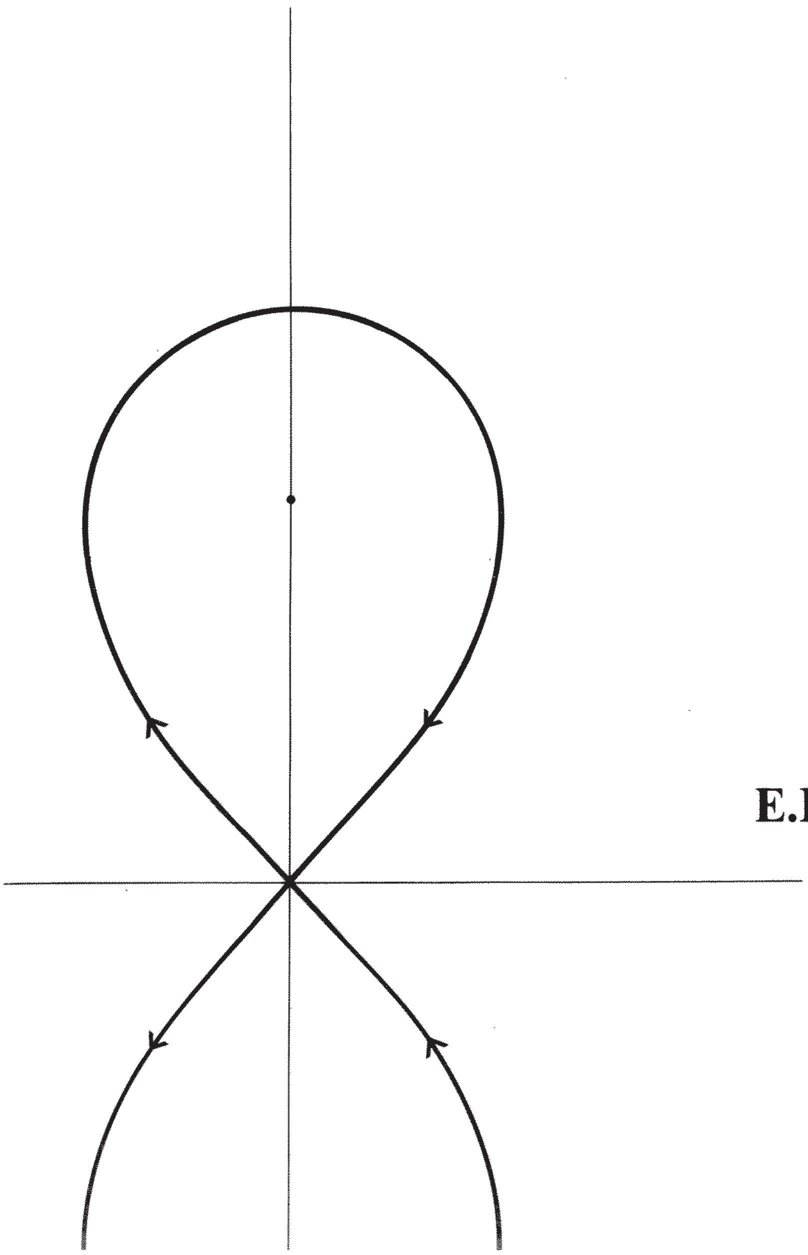
Asymptotic expansions and integral formulas for eigenfunctions on a semisimple Lie group



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Asymptotic expansions and integral formulas for eigenfunctions on a semisimple Lie group

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Summary

Let G be a connected real semisimple Lie group with finite centre and let G = KAN be an Iwasawa decomposition for G. If $\lambda \in \mathfrak{a}_c^*$ (the complexified dual of the Lie algebra $\mathfrak{a}$ of A), ϕ_λ denotes the elementary spherical function defined by the well known integral formula

$$\phi_\lambda(x) = \int_K e^{(i\lambda-\rho)H(xk)}dk \qquad (x \in G). \tag{1}$$

Let $\mathfrak{g}$ be the Lie algebra of G, and let W be the Weyl group of the pair $(\mathfrak{g},\mathfrak{a})$. Moreover, let Δ^+ be the set of positive roots of $(\mathfrak{g},\mathfrak{a})$ associated with the above Iwasawa decomposition and let $L = \mathbb{N}.\Delta^+$. The following formula of Harish-Chandra ([2]) describes the asymptotics of $\phi_\lambda(a)$ as $\log a \to \infty$ in the positive Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$.

$$\phi_\lambda(a) = \sum_{w \in W} c(w\lambda)e^{(iw\lambda-\rho)(\log a)}\Phi''(w\lambda,a), \tag{2}$$

$$\Phi''(\lambda,a) = \sum_{\mu \in L} \Gamma_\mu(\lambda)e^{-\mu(\log a)} \tag{3}$$

One of the basic results of this thesis is that the summands of (2) can be expressed as integrals similar to (1) but taken over compact cycles of real dimension dim(K) in a complexification K_c of the group K. The functions $\Phi''(w\lambda,a)$ can be written as integrals over compact cycles as well; since the integrands are holomorphic in the variables $e^{-\alpha(\log a)}$ $(\alpha \in \Delta^+)$, we obtain (3) from a convergent power series. The cycles are constructed by means of a holomorphic extension of the procedure used by Gindikin-Karpelevič ([1]) to derive the product formula for the c-function.

Similar integral formulas enable us to obtain the asymptotic expansions of Trombi-Varadarajan ([1]) as converging series. Our research in this direction has not yet been completed; for instance we have not obtained the global uniformity of the estimates of Trombi-Varadarajan although in other respects our asymptotics are more precise.

In a future publication we hope to extend the techniques developed in this thesis to the case of Eisenstein integrals

associated with a minimal parabolic subgroup of G.

For more information concerning the contents of this thesis we refer the reader to the second part of the Introduction. The non-specialist might profit from its first part, where a review of some basic concepts is given.

Contents

Introduction

0.1 A review of basic facts

In this thesis we study the asymptotic behaviour of elementary spherical functions and related eigenfunctions on a connected real semisimple Lie group G with finite centre (we let $SL(n,\mathbb{R})$ serve as an example of such a group).

Let K be a maximal compact subgroup of G (in our example we may take $K = SO(n,\mathbb{R})$), and let $\mathbb{D}(G)^K$ denote the algebra of left G- and bi-K-invariant differential operators on G. In $\mathbb{R}^n$ the exponential functions can be characterized as the eigenfunctions $\phi: \mathbb{R}^n \to \mathbb{C}$ of all translation invariant (or, equivalently, constant coefficient) differential operators, normalized by $\phi(0) = 1$. In the harmonic analysis of bi-K-invariant (or spherical) functions on G, the analogon of this set of special functions is the set of C^∞ functions $\phi: G \to \mathbb{C}$ satisfying the following conditions:

(i) ϕ is spherical,

(ii) ϕ is a simultaneous eigenfunction of $\mathbb{D}(G)^K$,

(iii) $\phi(e) = 1$.

These functions are called the elementary spherical functions of the pair (G,K).

Let $\mathfrak{g}, \mathfrak{k}$ be the Lie algebras of G, K and let $\mathfrak{a}$ be a maximal abelian subalgebra of the orthocomplement $\mathfrak{s}$ of $\mathfrak{k}$ with respect to the Killing form. In the example $\mathfrak{g}$ consists of the matrices with trace 0, $\mathfrak{k}$ of the anti-symmetric and $\mathfrak{s}$ of the symmetric ones among them; for $\mathfrak{a}$ we may take the diagonal matrices with trace 0. It is known that all such $\mathfrak{a}$ are conjugate to each other by elements of K. Their common dimension is called the real rank of $\mathfrak{g}$. Now consider again condition (ii) in the definition of elementary spherical functions. An equivalent formulation is: there exists an algebra homomorphism $\Lambda: \mathbb{D}(G)^K \to \mathbb{C}$ such that $D\phi = \Lambda(D).\phi$ for all $D \in \mathbb{D}(G)^K$. It is known that the eigenvalue homomorphisms Λ can be parametrized by the complexified dual $\mathfrak{a}_c^*$ of the real linear space $\mathfrak{a}$. If $\lambda \in \mathfrak{a}_c^*$ then the corresponding eigenvalue is denoted by $D \to \gamma(D,i\lambda)$. Moreover, it can be proved that for every $\lambda \in \mathfrak{a}_c^*$ there exists a unique elementary spherical function which satisfies the system

$$D\phi = \gamma(D,i\lambda).\phi \qquad (D \in \mathbb{D}(G)^K). \qquad (1)$$

We denote this function by ϕ_λ. Thus the collection of elementary spherical functions is equal to $\{\phi_\lambda;\ \lambda \in \mathfrak{a}_c^*\}$.

The exponential map $\exp: \mathfrak{g} \to G$ is injective on $\mathfrak{a}$ and $A = \exp \mathfrak{a}$ is a closed abelian subgroup of G. We let log denote the inverse of $\exp|\mathfrak{a} : \mathfrak{a} \to A$. Because of the Cartan decomposition

$$G = KAK$$

each spherical function is determined by its restriction to A. In order to give a more detailed description of the above Cartan decomposition, we recall that the linear operators $\operatorname{ad} X: \mathfrak{g} \to \mathfrak{g}$ $(X \in \mathfrak{a})$ given by

$$\operatorname{ad} X: Y \to [X,Y]$$

have a simultaneous diagonalization with eigenvalues $\alpha(X)$ depending linearly on $X \in \mathfrak{a}$. The non-zero $\alpha \in \mathfrak{a}^*$ such that

$$\mathfrak{g}_\alpha = \ker(\mathrm{ad}\, X - \alpha(X).I)$$

is non-trivial are called the roots of the pair $(\mathfrak{g},\mathfrak{a})$. Together they constitute a finite subset Δ of $\mathfrak{a}^*$. The $\mathfrak{g}_\alpha$ $(\alpha \in \Delta)$ are called the root spaces, and one has the direct sum of vector spaces

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha .$$

The hyperplanes $\ker \alpha$ $(\alpha \in \Delta)$ in $\mathfrak{a}$ are called the root hyperplanes. For $X \in \mathfrak{a}$ the condition of being in their common complement

$$\alpha(X) \neq 0 \text{ for all } \alpha \in \Delta$$

is equivalent to the geometric condition that the two-sided K-orbit KaK through $a = \exp X$ has at a a tangent space complementary to A. The set $\mathfrak{a}'$ of these X is called the regular set in $\mathfrak{a}$, $A' = \exp(\mathfrak{a}')$ is called the regular set in A. The set

$$G' = KA'K$$

is an open dense subset of G. Two elements a_1, a_2 in A' are in the same two-sided K-orbit iff they are conjugate by an element k of the normalizer M^* of $\mathfrak{a}$ in K. The element k is determined uniquely up to right multiplication by an element of the centralizer M of $\mathfrak{a}$ in K. The group $W = M^*/M$ is finite and it turns out to act on $\mathfrak{a}$ as the group generated by the orthogonal reflections s_α in the root hyperplanes $\ker \alpha$ $(\alpha \in \Delta)$. W is called the Weyl group of the pair $(\mathfrak{g},\mathfrak{a})$. It acts simply transitively on the connected components of $\mathfrak{a}'$, the so called (open) Weyl chambers. Select such a chamber. From now on it will be called the positive Weyl chamber $\mathfrak{a}^+$ in $\mathfrak{a}$. Similarly we call $A^+ = \exp(\mathfrak{a}^+)$ the positive Weyl chamber in A. By what we said above we have

$$G' = KA^+K \tag{2}$$

and now the A^+-part $\pi(x)$ of an element $x \in G'$ is uniquely determined. The corresponding map $\pi: G' \to A^+$ is a smooth fibration of G'. Its fibres are the two-sided K-orbits and A^+ is a global cross-section. We view (2) as a polar decomposition for G' with radial component A^+. The situation is analogous to the classical polar decomposition of $\mathbb{R}^n \setminus \{0\}$, where an open half line emanating from the origin serves as a global cross-section for the fibration of $\mathbb{R}^n \setminus \{0\}$ into spheres (the orbits of the orthogonal group). In analogy with this classical situation we define for every differential operator D on G' a differential operator $\Delta(D)$ on A^+, called the "radial part" of D. Its action on a smooth function $\psi: A^+ \to \mathbb{C}$ is described as follows. First extend ψ to a bi-K-invariant function $\pi^*\psi$ on G' (so $\pi^*\psi$ is constant along the fibres of π). Then apply D and finally restrict the result again to A^+. In formula:

$$\Delta(D)\psi = D(\pi^*\psi)\big|_{A^+}.$$

In particular the differential equations (1) lead to the system

$$\Delta(D)\psi = \gamma(D,i\lambda).\psi \qquad (D \in \mathbb{D}(G)^K) \tag{3}$$

of differential equations on A^+ for the restriction $\phi_\lambda|A^+$, called the radial differential equations. Recall that at a point a of the boundary ∂A^+ of A^+ the tangent space of KaK is not complementary to A anymore. As a consequence certain coefficients of the radial differential equations become singular at this boundary. Note that $\partial\mathfrak{a}^+$ is built up from parts of the root hyperplanes (called the walls of $\mathfrak{a}^+$) and that $\partial A^+ = \exp(\partial\mathfrak{a}^+)$.

The radial differential equations are linear differential equations, they are partial if the real rank of $\mathfrak{g}$ is > 1. In [2] Harish-Chandra proved the remarkable fact that the solution space of (3) is of finite dimension $\leq |W|$ over $\mathbb{C}$. We will describe how he used this to obtain a series expansion for $\phi_\lambda|A^+$. Define the set Δ^+ of positive roots with respect to the choice of $\mathfrak{a}^+$ by

$$\Delta^+ = \{\alpha \in \Delta;\ \alpha(X) > 0 \text{ for all } X \in \mathfrak{a}^+\},$$

and write $L = \mathbb{N}.\Delta^+$ for the lattice generated by the positive roots. Harish-Chandra constructed series of the form

$$e^{(iw\lambda-\rho)(\log a)} \sum_{\mu \in L} \Gamma_\mu(w\lambda)e^{-\mu(\log a)}, \qquad \Gamma_0 \equiv 1 \qquad (4)$$

as solutions of the radial differential equations corresponding to the Casimir operator (a special second order differential operator in $\mathbb{D}(G)^K$). Here

$$\rho = \tfrac{1}{2} \sum_{\alpha \in \Delta^+} \dim(\mathfrak{g}_\alpha).\alpha$$

and we have written $w\lambda$ for $\lambda \circ w^{-1}$ ($w \in W$). More precisely, he showed that for every $w \in W$ there exists exactly one formal series like (4) satisfying the radial Casimir equation. Here the Γ_μ are rational functions $\mathfrak{a}_c^* \to \mathbb{C}$, determined by certain recurrence relations. Next, he showed that the series (4) converges for $a \in A^+$ and in fact defines a solution for the entire system (3) (here λ has to be suitably restricted, it must lie outside a certain locally finite union of hyperplanes in $\mathfrak{a}_c^*$). From the principal terms $\exp[(iw\lambda-\rho)(\log a)]$ of (4) one reads off that the functions (4) are linearly independent and therefore constitute a basis for the solution space of (3). Hence $\phi_\lambda|A^+$ can be written as a linear combination of the functions (4), and since it is known that $\phi_{w\lambda} = \phi_\lambda$ $(w \in W)$ it follows that

$$\phi_\lambda(a) = \sum_{w\in W} c(w\lambda)e^{(iw\lambda-\rho)(\log a)}\Phi''(w\lambda,a), \qquad (5)$$

$$\Phi''(\lambda,a) = \sum_{\mu\in L} \Gamma_\mu(\lambda)e^{-\mu(\log a)}.$$

From (5) one reads off the asymptotic behaviour of $\phi_\lambda(a)$ as $\log a$ tends radially to infinity in $\mathfrak{a}^+$. The coefficient $c(\lambda)$ is called the Harish-Chandra c-function. It is a meromorphic function $\mathfrak{a}_c^* \to \mathbb{C}$, and plays a basic role in Harish-Chandra's Plancherel theorem for the symmetric space G/K (for an elementary exposition we refer the reader to Helgason's survey article [4]). It has been explicitly computed by Gindikin and Karpelevič, who expressed it as a product of c-functions corresponding to certain real rank 1 subgroups of G (cf. GK [1]).

The elementary spherical function ϕ_λ can be given by an integral formula which we shall describe now. The space

$$\mathfrak{n} = \sum_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$$

is a nilpotent Lie subalgebra of $\mathfrak{g}$. Moreover $N = \exp \mathfrak{n}$ is a closed nilpotent subgroup of G. In our example $\mathfrak{n}$ can be taken as the set of upper triangular matrices with zeros on the diagonal, and then N consists of the upper triangular matrices with diagonal entries equal to 1. Now G admits the so called Iwasawa decomposition

$$G = KAN.$$

Here the map $K \times A \times N \to G$, $(k,a,n) \to kan$ is a real analytic diffeomorphism. We let h denote the corresponding projection $G \to A$ and put $H = \log h$. Thus H is determined by

$$x \in K \exp H(x)N \qquad (x \in G).$$

As has been proved by Harish-Chandra ([1]) we have

$$\phi_\lambda(x) = \int_K e^{(i\lambda-\rho)H(xk)}dk. \tag{6}$$

Here dk is the Haar measure of K normalized by $\int_K dk = 1$.

0.2 Introduction to the thesis

The basic idea of our work is that solutions of the system (3) of radial differential equations can be given by integrals like (6) but then over compact cycles of dimension dim(K) in the natural complexification K_c of the group K. The Haar measure dk has to be replaced by an invariant holomorphic differential form ω of dimension dim(K), and one has to use a holomorphic extension of the map $H: G \to \mathfrak{a}$. Since $H = \log h$ it is clear that we should expect that H does not have a global holomorphic extension, but a multi-valued analytic extension with branching locus. As a consequence the integrals over cycles will only locally define functions. In Chapter 1 we study the analytic extensions we need in the remainder of the thesis. A main result is that the map $H: G \to \mathfrak{a}$ has a multi-valued analytic extension with branching locus $S = G_c \setminus K_c A_c N_c$ (here the subscript c denotes the natural complexification of a group).

In Chapter 2 we discuss basic properties of integrals over compact smooth cycles and of the radial differential equations. In the last section we introduce the principle we already referred to above. If Γ is a compact smooth dim(K)-cycle in K_c, if $a_0 \in A^+$ and if the map $k \to H(a_0 k)$ has a branch H_Γ over Γ, then

$$\phi_{\Gamma,\lambda}(a) = \int_\Gamma e^{(i\lambda-\rho)H_\Gamma(ak)} \omega(k)$$

defines a solution of (3) when a is sufficiently close to a_0.

