$, *then* $\mathrm{Ad}(a^{-1})U \in \mathfrak{q}$ *if and only if* $U \in \mathfrak{a}_{pq}$.

Proof. The if part is obvious. For the converse, suppose that $U \in \mathfrak{p}$. Using the decompositions (1.1), we may write

$$U = U_L + \sum_{\alpha \in \Delta^+} (U_+^\alpha - \theta U_+^\alpha) + (U_-^\alpha - \theta U_-^\alpha),$$

with $U_L \in \mathfrak{l} \cap \mathfrak{p} \cap \mathfrak{q} = \mathfrak{a}_{pq}$, $U_+^\alpha \in \mathfrak{g}_+^\alpha$, $U_-^\alpha \in \mathfrak{g}_-^\alpha$. Using that $\tau = \theta$ on $\mathfrak{g}_+$, whereas $\tau = -\theta$ on $\mathfrak{g}_-$, we find

$$\mathrm{Ad}(a^{-1})U = U_L + \sum (a^{-\alpha} U_+^\alpha - a^\alpha \tau U_+^\alpha) + (a^{-\alpha} U_-^\alpha + a^\alpha \tau U_-).$$

Since $a^\alpha \neq a^{-\alpha}$ for all $\alpha \in \Delta_+^+$, $\mathrm{Ad}(a^{-1})U \in \mathfrak{q}$ implies $U_+^\alpha = U_-^\alpha = 0$ for all $\alpha \in \Delta^+$. Hence $U \in \mathfrak{a}_{pq}$.

COROLLARY 5.8. *Let* $a \in A'_{pq}$, $X \in \mathfrak{a}_{pq}$, $w \in W(\Delta_+)$. *Then at all points of* wH_X *the value of* $F_{a,X}$ *and the signature and rank of its Hessian stay constant.*

Proof. From Lemma 5.6 it follows by continuity that the statement is true on $x_w H_X^0$ ($w \in W(\Delta_+)$). In view of Proposition 2.3 we have $H_X = H_X^0 Z_{K \cap H}(\mathfrak{a}_{pq})$. Moreover, by Corollary 3.2 the function $F_{a,X}$ is right $Z_{K \cap H}(\mathfrak{a}_{pq})$-invariant, and the proof is complete.

COROLLARY 5.9. *Let $a \in A'_{pq}$, $X \in \mathfrak{a}_{pq}$, $w \in W(\Delta_+)$. Then $F_{a,X}$ has a local maximum at the critical point $h \in wH_X$ if and only if*:

$$\alpha(X)\alpha(w^{-1}\log a) \geq 0 \quad \textit{for all } \alpha \in \Delta_+^+, \tag{5.3}$$

$$\alpha(X) \leq 0 \quad \textit{for all } \alpha \in \Delta_-^+. \tag{5.4}$$

Proof. Because of Corollary 5.8, $F_{a,X}$ has a local maximum at $h \in wH_X$ iff its Hessian at a representative x_w of w in $N_{K\cap H}(\mathfrak{a}_{pq})$ is negative definite transversally to wH_X. For this it is necessary and sufficient that all its eigenvalues are ≤ 0 (use Lemma 5.6).

By Lemma 5.5, the Hessian of $F_{a,X}$ at x_w is given by $F_{a,X}(x_w; UV) = \langle U, L_{a,X,x_w}(V)\rangle = \langle U, L'(V)\rangle$, where $L' = L_{a',X,e}$, $a' = a^{w^{-1}}$. In view of Lemma 5.2 we have

$$L'(V) = -\pi_{\mathfrak{h}} \circ \mathrm{Ad}(a^{w^{-1}})^{-1} \circ \mathrm{ad}(X) \circ E_{\mathfrak{k}}(V),$$

for $V \in \mathfrak{h}$. If $\alpha \in \Delta^+$, we put $\mathfrak{h}_+^\alpha = \{U + \theta U; U \in \mathfrak{g}_+^\alpha\}$ and $\mathfrak{h}_-^\alpha = \{U - \theta U; U \in \mathfrak{g}_-^\alpha\}$. Then

$$\mathfrak{h} = \mathfrak{h} \cap \mathfrak{l} \oplus \sum_{\alpha\in\Delta^+}{}^{\oplus} (\mathfrak{h}_+^\alpha \oplus \mathfrak{h}_-^\alpha).$$

We claim that L' diagonalizes over this decomposition.

Indeed, it is obvious that $L' = 0$ on $\mathfrak{h} \cap \mathfrak{l}$. Moreover, if $\alpha \in \Delta_-^+$, $U \in \mathfrak{g}_-^\alpha$, then $E_{\mathfrak{k}}(U - \theta U) = E_{\mathfrak{k}}(2U - (U + \theta U)) = -(U + \theta U)$. Also,

$$\begin{aligned}\mathrm{Ad}(a^{w^{-1}})^{-1} \circ \mathrm{ad}(X)(U + \theta U) &= \alpha(X)(a^{-w\alpha}U - a^{w\alpha}\theta U)\\ &= \alpha(X)(a^{-w\alpha}U + a^{w\alpha}\tau U).\end{aligned}$$

Since

$$a^{-w\alpha}U + a^{w\alpha}\tau U = p(U + \tau U) + q(U - \tau U),$$

with

$$p = \tfrac{1}{2}(a^{w\alpha} + a^{-w\alpha}), \quad q = \tfrac{1}{2}(a^{-w\alpha} - a^{w\alpha}),$$

it follows that

$$L'(U - \theta U) = \alpha(X)\cosh\alpha(w^{-1}\log a)(U - \theta U),$$

for $U \in \mathfrak{g}_-^\alpha$. A similar computation yields:

$$L'(U + \theta U) = \alpha(X)\sinh\alpha(w^{-1}\log a)(U + \theta U),$$

for $U \in \mathfrak{g}_+^\alpha$, whence the claim.

Taking into account that the Killing form is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$, we infer that the Hessian has all eigenvalues ≤ 0 iff (5.3, 4), thereby completing the proof.

6. Proof of the convexity theorem. We prove Theorem 1.1 by induction on the rank $\mathrm{rk}(\Delta)$ of Δ. If $\mathrm{rk}(\Delta) = 0$, then $E_{pq} \cdot \mathfrak{H} = 0$, and the theorem evidently holds. So let us assume that $\mathrm{rk}(\Delta) > 0$, and that the theorem has been proved already for groups of lower rank. In §2 we saw that this hypothesis implies that the theorem is also valid for lower rank groups of the Harish-Chandra class.

If $X \in \mathfrak{a}_{pq}$, we write $\Delta(X) = \{\alpha \in \Delta; \alpha(X) = 0\}$, $\Delta^+(X) = \Delta^+ \cap \Delta(X)$, etc.. Moreover, $W(\Delta_+(X))$ denotes the reflection group generated by the reflections in roots $\alpha \in \Delta_+(X)$. Put

$$\mathfrak{a}(X, Z) = \mathrm{conv}\big(W(\Delta_+(X)) \cdot Z\big) + \Gamma(\Delta_-^+(X)),$$

for X, $Z \in \mathfrak{a}_{pq}$. Then the assertion of Theorem 1.1 can be reformulated as

$$\mathrm{im}(F_a) = \mathfrak{a}(0, \log a). \tag{6.1}$$

We shall first prove (6.1) for $a \in A'_{pq}$. As a first step we have:

LEMMA 6.1. *Let $a \in A'_{pq}$. Then*

$$\mathrm{im}(F_a) \subset \mathfrak{a}(0, \log a).$$

Proof. By Lemma 4.4 the map $F_a\colon H \to \mathfrak{a}_{pq}$ is submersive except at points of

$$\mathscr{C} = \bigcup_{w \in W(\Delta_+)} \bigcup_{X \in \mathfrak{a}_{pq} \setminus \{0\}} wH_X.$$