The next idea is to obtain the basis (4) of the solution space of (3) as integrals over cycles. In Chapters 3 and 4 cycles Γ_w ($w \in W$) are constructed so that the corresponding functions $\phi_{w,\lambda}$ are defined in a suitable "neighbourhood of infinity" in A^+, and are equal to (4) up to certain non-zero scalars. This leads to the formula

$$\phi_\lambda(a) = \sum_{w \in W} d(w\lambda)^{-1} \int_{\Gamma_w} e^{(i\lambda-\rho)H_{0,w}(ak)} \omega(k), \qquad (7)$$

where $H_{0,w}$ denotes a certain branch of H, and where d is the simple holomorphic function $\mathfrak{a}_c^* \to \mathbb{C}$ given by

$$d(\lambda) = \prod_{\alpha \in \Delta^{++}} (\exp [2\pi(\lambda,\alpha)(\alpha,\alpha)^{-1}] - 1)$$

(here Δ^{++} denotes the set of indivisible positive roots).

The construction of the cycles Γ_w is carried out as follows. First, in Chapter 3, they are constructed in the real rank 1 case (i.e. dim $\mathfrak{a} = 1$) by making use of explicit computations. In this case the flag manifold K/M is a sphere. This geometric simplicity allowed us to prove (7) directly, without making any use of the radial differential equations. Here the cycles are defined in a complexification of K/M. Originally we tried to treat the general case in a similar way but we did not succeed in understanding the geometric structure well enough. We even gave up our original plan to define all cycles in a complexification of K/M since this caused difficulties in the constructions of Chapter 5. The main problem in the general case is to get control over the branching locus S of the map H. This is solved by using a complex analytic extension of the procedure Gindikin and Karpelevič used to derive the product formula for the c-function (cf. [1]). The cycles Γ_w are obtained as products of rank 1 cycles.

The construction of the cycles itself leads to a formula

$$\phi_{w,\lambda}(a) = e^{(iw\lambda-\rho)(\log a)}\Phi'_w(\lambda,(e^{-\alpha(\log a)})_{\alpha\in\Delta^{++}}). \quad (8)$$

Here the function Φ'_w is also given as an integral over a compact cycle. The integrand is holomorphic in the variable λ and holomorphic in a neighbourhood of 0 in the variables $e^{-\alpha(\log a)}$, and therefore the same holds for Φ'_w. Using the power series expansion of Φ_w' in the second variable we obtain a converging series expansion for $\phi_{w,\lambda}$ which is asymptotic if a tends to infinity. Thus Harish-Chandra's expansion (4) is obtained as a converging series.

In Chapter 5 we derive formulas similar to (7), but valid for a varying in certain neighbourhoods of the boundary of $\overline{A^+}$. Again this leads to converging series for ϕ_λ, describing the asymptotics along the walls of $\overline{A^+}$. For more information about the contents of Chapters 4 and 5 we refer the reader to the introductions of these chapters.

Observe that formula (7) breaks down at the zeros of the functions $\lambda \to d(w\lambda)$ $(w \in W)$. Nevertheless for such λ, ϕ_λ can be written as a sum of integrals too. This is the subject of the last chapter. In particular, asymptotic expansions for $\Xi = \phi_0$ are obtained in this way.

When we wrote this thesis we had generalizations to Eisenstein integrals associated with a minimal parabolic subgroup in mind. Therefore we have not used the symmetry $\phi_\lambda = \phi_{w\lambda}$ $(w \in W)$ as long as we could. For instance in Chapter 4 it would have sufficed to construct merely the cycle Γ_I and to use the functions $\phi_{I,w\lambda}$ instead of the functions $\phi_{w,\lambda}$ $(w \in W)$. In fact now we prove that $\phi_{I,w\lambda} = \phi_{w,\lambda}$. Also it seemed more natural to construct different cycles Γ_w. As the reader will see each cycle Γ_w is "based" at a representative $\overline{w}$ of w in M^*; as a consequence the factors $\exp[(iw\lambda-\rho)(\log a)]$ in front of the right hand side of (8) appear in a natural way.

Chapter 1

Analytic extensions

1.1 Preliminaries

In this thesis G is a connected real semisimple Lie group with finite centre (actually from p. 3 on it will be assumed that the centre is trivial) and K is a maximal compact subgroup of G. We denote their Lie algebras by $\mathfrak{g}$ and $\mathfrak{k}$ respectively. The ortho-complement of $\mathfrak{k}$ with respect to the Killing form $\langle\ ,\ \rangle$ of $\mathfrak{g}$ is denoted by $\mathfrak{s}$, and the Cartan involution corresponding to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ is denoted by θ.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{s}$, Δ the set of roots of the pair $(\mathfrak{g},\mathfrak{a})$, W the corresponding Weyl group. Fix a choice Δ^+ of positive roots, and let $\mathfrak{a}^+$ be the corresponding positive Weyl chamber in $\mathfrak{a}$.

Roots $\alpha \in \Delta$ with $\frac{1}{2}\alpha \notin \Delta$ are called indivisible; let Δ^{++} denote the set of indivisible positive roots. If $\mu \in \mathfrak{a}^*$ ($\mathfrak{a}^*$ denotes the dual of the real linear space $\mathfrak{a}$) we write $\mathfrak{g}_\mu$ for the space $\bigcap_{H\in\mathfrak{a}} \ker(\operatorname{ad} H - \mu(H)I)$ (ad denotes the adjoint representation of $\mathfrak{g}$). If $\alpha \in \Delta^{++}$ we write $\mathfrak{n}_\alpha$ for $\mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$ and $\overline{\mathfrak{n}}_\alpha$ for $\mathfrak{g}_{-\alpha} + \mathfrak{g}_{-2\alpha}$, and finally we write:

$$\mathfrak{n} = \sum_{\alpha\in\Delta^+} \mathfrak{g}_\alpha, \qquad \overline{\mathfrak{n}} = \sum_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha}.$$

The sets $A = \exp \mathfrak{a}$, $N = \exp \mathfrak{n}$, $\overline{N} = \exp \overline{\mathfrak{n}}$ are closed subgroups of G with Lie algebras $\mathfrak{a}$, $\mathfrak{n}$, $\overline{\mathfrak{n}}$ respectively. Let M,$\mathfrak{m}$ be the centralizers of $\mathfrak{a}$ in K,$\mathfrak{k}$ respectively; $\mathfrak{m}$ is the Lie algebra of M.

G admits the Iwasawa decomposition G = KAN. Here the map $K\times A\times N \to G$, $(k,a,n) \to kan$ is a real analytic diffeomorphism. The maps κ, h, ν of G into K, A, N respectively are defined by:

$$x = \kappa(x)h(x)\nu(x) \qquad (x \in G).$$

Thus κ, h, ν are real analytic maps $G \to K$, $G \to A$, $G \to N$. The exponential map exp: $\mathfrak{a} \to A$ is a real analytic diffeomorphism, its inverse is denoted by log. We define the map H: $G \to \mathfrak{a}$ by:

$$H(x) = \log h(x) \qquad (x \in G).$$

Let $\mathbb{D}(G)^K$ be the algebra of left G- right K-invariant differential operators on G. A C^∞ function ϕ: $G \to \mathbb{C}$ which is bi-K-invariant, a simultaneous eigenfunction for $\mathbb{D}(G)^K$, and satisfies $\phi(e) = 1$, is called an elementary spherical function of the pair (G,K). As Harish-Chandra proved in his paper [1], the elementary spherical functions of (G,K) are the functions ϕ_λ ($\lambda \in \mathfrak{a}_c^*$, the complexified dual of $\mathfrak{a}$) defined by:

$$\phi_\lambda(x) = \int_K e^{(i\lambda-\rho)H(xk)}dk \qquad (x \in G). \tag{1}$$

Here $\rho = \frac{1}{2} \Sigma_\alpha\, m(\alpha)\alpha$ (summation over Δ^+), $m(\alpha) = \dim \mathfrak{g}_\alpha$, and dk is the Haar measure of K normalized by $\int_K dk = 1$.
One has $\phi_\lambda = \phi_\mu$ iff λ, μ are conjugate under W (cf. Harish-Chandra [1]).

We write Aut($\mathfrak{l}$) for the group of automorphisms of a Lie algebra $\mathfrak{l}$. If L is a Lie group, its component of the identity is denoted by L^0. Let Ad be the adjoint representation of G in $\mathfrak{g}$. Since G is a connected semisimple Lie group, Ad is a Lie group homomorphism of G onto $(\mathrm{Aut}\,\mathfrak{g})^0$; its kernel is Z(G), the centre of G. Now $Z(G) \subset K$, so by the bi-K-invariance of the ϕ_λ, we may pass to Ad G = $(\mathrm{Aut}\,\mathfrak{g})^0$ and study the elementary spherical functions of the pair (Ad G, Ad K). The Lie algebra ad $\mathfrak{g}$ of Ad G is isomorphic to $\mathfrak{g}$ under ad and so there is no loss of generality if we assume that $Z(G) = \{e\}$.

Remark. In the remainder of this thesis it will be assumed that the centre of G is trivial.

This case has the advantage that G is isomorphic to $(\mathrm{Aut}\ \mathfrak{g})^0$ under Ad. Denote the complexification of the Lie algebra $\mathfrak{g}$ by $\mathfrak{g}_c$. $(\mathrm{Aut}\ \mathfrak{g})^0$ embeds in $(\mathrm{Aut}\ \mathfrak{g}_c)^0$ under the map $L \to L_c$, $L(\mathfrak{g},\mathfrak{g}) \to L(\mathfrak{g}_c,\mathfrak{g}_c)$ (If $L \in L(\mathfrak{g},\mathfrak{g})$, L_c denotes its complexification $\mathfrak{g}_c \to \mathfrak{g}_c$). $(\mathrm{Aut}\ \mathfrak{g}_c)^0$ is a connected complex semisimple Lie group with trivial centre and with Lie algebra $(\mathrm{ad}\ \mathfrak{g})_c$, and $(\mathrm{Aut}\ \mathfrak{g})^0$ is the connected analytic subgroup of $(\mathrm{Aut}\ \mathfrak{g}_c)^0$ with Lie algebra ad $\mathfrak{g}$. This shows that without loss of generality we may assume that we work already in a connected complex semisimple Lie group G_c with trivial centre and with Lie algebra $\mathfrak{g}_c$, and that G is the analytic subgroup of G_c generated by $\exp(\mathfrak{g})$.

1.2 Analytic extensions of G, K, A, M, N, $\overline{N}$

If E is a real linear subspace of $\mathfrak{g}$, we denote its complexification in $\mathfrak{g}_c$ by E_c. With this notation let K_c, A_c, N_c, $\overline{N}_c$ be the connected subgroups of G_c with Lie algebras $\mathfrak{k}_c$, $\mathfrak{a}_c$, $\mathfrak{n}_c$, $\overline{\mathfrak{n}}_c$ respectively. In this section we shall study the images of these groups under the isomorphism Ad: $G_c \to (\mathrm{Aut}\ \mathfrak{g}_c)^0$ (the adjoint representation of G_c in $\mathfrak{g}_c$).

We define the inner product (,) on $\mathfrak{g}$ by

$$(X,Y) = -\langle X,\theta Y\rangle \qquad (X,Y \in \mathfrak{g}). \tag{2}$$

We denote its extension to a complex bilinear form $\mathfrak{g}_c \times \mathfrak{g}_c \to \mathbb{C}$ by (,) as well. Provide $\mathfrak{a}^*$ with a lexicographic ordering with respect to some choice of linear coordinates, and let $\alpha_1,\ldots,\alpha_d$ be the induced ordering of Δ^+. Select a (,)-orthonormal basis $(e_k)_{k=1,\ldots,n}$ ($n = \dim \mathfrak{g}$) for $\mathfrak{g}$, subordinate to the direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_{\alpha_d} + \ldots + \mathfrak{g}_{\alpha_1} + \mathfrak{m} + \mathfrak{a} + \mathfrak{g}_{-\alpha_1} + \ldots + \mathfrak{g}_{-\alpha_d} \tag{3}$$

and such that the ordering $e_1,\ldots,e_n$ is compatible with the ordering of the spaces at the right hand side of (3). We identify any map $L \in L(\mathfrak{g}_c,\mathfrak{g}_c)$ with its matrix with respect to the basis (e_k), and we write L_{ij} for the entry of that matrix in the i-th row and j-th column. Let $\underline{G} = (GL(\mathfrak{g}))^0$, and let $\underline{K}$ be the subgroup of (,)-orthogonal maps in $\underline{G}$. Denote the group of diagonal matrices with positive diagonal entries by $\underline{A}$, and the group of upper (lower) triangular matrices with diagonal entries equal to 1 by $\underline{N}$ ($\underline{\overline{N}}$). We write $\underline{\mathfrak{g}}$, $\underline{\mathfrak{k}}$, $\underline{\mathfrak{a}}$, $\underline{\mathfrak{n}}$, $\underline{\overline{\mathfrak{n}}}$ for the Lie algebras of $\underline{G}$, $\underline{K}$, $\underline{A}$, $\underline{N}$, $\underline{\overline{N}}$ respectively.

<u>Proposition 1.1</u> If Q is any of the groups K, A, N, $\overline{N}$ and if $\mathfrak{q}$ denotes its Lie algebra then:

$$\mathrm{ad}(\mathfrak{q}) = \underline{\mathfrak{q}} \cap \mathrm{ad}(\mathfrak{g}), \quad \mathrm{Ad}(Q) = \underline{Q} \cap \mathrm{Ad}(G).$$

For a proof and a more detailed discussion of the above construction we refer the reader to Wallach [1, Ch. 7].

Observe that $\underline{\mathfrak{k}}$ consists of the real anti-symmetric matrices in $L(\mathfrak{g},\mathfrak{g})$, $\underline{\mathfrak{a}}$ consists of the real diagonal matrices, and $\underline{\mathfrak{n}}$ ($\underline{\overline{\mathfrak{n}}}$) consists of the real upper (lower) triangular matrices with diagonal entries equal to 0.

We denote the complexifications of $\underline{\mathfrak{k}}$, $\underline{\mathfrak{a}}$, $\underline{\mathfrak{n}}$, $\underline{\overline{\mathfrak{n}}}$ in $L(\mathfrak{g}_c,\mathfrak{g}_c)$ by $\underline{\mathfrak{k}}_c$, $\underline{\mathfrak{a}}_c$, $\underline{\mathfrak{n}}_c$, $\underline{\overline{\mathfrak{n}}}_c$ respectively. Thus $\underline{\mathfrak{k}}_c$ consists of all complex anti-symmetric matrices, $\underline{\mathfrak{a}}_c$ consists of the complex diagonal matrices and

$\underline{\mathfrak{n}}_c$ ($\overline{\underline{\mathfrak{n}}}_c$) consists of the complex upper (lower) triangular matrices with zeros on the diagonal. Let $\underline{K}_c$, $\underline{A}_c$, $\underline{N}_c$, $\overline{\underline{N}}_c$ be the connected subgroups of $GL(\mathfrak{g}_c)$ with Lie algebras $\underline{\mathfrak{k}}_c$, $\underline{\mathfrak{a}}_c$, $\underline{\mathfrak{n}}_c$, $\overline{\underline{\mathfrak{n}}}_c$ respectively. Then $\underline{K}_c$ consists of the complex matrices M with $M'M = M\,M' = I$ and $\det(M) = 1$ (where M' denotes the matrix defined by $(M')_{ij} = M_{ji}$), $\underline{A}_c$ consists of the complex diagonal matrices with non-zero diagonal entries, and $\underline{N}_c$ ($\overline{\underline{N}}_c$) consists of the complex upper (lower) triangular matrices with diagonal entries equal to 1.

Since $\underline{K}_c \cap Ad(G_c)$ has $\underline{\mathfrak{k}}_c \cap ad(\mathfrak{g}_c) = ad(\mathfrak{k}_c)$ as its Lie algebra, it follows that $Ad(K_c) = (\underline{K}_c \cap Ad\ G_c)^0$.
Furthermore $\exp\colon L(\mathfrak{g}_c,\mathfrak{g}_c) \to GL(\mathfrak{g}_c)$ maps $\underline{\mathfrak{a}}_c$ onto $\underline{A}_c$, and it maps $\underline{\mathfrak{n}}_c$ and $\overline{\underline{\mathfrak{n}}}_c$ diffeomorphically onto $\underline{N}_c$ and $\overline{\underline{N}}_c$ respectively. Thus we obtain the following proposition.

Proposition 1.2. We have

$$Ad(K_c) = (\underline{K}_c \cap Ad(G_c))^0,$$

and if Q is any of the groups A, N, $\overline{N}$ we have:

$$Ad(Q_c) = \underline{Q}_c \cap Ad(G_c).$$

Moreover the maps $\exp\colon \mathfrak{n}_c \to N_c$ and $\exp\colon \overline{\mathfrak{n}}_c \to \overline{N}_c$ are diffeomorphisms, and exp maps $\mathfrak{a}_c$ onto A_c.

1.3 Analytic extensions of κ, h, ν

The map $K\times A\times N \to G$, $(k,a,n) \to kan$ is a real analytic diffeomorphism. Let $\underline{\kappa}\colon \underline{G} \to \underline{K}$, $\underline{h}\colon \underline{G} \to \underline{A}$, $\underline{\nu}\colon \underline{G} \to \underline{N}$ be the maps defined by:

$$x = \underline{\kappa}(x)\underline{h}(x)\underline{\nu}(x) \qquad (x \in \underline{G}).$$

Then $\underline{\kappa}$, $\underline{h}$, $\underline{\nu}$ are real analytic maps and in view of Proposition 1.1

the following diagrams commute.

$$\begin{array}{ccc} G & \xrightarrow{Ad} & \underline{G} \\ \kappa\downarrow & & \downarrow\underline{\kappa} \\ K & \xrightarrow{Ad} & \underline{K} \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{Ad} & \underline{G} \\ h\downarrow & & \downarrow\underline{h} \\ A & \xrightarrow{Ad} & \underline{A} \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{Ad} & \underline{G} \\ \nu\downarrow & & \downarrow\underline{\nu} \\ N & \xrightarrow{Ad} & \underline{N} \end{array} \qquad (4)$$

In this section we will study analytic extensions of $\underline{\kappa}$, $\underline{h}$, $\underline{\nu}$. Restriction of the obtained extensions to $Ad(G_c)$ will lead to extensions of κ, h, ν.

If $k \in \underline{K}$, $a \in \underline{A}$, $n \in \underline{N}$, $x = kan$, we have that $x'x = n'a'k'kan = n'a^2n \in \overline{\underline{N}}\ \underline{A}\ \underline{N}$. We shall first study the set $\overline{\underline{N}}_c\underline{A}_c\underline{N}_c$ in $L(\mathfrak{g}_c,\mathfrak{g}_c)$. A matrix $y \in L(\mathfrak{g}_c,\mathfrak{g}_c)$ belongs to $\overline{\underline{N}}_c\underline{A}_c\underline{N}_c$ iff there exists a matrix $u \in \underline{N}_c$ such that yu is lower triangular with non zero diagonal elements. Translating the latter statement into equations for the matrix coefficients u_{jk} and applying Cramer's rule to the obtained equations we obtain that y belongs to $\overline{\underline{N}}_c\underline{A}_c\underline{N}_c$ iff all minors

$$D_k(y) = \det\left((y_{ij})_{1\leqslant i,j\leqslant k} \right) \qquad (1 \leqslant k \leqslant n)$$

are different from zero, and then there exists a unique $u = u(y) \in \underline{N}_c$ such that $yu(y) \in \overline{\underline{N}}_c\underline{A}_c$. Its entries are given by

$$u_{jk}(y) = \frac{D_{k,j}(y)}{D_{k-1}(y)} \qquad (1 \leqslant j < k \leqslant n), \qquad (5)$$

where $D_{k,j}(y)$ is the minor of $D_k(y)$ obtained by omitting the k-th row and j-th column. Moreover, if $1 \leqslant k \leqslant n$, then the minor consisting of the first k rows and columns of $yu(y)$ is equal to $D_k(y)$. Writing $v_i(y)$ for the i-th diagonal entry of $yu(y)$ we thus have:

$$v_1(y) = D_1(y) = y_{11}, \tag{6}$$

$$v_k(y) = \frac{D_k(y)}{D_{k-1}(y)} \qquad (1 < k \leqslant n). \tag{7}$$

We have proved:

Lemma 1.3. Let $D: L(\mathfrak{g}_c, \mathfrak{g}_c) \to \mathbb{C}$ be the polynomial function given by:

$$D(y) = D_1(y) \;.\; \dots \;.\; D_n(y).$$

Then the polynomial map $\underline{\bar{N}}_c \times \underline{A}_c \times \underline{N}_c \to \underline{G}_c$, $(\bar{n}, a, n) \to \bar{n}an$ is a diffeomorphism onto $L(\mathfrak{g}_c, \mathfrak{g}_c) \setminus D^{-1}(0)$. Its inverse is the rational map

$$y \to (yu(y)v(y)^{-1}, \;\; v(y), \;\; u(y)^{-1}),$$

where $v(y)$ denotes the diagonal matrix with entries $v_i(y)$ $(i = 1,\dots,n)$, and where $u(y)$, $v_i(y)$ are given by the formulas (5), (6) and (7).

Remark. The above computations can be found in more detail in Gelfand-Neumark [1, Ch. I, §3]

Now consider the map $g: L(\mathfrak{g}_c, \mathfrak{g}_c) \to L(\mathfrak{g}_c, \mathfrak{g}_c)$, defined by $g(x) = x'x$. Observe that $g(x)$ is equal to the Gram matrix

$$g(x) = (\, (x(e_i),\; x(e_j))\,)_{1 \leqslant i,j \leqslant n}.$$

Obviously, g maps $\underline{G}$ onto the set of positive definite symmetric matrices in $L(\mathfrak{g}, \mathfrak{g})$. Hence, if $x \in \underline{G}$, then all minors $D_k(g(x))$ $(1 \leqslant k \leqslant n)$ are strictly positive, and by the above lemma there exists a unique $(\bar{n}, b, n) \in \underline{\bar{N}} \times \underline{A} \times \underline{N}$ such that $y = \bar{n}bn$. By symmetry of

$g(x)$ we must have $\bar{n} = n'$. Let a be the unique element of $\underline{A}$ such that $b = a^2$, and write $k = xn^{-1}a^{-1}$. Then $k' = a^{-1}(n')^{-1}x' = $ $= a^{-1}(n')^{-1}g(x)x^{-1} = anx^{-1} = k^{-1}$, and $\det k > 0$, showing that $k \in \underline{K}$. This proves that the map $\underline{K}\times\underline{A}\times\underline{N} \to \underline{G}$, $(k,a,n) \to kan$ is a diffeomorphism (as was said before), and the corresponding maps $\underline{\kappa}$, $\underline{h}$, $\underline{\nu}$ are determined by:

$$\begin{aligned} \underline{\nu}(x) &= u(g(x))^{-1}, \\ \underline{h}(x)^2 &= v(g(x)), \\ \underline{\kappa}(x)\underline{h}(x) &= x\, u(g(x)). \end{aligned} \qquad (8)$$

Let the polynomial function F on $L(\mathfrak{g}_c,\mathfrak{g}_c)$ be defined by

$$\underline{F}(x) = D(g(x)) \qquad (x \in L(\mathfrak{g}_c,\mathfrak{g}_c)),$$

then by the above discussion we have:

Theorem 1.4. The maps $\underline{F}.\underline{\kappa}.\underline{h}$, $\underline{F}.\underline{h}^2$ and $\underline{F}.\underline{\nu}$ are polynomials in the entries $(x(e_i), x(e_j))$ of the Gram matrix $g(x)$. Consequently, $\underline{\kappa}.\underline{h}$, $\underline{h}^2$, $\underline{h}^2$ and $\underline{\nu}$ extend holomorphically to $\underline{G}_c \setminus \underline{F}^{-1}(0)$. Moreover, writing b_i for the i-th diagonal entry of $\underline{h}^2$, we have:

$$b_1(x) = D_1(g(x)), \qquad (9)$$

$$b_k(x) = \frac{D_k(g(x))}{D_{k-1}(g(x))} \qquad (1 < k \leq n). \qquad (10)$$

Remark. By the same type of computations, formulas similar to (9), (10) are obtained in Gelfand-Neumark [1, Ch. II, §8, (8.15)]. The above computation of $\underline{h}$ has also been used by T.S. Bhanu Murti to determine the Harish-Chandra c-function explicitly for the groups $SL(n,\mathbb{R})$ and $Sp(n,\mathbb{R})$ (cf. Bhanu Murti [1], [2]).

The following theorem is about multi-valued analytic extensions of the maps $\underline{\kappa}$: $\underline{G} \to \underline{K}$, $\underline{h}$: $\underline{G} \to \underline{A}$ to $\underline{G}_c \backslash \underline{F}^{-1}(0)$. The discrete group $\underline{M}$ defined by

$$\underline{M} = \underline{K} \cap \exp i\underline{\mathfrak{a}}$$

plays a main role in it. For the terminology of multi-valued analytic maps we refer the reader to the appendix to this chapter.

<u>Theorem 1.5</u>. The maps $\underline{\kappa}$: $\underline{G} \to \underline{K}$, $\underline{h}$: $\underline{G} \to \underline{A}$ have extensions to multi-valued analytic maps $\underline{\kappa}_c$: $\underline{G}_c \backslash \underline{F}^{-1}(0) \to \underline{K}_c$, $\underline{h}_c$: $\underline{G}_c \backslash \underline{F}^{-1}(0) \to \underline{A}_c$ (with respect to the base point I). Moreover, if $\underline{\kappa}_1$ and $\underline{\kappa}_2$ are two branches of $\underline{\kappa}_c$ at a point $x_0 \in \underline{G}_c \backslash \underline{F}^{-1}(0)$, and if $\underline{h}_1$, $\underline{h}_2$ are the corresponding branches of $\underline{h}_c$ at x_0, then $\underline{\kappa}_1 = \underline{\kappa}_2 d$ and $\underline{h}_1 = \underline{h}_2 d = d\underline{h}_2$ for some $d \in \underline{M}$.

<u>Remark</u>. In particular the monodromy groups associated with the multi-valued maps $\underline{h}_c$, $\underline{\kappa}_c$ are isomorphic to subgroups of $\underline{M}$. Observe that the group $\underline{M}$ consists of the diagonal matrices with diagonal entries equal to $\pm$ 1, with an even number of - signs.

<u>Proof of Theorem 1.5</u>. Let $\sqrt{\ }$ be the extension of the map $x \to x^{\frac{1}{2}}$, $\mathbb{R}^+ \to \mathbb{R}^+$ to a multi-valued map $\mathbb{C}\backslash\{0\} \to \mathbb{C}$ with base point 1. Writing $b_{k,c}$ for the holomorphic extensions of the functions b_k $(1 \leqslant k \leqslant n)$ given in Theorem 1.4 we define the multi-valued analytic map $\underline{h}_c$: $\underline{G}_c \backslash \underline{F}^{-1}(0) \to \underline{A}_c$ by:

$$\underline{h}_c(x) = \begin{pmatrix} \sqrt{b_{1,c}(x)} & & \theta \\ & \ddots & \\ \theta & & \sqrt{b_{n,c}(x)} \end{pmatrix} . \qquad (11)$$

Obviously $\underline{h}_c$ is the multi-valued analytic extension of $\underline{h}$. Writing $\underline{\nu}_c$ for the holomorphic extension of $\underline{\nu}$ to $\underline{G}_c \setminus \underline{F}^{-1}(0)$, we define the multi-valued analytic map $\underline{\kappa}_c$: $\underline{G}_c \setminus \underline{F}^{-1}(0) \to L(\mathfrak{g}_c, \mathfrak{g}_c)$ by

$$\underline{\kappa}_c(x) = x\ \underline{\nu}_c(x)^{-1} \underline{h}_c(x)^{-1}. \tag{12}$$

It is the multi-valued analytic extension of $\underline{\kappa}$ to $\underline{G}_c \setminus \underline{F}^{-1}(0)$. $\underline{K}_c$ is a Zariski closed subset of $L(\mathfrak{g}_c, \mathfrak{g}_c)$. Therefore, if I is the ideal of polynomial functions vanishing on $\underline{K}_c$, we have $\underline{K}_c = I^{-1}(0)$. If $f \in I$, then $(f \circ \underline{\kappa})(x) = 0$ for all $x \in \underline{G}$. Now $\underline{G}_c$ is connected, hence the complement $\underline{G}_c \setminus \underline{F}^{-1}(0)$ of the analytic null set $\underline{F}^{-1}(0)$ is (this local property is an easy consequence of the Weierstrass preparation theorem, cf. Griffiths-Harris [1, p. 8]). By analytic continuation it follows that $f \circ \underline{\kappa}_c \equiv 0$ on $\underline{G}_c \setminus \underline{F}^{-1}(0)$. Consequently $\underline{\kappa}_c$ maps $\underline{G}_c \setminus \underline{F}^{-1}(0)$ into $\underline{K}_c$.

Now let $x_0 \in \underline{G}_c \setminus \underline{F}^{-1}(0)$, and let c_1, c_2 be two continuous curves $[0,1] \to \underline{G}_c \setminus \underline{F}^{-1}(0)$ with $c_i(0) = I$, $c_i(1) = x_0$ $(i = 1,2)$. Write $a_{k,c}$ for the multi-valued analytic function $\sqrt{b_{k,c}}$ $(1 \leq k \leq n)$, and let $a_{k,i}$, $\underline{\kappa}_i$, $\underline{h}_i$ denote the branches of $a_{k,c}$, $\underline{\kappa}_c$, $\underline{h}_c$ obtained by continuation of a_k, $\underline{\kappa}$, $\underline{h}$ along c_i $(1 \leq k \leq n,\ i = 1,2)$. We have that $a_{k,1} = d_k a_{k,2}$ with $d_k = \pm 1$ $(1 \leq k \leq n)$, hence, writing d for the diagonal matrix with k-th diagonal entry d_k we obtain that $\underline{h}_1 = d\underline{h}_2 = \underline{h}_2 d$. Obviously $d \in \exp(i\underline{\mathfrak{a}})$. On the other hand, since $\underline{\kappa}_c \underline{h}_c$ is single valued, we have $\underline{\kappa}_1 = \underline{\kappa}_2 d^{-1} = \underline{\kappa}_2 d$. In particular it follows that $\det(d) = 1$. Consequently $d \in \underline{M}$.

We now turn our attention to the maps κ, h, ν. Let $M = \mathrm{Ad}^{-1}(\underline{M})$. Then obviously

$$M = K \cap \exp i\mathfrak{a} = K \cap A_c.$$

The following properties of M will be of use later. First we have $M = MM^0$ (cf. Warner [1, pp. 28,29]). Next we have the following characterization of the set $\{H \in \mathfrak{a}_c;\ \exp H \in M\}$ (cf. for instance Warner [1, p. 213]). If $\alpha \in \Delta^{++}$, let $H_{\alpha,0}$ be the element of $\mathfrak{a}$, $\langle\ ,\ \rangle$-orthogonal to ker α, with $\alpha(H_{\alpha,0}) = 1$. Then:

the lattice $\{H \in \mathfrak{a}_c;\ \exp H \in M\}$ is generated over $\mathbb{Z}$ by the vectors $2\pi i H_{\alpha,0}$ $(\alpha \in \Delta^{++})$.

Theorem 1.6. Let S be the null set of $F = \underline{F} \circ \mathrm{Ad}$ in G_c. Then:

(i) The maps $\kappa: G \to K$, $h: G \to A$, $\nu: G \to N$ have extensions to multi-valued analytic maps $\kappa_c: G_c \backslash S \to K_c$, $h_c: G_c \backslash S \to A_c$ and $\nu_c: G_c \backslash S \to N_c$ respectively (all with respect to the base point e).

(ii) The maps $\kappa_c h_c$, h_c^2 and ν_c are single valued. Moreover, if κ_1 and κ_2 are two branches of κ_c at a point $x_0 \in G_c \backslash S$, and if h_1, h_2 are the corresponding branches of h_c at x_0, then $\kappa_1 = \kappa_2 d$, $h_1 = dh_2 = h_2 d$ for some $d \in M$.

Proof. We may pass to $\mathrm{Ad}(G_c) = (\mathrm{Aut}\ \mathfrak{g}_c)^0$ and it suffices to prove that $\underline{\kappa}_c$, $\underline{h}_c$, $\underline{\nu}_c$ map $\mathrm{Ad}(G_c) \backslash \underline{F}^{-1}(0)$ into $(\mathrm{Ad}(G_c) \cap \underline{K}_c)^0 = \mathrm{Ad}(K_c)$, $\mathrm{Ad}(G_c) \cap \underline{A}_c = \mathrm{Ad}(A_c)$ and $\mathrm{Ad}(G_c) \cap \underline{N}_c$ respectively. Now $\mathrm{Ad}(G_c) \backslash \underline{F}^{-1}(0)$ is connected, and $\underline{\kappa}$, $\underline{h}$, $\underline{\nu}$ map $\mathrm{Ad}(G)$ into the Zariski closed subset $\mathrm{Aut}(\mathfrak{g}_c)$ of $GL(\mathfrak{g}_c)$. It follows that $\underline{\kappa}_c$, $\underline{h}_c$, $\underline{\nu}_c$ map $\mathrm{Ad}(G_c) \backslash \underline{F}^{-1}(0)$ into $\mathrm{Aut}(\mathfrak{g}_c)$. In view of Theorem 1.5 and the connectedness of $\mathrm{Ad}(G_c) \backslash \underline{F}^{-1}(0)$ we obtain that $\underline{\kappa}_c$, $\underline{h}_c$, $\underline{\nu}_c$ map $\mathrm{Ad}(G_c) \backslash \underline{F}^{-1}(0)$ into $(\mathrm{Ad}(G_c) \cap \underline{K}_c)^0$, $\mathrm{Ad}(G_c) \cap \underline{A}_c$, $\mathrm{Ad}(G_c) \cap \underline{N}_c$ respectively. We complete the proof by the observation that $\mathrm{Ad}(M) = \mathrm{Ad}(G_c) \cap \underline{M}$.

Remark. As we will show in Chapter 4, the monodromy groups associated with κ_c and h_c are isomorphic to the full group M.

The map exp: $\underline{\mathfrak{a}}_c \to \underline{A}_c$ is a covering. Therefore the inverse log: $\underline{A} \to \underline{\mathfrak{a}}$ of exp: $\underline{\mathfrak{a}} \to \underline{A}$ has a multi-valued analytic extension $\underline{A}_c \to \underline{\mathfrak{a}}_c$ (with respect to the base point I); we denote it by $\underline{\log}_c$. Obviously $\exp \circ \underline{\log}_c = \mathrm{id}(\underline{A}_c)$. It follows that the map exp: $\mathrm{ad}(\mathfrak{a}_c) \to \mathrm{Ad}(A_c)$ is a covering, and since ad and Ad are diffeomorphisms such that $\exp \circ \mathrm{ad} = \mathrm{Ad} \circ \exp$, it follows that exp: $\mathfrak{a}_c \to A_c$ is a covering. The inverse log: $A \to \mathfrak{a}$ of exp: $\mathfrak{a} \to A$ therefore has an extension to a multi-valued analytic map $A_c \to \mathfrak{a}_c$ (with respect to the base point e); it is denoted by $\log_c$. We obviously have $\exp \circ \log_c = \mathrm{id}(A_c)$ and $\mathrm{ad} \circ \log_c = \underline{\log}_c \circ \mathrm{Ad}$. Writing $\underline{H}_c$ for the map $\underline{\log}_c \circ \underline{h}_c$ and H_c for the map $\log_c \circ h_c$ we obtain the following theorem.

Theorem 1.7. The map $\underline{H}_c$: $\underline{G}_c \backslash \underline{F}^{-1}(0) \to \underline{\mathfrak{a}}_c$ is a multi-valued analytic extension of $\underline{H}$: $\underline{G} \to \underline{\mathfrak{a}}$ (with respect to the base point I), H_c: $G_c \backslash S \to \mathfrak{a}$ is a multi-valued analytic extension of H: $G \to \mathfrak{a}$ (with base point e).