Being a finite union of lower dimensional closed submanifolds of $H, \mathscr{C}$ has a complement which is open and dense in H. Therefore $\mathrm{im}(F_a)$ has dense interior. Moreover, $\mathrm{im}(F_a)$ being closed (Corollary 3.7), a point Z of the boundary $\partial\, \mathrm{im}(F_a)$ of $\mathrm{im}(F_a)$ must be the image $F_a(h)$ of some $h \in \mathscr{C}$. Write $h = x_w h'$, with x_w a representative of $w \in W(\Delta_+)$ in $N_{K \cap H}(\mathfrak{a}_{pq})$, and $h' \in H_X$, $X \in \mathfrak{a}_{pq} \setminus \{0\}$. Then $E_{pq} \circ \mathfrak{H}(ah) = E_{pq} \circ \mathfrak{H}(a^w h')$ which by the induction hypothesis is contained in $\mathfrak{a}(X, w^{-1}(\log a))$ (cf. §2). Now put

$$\mathscr{B} = \bigcup_{w \in W(\Delta_+)} \bigcup_{X \in \mathfrak{a}_{pq} \setminus \{0\}} \mathfrak{a}\big(X, w^{-1}(\log a)\big).$$

Then from the above reasoning we infer that

$$\partial\, \mathrm{im}(F_a) \subset F_a(\mathscr{C}) \subset \mathscr{B}.$$

It follows that every component of $\mathfrak{a}_{pq} \setminus \mathscr{B}$ must be entirely contained in the set $F_a(H)$, or have empty intersection with it. Now clearly $\mathscr{B} \subset \mathfrak{a}(0, \log a)$. In view of Lemma 3.9, $\mathrm{im}(F_a)$ does not contain the connected set $\mathfrak{a}_{pq} \setminus \mathfrak{a}(0, \log a)$. Therefore $\mathrm{im}(F_a) \setminus \mathfrak{a}(0, \log a) = \varnothing$ and the assertion follows.

LEMMA 6.2. *Let $a \in A'_{pq}$, $X \in \mathfrak{a}_{pq}$. If $F_{a,X}$ has a local maximum at $h \in H$, then $\langle U, X\rangle \leq F_{a,X}(h)$ for all $U \in \mathfrak{a}(0, \log a)$.*

Proof. Suppose $F_{a,X}$ has a local maximum in $h \in H$. Then h is a critical point, hence of the form $x_w h'$, with x_w a representative of $w \in W(\Delta_+)$ in $N_{K \cap H}(\mathfrak{a}_{pq})$, and $h' \in H_X$. Moreover, by Corollary 5.9 we must have

$$\alpha(X)\alpha(w^{-1} \log a) \geq 0 \quad \text{for all } \alpha \in \Delta_+^+,$$

$$\alpha(X) \leq 0 \quad \text{for all } \alpha \in \Delta_-^+.$$

In Proposition 6.3 below we deduce that the first statement implies that $\langle X, Z\rangle \leq \langle X, w^{-1}(\log a)\rangle$ for all $Z \in \mathrm{conv}(W(\Delta_+) \cdot \log a)$. Moreover, the second statement implies that $\langle X, Y\rangle \leq 0$ for all $Y \in \Gamma(\Delta_-^+)$. Hence

$$\langle X, U\rangle \leq \langle X, w^{-1}(\log a)\rangle,$$

for every $U \in \mathfrak{a}(0, \log a)$. Since $F_{a,X}(x_w h') = F_{a,X}(x_w) = \langle X, w^{-1}(\log a)\rangle$, the assertion now follows.

PROPOSITION 6.3. *Let X, $Y \in \mathfrak{a}_{pq}$ be such that $\alpha(X)\alpha(Y) \geq 0$ for all $\alpha \in \Delta_+$. Then $\langle X, uY\rangle \leq \langle X, Y\rangle$ for all $u \in W(\Delta_+)$.*

Proof. Let E be the subspace of $\mathfrak{a}_{pq}$ spanned by H_α, $\alpha \in \Delta_+$. Then $\Delta_+ = \Delta(\mathfrak{g}_+, \mathfrak{a}_{pq})$ is a (possibly non-reduced) root system on E. Moreover, since $W(\Delta_+)$ leaves E invariant and acts trivially on $E^\perp$, it suffices to prove the statement for X, $Y \in E$. But then it is well known that the hypothesis implies the existence of a closed Weyl chamber C such that X, $Y \in C$. The proposition now follows.

LEMMA 6.4. *If $a \in A'_{pq}$ then $\partial\, \mathrm{im}(F_a) \subset \partial\, \mathfrak{a}(0, \log a)$.*

Proof. Given $X \in \mathfrak{a}_{pq}$, write

$$\mathfrak{a}(X) = \sum_{\alpha \in \Delta(X)} \mathbf{R} \cdot H_\alpha.$$

Then for every $Z \in \mathfrak{a}_{pq}$, we have

$$\mathfrak{a}(X, Z) \subset Z + \mathfrak{a}(X).$$

By regularity of a, the set $\mathfrak{a}(X, w^{-1}(\log a))$ ($w \in W(\Delta_+)$) has non-empty interior $\mathring{\mathfrak{a}}(X, w^{-1}(\log a))$ in $w^{-1}(\log a) + \mathfrak{a}(X)$. Put $'\mathfrak{a}_{pq} = \{X \in \mathfrak{a}_{pq};\ \operatorname{rk}\Delta(X) = \operatorname{rk}\Delta - 1\}$. Then clearly

$$\mathscr{B} = \bigcup_{w \in W(\Delta_+)} \bigcup_{X \in '\mathfrak{a}_{pq}} \mathfrak{a}\big(X, w^{-1}(\log a)\big).$$

Moreover,

$$\mathring{\mathscr{B}} = \bigcup_{w \in W(\Delta_+)} \bigcup_{X \in '\mathfrak{a}_{pq}} \mathring{\mathfrak{a}}\big(X, w^{-1}(\log a)\big)$$

is dense in $\mathscr{B}$. Since $\operatorname{im}(F_a)$ is the closure of a union of connected components of $\mathfrak{a}_{pq} \setminus \mathscr{B}$, it follows that $\partial \operatorname{im}(F_a) \cap \mathring{\mathscr{B}}$ is dense in $\partial \operatorname{im}(F_a)$. Therefore it suffices to show that $\partial \operatorname{im}(F_a) \cap \mathring{\mathscr{B}} \subset \partial \mathfrak{a}(0, \log a)$.

Let $Z \in \partial \operatorname{im}(F_a) \cap \mathring{\mathscr{B}}$. Then there exist $w \in W(\Delta_+)$ and $X \in {'\mathfrak{a}_{pq}}$ such that $Z \in \mathring{\mathfrak{a}}(X, w^{-1}(\log a))$. Moreover, by the induction hypothesis there exists a $h \in H_X$ such that $Z = E_{pq} \circ \mathfrak{H}(a^{w^{-1}}h) = F_a(x_w h)$. Multiplying X by -1 if necessary, we can arrange that X is an outward normal to $F_a(H)$. Thus, $F_{a,X}$ attains a local maximum at $x_w h$. By Proposition 6.2 it now follows that $Z = F_a(x_w h) \in \partial\, \mathfrak{a}(0, \log a)$.

Corollary 6.5. *If $a \in A'_{pq}$, then* $\operatorname{im}(F_a) = \mathfrak{a}(0, \log a)$.

Completion of the proof. Let $a \in A_{pq} \setminus A'_{pq}$, and select a sequence $\{a_n\}$ in A'_{pq} which converges to a. Then $\mathscr{A} = \{a\} \cup \{a_n\}$ is a compact subset of A_{pq}.

Let $h \in H$. Then

$$E_{pq} \circ H(a_n h) = U_n + V_n,$$

where $U_n \in \operatorname{conv}(W(\Delta_+) \cdot \log a_n)$ and $V_n \in \Gamma(\Delta_-^+)$. Clearly U_n varies in a compact subset of $\mathfrak{a}_{pq}$, and so does $E_{pq} \circ \mathfrak{H}(a_n h)$. It follows that $\{V_n\}$ is relatively compact in $\Gamma(\Delta_-^+)$. Passing to a subsequence if necessary, we may therefore assume that the sequences $\{U_n\}$ and $\{V_n\}$ converge, to say U and V respectively. Clearly $U \in \operatorname{conv}(W(\Delta_+) \cdot \log a)$, $V \in \Gamma(\Delta_-^+)$. On the other hand, $U + V = E_{pq} \circ \mathfrak{H}(ah)$, and we have shown that $E_{pq}(\mathfrak{H}(aH)) \subset \mathfrak{a}(0, \log a)$.