1.4 Some properties of the set S

Lemma 1.8.

$$GL(\mathfrak{g}_c)\backslash \underline{F}^{-1}(0) = \underline{K}_c \underline{A}_c \underline{N}_c, \tag{13}$$

$$G_c \backslash S = K_c A_c N_c. \tag{14}$$

Proof. If $(k,a,n) \in \underline{K}_c \times \underline{A}_c \times \underline{N}_c$, then

$$\underline{F}(kan) = \prod_{k=1}^{n} D_k(n'a^2n) = \prod_{k=1}^{n} (a_{11} \cdots a_{kk})^2 \neq 0.$$

This shows that $\underline{K}_c\underline{A}_c\underline{N}_c \subset GL(\mathfrak{g}_c)\setminus \underline{F}^{-1}(0)$. On the other hand the existence of the multi-valued analytic extenxions $\underline{\kappa}_c$, $\underline{h}_c$, $\underline{\nu}_c$ implies that $GL(\mathfrak{g}_c)\setminus \underline{F}^{-1}(0) \subset \underline{K}_c\underline{A}_c\underline{N}_c$. This proves (13). Formula (14) now follows from Proposition 1.2 and from $Ad(S) =$
$= Ad(G_c) \cap \underline{F}^{-1}(0)$.

The following two lemmas will be useful in Chapter 3, where we will have to determine the set S explicitly in the real rank 1 case, and in Chapter 4, where it will be necessary to compare S with the analogous subsets of lower dimensional subgroups with compatible Iwasawa decompositions.

Lemma 1.9. The map h_c^2 is a rational map $G_c \to A_c$. It is regular on $G_c\setminus S$. If (x_n) is any sequence in $G_c\setminus S$ converging to a point $x \in S$ then $\{h_c^2(x_n);\ n \in \mathbb{N}\}$ is not relatively compact in A_c.

Proof. The first statements follow readily from Theorem 1.4. As for the last statement we pass to $\underline{G}_c$ by Ad. So let (x_n) be a sequence in $\underline{G}_c\setminus \underline{F}^{-1}(0)$ converging to $x \in \underline{F}^{-1}(0)$. Let k be the smallest element of $\{1,\dots,n\}$ such that $D_k(g(x)) = 0$. By (9), (10) it follows that $(\underline{h}_c^2(x_n))_k \to 0$, hence $\{\underline{h}_c^2(x_n);\ n \in \mathbb{N}\}$ is not relatively compact in $\underline{A}_c$. It is now easy to complete the proof.

Lemma 1.10. Let L be a connected complex analytic submanifold of G_c, containing e. Then the set $L\setminus S$ is a connected dense open subset of L. Moreover, it can be characterized as follows. Fix a simply connected open neighbourhood $\mathcal{O}$ of e in G_c, disjoint from S, and let h_0 denote the holomorphic extension of $h|(\mathcal{O} \cap G)$ to $\mathcal{O}$. Then

$L \setminus S$ is the biggest open subset U of L such that:

(i) U is connected and contains $\mathcal{O} \cap L$.

(ii) The restriction $h_0 | (\mathcal{O} \cap L)$ of h_0 to $\mathcal{O} \cap L$ has a multi-valued analytic extension to U (with respect to the base point e).

<u>Proof</u>. The holomorphic function $F = \underline{F} \circ Ad$ is not identically zero on L. Therefore $L \setminus S = L \setminus F^{-1}(0)$ is a connected dense open subset of L.
If U_1 and U_2 are open subsets of L such that (i) and (ii) are satisfied, then the same holds for $U_1 \cup U_2$. Therefore there exists a biggest open subset U of L such that (i) and (ii). Since $h_c | (L \setminus S)$ extends $h_0 | (\mathcal{O} \cap L)$, $L \setminus S$ is contained in U. Assume that $L \setminus S \neq U$. Then $S \cap U \neq \emptyset$. Now $S \cap U$ is the null set of F in U, and F is not identically 0 on U. So $S \cap U$ is an analytic variety of positive codimension in U. Let x be a smooth point of $S \cap U$ (for the existence of such a point cf. Griffiths-Harris [1, pp. 20, 21]). From the local structure of $S \cap U$ at x it follows that there exists a point $x' \in U \setminus S$ and a continuous curve $c' : [0,1] \to U$ such that $c'(0) = x', c'(1) = x$ and $c'(t) \notin S$ for $0 \leq t < 1$. Hence, since $L \setminus S$ is connected and open in L, there exists a continuous curve $c: [0,1] \to L$, such that $c(0) = e$, $c(1) = x$ and $c(t) \notin S$ for $0 \leq t < 1$. Now $c([0,1]) \subset U$ and by definition of U $h_0 | (\mathcal{O} \cap L)$ has a continuation h_1 along c. It follows that $\{h_1^2(c(t)); t \in [0,1]\}$ is a compact subset of A_c. On the other hand h_1^2 must be the restriction of h_c^2 over $c | [0,1)$, hence $h_c^2(c[0,1))$ is relatively compact, contradictory to the assertion of Lemma 1.9. We conclude that $L \setminus S = U$.

Corollary 1.11. $G_c \backslash S$ is the biggest open subset U of G_c such that

(i) U is connected and contains G;

(ii) $h\colon G \to A$ has a multi-valued analytic extension to U.

1.5 The manifold G_c/P_c

Consider the subalgebra $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ of $\mathfrak{g}$; its normalizer P in G is equal to MAN. The normalizer P_c of $\mathfrak{p}_c$ in G_c is a parabolic subgroup of G_c, hence connected (cf. Humphreys [1, p.143]).

Proposition 1.12. Let M_c denote the centralizer of $\mathfrak{a}$ in K_c. Then $P_c = M_cA_cN_c$.

Proof. P_c is the normalizer of $\mathfrak{p}_c$ in G_c, hence $P_c \supset M_cA_cN_c$. P_c has Lie algebra $\mathfrak{p}_c = \mathfrak{m}_c + \mathfrak{a}_c + \mathfrak{n}_c$ and this is the Lie algebra of the closed subgroup $M_cA_cN_c$ as well; hence $M_cA_cN_c$ is an open subgroup of P_c. Since P_c is connected this completes the proof.

By the Iwasawa decomposition we have $K \cap P = M$, and therefore the map $K/M \to G/P$ induced by the inclusion $K \to G$ is injective. Again by the Iwasawa decomposition it follows that the map $M/K \to G/P$ is a diffeomorphism.

Consider the inclusion $K_c \to G_c$. If $(m,a,n) \in M_c \times A_c \times N_c$, $man \in K_c$ then $an \in K_c$, showing that $\mathrm{Ad}(an) \in (\underline{K}_c \cap \underline{A}_c\underline{N}_c) \subset \underline{A}_c$, whence $n = e$. It follows that $K_c \cap P_c = M_c$ and therefore the induced map $K_c/M_c \to G_c/P_c$ is injective. We have $K \cap M_c = M$ and $G \cap P_c = P$, so we may identify K/M and G/P with submanifolds of K_c/M_c and G_c/P_c

via the maps induced by the inclusions $K \to K_c$ and $G \to G_c$ respectively. With these identifications the following diagram commutes.

$$\begin{array}{ccc} K/M & \xrightarrow{\cong} & G/P \\ \downarrow & & \downarrow \\ K_c/M_c & \longrightarrow & G_c/P_c \end{array} \qquad (15)$$

From now on we shall identify via these maps.

Let π denote the canonical projection $G_c \to G_c/P_c$, and let $\mathcal{P} = \pi(S)$. In view of the right P_c-invariance of S (cf. Lemma 1.8.) it follows that $S = \pi^{-1}(\mathcal{P})$.

<u>Proposition 1.13</u>. $(G_c/P_c)\backslash\mathcal{P} = K_c/M_c$, $\mathcal{P}$ is a closed left K_c-invariant subset of G_c/P_c, and K_c/M_c is a dense open subset of G_c/P_c.

<u>Proof</u>. Since $S = \pi^{-1}(\mathcal{P})$ the identity follows from the fact that $K_cA_cN_c$ is the complement of S in G_c. Moreover, since $K_cA_cN_c$ is a dense open subset of G_c, $K_c/M_c = \pi(K_cA_cN_c)$ is a dense open subset of G_c/P_c. Hence $\mathcal{P}$ is closed; its left K_c-invariance follows immediately from the left K_c-invariance of S.

The complex analytic manifold G_c/P_c is compact (it is even projective, cf. Humphreys [1, p. 135]). We define the map $\chi\colon \overline{N}_c \to G_c/P_c$ by

$$\chi(\overline{n}) = \overline{n}P_c.$$

As is well known, χ is a complex analytic diffeomorphism onto a dense open subset of G_c/P_c.

If $x \in G_c$, let λ_x denote the left multiplication by x in G_c/P_c and define $\chi_x = \lambda_x \circ \chi$. Thus χ_x is a complex analytic diffeomorphism of $\overline{N}_c$ onto a dense open subset of G_c/P_c.

We write M^* for the normalizer of $\mathfrak{a}$ in K. The Weyl group $W = M^*/M$ embeds naturally as a finite subset in $K/M \subset G_c/P_c$. Let from now on for each $w \in W$ a representative $\overline{w} \in M^*$ be fixed. The sets $\chi_{\overline{w}}(\overline{N})$ ($w \in W$) are dense open subsets of G/P; they are independent of the particular choice of representatives and by the Bruhat decomposition of G/P they cover G/P. We call the maps $\chi_{\overline{w}}^{-1}: \overline{w}\,\overline{N}_c P_c \to \overline{N}_c$ Bruhat charts of G_c/P_c.

Lemma 1.14. There exists a polynomial function $p: \overline{\mathfrak{n}}_c \to \mathbb{C}$ such that for any $k \in K_c$ we have:

$$\chi_k^{-1}(\mathcal{P}) = S \cap \overline{N}_c = \exp(p^{-1}(0)).$$

Proof. In view of the left K_c-invariance of $\mathcal{P}$ we have $\chi_k^{-1}(\mathcal{P}) = \chi^{-1}(\mathcal{P}) = \pi^{-1}(\mathcal{P}) \cap \overline{N}_c = S \cap \overline{N}_c$. The function

$$p = \underline{F} \circ \mathrm{Ad} \circ (\exp|_{\overline{\mathfrak{n}}_c}) = \underline{F} \circ \exp \circ (\mathrm{ad}|_{\overline{\mathfrak{n}}_c})$$

is polynomial and we have $S \cap \overline{N}_c = \exp(p^{-1}(0))$.

At this stage we shall rewrite the integral in formula (1) as an oriented integral of a differential form over K/M. First we recall a lemma.

Lemma 1.15. Let G be a real (complex) analytic Lie group and let H be a closed real (complex) analytic subgroup. Let m be the dimension of the real (complex) analytic manifold G/H. Then the following conditions are equivalent.

(i) G/H has a non-zero G-invariant real analytic (holomorphic) differential m-form $\overline{\omega}$.

(ii) $\det(Ad_G(h)) = \det(Ad_H(h))$ for any $h \in H$.

If these conditions are satisfied, then $\overline{\omega}$ is unique up to a real (complex) non-zero factor.

For a proof of the real analytic part of this lemma, we refer the reader to Helgason [1, p. 386]. The complex analytic part is proved analogously. Recall that for $h \in H$ we have $Ad_H(h) = Ad_G(h)|\mathfrak{h}$, where $\mathfrak{h}$ denotes the Lie algebra of H.

Lemma 1.16. The manifold K/M has a non-zero K-invariant differential m-form $\overline{\omega}$ (here $m = \dim \overline{\mathfrak{n}} = \dim (K/M)$).

Proof. To begin with, $M = MM^0$ (cf. Warner [1, pp. 28,29]). This shows that for each $m \in M$ we have $\det (Ad(m)|_{\mathfrak{m}}) = 1$ and hence $\det (Ad_M(m)) = 1$. K is a connected compact group and therefore $\det(Ad_K(k)) = 1$ for $k \in K$. By Lemma 1.15 this completes the proof.

Corollary 1.17. K/M has a K-invariant orientation.

From now on let us fix an orientation of K/M, and an orientation of $\overline{N}$ such that $\chi|_{\overline{N}}$ is orientation preserving. Let $\overline{\omega}$ denote the K-invariant differential m-form on K/M such that:

$$\int_{K/M} \overline{\omega} = 1$$

(throughout this thesis we assume all integrals over forms to be oriented).

Now consider the map $G \times K \to \mathfrak{a}$, $(x,k) \to H(xk)$. Since M centralizes A and normalizes N this map is right M-invariant in its second variable. We denote the induced map $G \times K/M \to \mathfrak{a}$ by

$H(.,.)$. It is given by $H(x,kM) = H(xk)$. We may rewrite formula (1) as:

$$\phi_\lambda(x) = \int_{K/M} e^{(i\lambda-\rho)H(x,y)}\,\bar{\omega}(y) \qquad (x \in G).$$

In the remainder of this section we shall study complex analytic extensions of $\bar{\omega}$ and $H(.,.)$.

Lemma 1.18. The form $\bar{\omega}$ has a unique extension to a holomorphic m-form $\bar{\omega}_c$ on K_c/M_c. $\bar{\omega}_c$ is left K_c-invariant.

Proof. $P_c = M_cA_cN_c \cong M_cA_c \times N_c$ is connected, and therefore M_cA_c is a connected subgroup of G_c.

If $m \in M$, then $\det(\mathrm{Ad}(m)|\mathfrak{m}_c) = \det(\mathrm{Ad}_M(m)) = 1$ (here the first determinant is the (complex multilinear) determinant of the complex linear map $\mathrm{Ad}(m)|\mathfrak{m}_c$ whereas the second is the determinant of the real linear map $\mathrm{Ad}_M(m)\colon \mathfrak{m} \to \mathfrak{m}$; cf. also the proof of Lemma 1.16). Hence if $l \in MA$, then $\det(\mathrm{Ad}(l)|\mathfrak{m}_c) = 1$. Since M_cA_c is connected it follows by analytic continuation that $\det(\mathrm{Ad}(l)|\mathfrak{m}_c) = 1$ for $l \in M_cA_c$; in particular this holds for $l \in M_c$.

If $k \in K$, then $\det(\mathrm{Ad}(k)|\mathfrak{k}_c) = \det(\mathrm{Ad}_K(k)) = 1$. By analytic continuation we have $\det(\mathrm{Ad}(k)|\mathfrak{k}_c) = 1$ for $k \in K_c$, so in particular this holds for $k \in M_c$.

By Lemma 1.15 there exists a non-zero left K_c-invariant holomorphic m-form $\bar{\omega}_c'$ on K_c/M_c. Its pull back to K/M under the natural embedding is a non-zero K-invariant complex valued m-form on K/M hence equal to $C.\bar{\omega}$ for some $C \in \mathbb{C}$, $C \neq 0$. The form $\bar{\omega}_c = C^{-1}\bar{\omega}_c'$ satisfies all requirements.

By the same argument as in Harish-Chandra [2, p. 287] there exists a unique invariant differential m-form Ω on $\overline{N}$ such that

$$(\chi|_{\overline{N}})^{*}(\overline{\omega})_{\overline{n}} = e^{-2\rho H(\overline{n})}\Omega(\overline{n}) \qquad (16)$$

(here $(\chi|_{\overline{N}})^{*}(\overline{\omega})$ denotes the pull-back of $\overline{\omega}$ under $\chi|_{\overline{N}}$).

Since $\chi(\overline{N})$ is a dense open subset in K/M, its complement is of measure zero and hence

$$\int_{\overline{N}} e^{-2\rho H(\overline{n})}\Omega(\overline{n}) = \int_{K/M} \overline{\omega} = 1,$$

the integrals being absolutely convergent. Let Ω_c denote the extension of Ω to a holomorphic differential m-form on $\overline{N}_c$. We have the following holomorphic version of (16).

Lemma 1.19. The function $\overline{n} \to \exp(-2\rho H_c(\overline{n}))$, $\overline{N}_c \backslash S \to \mathbb{C}$ is single valued. Moreover if $k \in K_c$, then

$$(\chi_k|_{\overline{N}_c\backslash S})^{*}(\overline{\omega}_c) = e^{-2\rho H_c(.)}\Omega_c \qquad (17)$$

Proof. In view of the left K_c-invariance of ω_c it suffices to prove (17) for $k = e$. But then (17) follows from (16) by analytic continuation. By abuse of language we write $(\chi_k)^{*}(\overline{\omega}_c)$ for the left hand side of (17). Since $\chi^{*}(\overline{\omega}_c)$ and Ω_c are nowhere vanishing holomorphic m-forms on $\overline{N}_c\backslash S$ we obtain that the map $\overline{n} \to \exp(-2\rho H_c(\overline{n}))$ is single valued.

Remark 1. The first assertion of Lemma 1.19 can also be proved by the following direct argument. If $H \in \mathfrak{a}_c$ is such that $\exp H \in K$, then $\phi = \exp \operatorname{ad} H$ normalizes $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$ and we have $1 = \det(\phi|\mathfrak{g}) = \det(\phi|\mathfrak{k}) = \det(\phi|\mathfrak{a})$. So, by the direct sum decompostion $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$, we obtain that $\det(\phi|\mathfrak{n}) = 1$. On the other hand $\det((\exp \operatorname{ad}(H))|\mathfrak{n}) = \exp(2\rho(H))$. It follows that

$$e^{2\rho(H)} = 1 \text{ for } H \in \mathfrak{a}_c \text{ with } \exp H \in M.$$

In view of Theorem 1.6 this implies that $\bar{n} \to \exp(-2\rho H_c(\bar{n}))$ is single valued.

Remark 2. If $\mathfrak{g}$ is a real rank 1 algebra (i.e. $\dim \mathfrak{a} = 1$) and if we use the notations of Chapter 3, the function $\exp(-2\rho H)$ is given by:

$$e^{-2\rho(H)(\bar{n})} = \{(1 + c(X,X))^2 + 4c(Y,Y)\}^{-(\frac{1}{2}m(\alpha)+m(2\alpha))}$$

(where $\bar{n} = \exp(X + Y)$, $X \in \mathfrak{g}_{-\alpha}$, $Y \in \mathfrak{g}_{-2\alpha}$). Hence it follows from Lemma 1.19 that we must have $m(2\alpha) = 0$ or else $m(\alpha)$ even. This is in agreement with Proposition 2.3 of Araki [1].

We now turn our attention to the map $H(.,.)$: $G\times(K/M) \to \mathfrak{a}$. In view of the identification of K/M with G/P we may consider it as a map $G\times(G/P) \to \mathfrak{a}$. As such it is given by

$$H(x,yP) = H(x\kappa(y)) \qquad (x,y \in G). \tag{18}$$

Theorem 1.20. Let

$$\mathcal{P}_2 = \{(x,y) \in G_c\times(G_c/P_c);\ y \in \mathcal{P} \text{ or } \lambda_x(y) \in \mathcal{P}\},$$

then $(G_c\times(G_c/P_c))\backslash\mathcal{P}_2$ is a connected dense open subset of $G_c\times(G_c/P_c)$. The map $H(.,.)$: $G\times(G/P) \to \mathfrak{a}$ has an extension to a multi-valued analytic map $H_c(.,.)$: $(G_c\times(G_c/P_c))\backslash\mathcal{P}_2 \to \mathfrak{a}$ (with respect to the base point (e,P_c)).