For the converse, let $W \in \mathfrak{a}(0, \log a)$ and write $W = U + V$, with $U \in \operatorname{conv}(W(\Delta_+) \cdot \log a)$ and $V \in \Gamma(\Delta_-^+)$. Then there exists a sequence $\{U_n\}$ in $\mathfrak{a}_{pq}$ which converges to U, and such that $U_n \in \operatorname{conv}(W(\Delta_+) \cdot \log a_n)$ for all n. By Corollary 6.5 there exists a sequence $\{h_n\}$ in H such that $E_{pq} \circ \mathfrak{H}(a_n h_n) = U_n + V$, and by Lemma 3.3 the set $\{h_n\}$ must be relatively compact in H. Passing to a subsequence if necessary, we may therefore assume that h_n converges to a point $h \in H$. It follows that $E_{pq} \circ \mathfrak{H}(ah) = \lim E_{pq} \circ \mathfrak{H}(a_n h_n) = \lim(U_n + V) = U + V$ and the proof is complete.

Appendix A. The group case. Let G be a connected real semisimple Lie group with finite centre. It may be viewed as a symmetric space in the following way. Let $'G = G \times G$, $'\tau$: $'G \to 'G$ the involution given by $'\tau(x, y) = (y, x)$. Then $'H = d(G)$, the diagonal in $G \times G$, and the map $G \times G \to G, (x, y) \mapsto xy^{-1}$ induces a diffeomorphism $'G/'H \simeq G$.

In this appendix we reformulate Theorem 1.1 for the symmetric pair $('G, 'H)$ in terms of the structure of G. If not specified, our notations have an obvious meaning.

Let θ be a Cartan involution for G. Then $'\theta = \theta \times \theta$ is a Cartan involution for $'G$ which commutes with $'\tau$. Thus, on the Lie algebra level we have $'\mathfrak{p} = \mathfrak{p} \times \mathfrak{p}$, $'\mathfrak{q} = \delta(\mathfrak{g})$, $'\mathfrak{p} \cap '\mathfrak{q} = \delta(\mathfrak{p})$, where we have used the notation $\delta(\mathfrak{g})$ for the subset $\{(X, -X); x \in \mathfrak{g}\}$ of $'\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}$, etc.

Let $\mathfrak{a}_p$ be maximal abelian in $\mathfrak{p}$ and put $'\mathfrak{a}_p = \mathfrak{a}_p \times \mathfrak{a}_p$ and $'\mathfrak{a}_{pq} = \delta(\mathfrak{a}_p)$. Let $\mathfrak{j}$: $\mathfrak{a}_p \to '\mathfrak{a}_{pq}$ be the linear isomorphism given by $\mathfrak{j}(X) = (X, -X)$. Then the projection $'E_{pq}$: $'\mathfrak{a}_p \to '\mathfrak{a}_{pq}$ is given by

$$'E_{pq}(X, Y) = \mathfrak{j}(\tfrac{1}{2}(X - Y)).$$

Moreover, with obvious notations, $'\Delta_{pq} = \mathfrak{j}^{*-1}(\Delta_p)$, and if π_i denotes the projection of $'\mathfrak{a}_p$ on the ith coordinate $(i = 1, 2)$, then $'\Delta_p = \pi_1^*\Delta_p \cup \pi_2^*\Delta_p$. Let Δ_p^+ be a choice of positive roots for Δ_p,

$$G = KA_pN$$

the associated Iwasawa decomposition and $\mathfrak{H}$: $G \to \mathfrak{a}_p$ the corresponding Iwasawa projection. Then $'\Delta_{pq}^+ = \mathfrak{j}^{*-1}(\Delta_p^+)$ and $'\Delta_p^+ = \pi_1^*(\Delta_p^+) \cup \pi_2^*(-\Delta_p^+)$ are compatible choices of positive roots. The associated Iwasawa decomposition for $'G$ is $'G = 'K'A_p'N$, where $'K = K \times K$, $'A_p = A_p \times A_p$, $'N = N \times \overline{N}$. The associated projection $'\mathfrak{H}$: $'G \to '\mathfrak{a}_p$ is given by $'\mathfrak{H}(x, y) = (\mathfrak{H}(x), -\mathfrak{H}(\theta y))$, so that

$$'E_{pq} \circ '\mathfrak{H}(x, y) = \tfrac{1}{2}\mathfrak{j}(\mathfrak{H}(x) + \mathfrak{H}(\theta y)).$$

It is now straightforward to derive the following equivalent formulation of Theorem 1.1 in terms of G's structure. Let W denote the Weyl group of $\mathfrak{a}_p$ in $\mathfrak{g}$. If $\alpha \in \Delta_p^+$, we let H_α denote the element of $\mathfrak{a}_p \cap (\ker \alpha)^\perp$ with $\alpha(H_\alpha) = 1$, and write

$$\Gamma(\Delta_p^+) = \sum_{\alpha \in \Delta_p^+} \mathbf{R}_+ H_\alpha.$$

THEOREM A.1. *Let G be a connected real semisimple Lie group with finite centre, $G = KA_pN$ an Iwasawa decomposition for G, and $\mathfrak{H}$: $G \to \mathfrak{a}_p$ the corresponding projection. If $a \in A_p$, then the image of the map Ψ_a: $G \to \mathfrak{a}_p$ given by*

$$\Psi_a(x) = \tfrac{1}{2}(\mathfrak{H}(ax) + \mathfrak{H}(a\theta x))$$

is equal to

$$\mathrm{im}(\Psi_a) = \mathrm{conv}(W \cdot \log a) + \Gamma(\Delta_p^+).$$

In particular, putting $a = e$, and using the Iwasawa decomposition $G = KA_pN$, one easily finds

$$\mathfrak{H}(\overline{N}) = \Gamma(\Delta_p^+).$$

Moreover, Lemma 3.3 implies that the map $\mathfrak{H}$: $\overline{N} \to \mathfrak{a}_p$is proper. Now these facts can be checked independently as follows.

By **[12]** (cf. also **[25]**), there exist a diffeomorphism Φ: $X_{\alpha \in P}\overline{N}_\alpha \to \overline{N}$, such that

$$\mathfrak{H} \circ \Phi\big((\bar{n}_\alpha)_{\alpha \in P}\big) = \sum_{\alpha \in P} \mathfrak{H}(\bar{n}_\alpha). \tag{A.1}$$

Here the Cartesian product extends over the set P of indivisible roots in Δ_p^+. Moreover, $\overline{N}_\alpha = \overline{N} \cap G_\alpha$, where G_α is a closed semisimple subgroup of G whose Lie algebra is the real rank one algebra generated by $\mathfrak{g}^{-2\alpha}$, $\mathfrak{g}^{-\alpha}$, $\mathfrak{g}^{\alpha}$, $\mathfrak{g}^{2\alpha}$. The Iwasawa decomposition of G induces the Iwasawa decompositions $G_\alpha = K_\alpha A_{p,\alpha} N_\alpha$ with $K_\alpha = K \cap G_\alpha$, etc. Thus we see that by (A.1) the above statements for the map $\mathfrak{H}$: $\overline{N} \to \mathfrak{a}_p$ reduce to the corresponding statements for the maps $\mathfrak{H}$: $\overline{N}_\alpha \to \mathfrak{a}_{p,\alpha}$. The latter statements can be checked to be true from the explicit formula for the Iwasawa projection of a real rank one group (cf. **[16]**, **[25]**).

The following independent proof of Theorem A.1 was communicated to me by T. H. Koornwinder.

Independent proof of Theorem A.1. Clearly it suffices to prove the theorem for $\log a \in -\mathrm{cl}(\mathfrak{a}_p^+)$. The key observation is that in that case we have

$$\mathrm{conv}(W \cdot \log a) + \Gamma(\Delta_p^+) = \log a + \Gamma(\Delta_p^+). \tag{A.2}$$

Now if $k \in K$, $b \in A_p$, $n \in N$, then

$$\begin{aligned}\tfrac{1}{2}&\big[\mathfrak{H}(akbn) + \mathfrak{H}(akb^{-1}\theta n)\big]\\ &= \tfrac{1}{2}\big[\mathfrak{H}(ak) + \log b + \mathfrak{H}\big(ak\kappa(b^{-1}\theta n)\big) + \mathfrak{H}(b^{-1}\theta n)\big]\\ &= \tfrac{1}{2}\big[\mathfrak{H}(ak) + \mathfrak{H}\big(ak\kappa(b^{-1}\theta n)\big) + \mathfrak{H}(b^{-1}\theta nb)\big].\end{aligned}$$

By the above we have $\mathfrak{H}(b^{-1}\theta nb) \in \mathfrak{H}(\overline{N}) = \Gamma(\Delta_p^+)$. Moreover, $\frac{1}{2}[\mathfrak{H}(ak) + \mathfrak{H}(ak\kappa(b^{-1}\theta n))] \in \mathrm{conv}(W \cdot \log a)$ by Kostant's convexity theorem. It follows that $\mathrm{im}(\Psi_a)$ is contained in the set (A.2).