Proof. Starting point of our proof is formula (18). We first consider the map $G\times G \to \mathfrak{a}$, $(x,y) \to H(x\kappa(y))$. Writing ϕ for the map $G\times G \to G$, $(x,y) \to x\kappa(y)$, this map is equal to $H\circ\phi$. Let

$$S_2 = \{(x,y) \in G_c \times G_c;\ y \in S \text{ or } xy \in S\}.$$

Then ϕ has the multi-valued extension ϕ_c: $(G_c \times G_c) \setminus S_2 \to G_c$ defined by $\phi_c(x,y) = x\kappa_c(y)$. If $(x,y) \notin S_2$ we have $xy \notin S$, and since $x\kappa_c(y) \equiv xy \bmod P_c$ it follows that $x\kappa_c(y) \notin S$. Therefore the image of ϕ_c is contained in $G_c \setminus S$ and $H_c \circ \phi_c$ is a well defined multi-valued analytic map $(G_c \times G_c) \setminus S_2 \to \mathfrak{a}_c$; it is the extension of $H \circ \phi$. S_2 is the null set of the analytic function $G_c \times G_c \to \mathbb{C}$, $(x,y) \to F(y).F(xy)$, hence its complement is a connected dense open subset of $G_c \times G_c$. Let π_2 denote the map $G_c \times G_c \to G_c \times (G_c/P_c)$, $(x,y) \to (x, yP_c)$. The set S_2 is right P_c-invariant in the second coordinate and hence the complement of $\mathcal{P}_2 = \pi_2(S_2)$ is a connected dense open subset of $G_c \times (G_c/P_c)$.

The map $H_c \circ \phi_c$: $(G_c \times G_c) \setminus S_2 \to \mathfrak{a}_c$ is right P_c-invariant in its second variable and therefore there exists a unique multi-valued analytic map $H_c(.,.)$: $(G_c \times (G_c/P_c)) \setminus \mathcal{P}_2 \to \mathfrak{a}_c$ such that:

$$H_c(.,.) \circ \pi_2 = H_c \circ \phi_c.$$

In view of formula (18), $H_c(.,.)$ is the multi-valued analytic extension of $H(.,.)$.

We write $h_c(.,.)$ for the multi-valued analytic map $\exp \circ H_c(.,.)$: $(G_c \times (G_c/P_c)) \setminus \mathcal{P}_2 \to A_c$. It is the multi-valued analytic extension of $h(.,.) = \exp \circ H(.,.)$.

Remark. In this chapter we have used the subscript c to distinguish between a real analytic functon (or differential form) and its holomorphic or multi-valued analytic extension. In order to keep our notations simple we shall omit the subscript c in the remainder of this thesis.

Appendix to Chapter 1
Multi-valued analytic maps

Let X be a connected complex analytic manifold. A covering $p: E \to X$ together with points $e \in E$, $a \in X$ such that $p(e) = a$ is called a covering with base points of X. We write $p: (E,e) \to (X,a)$ for such a covering. Let a point $a \in X$ be fixed from now on.

Fix a universal covering $\pi: (\tilde{X},\alpha) \to (X,a)$ with base points of X. An analytic map f of $\tilde{X}$ into a complex analytic manifold Z is called a multi-valued analytic map of $\tilde{X}$ into Z. Let us denote the germ of an analytic map F at a point z by F_z. Then $f_0 = f_\alpha \circ (\pi_\alpha)^{-1}$ is the germ of an analytic map at a, and the multi-valued analytic map f is said to be the multi-valued analytic extension of f_0. If we work with multi-valued maps defined on a complex analytic manifold this will always be done with respect to a fixed point, called the base point of the function. Thus we may speak of multi-valued analytic extensions.

We prefer to use the terminology of multi-valued analytic maps rather than to introduce universal covering spaces with base points. On the one hand the notations remain simple this way, on the other hand it is not sufficient to work merely with universal covering spaces if one wishes to integrate a branch of a multi-valued analytic function over a smooth cycle; for such purposes the so called covering space associated with the multi-valued analytic function (the analogon of the Riemann surface) must be introduced.

In the remainder of this appendix we fix some more terminology.

Branch at a point. If $x \in X$, $\xi \in \pi^{-1}(x)$ then the germ $f_1 = f_\xi \circ (\pi_\xi)^{-1}$ is called a branch of f at x. Let $y \in X$ and let $k: [0,1] \to X$ be a continuous curve with $k(0) = x$, $k(1) = y$. Then k has a unique lifting to a curve $\tilde{k}$: $[0,1] \to \tilde{X}$ with $\pi \circ \tilde{k} = k$, $\tilde{k}(0) = \xi$. Let $\eta = \tilde{k}(1)$, then $f_\eta \circ (\pi_\eta)^{-1}$ is called the branch of f at y obtained by continuation of f_1 along k.

Composition. If Z,Z' are connected complex analytic manifolds, g: Z → Z' an analytic map and f: X → Z a multi-valued analytic map, $g \circ f$ is a well defnied multi-valued analytic map X → Z'. If g is also a multi-valued analytic map (with base point $c = f(\alpha)$) we define $g \circ f$ as follows. f is actually an analytic map $\tilde{X} \to Z$. Let p: $(\tilde{Z},\gamma) \to (Z,c)$ be a universal covering with base points. Since $\tilde{X}$ is simply connected, there exists a unique analytic map $\tilde{f}$: $\tilde{X} \to \tilde{Z}$ such that $f = p \circ \tilde{f}$ and $\tilde{f}(\alpha) = \gamma$, called the lifting of f. The map $g \circ \tilde{f}$: $\tilde{X} \to Z'$ is a multi-valued analytic map on X, it is called the composition of g and f and denoted by $g \circ f$. In particular, if F: X → Z is an analytic map, then we identify F with the multi-valued analytic map $F \circ \pi$: $\tilde{X} \to Z$ on X. Thus if g: Z → Z' is a multi-valued analytic map with base point $c = F(a)$, the composition $g \circ F$ is defined in the above sense. It is a multi-valued analytic map X → Z' with base point a.

Restriction. Let f: X → Z be a multi-valued analytic map with base point a. If Y is a connected complex analytic submanifold of X containing a, then the inclusion ι: Y → X is an analytic map with $\iota(a) = a$. The multi-valued analytic map $f \circ \iota$ (with base point a) is called the restriction of f to Y. It is denoted by $f|Y$.

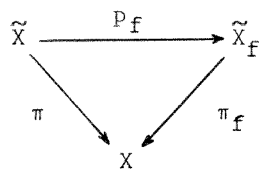
Single valued maps. If F: X → Z is an analytic map, then the analytic map $G = F \circ \pi$: $\tilde{X} \to Z$ satisfies $G_\xi \circ (\pi_\xi)^{-1} = F_x = G_\eta \circ (\pi_\eta)^{-1}$ for all $x \in X$, $\xi, \eta \in \pi^{-1}(x)$. Conversely if G: $\tilde{X} \to Z$ is an analytic map such that $G_\xi \circ (\pi_\xi)^{-1} = G_\eta \circ (\pi_\eta)^{-1}$ for all $\xi, \eta \in \tilde{X}$ with

$\pi(\xi) = \pi(\eta)$, then there exists an analytic map $F: X \to Z$ such that $G = F \circ \pi$. Such a map G will be called single valued on X.

<u>Definition of $\tilde{X}_f$</u> (analogon of the Riemann surface). Let $f: \tilde{X} \to Z$ be an analytic map. The relation $\sim$ in X defined by:

$$\xi \sim \eta \text{ iff } \pi(\xi) = \pi(\eta) \text{ and } f_\xi \circ (\pi_\xi)^{-1} = f_\eta \circ (\pi_\eta)^{-1}$$

is an equivalence relation. Let $\tilde{X}_f$ be the set of equivalence classes, $p_f: \tilde{X} \to \tilde{X}_f$ the canonical projection, and let π_f be the map $\tilde{X}_f \to X$ that makes the diagram



commutative. $\tilde{X}_f$ has a unique structure of complex analytic manifold that makes $\pi_f: \tilde{X}_f \to X$ an analytic covering. With this structure $p_f: \tilde{X} \to \tilde{X}_f$ is an analytic covering as well. The covering $\pi_f: (\tilde{X}_f, p_f(\alpha)) \to (X,a)$ with base points is called the covering associated with the multi-valued analytic map f. The analytic map $\tilde{X}_f \to Z$, $p_f(\xi) \to f(\xi)$ is denoted by $\tilde{f}$.

<u>Branch over a continuous map</u>. With the notations of the preceding alinea let T be a topological space and let $\tau: T \to X$ be a continuous map. The multi-valued analytic map $f: X \to Z$ is said to have a branch over τ iff there exists a continuous map $\tau: T \to X_f$ such that $\pi_f \circ \tilde{\tau} = \tau$. The germ f_τ of $\tilde{f}$ at $\text{im}(\tilde{\tau})$ is called a branch of f over τ. Now let T' be a subspace of T and let $f_{\tau'}$ be a branch of f over $\tau' = \tau | T'$. So $f_{\tau'}$ is the germ of $\tilde{f}$ at $\tilde{\tau}'(T')$ where $\tilde{\tau}': T' \to \tilde{X}_f$ is a continuous map with $\pi_f \circ \tilde{\tau}' = \tau'$. If $\tilde{\tau}$ is a lifting

of τ such that $\tilde{\tau}|T' = \tilde{\tau}'$ then the germ f_{τ} of $\tilde{f}$ at $\mathrm{im}(\tilde{\tau})$ is said to be the branch of f over τ that extends $f_{\tau'}$ (or restricts to $f_{\tau'}$ over τ'). Finally if S is a subset of X, then f is said to have a branch over S if it has a branch over the inclusion $\sigma: S \to X$; a branch of f over σ is also called a branch of f over S. If S' is a subset of S, a branch of f over S is said to extend a branch over S' if the corresponding branch over σ extends the corresponding branch over $\sigma|S'$.

Chapter 2

Integral representations of solutions of the radial differential equations

2.1 Smooth Cycles

For our purposes it is convenient to work with the following notion of smooth cycle. Let X be a C^∞ manifold of dimension $n \geqslant 0$ and let $0 \leqslant q \leqslant n$. We denote the space of C^∞ differential q-forms (with values in $\mathbb{C}$) on X by $\Omega^q(X)$. A C^∞ map γ of a connected compact oriented manifold Y of dimension q into X will be called a smooth q-cycle in X. γ determines the linear form $\int_\gamma$ on $\Omega^q(X)$, defined by

$$\int_\gamma \alpha = \int_Y \gamma^*(\alpha) \qquad (\alpha \in \Omega^q(X)).$$

In fact $\int_\gamma$ is a current on X, and in the homological sense it is a cycle (cf. de Rham [1]).

Two smooth q-cycles γ_0, γ_1: $Y \to X$ are called (smoothly) homotopic if there exists a C^∞ map γ: $[0,1] \times Y \to X$ such that $\gamma_i(y) = \gamma(i,y)$ for $i=0,1$, $y \in Y$. The map γ is called a homotopy, we write γ_t ($t \in [0,1]$) for the cycle $Y \to X$, $y \to \gamma(t,y)$.

<u>Lemma 2.1</u>. Let X be a C^∞ manifold of dimension $n \geqslant 0$ and let $0 \leqslant q \leqslant n$. Let γ_0, γ_1: $Y \to X$ be homotopic smooth q-cycles in X. Then for any closed form $\alpha \in \Omega^q(X)$ we have

$$\int_{\gamma_1} \alpha = \int_{\gamma_2} \alpha$$

For a proof of this lemma we refer the reader to Guillemin-Pollack [1, p.186] (and for a more general lemma to de Rham [1, § 14]).

Lemma 2.1 will be one of the main tools in our study of the spherical functions. We need it in the following setting. Let X be a complex analytic manifold of complex dimension $m \geqslant 0$ and let α be a holomorphic differential m-form on X. In local coordinates $z_1,\dots,z_m$ the form α can be written as

$$\alpha = a.dz_1\wedge..\wedge dz_m,$$

where a is a holomorphic function. In view of the Cauchy-Riemann equations for a we have

$$d\alpha = \sum_{j=1}^{m} \frac{\partial a}{\partial \bar{z}_j} d\bar{z}_j\wedge dz_1\wedge..\wedge dz_n = 0,$$

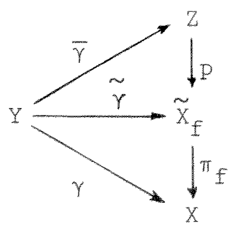
hence α is closed as an element of $\Omega^m(X)$. Consequently, if γ_0, γ_1: $Y \to X$ are homotopic smooth m-cycles (so Y has real dimension m) in X, we have

$$\int_{\gamma_0} \alpha = \int_{\gamma_1} \alpha.$$

We conclude this section with a discussion of integrals of multi-valued analytic functions. Let f be a multi-valued analytic function on X with respect to a fixed base point x_0, and let γ: $Y \to X$ be a smooth m-cycle in X such that f has a branch f_γ over γ (for this terminology see the appendix to Chapter 1). Thus, if we write π_f: $(\tilde{X}_f,\xi_{0,f}) \to (X,x_0)$ for the covering associated with f and $\tilde{f}$ for the corresponding analytic function $\tilde{X}_f \to \mathbb{C}$, then γ has a lifting $\tilde{\gamma}$: $Y \to \tilde{X}_f$ such that f_γ is the germ of $\tilde{f}$ at $\tilde{\gamma}(Y)$. Now $\tilde{\gamma}$ is a smooth m-cycle in $\tilde{X}_f$ and $\tilde{f}\pi_f^*(\alpha)$ is a holomorphic m-form on $\tilde{X}_f$. We define

$$\int_\gamma f_\gamma\alpha = \int_{\tilde{\gamma}} \tilde{f}\pi_f^*(\alpha). \qquad (1)$$

Observe that if $p\colon (Z,\zeta_0) \to (\tilde{X}_f,\xi_{0,f})$ is a covering with base points such that $\tilde{\gamma}$ has a lifting $\bar{\gamma}\colon Y \to Z$, then the diagram



commutes, and writing $\bar{p} = \pi_f \circ p$, $\bar{f} = \tilde{f} \circ p$ we have that:

$$\int_\gamma f_\gamma \alpha = \int_{\bar{\gamma}} \bar{f}\, \bar{p}^*(\alpha).$$

<u>Lemma 2.2</u>. Let X be a complex analytic manifold of complex dimension m, let α be a holomorphic m-form and let f be a multi-valued analytic function on X. Let $\gamma\colon [0,1]\times Y \to X$ be a homotopy of smooth m-cycles such that f has a branch f_0 over γ_0. Then f_0 extends to a branch f_γ over γ and we have

$$\int_{\gamma_0} f_\gamma \alpha = \int_{\gamma_1} f_\gamma \alpha. \qquad (2)$$

<u>Proof</u>. Using the same notations as above, γ_0 has a lifting to a cycle $\tilde{\gamma}_0\colon Y \to \tilde{X}_f$ such that f_0 is the germ of $\tilde{f}$ at $\tilde{\gamma}_0(Y)$. Now γ has a lifting to a C^∞ map $\tilde{\gamma}\colon [0,1]\times Y \to \tilde{X}_f$ such that $\tilde{\gamma}(0,y) = \tilde{\gamma}_0(y)$ for $y \in Y$. Hence f_0 extends to a branch f_γ of f over γ. Moreover, by the preceding discussion it follows that

$$\int_{\tilde{\gamma}_0} \tilde{f}\pi_f^*(\alpha) = \int_{\tilde{\gamma}_1} \tilde{f}\pi_f^*(\alpha).$$

By definition (1) and the remark following that definition, this implies (2).

2.2. The radial differential equations

The group G admits the decomposition

$$G = K\overline{A}^{+}K$$

(here $\overline{A}^{+}$ denotes the closure of $A^{+} = \exp(\mathfrak{a}^{+})$ in A). If $x \in G$, then there exist $k_1, k_2 \in K$, $a \in \overline{A}^{+}$ such that $x = k_1 a k_2$; here a is uniquely determined by the property $x \in KaK$. The set $KA^{+}K$ is open and dense in G, we denote it by G'. Let the map $\pi: G' \to A^{+}$ be defined by $x \in K\pi(x)K$ $(x \in G)$. Then π is a real analytic fibration with fibre diffeomorphic to $(K/M)\times K$. Writing i for the inclusion $A^{+} \to G$ we have $\pi \circ i = \mathrm{id}(A^{+})$.

Now let $D \in \mathrm{Diff}(G)$ (if X is a C^{∞} manifold, we write Diff(X) for the algebra of linear differential operators with C^{∞} coefficients on X). The operator $\Delta(D): C^{\infty}(A^{+}) \to C^{\infty}(A^{+})$ defined by

$$\Delta(D) = i^{*} \circ D \circ \pi^{*}$$

is linear, continuous and support preserving, hence an element of $\mathrm{Diff}(A^{+})$. It is called the radial part of D.

If $\mathfrak{l}$ is any Lie algebra over $\mathbb{C}$, we write $U(\mathfrak{l})$ for its universal enveloping algebra. Now consider $U(\mathfrak{g}_c)$, and let ∂ be the $\mathbb{C}$-algebra homomorphism of $U(\mathfrak{g}_c)$ into $\mathbb{D}(G)$, the algebra of left invariant differential operators on G, determined by

$$(\partial(X)f)(x) = \frac{d}{dt} f(x \exp tX)\Big|_{t=0}$$

for $X \in \mathfrak{g}$, $f \in C^{\infty}(G)$, $x \in G$. As is well known, ∂ is an isomorphism, and we shall identify $U(\mathfrak{g}_c)$ with $\mathbb{D}(G)$ via this isomorphism. Thus, we shall also write u for $\partial(u)$ ($u \in U(\mathfrak{g}_c)$). Let $U(\mathfrak{g}_c)^K$ be the set of Ad(K)-invariant elements of $U(\mathfrak{g}_c)$.

Proposition 2.3. The map Δ: $U(\mathfrak{g}_c)^K \to \mathrm{Diff}(A^+)$ is a homomorphism of algebras.

Proof. This follows from the fact that $\pi^* i^*$ is the identity on bi-K-invariant functions, whereas the elements of $U(\mathfrak{g}_c)^K$ are bi-K-invariant as differential operators.