On the other hand, if $n \in N$, then $\frac{1}{2}[\mathfrak{H}(an) + \mathfrak{H}(a\theta n)] = \log a + \frac{1}{2}\mathfrak{H}(a\theta n a^{-1})$. Hence $\operatorname{im}(\Psi_a) \supset \log a + \Gamma(\Delta_p^+)$ and the proof is complete.

Appendix B. Holomorphic continuation of a decomposition.

B.1. *Introduction.* We assume that $\mathfrak{g}$ is semisimple, G its adjoint group. Let G_c be the adjoint group of the complexified Lie algebra $\mathfrak{g}_c$. In this appendix we study the holomorphic continuation to G_c of the decomposition $G = KL_0A_{pq}N_Q$ (cf. Prop. 3.1). The main result, Theorem B.1.2, generalizes a result of [2] on the holomorphic continuation of the Iwasawa decomposition. In Section B.4 it is used to prove Lemma 3.4.

Let L_c be the centralizer of $\mathfrak{a}_{pq}$ in G_c, Q_c the normalizer of $\mathfrak{l}_c + \mathfrak{n}_{Qc}$ in G_c and $N_{Qc} = \exp(\mathfrak{n}_{Qc})$. Then it is well known that L_c, Q_c and N_{Qc} are algebraic and connected. Moreover, Q_c is a parabolic subgroup with Levi decomposition $Q_c = L_cN_{Qc}$. Let K_c, A_{pc}, A_{pqc}, L_{0c} be the connected analytic subgroups of G_c with Lie algebras $\mathfrak{k}_c$, $\mathfrak{a}_{pc}$, $\mathfrak{a}_{pqc}$, $\mathfrak{l}_{0c}$.

Proposition B.1.1. *The groups K_c, A_{pc}, A_{pqc}, L_{0c} are the identity components (for the usual topology) of algebraic subgroups of G_c.*

Remark. If we speak about connected components, it will always be with respect to the usual (i.e. non-Zariski) topology.

Proof. The holomorphic involutions of G_c whose differentials at the identity are θ and τ, are denoted by the same symbols. Define

$$'K_c = \{x \in G_c;\, \theta x = x\},$$
$$'A_{pc} = \left\{x \in G_c;\, \theta x = x^{-1}, x\big|\mathfrak{g}^\alpha \in \mathbf{C}\cdot \operatorname{Id}(\mathfrak{g}^\alpha) \text{ for } \alpha \in \Delta_p\right\}.$$
$$'A_{pqc} = \left\{x \in {'A_{pc}};\, \tau(x) = x^{-1}\right\}.$$

Then K_c, A_{pc}, A_{pqc} are the identity components of the algebraic subgroups $'K_c$, $'A_{pc}$, $'A_{pqc}$. As for the remaining assertion, we claim that L_{0c} is the identity component of

$$'L_{0c} = \left\{x \in L_c;\, \det(x|\mathfrak{g}^\alpha) = 1 \text{ for } \alpha \in \Delta^+\right\}.$$

To prove this it suffices to show that $\mathfrak{l}_0$ equals

$$'\mathfrak{l}_0 = \left\{X \in \mathfrak{l};\, \operatorname{tr}(\operatorname{ad}(X)|\mathfrak{g}^\alpha) = 0 \text{ for } \alpha \in \Delta^+\right\}.$$

Since $\bigcap\{\ker\alpha;\, \alpha \in \Delta^+\} = \varnothing$, we have $'\mathfrak{l}_0 \cap \mathfrak{a}_{pq} = 0$. Hence it suffices to show that $\mathfrak{l}_0 \subset {'\mathfrak{l}_0}$.

If $\alpha \in \Delta$, we write $\Delta_p(\alpha) = \{\beta \in \Delta_p; \beta \,|\, \mathfrak{a}_{pq} = \alpha\}$. Thus, if $\alpha \in \Delta$, then

$$\mathfrak{g}^\alpha = \sum_{\beta \in \Delta_p(\alpha)} \mathfrak{g}^\beta.$$

Let $t_\alpha(X) = \mathrm{tr}(\mathrm{ad}(X) \,|\, \mathfrak{g}^\alpha)$, for $X \in \mathfrak{l}$. Since $\mathfrak{l} \cap \mathfrak{k}$ acts by skew symmetric transformations on $\mathfrak{g}^\alpha$, it follows that $t_\alpha = 0$ on $\mathfrak{l} \cap \mathfrak{k}$. Moreover, for $X \in \mathfrak{a}_{ph}$ we have

$$t_\alpha(X) = \sum_{\beta \in \Delta_p(\alpha)} \beta(X) \dim(\mathfrak{g}^\beta).$$

Since $\Delta_p(\alpha) = -\Delta_p(-\alpha)$, it follows that $t_\alpha = -t_{-\alpha}$ on $\mathfrak{a}_{ph}$. On the other hand, if $X \in \mathfrak{a}_{ph}$, then $\tau X = X$, so that $t_\alpha(X) = t_\alpha(\tau X) = \mathrm{tr}(\tau \circ \mathrm{ad}(X) \circ \tau^{-1} \,|\, \mathfrak{g}^\alpha) = \mathrm{tr}(\mathrm{ad}(X) \,|\, \mathfrak{g}^{-\alpha}) = t_{-\alpha}(X)$. Hence $t_\alpha = 0$ on $\mathfrak{a}_{ph}$. Since obviously $t_\alpha(k \cdot X) = t_\alpha(X)$ for $X \in \mathfrak{l}$, $k \in L \cap K$, this implies that $t_\alpha = 0$ on $(L \cap K) \cdot \mathfrak{a}_{ph} = \mathfrak{l}_{ph}$, hence on l_0. We conclude that $\mathfrak{l}_0 \subset {}'\mathfrak{l}_0$.

Let S_Q be the complement of $K_c Q_c$ in G_c. Then S_Q may be identified with the union of the lower dimensional K_c-orbits on the flag manifold G_c/Q_c. Inspecting the proof of Proposition 3.1, one readily checks that the maps $\lambda\colon G \to (K \cap L_0)\backslash L_0$, $h_q\colon G \to A_{pq}$, $\nu_Q\colon G \to N_Q$ defined by

$$x \in K\lambda(x) h_q(x) \nu_Q(x) \qquad (x \in G)$$

are real analytic. The main result of this appendix is the following.

THEOREM B.1.2. *The set S_Q is algebraic. The maps λ, h_q and ν_Q extend to multi-valued holomorphic maps $G_c - S_Q \to (K_c \cap L_{0c})\backslash L_{0c}$, A_{pqc}, N_{Qc}. The map ν_Q is rational and there exists an integer $m > 0$ such that h_q^m is rational. Moreover, if $\{x_k\}$ is a sequence in $G_c - S_Q$ converging to a point $x \in S_Q$, then $\{h_q^m(x_k); k \in \mathbf{N}\}$ is not relatively compact in A_{pqc}.*

REMARK. By a multi-valued holomorphic map from a connected complex analytic manifold X into a complex analytic manifold Y, we mean a holomorphic map from the universal covering $\tilde{X}$ of X into Y.

Loosely said, the line of proof is as follows. Suppose $x \in K_c lan$. Then $(\theta x)^{-1}x = (\theta n)^{-1}(\theta l)^{-1} l a^2 n$. Now l, a, n can be solved from this by using properties of the $\overline{N}_{Qc} L_{0c} A_{pqc} N_{Qc}$-decomposition (here $\overline{N}_{Qc} = \theta(N_{Qc})$). The latter decomposition is studied as follows. First we construct an embedding of G_c in the matrix group $\mathrm{Gl}(n, \mathbf{C})$ (here $n = \dim \mathfrak{g}_c$). Then, in the next section, we generalize certain matrix computations which go back to [**11**, Ch. 2, §8].

The proof is completed in §B.3.