For a more detailed discussion of the radial part of a differential operator, we refer the reader to Harish-Chandra [2], or to Helgason [3].

Consider again the algebra $U(\mathfrak{g}_c)$. In view of the Iwasawa decomposition $\mathfrak{g}_c = \mathfrak{k}_c + \mathfrak{a}_c + \mathfrak{n}_c$, we have a direct sum decomposition

$$U(\mathfrak{g}_c) = (\mathfrak{k}_c U(\mathfrak{g}_c) + U(\mathfrak{g}_c)\mathfrak{n}_c) \oplus U(\mathfrak{a}_c). \qquad (3)$$

Let γ' be the corresponding projection $U(\mathfrak{g}_c) \to U(\mathfrak{a}_c)$. Since $\mathfrak{a}_c$ is abelian, $U(\mathfrak{a}_c)$ is canonically isomorphic to the symmetric algebra $S(\mathfrak{a}_c)$ of $\mathfrak{a}_c$. On the other hand $S(\mathfrak{a}_c)$ is canonically isomorphic to the algebra $P(\mathfrak{a}_c^*)$ of polynomial functions $\mathfrak{a}_c^* \to \mathbb{C}$. Under this isomorphism $X \in \mathfrak{a}_c$ corresponds to the linear function $\nu \to \nu(X)$, $\mathfrak{a}_c^* \to \mathbb{C}$. Let $T_{-\rho}$ be the automorphism of $P(\mathfrak{a}_c^*)$ defined by $(T_{-\rho}f)(\nu) = f(\nu-\rho)$. Writing $\gamma = (T_{-\rho} \circ \gamma')|U(\mathfrak{g}_c)^K$ we have the following lemma, due to Harish-Chandra.

Lemma 2.4. The map γ is an algebra homomorphism of $U(\mathfrak{g}_c)^K$ onto $I(\mathfrak{a}_c)$, the algebra of W-invariants in $S(\mathfrak{a}_c)$.

Theorem 2.5. (Harish-Chandra). The elementary spherical functions of the pair (G,K) are the functions ϕ_λ ($\lambda \in \mathfrak{a}_c^*$), given by formula (1), Ch. 1. Moreover, if $\lambda \in \mathfrak{a}_c^*$, then

$$D\phi_\lambda = \gamma(D,i\lambda)\phi_\lambda \qquad (D \in U(\mathfrak{g}_c)^K).$$

For proofs of the lemmas 2.4 and 2.5 we refer the reader to Harish-Chandra [1], or to Helgason [1, Ch. X].

Corollary 2.6. If $\lambda \in \mathfrak{a}_c^*$, then the restriction $\phi_\lambda | A^+$ of ϕ_λ to A^+ satisfies the system of radial differential equations

$$\Delta(D)\phi = \gamma(D,i\lambda)\phi \qquad (D \in U(\mathfrak{g}_c)^K). \qquad (4)$$

We end this section with a result of Harish-Chandra (cf. [2]) concerning the system (4) of radial differential equations. The following lemma (cf. Harish-Chandra [2, p. 251]) plays a crucial role in it.

Lemma 2.7. There exist homogeneous elements P_w ($w \in W$) in $S(\mathfrak{a}_c)$ such that $P_I = 1$ and such that $S(\mathfrak{a}_c)$ is a free $I(\mathfrak{a}_c)$-module of rank $\#W$, with free basis $(P_w)_{w\in W}$.

Theorem 2.8. (Harish-Chandra). Let $\lambda \in \mathfrak{a}_c^*$, let $U \subset A^+$ be a connected open set, and let $a_0 \in U$. Then the space $E_\lambda(U)$ of C^∞ functions ϕ: $U \to \mathbb{C}$ satisfying (4) consists of real analytic functions. Moreover the linear map

$$P(a_0): E_\lambda(U) \to \mathbb{C}^W, \qquad \psi \to ((P_w\psi)(a_0))_{w\in W} \qquad (5)$$

is injective. consequently $\dim E_\lambda(U) \leqslant \#W$.

2.3 Integral representations of solutions

In this section we introduce the method to obtain local solutions of the system (4) of radial differential equations. Our principal result is the following theorem.

Theorem 2.9. Let $\Gamma: Y \to K_c$ be a smooth dim(K)-cycle and let $a_0 \in A^+$ be such that $\mathrm{im}(\lambda(a_0)\circ\Gamma) \cap S = \emptyset$, and such that the multi-valued map $H: G_c\backslash S \to \mathfrak{a}_c$ has a branch H_0 over $\lambda(a_0)\circ\Gamma$. Then there exists an open neighbourhood A of a_0 in A^+ such that $\mathrm{im}(\lambda(a)\circ\Gamma) \cap S = \emptyset$ for $a \in A$, and such that H_0 extends to a branch H_Γ over the map $(a,y) \to a\Gamma(y)$, $A\times Y \to G_c\backslash S$. Moreover, the function $\phi: A \to \mathbb{C}$ defined by

$$\phi(a) = \int_\Gamma e^{(i\lambda-\rho)H_\Gamma(ak)}\,\omega(k)$$

satisfies the radial differential equations (4) on A.

We shall prove Theorem 2.9 by adapting Harish-Chandra's method of proof for the cycle $K \to K_c$, $k \to k$ (cf. [2, Lemma 3]) to our situation. The key idea is that instead of the bi-K-invariance of the Haar measure dk on K we may use the bi-K-invariance of the form ω together with the invariance of the homotopy class of a smooth cycle under right or left translation by K. The following lemma is based on this idea.

Lemma 2.10. Let V be some connected open subset of G_c, and let $f: V \to \mathbb{C}$ be a multi-valued analytic function with base point $x_1 \in V$. Suppose that $x_0 \in V$, and let $\Gamma: Y \to K_c$ be a smooth dim(K)-cycle such that $\mathrm{im}(\rho(x_0)\circ\Gamma) \subset V$, and such that f has a branch f_0 over $\rho(x_0)\circ\Gamma$. Then there exists an open neighbourhood U of x_0 in V, such that $\mathrm{im}(\rho(x)\circ\Gamma) \subset V$ for all $x \in U$, and such that f_0 extends to a branch f_Γ over the map $U \times Y \to V$, $(x,y) \to \rho(x)(\Gamma(y))$. Moreover, the function $\psi: U \to \mathbb{C}$ defined by

$$\psi(x) = \int_\Gamma f_\Gamma(kx)\omega(k) \tag{6}$$

is holomorphic and locally left K-invariant.

Remark 1. By saying that $\psi: U \to \mathbb{C}$ is locally left K-invariant we mean that for every $x' \in U$ there exists an open neighbourhood U' of x' in U and an open neighbourhood W of e in K, such that $WU' \subset U$, and such that $\psi(kx) = \psi(x)$ for all $k \in W$, $x \in U'$. The qualifications locally left K-invariant or locally bi-K-invariant are to be interpreted analogously.

Remark 2. If, in the above lemma, the symbol ρ for right multiplication is replaced by λ everywhere, then the function $\psi': U \to \mathbb{C}$ defined by

$$\psi'(x) = \int_\Gamma f_\Gamma(xk)\omega(k)$$

is holomorphic and locally right K-invariant.

Proof of Lemma 2.10. By compactness of $\mathrm{im}(\Gamma)$ there exists a simply connected open neighbourhood U of x_0, such that $\mathrm{im}(\rho(x)\circ\Gamma) \subset V$ for $x \in U$. Now, since U is simply connected, f_0 extends to a branch f_Γ over the map $U \times Y \to V$, $(x,y) \to \Gamma(y)x$. Consequently ψ is a well defined holomorphic function $U \to \mathbb{C}$. To establish its local left K-invariance, fix $x' \in U$. There exist connected open neighbourhoods U' and W of x' and e in V and K respectively, such that $WU' \subset U$. Now if $k \in K$, then $\rho(k^{-1})^*\omega = \omega$, and therefore if $x \in U'$, $k \in W$, we have

$$\psi(kx) = \int_\Gamma f_\Gamma(k'kx)\omega(k')$$
$$= \int_{\rho(k)\circ\Gamma} f_\Gamma(k'x)\omega(k'). \qquad (7)$$

Select a C^∞ curve $c\colon [0,1] \to W$ such that $c(0) = e$, $c(1) = k$. The map $[0,1]\times Y \to K_c$, $(t,y) \to \Gamma(y)c(t)$ is a smooth homotopy of $\dim(K)$-cycles in K_c. The branch f_Γ is defined over this homotopy, and therefore by Lemma 2.2, the last integral in (7) equals the integral in (6). This shows that $\psi(kx) = \psi(x)$ for $x \in U'$, $k \in W$, hence completes the proof.

If $u \in U(\mathfrak{a}_c)$, then the left invariant differential operator $\partial(u)$ naturally induces a left G_c-invariant holomorphic differential operator on G_c. Let us describe how a holomorphic linear differential operator D acts on a multi-valued analytic function f defined on a complex analytic manifold X. Let a be the base point of f, and let $\pi\colon (\tilde{X},\alpha) \to (X,a)$ be the universal covering with base points. There exists a unique holomorphic differential operator $\tilde{D}$ on $\tilde{X}$ such that

$$\tilde{D}\circ\pi^* = \pi^*\circ D$$

on C^∞ functions $X \to \mathbb{C}$. Considering f as a holomorphic function on $\tilde{X}$, we write Df for the holomorphic function $\tilde{D}f$ on $\tilde{X}$, viewed as a multi-valued analytic function on X. We now come to a lemma that has Theorem 2.9 as an easy corollary.

<u>Lemma 2.11</u>. Let $\Gamma\colon Y \to K_c$ be a smooth $\dim(K)$-cycle and let $x_0 \in G_c$ such that $\mathrm{im}(\lambda(x_0)\circ\Gamma) \cap S = \emptyset$, and such that $H\colon G_c\backslash S \to \mathfrak{a}_c$ has a branch H_0 over $\lambda(x_0)\circ\Gamma$. Then there exists an open neighbourhood U of x_0 in G_c such that $\mathrm{im}(\lambda(x)\circ\Gamma) \cap S = \emptyset$ for $x \in U$, and

such that H_0 extends to a branch H_Γ over the map $(x,y) \to x\Gamma(y)$, $U\times Y \to G_c\backslash S$. Moreover, the function $\psi: U \to \mathbb{C}$ defined by

$$\psi(x) = \int_\Gamma e^{(i\lambda-\rho)H_\Gamma(xk)}\omega(k) \tag{8}$$

is holomorphic, locally bi-K-invariant, and it satisfies the system of differential equations

$$D\psi = \gamma(D,i\lambda)\psi \qquad (D \in U(\mathfrak{g}_c)^K) \tag{9}$$

on U.

<u>Proof</u>. By Remark 2 following Lemma 2.10 the function ψ is holomorphic and locally right K-invariant. On the other hand, by analytic continuation we see that each branch of the multi-valued analytic map $H: G_c\backslash S \to \mathfrak{a}_c$ is locally left K-invariant. Consequently ψ is locally left K-invariant.

To prove the last statement, fix $D \in U(\mathfrak{g}_c)^K$. Writing f_Γ for the branch $\exp(\,(i\lambda-\rho)H_\Gamma(.))$ over the map $(x,y) \to x\Gamma(y)$, $U\times Y \to G_c\backslash S$, we have:

$$\begin{aligned} D\psi(x) &= \int_\Gamma D(\rho_k^* f_\Gamma)(x).\omega(k) \\ &= \int_\Gamma (Df_\Gamma)(xk).\omega(k). \end{aligned}$$

By (3) we may write $D = u + v + D_0$ with $u \in \mathfrak{k}_c U(\mathfrak{g}_c)$, $v \in U(\mathfrak{g}_c)\mathfrak{n}_c$, $D_0 \in U(\mathfrak{a}_c)$. We claim that

$$\int_\Gamma (uf_\Gamma)(xk).\omega(k) = 0. \tag{10}$$

To see this, let $X \in \mathfrak{k}$, $w \in U(\mathfrak{g}_c)$. Then

$$(Xwf_\Gamma)(xk) = \frac{d}{dt}\,(wf_\Gamma)(xk\ \exp tX)\Big|_{t=0}.$$

By Lemma 2.10 the function $t \to \int_\Gamma (wf_\Gamma)(xk \exp tX).\omega(k)$ is defined for t in some interval containing 0, and there it is constant. Consequently

$$\int_\Gamma (Xwf_\Gamma)(xk).\omega(k) = 0$$

and this proves (10). On the other hand, the multi-valued analytic map $H\colon G_c\backslash S \to \mathfrak{a}_c$ is right N-invariant, whence $vf_\Gamma \equiv 0$. Therefore

$$D\psi(x) = \int_\Gamma (D_0 f_\Gamma)(xk).\omega(k).$$

Finally, applying the proposition below, by analytic continuation we obtain that $D_0 f_\Gamma = \gamma(D,i\lambda)f_\Gamma$, and so, (9) follows.

Proposition 2.12. Let f_λ be the function $x \to \exp((i\lambda-\rho)H(x))$, $G \to \mathbb{C}$. If $p \in P(\mathfrak{a}_c^*) \cong S(\mathfrak{a}_c) \cong U(\mathfrak{a}_c)$, then

$$\partial(p)f_\lambda = p(i\lambda-\rho).f_\lambda \tag{11}$$

Proof. It suffices to prove (11) for $p = X \in \mathfrak{a}$, but then the assertion follows straightforwardly from $H(x \exp tX) =$ $= H(x) + tX$ $(x \in G,\ t \in \mathbb{R})$.

Chapter 3

The rank 1 case

3.1 Preliminaries

In this chapter we assume that $\dim\mathfrak{a} = 1$; we then have $\#\Delta^{++} = 1$. We denote the element of Δ^{++} by α. The Weyl group W consists of two elements, I and w. The action of the latter on $\mathfrak{a}$ is given by $w(H) = -H$ $(H \in \mathfrak{a})$.

G/P is a compact manifold of dimension $m = m(\alpha) + m(2\alpha)$. It has a Bruhat decomposition: G/P is the disjoint union of the two Bruhat cells $\overline{N}P = \chi(\overline{N})$ and $\overline{w}P$ (recall that $\overline{w}$ is a fixed representative of w in M^*, the normalizer of $\mathfrak{a}$ in K). Moreover, $\overline{N}P \cup \overline{w}P = G/P$ implies $\overline{w}\,\overline{N}P \cup P = G/P$, and so, $\chi_{\overline{w}}$ (for simplicity we shall write χ_w from now on) maps $\overline{N}$ diffeo-morpically onto the complement of eP in G/P. From these facts it follows that G/P is an m-sphere.

Let R be a positive real number (in the next section we shall impose a condition on its magnitude). Define the compact ball B_I in $\mathfrak{g}_{-\alpha}\times\mathfrak{g}_{-2\alpha}$ by

$$B_I = \{(X,Y) \in \mathfrak{g}_{-\alpha}\times\mathfrak{g}_{-2\alpha};\ (X,X) + (Y,Y) \leqslant R\}$$

(here $(\ ,\ )$ denotes the inner product defined in section 1.2). The map $E\colon \mathfrak{g}_{-\alpha}\times\mathfrak{g}_{-2\alpha} \to \overline{N}$, $(X,Y) \to \exp(X+Y)$ is a diffeomorphism, hence $\chi E(B_I)$ is a compact m-dimensional submanifold of G/P with boundary $\chi E(\partial B_I)$. Its complement $C\chi E(B_I)$ is an open subset of G/P that does not contain eP hence is contained in $\chi_w(\overline{N})$. Similarly $C\chi E(B_I^{\mathrm{int}})$ is a compact subset of $\chi_w(\overline{N})$. We obviously have

$$\overline{C\chi E(B_I)} = C\chi E(B_I^{int})$$

and therefore $C\chi E(B_I^{int})$ is a compact m-dimensional submanifold of $\chi_w(\bar{N})$ with boundary $\chi E(\partial B_I)$. Let B_w be the unique compact submanifold with boundary of $\mathfrak{g}_{-\alpha}\times\mathfrak{g}_{-2\alpha}$ such that $\chi_w E(B_w) = C\chi E(B_I^{int})$. The disjoint union $\chi E(B_I^{int}) \cup \chi_w E(B_w^{int})$ has the set $\chi E(\partial B_I)$ of measure zero as its complement in G/P. Providing $\chi E(B_I)$ and $\chi_w E(B_w)$ with the orientations induced by the orientation of G/P we thus have:

$$\phi_\lambda(x) = \int_{\chi E(B_I)} e^{(i\lambda-\rho)H(x,y)}\overline{\omega}(y) + \int_{\chi_w E(B_w)} e^{(i\lambda-\rho)H(x,y)}\overline{\omega}(y). \tag{1}$$

In the next section we shall extend the manifolds $\chi E(B_I)$ and $\chi_w E(B_w)$ to cycles Γ_I and Γ_w in the complex flag manifold G_c/P_c. Writing H_0 for the element of $\mathfrak{a}$ with $\alpha(H_0) = 1$, and writing $a(\tau)$ $(\tau \in \mathbb{C})$ for $\exp(\tau H_0)$, the cycles $\Gamma_I\colon \partial([0,\pi]\times B_I) \to G_c/P_c$ and $\Gamma_w\colon \partial([0,\pi]\times B_w) \to G_c/P_c$ will be defined by the following formulas:

$$\Gamma_I(t,(X,Y)) = a(e^{-it}).\chi E(X,Y),$$
$$\Gamma_w(t,(X,Y)) = a(e^{-it}).\chi_w E(X,Y)$$

(we use both notations λ_x and x. for the left multiplication by an element $x \in G_c$ in G_c/P_c). Observe that the cycles Γ_I,Γ_w are not smooth. We will show that the real branches of $H(a,\cdot)$ over $\chi E(B_I)$ and $\chi_w E(B_w)$ extend to branches $H_I(a,.)$ and $H_w(a,.)$ over the continuous maps Γ_I and Γ_w respectively. Consider the integrals of $\exp[(i\lambda-\rho)H_s(a,.)]$ over Γ_s $(s = I,w)$. The contributions of these integrals over $\Gamma_I|([0,\pi]\times\partial B_I)$ and $\Gamma_w|([0,\pi]\times\partial B_w)$ cancel each other. On the other hand the contributions of the integrals over $\Gamma_s|(\{\pi\}\times B_s)$ $(s = I,w)$ are equal to a factor $\exp[2\pi\lambda(H_0)]$

times the integrals over $\chi E(B_I)$ and $\chi_w E(B_w)$ in (1) respectively. This finally leads to the formula:

$$(e^{2\pi\lambda(H_0)}-1)\phi_\lambda(a) = \sum_{s=I,w} \int_{\Gamma_s} e^{(i\lambda-\rho)H_s(a,y)}\bar{\omega}(y). \qquad (2)$$

3.2 Construction of the cycles Γ_I, Γ_w

Let H_0 be the element of $\mathfrak{a}$ such that $\alpha(H_0) = 1$. The following formula for the map $H\colon \overline{N} \to \mathfrak{a}$ has been found independently by Helgason [2, p. 59] and Schiffmann [1, p. 24].