Let $<$ be the ordering of $\mathfrak{a}_{pq}^*$ which is lexicographic in the coordinates relative to the simple roots of Δ^+, and let $\alpha_s < \cdots < \alpha_1$ be the corresponding enumeration of the elements of Δ^+. For every $1 \leq j \leq s$, we put $\mathfrak{g}_j = \mathfrak{g}^{\alpha_j}$. Moreover, we write $\mathfrak{g}_{s+1} = \mathfrak{l}_0$, $\mathfrak{g}_{s+2} = \mathfrak{a}_{pq}$, $\mathfrak{g}_{s+2+j} = \theta\mathfrak{g}_{s+1-j}$ for $1 \leq j \leq s$. Now let $(\cdot, \cdot)$ be the positive definite inner product on $\mathfrak{g}$ defined by $(X, Y) = -\langle X, \theta Y\rangle$ for $X, Y \in \mathfrak{g}$. Then

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_t \tag{B.1}$$

(where $t = 2s + 2$) is an orthogonal direct sum decomposition. Select an orthonormal basis $(e_i; 1 \leq i \leq n)$ of $\mathfrak{g}$ which is subordinate to (B.1) and such that the ordering $e_1, \ldots, e_n$ of its elements is compatible with the ordering of the sum in (B.1).

If $1 \leq j \leq t$, let $d_j = \dim(\mathfrak{g}_j)$, and let P_j denote the orthogonal projection $\mathfrak{g} \to \mathfrak{g}_j$. In the sequel we shall identify real linear maps with their complex linear extensions. Also, given a linear endomorphism $X \in \mathrm{End}(\mathfrak{g}_c)$, we let X_{ij} denote the $d_i \times d_j$-matrix of the linear map $(P_i \circ X) \mid \mathfrak{g}_j$ from $\mathfrak{g}_j$ into $\mathfrak{g}_i$ and we identify X with the matrix of blocks $(X_{ij}; 1 \leq i, j \leq t)$. With these notations the composition of endomorphisms corresponds to matrix multiplications in the usual way:

$$(XY)_{ik} = \sum_{1 \leq j \leq t} X_{ij} Y_{jk},$$

for $X, Y \in \mathrm{End}(\mathfrak{g}_c)$, $1 \leq i, k \leq t$.

Now let

$$\underline{\mathfrak{n}}_Q = \{X \in \mathrm{End}(\mathfrak{g}); X_{ij} = 0 \text{ for } 1 \leq j \leq i \leq t\},$$

$$\underline{\bar{\mathfrak{n}}}_Q = \{X \in \mathrm{End}(\mathfrak{g}); X_{ij} = 0 \text{ for } 1 \leq i \leq j \leq t\},$$

$$\underline{\mathfrak{l}} = \{X \in \mathrm{End}(\mathfrak{g}); X_{ij} = 0 \text{ for } i \neq j\}.$$

Then clearly $\mathrm{End}(\mathfrak{g}) = \underline{\bar{\mathfrak{n}}} \oplus \underline{\mathfrak{l}} \oplus \underline{\mathfrak{n}}_Q$. Moreover, $\underline{\mathfrak{l}} = \underline{\mathfrak{l}}_0 \oplus \underline{\mathfrak{a}}_{pq}$, where

$$\underline{\mathfrak{l}}_0 = \{X \in \underline{\mathfrak{l}}; \mathrm{tr}(X_{ii}) = 0 \text{ for } 1 \leq i \leq t\},$$

$$\underline{\mathfrak{a}}_{pq} = \{X \in \underline{\mathfrak{l}}; X_{jj} \in \mathbf{C} \cdot I_j \text{ for } 1 \leq j \leq t\}.$$

Here we have written I_j for the identity matrix of size $d_j \times d_j$.

Consequently

$$\mathrm{End}(\mathfrak{g}) = \underline{\bar{\mathfrak{n}}}_Q \oplus \underline{\mathfrak{l}}_0 \oplus \underline{\mathfrak{a}}_{pq} \oplus \underline{\mathfrak{n}}_Q. \tag{B.2}$$

PROPOSITION B.1.3. *Let* $\mathfrak{B}$ *be any of the algebras* $\bar{\mathfrak{n}}_Q$, $\mathfrak{l}_0$, $\mathfrak{a}_{pq}$, $\mathfrak{n}_Q$. *Then* $\mathrm{ad}(\mathfrak{B}) = \mathrm{ad}(\mathfrak{g}) \cap \underline{\mathfrak{B}}$.

Proof. The inclusions $\mathrm{ad}(\mathfrak{B}) \subset \mathrm{ad}(\mathfrak{g}) \cap \mathfrak{B}$ are obvious (see also the proof of Prop. B.1.1). Therefore the direct sum decomposition $\mathrm{ad}(\mathfrak{g}) = \mathrm{ad}(\bar{\mathfrak{n}}_Q) \oplus \mathrm{ad}(\mathfrak{l}_0) \oplus \mathrm{ad}(\mathfrak{a}_{pq}) \oplus \mathrm{ad}(\mathfrak{n}_Q)$ is compatible with (B.2). The latter sum being direct, the inclusions must be equalities.

Now let $\underline{G} = \mathrm{GL}(\mathfrak{g})$, $\underline{G}_c = \mathrm{GL}(\mathfrak{g}_c)$, and put

$$\underline{N}_{Qc} = \{x \in \underline{G}_c;\, x - I \in \underline{\mathfrak{n}}_{Qc}\},$$

$$\underline{\bar{N}}_{Qc} = \{x \in \underline{G}_c;\, x - I \in \underline{\bar{\mathfrak{n}}}_{Qc}\},$$

$$\underline{L}_c = \{x \in \underline{G}_c;\, x_{ij} = 0 \text{ if } i \neq j\},$$

$$\underline{L}_{0c} = \{x \in \underline{L}_c;\, \det(x_{jj}) = 1 \text{ for } 1 \leq j \leq t\},$$

$$\underline{A}_{pqc} = \{x \in \underline{L}_c;\, x_{jj} \in \mathbf{C} \cdot I_j \text{ for } 1 \leq j \leq t\}.$$

These are algebraic subgroups of $\underline{G}_c$ with Lie algebras $\underline{\mathfrak{n}}_{Qc}$, $\underline{\bar{\mathfrak{n}}}_{Qc}$, $\underline{\mathfrak{l}}_c$, $\underline{\mathfrak{l}}_{0c}$, $\underline{\mathfrak{a}}_{pqc}$ respectively. The following corollary is now immediate.

COROLLARY B.1.4. *Let B be any of the groups N_Q, $\bar{N}_Q$, L_0 or A_{pq}. Then $B_c = (G_c \cap \underline{B}_c)^0$. In particular, B_c is the identity component (with respect to the usual topology) of an algebraic subgroup of* $\mathrm{GL}(\mathfrak{g}_c)$.

B.2. *Decompositions in* $\mathrm{GL}(\mathfrak{g}_c)$. If $1 \leq k \leq t$ we define the polynomial function $D_k\colon \underline{G}_c \to \mathbf{C}$ by

$$D_k(x) = \det(x_{ij};\, 1 \leq i, j \leq k),$$

for $x \in \underline{G}_c$. Moreover, we let $D_0 \equiv 1$ and $D = D_1 \cdots D_t$.

LEMMA B.2.1. *Let $1 \leq k \leq t$. If $x \in \underline{G}_c$, $\bar{n} \in \underline{\bar{N}}_{Qc}$, $l \in \underline{L}_c$, $n \in \underline{N}_{Qc}$, then*

$$D_k(\bar{n}xln) = D_k(\bar{n}lxn) = D_k(x)D_k(l).$$

Proof. If $1 \leq k \leq t$, $x \in \mathrm{End}(\mathfrak{g}_c)$, let $\mathfrak{m}_k(x)$ denote the matrix $(x_{ij};\, 1 \leq i, j \leq k)$. Then an easy matrix computation yields $\mathfrak{m}_k(\bar{n}xln) = \mathfrak{m}_k(\bar{n})\mathfrak{m}_k(x)\mathfrak{m}_k(l)\mathfrak{m}_k(n)$ and $\mathfrak{m}_k(lx) = \mathfrak{m}_k(l)\mathfrak{m}_k(x)$. The assertion now follows by taking determinants.

If $1 \leq k \leq t$, we define the subgroup G_k of $\underline{G}_c$ by

$$G_k = \{x \in \underline{G}_c;\, x_{ij} = 0 \text{ for } 1 \leq j < k, i > j\}.$$

Thus $G_1 = \underline{G}_c$ and $G_t = \underline{L}_c\underline{N}_{Qc}$. Moreover, we define the subgroup $\overline{N}_k$ of $\underline{\overline{N}}_{Qc}$ by

$$\overline{N}_k = \left\{x \in \underline{\overline{N}}_{Qc};\, x_{ij} = 0 \text{ for } j \neq i, k\right\}.$$

LEMMA B.2.2. *Let $1 \leq k < t$, $y \in G_k$, $D_k(y) \neq 0$. Then there exists a unique $W_k(y) \in \overline{N}_k$ such that $W_k(y)y \in G_{k+1}$. Moreover, the map $y \mapsto D_k(y)W_k(y)$ is polynomial (in the entries of y).*

Proof. The uniqueness follows from the fact that $\overline{N}_k \cap G_{k+1} = \{I\}$.

The existence is proved by sweeping the kth column $(y_{\cdot k})$ of y. This amounts to left multiplication by an element of $\overline{N}_k$. More precisely, let E_k be the space of linar maps from $\mathfrak{g}_{kc}$ into $\mathfrak{g}_{k+1c} \oplus \cdots \oplus \mathfrak{g}_{tc}$. If $a \in E_k$, we put $a_j = P_j \circ a$ for $k+1 \leq j \leq t$ and identify a with its matrix

$$\begin{pmatrix} a_{k+1} \\ \vdots \\ a_t \end{pmatrix}$$

Also, we let $w_k(a)$ denote the element of $\overline{N}_k$ whose kth column x is given by $x_j = 0$ for $1 \leq j < k$, $x_k = I_k$, $x_j = a_j$ for $k < j \leq t$. If E_k is viewed as an abelian group for the addition, then the map $w_k\colon E_k \to \overline{N}_k$ thus defined is a group isomorphism.

If $y \in G_k$, $D_k(y) \neq 0$, then clearly $\det(y_{kk}) \neq 0$. Put

$$\alpha_k(y) = -\begin{pmatrix} y_{k+1\,k} \\ \vdots \\ y_{tk} \end{pmatrix} \cdot \left(y_{kk}^{-1}\right).$$

Then $w_k(\alpha_k(y))y \in G_{k+1}$. Hence $W_k(y) = w_k(\alpha_k(y))$. Since $D_k(y)\det(y_{kk})^{-1} = D_{k-1}(y)$, it follows that $D_k(y)W_k(y)$ is polynomial in the entries of y.

COROLLARY B.2.3. *Let $y \in \underline{G}_c$, $D(y) \neq 0$. Then there exist unique $U(y) \in \underline{\overline{N}}_{Qc}$, $\mathscr{L}(y) \in \underline{L}_c$ and $V(y) \in \underline{N}_{Qc}$ such that $y = U(y)\mathscr{L}(y)V(y)$. The maps U, $\mathscr{L}$ and V are rational.*

Proof. In view of Lemma B.2.1, the polynomial function D is left $\underline{\overline{N}}_{Qc}$-invariant. Therefore we may apply Lemma B.2.2 repeatedly and infer that for $y \in \underline{G}_c - D^{-1}(0)$ there exists a $W(y) \in \underline{\overline{N}}_{Qc}$ such that $W(y)y \in \underline{L}_c\underline{N}_{Qc}$. It is unique because $\underline{\overline{N}}_{Qc} \cap \underline{L}_c\underline{N}_{Qc} = \{I\}$. Clearly $W(y)$ is rational in the entries of y and therefore $U(y) = W(y)^{-1}$ is. The proof is completed by the easy observation that the map $\underline{L}_c \times \underline{N}_{Qc} \to \underline{L}_c\underline{N}_{Qc}$, $(l, n) \mapsto ln$ is a diffeomorphism with rational inverse.

We end this section with a proposition which will be needed in the next section. If $1 \leq j \leq t$, $d_j \neq 0$, let the function $\lambda_j \colon \underline{A}_{pqc} \to \mathbf{C}^*$ be defined by

$$x|\mathfrak{g}_j = \lambda_j(x) \cdot I_j$$

for $x \in \underline{A}_{pqc}$. It might occur that $d_j = 0$ for some j. This only happens when $\mathfrak{l}_0 = 0$, $j = s + 1$. In that case we define $\lambda_{s+1} \equiv 1$. Observe that the latter equality holds in any case.

PROPOSITION B.2.4. *If $x = ulbv$, with $u \in \underline{\overline{N}}_{Qc}$, $l \in \underline{L}_{0c}$, $b \in \underline{A}_{pqc}$, $v \in \underline{N}_{Qc}$, then*

$$\lambda_j(b)^{d_j} = D_j(x)/D_{j-1}(x),$$

for $1 \leq j \leq t$.

Proof. In view of Lemma B.2.1 and the definition of $\underline{L}_{0c}$, we have $D_j(x) = D_j(b)D_j(l) = D_j(b)$. But obviously

$$D_k(b) = \prod_{1 \leq j \leq k} \lambda_j(b)^{d_j} \qquad (1 \leq k \leq t).$$

from which the assertion follows (recall that $D_0 \equiv 1$).

B.3. *Proof of the main result.* In this section we complete the proof of Theorem B.1.2. We start with some results on the $\overline{N}_{Qc}L_{0c}A_{pqc}N_{Qc}$-decomposition.

LEMMA B.3.1. *The map $p \colon L_{0c} \times A_{pqc} \to L_c$, $(l, a) \mapsto la$ is a finite covering.*

Proof. By a standard argument we infer that p is a covering with fibre $p^{-1}(e) = L_{0c} \cap A_{pqc}$ (recall that L_c is connected).

From Proposition B.2.4 we deduce that $\underline{L}_{0c} \cap \underline{A}_{pqc}$ consists of elements $b \in \underline{A}_{pqc}$ with

$$\lambda_j(b)^{d_j} = 1$$

for $1 \leq j \leq t$. Hence $\underline{L}_{0c} \cap \underline{A}_{pqc}$ is finite. In view of Corollary B.1.4, $p^{-1}(e)$ is contained in $\underline{L}_{0c} \cap \underline{A}_{pqc}$, hence finite.

LEMMA B.3.2. *The map $\gamma \colon \overline{N}_{Qc} \times L_{0c} \times A_{pqc} \times N_{Qc} \to G_c - D^{-1}(0)$, $(\bar{n}, l, a, n) \mapsto \bar{n}lan$ is a finite covering.*

Proof. By Corollary B.2.3 the map ψ: $(\bar{n}, l, n) \mapsto \bar{n}ln$ from $\underline{\mathscr{M}} = \underline{\bar{N}}_{Qc} \times \underline{L}_c \times \underline{N}_{Qc}$ onto $\underline{G}_c - D^{-1}(0)$ is a diffeomorphism. Since D is not identically zero on G_c, $G_c - D^{-1}(0)$ is connected. In view of Proposition B.3.1 it therefore suffices to prove that ψ^{-1} maps $G_c - D^{-1}(0)$ onto $\mathscr{M} = \bar{N}_{Qc} \times L_c \times N_{Qc}$. Now clearly $\psi^{-1}(G_c - D^{-1}(0)) \supset \mathscr{M}$. Since ψ is a diffeomorphism, it follows by comparison of dimensions that there exists an open neighbourhood U of (e, e, e) in $\mathscr{M}$, such that $V = \psi(U)$ is an open neighbourhood of e in $G_c - D^{-1}(0)$. Hence ψ^{-1} maps V into $\mathscr{M}$. By analytic continuation, the holomorphic map ψ^{-1} maps the connected complex analytic manifold $G_c - D^{-1}(0)$ into the Zariski closure $\mathscr{C}$ of $\mathscr{M}$. By connectedness, $\psi^{-1}(G_c - G^{-1}(0))$ is contained in the identity component $\mathscr{C}^0$ of the linear algebraic group $\mathscr{C}$ (with respect to the usual topology). Finally, by Corollary B.1.4, $\mathscr{C}^0 = \mathscr{M}$, so that $\psi^{-1}(G_c - D^{-1}(0)) \subset \mathscr{M}$.

The map γ is a local diffeomorphism, so has a local inverse (u, l, b, v) mapping e onto (e, e, e, e). Since γ is a covering, this local inverse has a multi-valued holomorphic extension to $G_c - S_Q$. We denote it by the same symbol (u, l, b, v).