Lemma 3.1. Let $\bar{n} = \exp(X+Y)$, $X \in \mathfrak{g}_{-\alpha}$, $Y \in \mathfrak{g}_{-2\alpha}$. Then:

$$H(\bar{n}) = \tfrac{1}{2}\log[(1+c(X,X))^2 + 4c(Y,Y)]H_0 \qquad (3)$$

where $c^{-1} = 4(m(\alpha) + m(2\alpha))$.

We denote the holomorphic diffeomorphism $\mathfrak{g}_{-\alpha,c}\times\mathfrak{g}_{-2\alpha,c} \to \overline{N}_c$, $(X,Y) \to \exp(X+Y)$ by E too. Let q be the polynomial function $\mathfrak{g}_{-\alpha,c}\times\mathfrak{g}_{-2\alpha,c} \to \mathbb{C}$ defined by:

$$q(X,Y) = (1 + c(X,X))^2 + 4c(Y,Y) \qquad (4)$$

From (3) it follows that $\overline{N}_c \setminus E(q^{-1}(0))$ is the biggest open subset of $\overline{N}_c$ with the property that $H\colon \overline{N} \to \mathfrak{a}$ has a multi-valued analytic extension to it. Hence by Lemma 1.8 we see that $\overline{N}_c \cap S = E(q^{-1}(0))$.

Example. Let $G = SL(2,\mathbb{R})$, $K = SO(2,\mathbb{R})$. Define

$$H_0 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and let $\mathfrak{a} = \mathbb{R}H_0$. Let α be the root of the pair $(\mathfrak{g},\mathfrak{a})$ with $\alpha(H_0) = 1$. Writing

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

we have $\mathfrak{n} = \mathbb{R}X$, $\overline{\mathfrak{n}} = \mathbb{R}Y$. We identify $\overline{N}_c$ with $\mathbb{C}$ (with the addition) via the group isomorphism

$$\mathbb{C} \to \overline{N}_c, \; z \to \exp(zY) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}.$$

The manifold G_c/P_c is diffeomorphic to the Riemann sphere, the charts χ^{-1} and χ_w^{-1} form an atlas for G_c/P_c, and the transition map $\chi_w^{-1}\circ\chi\colon \mathbb{C}\setminus\{0\} \to \mathbb{C}\setminus\{0\}$ is given by

$$\chi_w^{-1}\circ\chi(z) = -\frac{1}{z} \qquad (z \in \mathbb{C}\setminus\{0\}).$$

Moreover, $H\colon \overline{N} \to \mathfrak{a}$ is given by

$$H(x) = \log(1 + x^2).H_0 \qquad (x \in \mathbb{R}),$$

and so $S \cap \overline{N}_c = \{-i,+i\}$. Finally, we have

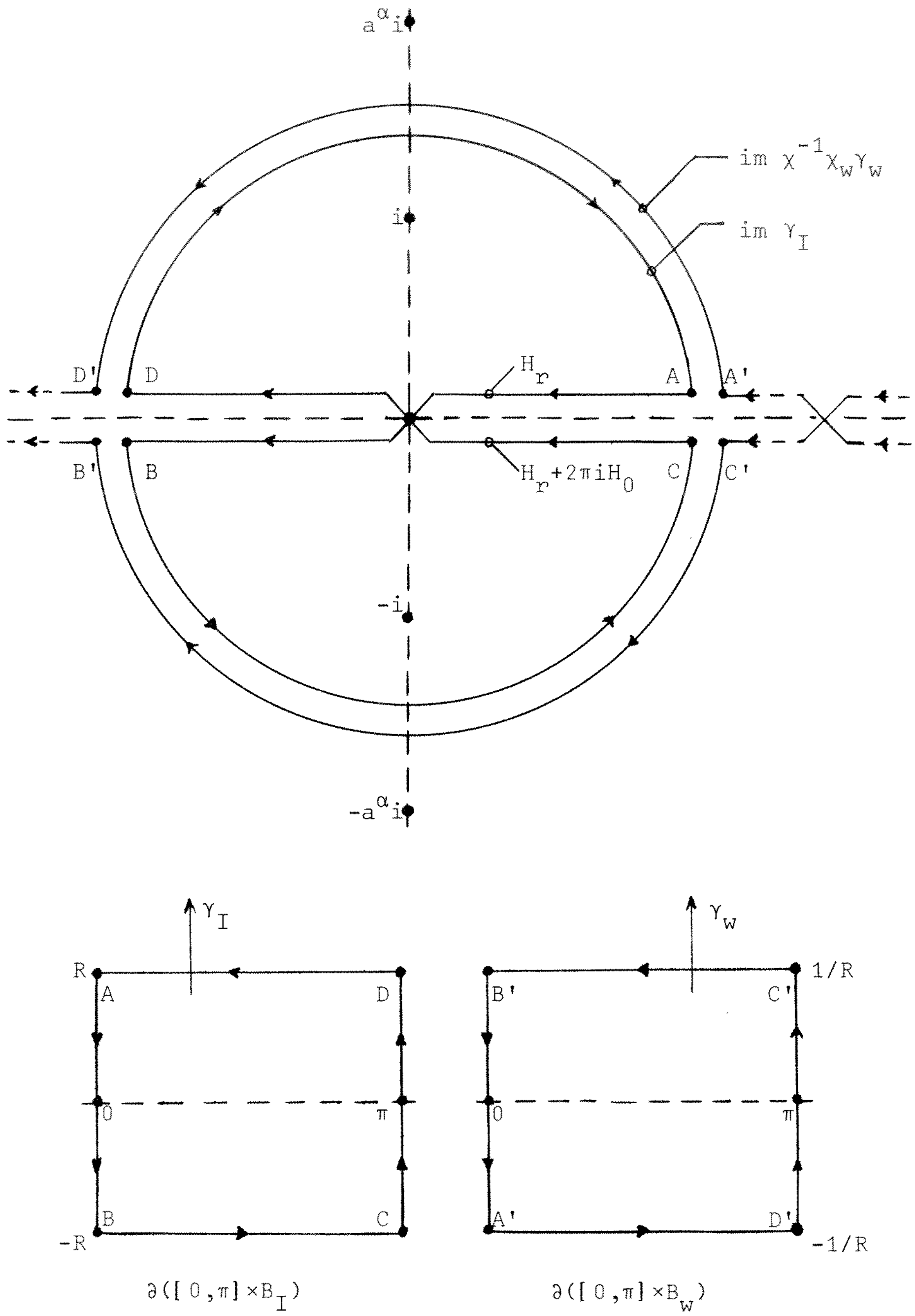
$$\chi^*(\overline{\omega}) = \frac{1}{\pi} \cdot \frac{dz}{1 + z^2} \,.$$

In the picture on p. 3-5 we have indicated some concepts that will be introduced in the remainder of this chapter, specialized to the case of $SL(2,\mathbb{R})$. We hope it will help the reader to find his (her) way through this chapter.

Let us return to the general case. Fix $(X,Y) \in \mathfrak{g}_{-\alpha}\times\mathfrak{g}_{-2\alpha}$, $(X,Y) \neq (0,0)$, and consider the polynomial function $r\colon \mathbb{C} \to \mathbb{C}$ defined by:

$$r(z) = q(zX,z^2Y). \qquad (5)$$

Figure 3.1.



$a^\alpha i$
i
$-i$
$-a^\alpha i$
im $\chi^{-1}\chi_w\gamma_w$
im γ_I
H_r
$H_r + 2\pi i H_0$
A
A'
B
B'
C
C'
D
D'
γ_I
R
-R
0
π
$\partial([0,\pi]\times B_I)$
γ_w
1/R
-1/R
$\partial([0,\pi]\times B_w)$

Writing $a = c(X,X)$ and $b = 4c(Y,Y)$ we have $a \geqslant 0$, $b \geqslant 0$ and

$$r(z) = (1 + az^2)^2 + bz^4.$$

Let $\sqrt{a + i\sqrt{b}}$ denote the square root of $a + i\sqrt{b}$ that has its argument in the interval $[0, \frac{\pi}{4}]$, and let

$$\zeta = i/\sqrt{a + i\sqrt{b}},$$

then we obtain

$$r(z) = (1+\zeta^{-1}z)(1-\zeta^{-1}z)(1+\bar{\zeta}^{-1}z)(1-\bar{\zeta}^{-1}z) \tag{6}$$

(here $\bar{\zeta}$ denotes the complex conjugate of ζ). It follows that r has roots ζ, $-\zeta$, $\bar{\zeta}$ and $-\bar{\zeta}$. Set $R_0 = (X,X) + (Y,Y)$. Observe that:

$$|\zeta|^4 = (a^2+b)^{-1} = (c^2(X,X)^2 + 4c(Y,Y))^{-1}.$$

Since $(X,X) \geqslant \frac{1}{2}R_0$ or $(Y,Y) \geqslant \frac{1}{2}R_0$ we obtain that:

$$|\zeta|^4 \leqslant \max\left(\frac{4}{R_0^2\, C^2}, \frac{2}{4R_0 C}\right). \tag{7}$$

Furthermore $\arg(\zeta) = \frac{\pi}{2} - \frac{1}{2}\arg(a + i\sqrt{b})$, showing that:

$$\frac{\pi}{4} \leqslant \arg(\zeta) \leqslant \frac{\pi}{2}.$$

Moreover, $\arg(\zeta) = \frac{\pi}{4}$ iff $(X,X) = 0$ and $\arg(\zeta) = \frac{\pi}{2}$ iff $(Y,Y) = 0$.

Now fix R (cf. Section 3.1) such that $|\zeta| < 1$ for all $(X,Y) \in \mathfrak{g}_{-\alpha} \times \mathfrak{g}_{-2\alpha}$ with $(X,X) + (Y,Y) = R$. This is possible in view of (7). Recall that B_I denotes the compact ball $(X,X) + (Y,Y) \leqslant R$ in $\mathfrak{g}_{-\alpha} \times \mathfrak{g}_{-2\alpha}$ (cf. Section 3.1), and that $\bar{N}$ and G/P are oriented such that $\chi\colon \bar{N} \to G/P$ is orientation preserving (cf. Section 1.5). Give $\mathfrak{g}_{-\alpha} \times \mathfrak{g}_{-2\alpha}$ the orientation that turns $\mathcal{E}$ into an orientation preserving map and provide B_I with the induced orientation. Give $[0,\pi] \times B_I$ the product orientation and orient $\partial([0,\pi] \times B_I)$ according to the outward normal. We define the map $\gamma_I: \partial([0,\pi]\times B_I) \to \overline{N}_c$ by:

$$\gamma_I(t,(X,Y)) = a(e^{-it})E(X,Y)a(a^{-it})^{-1}$$

(here $a(\tau) = \exp(\tau H_0)$ for $\tau \in \mathbb{C}$). It easily follows that:

$$\gamma_I(t,(X,Y)) = E(e^{it}X,e^{2it}Y). \qquad (8)$$

If $(X,Y) \in \partial B_I$ is fixed we have $q(e^{it}X,e^{2it}Y) = r(e^{it}) \neq 0$ whence $\gamma_I(t,(X,Y)) \notin S$ for $(X,Y) \in \partial B_I$, $t \in [0,\pi]$. Since γ_I maps $\{0\}\times B_I$ and $\{\pi\}\times B_I$ into $\overline{N}$ it follows that $\mathrm{im}(\gamma_I) \subset \overline{N}_c\backslash S$.

<u>Lemma 3.2</u>. The multi-valued analytic map $H: \overline{N}_c\backslash S \to \mathfrak{a}_c$ has a branch H_I over γ_I that equals the real branch H_r over $\gamma_I|(\{0\}\times B_I)$. Moreover

$$H_I(.) = H_r(.) + 2\pi i H_0 \qquad \text{over} \qquad \gamma_I|(\{\pi\}\times B_I).$$

<u>Proof</u>. As we observed above, $\mathrm{im}(\gamma_I) \subset \overline{N}_c\backslash S$. Fix $(X,Y) \in \partial B_I$ and consider the curve $c: [0,\pi] \to \overline{N}_c$, $t \to \gamma_I(t,(X,Y))$. From (8), (5), (4) and (3) it follows that:

$$H(c(t)) = \tfrac{1}{2}\log(r(e^{it}))H_0.$$

From (6) it follows that the argument of $r(e^{it})$ increases with 4π if t increases from 0 to π. This shows that the branch of H at $(-X,Y) = \gamma_I(\pi,(X,Y))$ obtained by continuation of the real branch H_r along c is equal to $H_r + 2\pi i H_0$. From this it follows that H has a branch H_I over γ_I that restricts to the branch H_r over $\gamma_I|(\{0\}\times B_I)$. Over $\gamma_I(\{\pi\}\times B_I)$ this branch is equal to $H_r + 2\pi i H_0$, and at a point $\gamma_I(t_0,(X,Y))$ ($t_0 \in [0,\pi]$, $(X,Y) \in \partial B_I$) it is equal to the branch of H obtained by continuation of H_r along the curve $t \to \gamma_I(t,(X,Y))$, $[0,t_0] \to \overline{N}_c\backslash S$.

Define the map $\Gamma_I: \partial([0,\pi]\times B_I) \to G_c/P_c$ by $\Gamma_I = \chi\circ\gamma_I$. If $x \in G_c$ we write $Ad(x)$ for the conjugation $g \to xgx^{-1}$, $G_c \to G_c$; we write $Ad(x,g)$ for $Ad(x)(g) = xgx^{-1}$. With this notation we have $\lambda_a\circ\chi = \chi\circ Ad(a)$ for $a \in A_c$, and hence indeed we have:

$$\Gamma_I(t,(X,Y)) = a(e^{-it}).\chi E(X,Y).$$

If $a \in A$, $\mu \in \mathfrak{a}_c^*$ let us write a^μ for $\exp(\mu(\log a))$.

<u>Lemma 3.3</u>. There exists a constant $C_I > 0$ such that for all $a \in A$ with $a^\alpha > C_I$ we have:

(i) $\mathrm{im}(\Gamma_I) \cap (P \cup a^{-1}\cdot P) = \emptyset$;

(ii) the multi-valued analytic map $y \to H(a,y)$ (cf. Theorem 1.20) has a branch $H_I(a,\cdot)$ over Γ_I that restricts to the real branch $H_r(a,\cdot)$ over $\Gamma_I|(\{0\}\times B_I)$ and to the branch $H_r(a,\cdot) - 2\pi i H_0$ over $\Gamma_I|(\{\pi\}\times B_I)$.

<u>Proof</u>. If $a \in A$, $\bar{n} \in \bar{N}$ we have $H(a\kappa(\bar{n})) = H(a\bar{n}) - H(\bar{n})$, hence:

$$H(a,\chi(\bar{n})) = H(a\bar{n}a^{-1}) - H(\bar{n}) + \log a.$$

Select an open neighbourhood V of e in $\bar{N}_c$ such that the real branch H_r of $H: \bar{N}_c\backslash S \to \mathfrak{a}_c$ extends holomorphically to V. In view of the compactness of $\mathrm{im}(\gamma_I)$ there exists a constant $C_I > 0$ such that $a\bar{n}a^{-1} \in V$ for all $a \in A$ with $a^\alpha > C_I$ and for all $\bar{n} \in \mathrm{im}(\gamma_I)$. This shows that for $a \in A$ with $a^\alpha > C_I$ we have $\mathrm{im}(\gamma_I) \cap a^{-1}Sa = \emptyset$. By Lemma 3.2 we have $\mathrm{im}(\gamma_I) \cap S = \emptyset$. Hence (i). Moreover if $a \in A$, $a^\alpha > C_I$ then the multi-valued analytic map $H(a,\cdot)$ has a branch $H_I(a,\cdot)$ over Γ_I, defined by:

$$H_I(a,\Gamma_I(y)) = H_r(a\gamma_I(y)a^{-1}) - H_I(\gamma_I(y)) + \log a \qquad (9)$$

(here $H_I(\cdot)$ denotes the branch given by Lemma 3.2). Consequently (ii) follows straightforwardly from the assertions of Lemma 3.2.

We end this section with a proof of (2). First we have to discuss briefly integrals of the type occurring in (2). So let B be an oriented m-dimensional C^∞ manifold with boundary, diffeomorphic to a compact ball in $\mathbb{R}^m$. Let $a,b \in \mathbb{R}$, $a < b$, and set $I = [a,b]$. Give $C = I\times B$ the product orientation and orient

$$\partial C = (\{a\}\times B) \cup (\{b\}\times B) \cup (I\times\partial B)$$

according to the outward normal vector. Now let X be a C^∞ manifold. A C^∞ map $\gamma\colon \partial C \to X$ will be called a cylinder cycle of dimension m in X. As in Chapter 2 we define the linear form $\int_\gamma \colon \Omega^m(X) \to \mathbb{C}$ by

$$\int_\gamma \alpha = \int_{\partial C} \gamma^*(\alpha)$$
$$= \int_{\{a\}\times B} \gamma^*(\alpha) + \int_{\{b\}\times B} \gamma^*(\alpha) + \int_{I\times\partial B} \gamma^*(\alpha).$$

We define integrals of type (2.1) in a similar way.

Recall the definition of the compact submanifold B_w of $\mathfrak{g}_{-\alpha}\times\mathfrak{g}_{-2\alpha}$ with boundary (cf. Section 3.1), and provide B_w with the orientation of the ambient space. Give $[0,\pi]\times B_w$ the product orientation and orient $\partial([0,\pi]\times B_w)$ according to the outward normal. Let Γ_w be the cylinder cycle $\partial([0,\pi]\times B_w) \to G_c/P_c$ defined by

$$\Gamma_w(t,(X,Y)) = a(e^{-it}).\chi_w E(X,Y). \qquad (10)$$

<u>Lemma 3.4</u>. Let C_I be as in Lemma 3.3. Then for all $a \in A$ with $a^\alpha > C_I$ we have:

(i) $\mathrm{im}(\Gamma_W) \cap (P \cup a^{-1}.P) = \emptyset$;

(ii) the multi-valued analytic map $H(a,.)$ has a branch $H_r(a,.)$ over Γ_W that restricts to the real branch $H_r(a,.)$ over $\Gamma_W|(\{0\}\times B_W)$ and to the branch $H_r(a,.) - 2\pi i H_0$ over $\Gamma_W|(\{\pi\}\times B_W)$.