PROPOSITION B.3.3. *Let $1 \leq j \leq t$. Then the map $\lambda_j^{d_j} \circ b$: $G_c - D^{-1}(0) \to \mathbf{C}^*$, $y \mapsto \lambda_j(b(y))^{d_j}$ is rational. In fact, if $y \in G_c - D^{-1}(0)$, then*

$$\lambda_j(b(y))^{d_j} = D_j(y)/D_{j-1}(y).$$

Proof. This follows immediately from Proposition B.2.4.

COROLLARY B.3.4. *Let μ be the least common multiple of $d_1, \ldots, \hat{d}_{s+1}, \ldots, d_t$. Then $\mu > 0$, and the maps v: $G_c - D^{-1}(0) \to N_{Qc}$ and b^μ: $G_c - D^{-1}(0) \to A_{pqc}$ are rational. Moreover, if $\{y_k\}$ is a sequence in $G_c - D^{-1}(0)$ converging to a point $y \in D^{-1}(0)$, then the set $\{b^\mu(y_k); k \in \mathbf{N}\}$ is not relatively compact in A_{pqc}.*

Proof. Obviously v is the restriction of V to $G_c - D^{-1}(0)$, hence rational (see Cor. B.2.3). Since $\underline{A}_{pqc}$ centralizes $\mathfrak{l}_{0c}$, we have $\lambda_{s+1} \equiv 1$. Hence the rationality of the map b^μ follows from Proposition B.3.3.

Now let j be the lowest index among $1, \ldots, t$ such that $D_j(y) = 0$. Then $D_{j-1}(y) \neq 0$ (recall that $D_0 \equiv 1$) and by Proposition B.3.3 it follows that we must have $d_j \neq 0$ and

$$\lambda_j(b^\mu(y_k)) = \lambda_j(b(y_k))^{d_j \cdot \mu/d_j} \to 0$$

as $k \to \infty$. Hence $\{\lambda_j(b^\mu(y_k)); k \in \mathbf{N}\}$ is not relatively compact in the subset $\lambda_j(A_{pqc})$ of $\mathbf{C} \setminus \{0\}$, so that the last assertion follows.

Before proceeding, we recall some facts that can essentially be found in [**20**, Thm. II.1.3 and Proof of Prop. IV.4.4]. Let B be any connected Lie group and σ an involution of B. Then B^σ denotes the fixed point set of σ. The set $\mathscr{T} = \{x \in B; \sigma(x) = x^{-1}\}$ is a smooth submanifold of B. Now B acts on $\mathscr{T}$ according to the rule $b \cdot x = \sigma(b)xb^{-1}$. By a computation of differentials one may check that all B-orbits are open in $\mathscr{T}$. Hence the connected identity component $\mathscr{S}_\sigma(B)$ of $\mathscr{T}$ is equal to the B-orbit through e:

$$\mathscr{S}_\sigma(B) = \{\sigma(b)b^{-1}; b \in B\}.$$

The manifold $\mathscr{S}_\sigma(B)$ is called the space of symmetric elements in B. The map $B \to \mathscr{S}_\sigma(B)$, $b \mapsto \sigma(b)^{-1}b$ induces a B-equivariant diffeomorphism $B^\sigma \setminus B \xrightarrow{\simeq} \mathscr{S}_\sigma(B)$. If C is any open subgroup of B^σ, then $|C \setminus B^\sigma| < \infty$ (cf. [**20**, Thm. IV. 3.4]) and the above map $B \to \mathscr{S}_\sigma(B)$ induces a finite covering $C \setminus B \to \mathscr{S}_\sigma(B)$.

Applying the above to G_c and L_{0c} together with the holomorphic continuation of the Cartan involution θ, we obtain finite coverings

$$X \to \mathscr{S}, \qquad X_L \to \mathscr{S}_L,$$

where

$$X = K_c \setminus G_c, \quad X_L = (K_c \cap L_{0c}) \setminus L_{0c}, \quad \mathscr{S} = \mathscr{S}_\theta(G_c),$$

$\mathscr{S}_L = \mathscr{S}_\theta(L_{0c})$.

Let us now return to the proof of Theorem B.1.2. If $x \in G_c$, we put $x' = (\theta x)^{-1}$. In view of Lemma B.3.2 the map $(l, b, n) \mapsto n'lbn$ maps $\mathscr{S}_L \times A_{pqc} \times N_{Qc}$ into $\mathscr{S} - D^{-1}(0)$.

PROPOSITION B.3.5. *The map* ε: $\mathscr{S}_L \times A_{pqc} \times N_{Qc} \to \mathscr{S} - D^{-1}(0)$, $(l, b, n) \mapsto n'lbn$ *is a finite covering.*

Proof. Consider the finite covering γ of Lemma B.3.2. One easily checks that $\gamma^{-1}(\mathscr{S} - D^{-1}(0))$ equals the smooth submanifold

$$T = \{(\bar{n}, l, b, n) \in \overline{N}_{Qc} \times L_{0c} \times A_{pqc} \times N_{Qc}; n = n', l = l'\}.$$

Let S be the connected component of T which contains (e, e, e, e). Then $\gamma | S$: $S \to \mathscr{S}(G_c) - D^{-1}(0)$ is a finite covering. Moreover, the map

$$i: \mathscr{S}_L \times A_{pqc} \times N_{Qc} \to \overline{N}_{Qc} \times L_{0c} \times A_{pqc} \times N_{Qc}, \quad (l, b, n) \mapsto (n', l, b, n)$$

maps $\mathscr{S}_L \times A_{pqc} \times N_{Qc}$ diffeomorphically onto S. Since $\varepsilon = (\gamma | S) \circ i$, the proposition follows.

Define the map

$$\delta\colon X_L \times A_{pqc} \times N_{Qc} \to \mathcal{S}_L \times A_{pqc} \times N_{Qc}$$

by $\delta(\bar{l}, a, n) = (l'l, a^2, n)$ (here $\bar{l}$ denotes the coset of l). Then clearly δ is a finite covering.

Consider the map $\vartheta\colon G_c \to \mathcal{S}$, $x \mapsto x'x$, and define the polynomial function $F\colon G_c \to \mathbf{C}$ by

$$F(x) = D(\vartheta x) = D(x'x).$$

Then F is left K_c-invariant, hence can be viewed as a function on X. Similarly, $F^{-1}(0)$ can be viewed as a subset of X. As such it is the preimage of $D^{-1}(0)$ under the finite covering $\bar{\vartheta}\colon X \to \mathcal{S}$ induced by ϑ. Being the complement of an analytic null set, $X - F^{-1}(0)$ is connected, so that the restriction of $\bar{\vartheta}$ to $X - F^{-1}(0)$ is a finite covering

$$\eta\colon X - F^{-1}(0) \to \mathcal{S} - D^{-1}(0).$$

Finally, if we define the map $\varphi\colon X_L \times A_{pqc} \times N_{Qc} \to X$ by $\varphi(\bar{l}, a, n) = K_c lan$, then $\bar{\vartheta} \circ \varphi = \varepsilon \circ \delta$, where ε is the map of Proposition B.3.5. Hence $\operatorname{im}(\varphi) \subset \bar{\vartheta}^{-1}(\operatorname{im} \varepsilon) = X - F^{-1}(0)$ and the following diagram commutes:

$$\begin{tikzcd}
X_L \times A_{pqc} \times N_{Qc} \arrow[r, "\varphi"] \arrow[d, "\delta"] & X - F^{-1}(0) \arrow[ddl, "\eta"] \\
\mathcal{S}_L \times A_{pqc} \times N_{Qc} \arrow[d, "\varepsilon"] & \\
\mathcal{S} - D^{-1}(0) &
\end{tikzcd}$$

Since δ, ε and η are finite coverings, we now have the following result.

PROPOSITION B.3.6. *The map* $\varphi\colon (K_c \cap L_{0c})\backslash L_{0c} \times A_{pqc} \times N_{Qc} \to K_c \backslash G_c - F^{-1}(0)$, $(\bar{l}, a, n) \mapsto K_c lan$ *is a finite covering.*

Proof of Theorem B.1.2. Let $\pi\colon G_c - F^{-1}(0) \to X - F^{-1}(0)$ be the restriction of the canonical map $G_c \to X$. By Lemma B.3.6 the map φ has a local inverse ζ_e mapping $\bar{e}$ onto $(\bar{e}, e, e)$. Since φ is a covering, ζ_e has a multivalued holomorphic extension $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ mapping $K_c \backslash G_c - F^{-1}(0)$ into $(K_c \cap L_{0c})\backslash L_{0c} \times A_{pqc} \times N_{Qc}$.

Locally at $K_c e$, we have $\zeta_e \circ \pi = (\lambda, h_q, \nu_Q)$, by definition of the real analytic maps on the right. Hence $\zeta_1 \circ \pi$, $\zeta_2 \circ \pi$, $\zeta_3 \circ \pi$ are the multi-valued holomorphic extensions of λ, h_q, ν_Q to $G_c - F^{-1}(0)$. Moreover,

$$\pi^{-1}\big(X - F^{-1}(0)\big) = \pi^{-1}(\operatorname{im}\varphi) = K_c L_{0c} A_{pqc} N_{Qc} = K_c Q_c,$$

and therefore $S_Q = G_c - K_c Q_c = F^{-1}(0)$ is algebraic.

As for the last three assertions, by holomorphic continuation it follows that

$$x'x = \nu_Q(x)'\lambda(x)'\lambda(x)h_q^2(x)\nu_Q(x)$$

for all $x \in G_c - D^{-1}(0)$. Consequently, with the notations preceding Proposition B.3.3.,

$$h_q^{2\mu}(x) = b(x'x)^{\mu}, \tag{B.3}$$

$$\nu_Q(x) = v(x'x). \tag{B.4}$$

Now put $m = 2\mu$. Then the last assertions readily follow by application of Corollary B.3.4.

We end this section with two related propositions, which will be useful in the next section.

PROPOSITION B.3.7. $G_c - S_Q = G_c^{\theta}Q_c$.

Proof. From [**19**, Proposition 1] it follows that $G_c^{\theta} \subset K_cL_c$. Hence $G_c^{\theta}Q_c = K_cQ_c$.

PROPOSITION B.3.8. *Let* $(a, n) \in A_{pqc} \times N_{Qc}$ *and assume that* $x \in G_c^{\theta}L_{0c}an$. *Then*

$$a^{2\mu} = h_q^{2\mu}(x),$$

$$n = \nu_Q(x).$$

Proof. It follows that $x'x \in n'\mathcal{S}_L a^2 n$. Hence, with the notations preceding Proposition B.3.3, we have $a^{2\mu} = b(x'x)^{\mu}$ and $n = v(x'x)$. The assertion now follows by comparison of these two formulas with (B.3, B.4).

B.4. *Proof of Lemma* 3.4. Obviously it suffices to prove the lemma under the assumption that G is the adjoint group of the semisimple algebra $\mathfrak{g}$. To make the lemma also available for groups of class $\mathfrak{H}$, we shall in fact work under the following somewhat weaker assumption.

(A) $\mathfrak{g}$ is a real semisimple Lie algebra and G is an open subgroup of the normalizer $G_{\mathbf{R}}$ of $\mathfrak{g}$ in the adjoint group G_c of $\mathfrak{g}_c$.

The proof goes by exploitation of the duality introduced by Berger [**6**] (and also used by [**9, 10, 22**]).

The space

$$\mathfrak{g}^d = i(\mathfrak{k} \cap \mathfrak{q}) \oplus (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{q}) \oplus i(\mathfrak{p} \cap \mathfrak{h})$$

is a subalgebra of $\mathfrak{g}_c$, called the real form dual to $\mathfrak{g}$. The restriction θ^d of the (complex) involution τ to $\mathfrak{g}^d$ is a Cartan involution for $\mathfrak{g}^d$, with associated eigenspace decomposition

$$\mathfrak{g}^d = \mathfrak{k}^d \oplus \mathfrak{p}^d,$$

where $\mathfrak{k}^d = \mathfrak{h}_c \cap \mathfrak{g}^d$, $\mathfrak{p}^d = \mathfrak{q}_c \cap \mathfrak{g}^d$ (read this as: the $\mathfrak{k}$ in the dual situation, etc.). Similarly, we put $\tau^d = \theta \,|\, \mathfrak{g}^d$, $\mathfrak{h}^d = \mathfrak{k}_c \cap \mathfrak{g}^d$, $\mathfrak{q}^d = \mathfrak{p}_c \cap \mathfrak{g}^d$. Let G^d, K^d, H^d be the connected analytic subgroups of G_c with Lie algebras $\mathfrak{g}^d$, $\mathfrak{k}^d$ and $\mathfrak{h}^d$ respectively. Moreover, let $\mathfrak{a}^d_{pq} = \mathfrak{a}_{pq}$, $A^d_{pq} = A_{pq}$, $L^d = L_c \cap G^d$, $\mathfrak{n}^d_Q = \mathfrak{n}_{Qc} \cap \mathfrak{g}^d$, $N^d_Q = \exp(\mathfrak{n}^d_Q)$, and define

$$L^d_0 = (K^d \cap L^d)\exp(\mathfrak{p}^d \cap \mathfrak{h}^d \cap \mathfrak{l}^d).$$

Then according to Proposition 3.1, we have $G^d = K^d L^d_0 A^d_{pq} N^d_Q$ with corresponding maps λ^d, h^d_q, ν^d_Q: $G^d \to (L^d_0 \cap K^d)\backslash L^d_0$, A^d_{pq}, N^d_Q determined by

$$\text{(B.5)} \qquad x \in K^d \lambda^d(x) h^d_q(x) \nu^d_Q(x).$$

The idea is now to view (3.3) and (B.5) as different real forms of the same multi-valued holomorphic decomposition.

Let H_c be the connected analytic subgroup of G_c with Lie algebra $\mathfrak{h}_c$. Set $S^d_Q = G_c - H_c L_c N_{Qc}$. Then according to Theorem B.1.1, the maps λ^d, h^d_q, ν^d_Q have multi-valued holomorphic extensions to maps $G_c - S^d_Q \to (H_c \cap L^d_{0c})\backslash L^d_{0c}$, A_{pqc}, N_{Qc}.

To complete the proof of Lemma 3.4, we need the following.

PROPOSITION B.4.1. *Under the assumption* (A), *the set* Ω *is a union of connected components of* $G - S^d_Q$.

Proof. The group $H \times Q$ acts on G_c according to the rule $(h, q) \cdot x = hxq^{-1}$, for $h \in H$, $q \in Q$, $x \in G_c$. In view of Proposition B.3.7, this action leaves $G_c - S^d_Q = G^\tau_c Q_c$ invariant. Moreover, by an easy computation of differentials at points of $G^\tau_c Q_c$, it follows that all $H \times Q$-orbits in $G_c - S^d_Q$ are submanifolds of real dimension $\dim(G)$. Hence $G - S^d_Q$ is a union of open $H \times Q$-orbits. Now Ω is just the $H \times Q$-orbit through e, hence open and closed in $G - S^d_Q$.

End of proof of Lemma 3.4. Let l_1, $l_2 \in L_0$, a_1, $a_2 \in A_{pq}$, n_1, $n_2 \in N_Q$ and assume that $Hl_1a_1n_1 = Hl_2a_2n_2$. Then $G^\tau_c l_1a_1n_1 = G^\tau_c l_2a_2n_2$ and using Proposition B.3.8 we infer that $n_1 = n_2$ and $a_1^{2\mu} = a_2^{2\mu}$. The map exp: $\mathfrak{a}_{pq} \to A_{pq}$ being a diffeomorphism it follows that $a_1 = a_2$. Hence $Hl_1 = Hl_2$ from which it is immediate that $(H \cap L_0)l_1 = (H \cap L_0)l_2$. This proves uniqueness and the maps l, a_{pq}, n_Q are well defined by (3.3).

By a standard computation of differentials it now follows that the real analytic map $(\bar{l}, a, n) \mapsto Hlan$ maps $((H \cap L_0)\backslash L_0) \times A_{pq} \times N_Q$ diffeomorphically onto the canonical image $\underline{\Omega}$ of Ω in $H\backslash G$. Its inverse ζ necessarily is a real analytic map. Now let π: $\Omega \to \underline{\Omega}$ be the restriction of the natural map $G \to H\backslash G$ to Ω. Then (l, a_{pq}, n_Q) equals $\zeta \circ \pi$, hence is real analytic.

Finally, by Proposition B.3.8 we have

$$a(x)^{2\mu} = h_q^d(x)^{2\mu},$$

for $x \in \Omega$. By Proposition B.4.1, $\partial\Omega$ is contained in S_Q^d and so the last assertion of the lemma follows from the corresponding assertion of Theorem B.1.2.

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