Proof. Let $B = \chi E(B_I)$. B is a compact m-dimensional submanifold of G/P with boundary $\partial B = \chi E(\partial B_I) = \chi_W E(\partial B_W)$ (cf. Section 3.1). Fix a point $y \in \partial B$ and write $y = \chi E(X_I,Y_I) = \chi_W E(X_W,Y_W)$ with $(X_I,Y_I) \in \partial B_I$, $(X_W,Y_W) \in \partial B_W$. Then we have:

$$\Gamma_I(t,(X_I,Y_I)) = a(e^{-it}).y = \Gamma_W(t,(X_W,Y_W)). \qquad (11)$$

This shows that $\Gamma_W([0,\pi]\times\partial B_W) = \Gamma_I([0,\pi]\times\partial B_I) \subset \mathrm{im}(\Gamma_I)$. Since both $\Gamma_W(\{0\}\times B_W)$ and $\Gamma_W(\{\pi\}\times B_W)$ are contained in G/P whereas $\lambda(a)^{-1}$ leaves G/P invariant, (i) follows.

Consider the curve c: $[0,\pi] \to G_c/P_c$, $t \to \Gamma_W(t,(X_W,Y_W))$. By (11) we have

$$c(t) = \Gamma_I(t,(X_I,Y_I)),$$

hence, by Lemma 3.3, the branch of $H(a,.)$ obtained by continuation of the real branch $H_r(a,.)$ along c is equal to $H_r(a,.) - 2\pi i H_0$. This proves (ii). Observe that at a point $\Gamma_W(t,(X_W,Y_W))$ ($t \in [0,\pi]$, $(X_W,Y_W) \in \partial B_W$) the branch $H_W(a,.)$ equals the branch $H_I(a,.)$ given by Lemma 3.3.

Theorem 3.5. Let $C_I, H_I(.,.)$ and $H_W(.,.)$ be as in the Lemmas 3.3 and 3.4. If $\lambda \in \mathfrak{a}_c^*$ and if $a \in A$ is such that $a^\alpha > C_I$, then:

$$(e^{2\pi\lambda(H_0)} - 1)\phi_\lambda(a) = \sum_{s=I,W} \int_{\Gamma_s} e^{(i\lambda-\rho)H_s(a,y)}\overline{\omega}(y). \qquad (12)$$

Proof. We prove formula (12) by decomposing the integrals at its right hand side in integrals over $\Gamma_s|(\{0\}\times B_s)$, $\Gamma_s(\{\pi\}\times B_s)$ and $\Gamma_s|([0,\pi]\times\partial B_s)$ $(s = I,w)$.

First observe that the integrals over $\Gamma_s|([0,\pi]\times\partial B_s)$ $(s = I,w)$ cancel each other. This is seen as follows. As is easily verified, the maps $\Gamma_s|((0,\pi)\times\partial B_s)$ are embeddings into G_c/P_c. Their images are the same m-dimensional C^∞ submanifold Σ of G_c/P_c. As we saw in the proof of Lemma 3.4 the branches $H_I(a,.)$ and $H_w(a,.)$ coincide over Σ. However, the orientation of Σ induced by Γ_I is the opposite of the one induced by Γ_w, and so indeed the integrals cancel.

Next consider $\partial([0,\pi]\times B_I) = (\{0\}\times B_I) \cup (\{\pi\}\times B_I) \cup ([0,\pi]\times\partial B_I)$ with the orientation corresponding to the outward normal. Since $\{0\}\times B_I \to B_I$, $(0,y) \to y$ is orientation reversing whereas $\chi\circ E$ is orientation preserving, it follows that $\Gamma_I|(\{0\}\times B_I)$ is an orientation reversing diffeomorphism of $\{0\}\times B_I$ onto $B = \chi E(B_I)$. Consequently:

$$\int_{\Gamma_I|(\{0\}\times B_I)} e^{(i\lambda-\rho)H_I(a,y)}\overline{\omega}(y) = -\int_{\chi E(B_I)} e^{(i\lambda-\rho)H_r(a,y)}\overline{\omega}(y). \quad (13)$$

On the other hand if $(X,Y) \in B_I$ then $\Gamma_I(\pi,(X,Y)) = \chi E(-X,Y)$. The map $\{\pi\}\times B_I \to B_I$, $(\pi,y) \to y$ is orientation preserving whereas $B_I \to B_I$, $(X,Y) \to (-X,Y)$ changes the orientation by $(-1)^{m(\alpha)}$. Hence $\Gamma_I|(\{\pi\}\times B_I)$ is a diffeomorphism onto $\chi E(B_I)$, changing the orientation by $(-1)^{m(\alpha)}$. Since $H_I(a,.)$ equals $H_r(a,.) - 2\pi i H_0$ over $\Gamma_I|(\{\pi\}\times B_I)$ whereas $(i\lambda-\rho)(-2\pi i H_0) = 2\pi\lambda(H_0) + \pi i(m(\alpha) + 2m(2\alpha))$ we obtain:

$$\int_{\Gamma_I|(\{\pi\}\times B_I)} e^{(i\lambda-\rho)H_I(a,y)}\overline{\omega}(y) = e^{2\pi\lambda(H_0)}\int_{\chi E(B_I)} e^{(i\lambda-\rho)H_r(a,y)}\overline{\omega}(y). \quad (14)$$

Similarly (13) and (14) hold with I replaced by w. Combining all these results with (1) we obtain (12).

3.3 Asymptotic behaviour of the integrals over Γ_I, Γ_w

In this section we shall study the asymptotic behaviour of the integrals in (11) when $a^\alpha \to +\infty$. We will obtain two theorems (3.6 and 3.10) that will enable us to compare formula (11) with Harish-Chandra's asymptotic expansion for $\phi_\lambda(a)$.

Let us first consider the integral

$$\int_{\Gamma_I} e^{(i\lambda-\rho)H_I(a,y)}\bar{\omega}(y) \tag{15}$$

(we assume that $a \in A$, $a^\alpha > C_I$, C_I as in Lemma 3.3). Consider formula (9) for $H_I(a,.)$. Since

$$\chi^*(\bar{\omega}) = e^{-2\rho H(.)}\Omega$$

(cf. (21), Section 1.5), it follows that the integral (15) is equal to

$$a^{i\lambda-\rho}\int_{\gamma_I} e^{(i\lambda-\rho)H_r(a\bar{n}a^{-1})}e^{-(i\lambda+\rho)H_I(\bar{n})}\Omega(\bar{n}). \tag{16}$$

If $z \in \mathbb{C}$, $\bar{n} = \exp(X + Y)$, $X \in \mathfrak{g}_{-\alpha,c}$, $Y \in \mathfrak{g}_{-2\alpha,c}$, we write $z.\bar{n} = \exp(zX + z^2Y)$. With this notation we have

$$a\bar{n}a^{-1} = a^{-\alpha}.\bar{n} \quad (a \in A,\ \bar{n} \in \bar{N}_c).$$

Recall that the real branch $H_r(.)$ of H extends holomorphically to the neighbourhood V of e in $\bar{N}_c$ (cf. the proof of Lemma 3.3). Select an open neighbourhood U_I of 0 in $\mathbb{C}$ such that $z.\bar{n} \in V$ for $z \in U_I$, $\bar{n} \in \mathrm{im}(\gamma_I)$, and such that $a^\alpha > C_I$ for $a \in A$ with $a^{-\alpha} \in U_I$.

Then:

$$\Psi_I(\lambda,z) = \int_{\gamma_I} e^{(i\lambda-\rho)H_r(z.\bar{n})} e^{-(i\lambda+\rho)H_I(\bar{n})} \Omega(\bar{n}) \qquad (17)$$

is well defined for $z \in U_I$, and the map $\Psi_I\colon \mathfrak{a}_c^* \times U_I \to \mathbb{C}$ is holomorphic in both variables. We have proved the following theorem.

<u>Theorem 3.6</u>. There exists an open neighbourhood U_I of 0 in $\mathbb{C}$ such that the map $\Psi_I\colon \mathfrak{a}_c^* \times U_I \to \mathbb{C}$ defined by (17) is holomorphic, and if $a \in A$ is such that $a^{-\alpha} \in U_I$, then $a^\alpha > C_I$ and:

$$\int_{\Gamma_I} e^{(i\lambda-\rho)H_I(a,y)} \bar{\omega}(y) = a^{(i\lambda-\rho)} \Psi_I(\lambda,a^{-\alpha}). \qquad (18)$$

Next we turn our attention to the integral

$$\int_{\Gamma_w} e^{(i\lambda-\rho)H_w(a,y)} \bar{\omega}(y). \qquad (19)$$

If $a \in A$, $\bar{n} \in \bar{N}$ then $H(a\bar{w}\kappa(\bar{n})) = H(a^{-1}\kappa(\bar{n})) = H(a^{-1}\bar{n}) - H(\bar{n})$, whence

$$H(a,\chi_w(\bar{n})) = H(a^{-1}\bar{n}a) - H(\bar{n}) - \log a. \qquad (20)$$

Define the cylinder cycle $\gamma_w\colon \partial([0,\pi]\times B_w) \to \bar{N}_c$ by

$$\chi_w(t,(X,Y)) = \exp(e^{-it}X + e^{-2it}Y). \qquad (21)$$

We obviously have $\Gamma_w = \chi_w \circ \gamma_w$. It follows that the multi-valued analytic map $\phi(a)\colon \bar{N}_c \backslash (S \cup aSa^{-1}) \to \mathfrak{a}_c$ defined by

$$\phi(a,\bar{n}) = H(a^{-1}\bar{n}a) - H(\bar{n})$$

has a branch $\phi_w(a)$ over γ_w such that the integral (19) is equal to:

$$a^{-i\lambda+\rho} \int_{\gamma_w} e^{(i\lambda-\rho)\phi_w(a,\bar{n})} \chi_w^*(\bar{\omega})(\bar{n}) \tag{22}$$

If $a \in A$ then the map $Ad(a)\colon \bar{N}_c \to \bar{N}_c$, $\bar{n} \to Ad(a,\bar{n}) = a\bar{n}a^{-1}$ is a holomorphic diffeomorphism, and we have

$$Ad(a)^*\Omega = a^{-2\rho}\Omega.$$

Taking formula (21), Section 1.5 into account it follows that the the multi-valued analytic map $\psi(a) = Ad(a)^*(\phi(a))$ has a branch $\psi_w(a)$ over $Ad(a^{-1})\circ\gamma_w$ such that the integral (22) is equal to

$$a^{-i\lambda-\rho} \int_{Ad(a^{-1})\circ\gamma_w} e^{(i\lambda-\rho)\psi_w(a,\bar{n})} e^{-2\rho H(a\bar{n}a^{-1})} \Omega(\bar{n}) \tag{23}$$

Since $\psi(a,\bar{n}) = H(\bar{n}) - H(a\bar{n}a^{-1})$ we can treat the integral (23) as in the proof of Theorem 3.6, if we manage to replace the cycle $Ad(a^{-1})\circ\gamma_w$ by a fixed cycle without changing the value of the integral. We will do this by means of a homotopy $Ad(a(t)^{-1})\circ\gamma_w$. The singular set of the integrand of (23) is equal to $(S \cup Ad(a^{-1})S) \cap \bar{N}_c$, so the image of our homotopy has to be disjoint from this set.

<u>Proposition 3.7</u>. If $b \in A$ then the following statements are equivalent.

(i) $\mathrm{im}(\gamma_w) \cap Ad(b)S = \emptyset$;

(ii) $\mathrm{im}(\gamma_I) \cap Ad(b^{-1})S = \emptyset$.

<u>Proof</u>. Since $\lambda(b^{-1})\circ\chi_w = \chi_w\circ Ad(b)$ it follows that (i) is equivalent to $\mathrm{im}(\Gamma_w) \cap \lambda(b^{-1})P = \emptyset$. Now $\lambda(b^{-1})$ leaves G/P invariant, and since $(G/P) \cap P = \emptyset$ whereas $\mathrm{im}(\Gamma_w)\setminus(G/P) = \mathrm{im}(\Gamma_I)\setminus(G/P)$ it follows that (i) is equivalent to $\mathrm{im}(\Gamma_I) \cap \lambda(b^{-1})P = \emptyset$. The assertion now follows from the fact that $\lambda(b^{-1})\circ\chi_I = \chi_I\circ Ad(b^{-1})$.

Proposition 3.8. If $b \in A$ is such that $b^\alpha \leqslant 1$ or $b^\alpha > C_I$ then $\mathrm{im}(\gamma_W) \cap Ad(b)S = \emptyset$.

Proof. By the preceding proposition it suffices to show that in both cases $\mathrm{im}(\gamma_I) \cap Ad(b^{-1})S = \emptyset$.

First let $b \in A$, $b^\alpha \leqslant 1$. Fix $(X,Y) \in \partial B_I$ and let $t \in [0,\pi]$. Then

$$Ad(b)\gamma_I(t,(X,Y)) = \exp\,(e^{it}b^{-\alpha}X + e^{2it}b^{-2\alpha}Y)$$

and since $(b^{-\alpha}X,b^{-\alpha}X) + (b^{-2\alpha}Y,b^{-2\alpha}Y) \geqslant (X,X) + (Y,Y) = R$ it follows that $Ad(b)\gamma_I(t,(X,Y)) \notin S$ (cf. Section 3.2). We now easily obtain that $\mathrm{im}(\gamma_I) \cap Ad(b^{-1})S = \emptyset$.

Next let $b \in A$, $b^\alpha > C_I$. By Lemma 3.2 we have that $\mathrm{im}(\gamma_I) \cap Ad(b^{-1})S = \emptyset$.

Corollary 3.9. Let $a,b \in A$ be such that $C_I < b^\alpha < a^\alpha$. If $x \in A$ is such that $b^\alpha \leqslant x^\alpha \leqslant a^\alpha$, then:

$$\mathrm{im}(Ad(x^{-1})\circ\gamma_W) \cap (S \cup Ad(a^{-1})S) = \emptyset.$$

Proof. Observe that $x^\alpha > C_I$, that $(xa^{-1})^\alpha \leqslant 1$ and apply Proposition 3.8.

Let us return to the integral (23). Select a $b \in A$ such that $b > C_I$. From now on we assume that $a \in A$, $a^\alpha > b^\alpha$. Consider the homotopy $Ad(a(t)^{-1})\circ\gamma_W$ $(\alpha(\log b) \leqslant t \leqslant \alpha(\log a))$. In view of Corollary 3.9 its image is disjoint from $S \cup Ad(a^{-1})S$ and so the branch $\psi_W(a)$ extends to a branch over this homotopy; we denote it by $\psi_W(a)$ too. The integral

$$\int_{(a(t)^{-1})\circ\gamma_w} e^{(i\lambda-\rho)\psi_w(a,\bar{n})} e^{-2\rho H(a\bar{n}a^{-1})} \Omega(\bar{n})$$

does not depend on t (this might be proved by approximating γ_w by a sequence of homotopic smooth cycles; cf. also de Rham [1, §14]). It follows that the integral (19) is equal to (23) with $Ad(a^{-1})\circ\gamma_w$ replaced by $Ad(b^{-1})\circ\gamma_w$.

Now select a constant $C_w > b^\alpha$ such that $a\bar{n}a^{-1} \in V$ if $a \in A$, $a^\alpha > C_w$ and $\bar{n} \in \mathrm{im}(Ad(b^{-1})\circ\gamma_w)$. If $a \in A$, $a^\alpha > C_w$ then $\psi(a)$ has the branch $\psi_w(a)$ over $Ad(b^{-1})\circ\gamma_w$ and so H has the branch H_w defined by

$$H_w = \psi_w(a) + H_r\circ Ad(a)$$

over $Ad(b^{-1})\circ\gamma_w$. Select an open neighbourhood U_w of 0 in $\mathbb{C}$ such that $z.\bar{n} \in V$ if $z \in U_w$, $\bar{n} \in \mathrm{im}(Ad(b^{-1})\circ\gamma_w)$ and such that $a^{-\alpha} \in U_w \Rightarrow a^\alpha > C_w$. Then for $\lambda \in \mathfrak{a}_c^*$, $z \in U_w$ the integral

$$\Psi_w(\lambda,z) = \int_{Ad(b^{-1})\circ\gamma_w} e^{(i\lambda-\rho)H_w(\bar{n})} e^{-(i\lambda+\rho)H_r(z.\bar{n})} \Omega(\bar{n}) \tag{24}$$

is well defined, and the function $\Psi_w: \mathfrak{a}_c^*\times U_w \to \mathbb{C}$ is holomorphic. Moreover, if $a^{-\alpha} \in U_w$ then the integral (19) is equal to

$$a^{-i\lambda-\rho}\Psi_w(\lambda,a^{-\alpha}).$$

We have proved the following theorem.

<u>Theorem 3.10</u>. There exists an open neighbourhood U_w of 0 in $\mathbb{C}$ and a holomorphic function $\Psi_w: \mathfrak{a}_c^*\times U_w \to \mathbb{C}$ (cf. formula (24)), such that for $a \in A$ with $a^{-\alpha} \in U_w$ we have $a^\alpha > C_I$, and:

$$\int_{\Gamma_w} e^{(i\lambda-\rho)H_w(a,y)} \bar{\omega}(y) = a^{-i\lambda-\rho}\Psi_w(\lambda,a^{-\alpha}). \tag{25}$$

3.4 Harish-Chandra's formula

We define the holomorphic function d: $\mathfrak{a}_c^* \to \mathbb{C}$ by

$$d(\lambda) = e^{2\pi\lambda(H_0)} - 1.$$

With this notation we have the following easy consequence of Theorems 3.5, 3.6, 3.10.

Corollary 3.11. Let $\lambda \in \mathfrak{a}_c^*$, $a \in A$, $a^{-\alpha} \in U_I \cap U_w$. Then

$$d(\lambda)\phi_\lambda(a) = \sum_{s=I,w} a^{is\lambda-\rho}\Psi_s(\lambda,a^{-\alpha}) .$$

Since Ψ_s $(s=I,w)$ is holomorphic on $\mathfrak{a}_c^* \times U_s$ it has a power series expansion

$$\Psi_s(\lambda,z) = \sum_{n=0}^{\infty} b_{s,n}(\lambda)z^n,$$

valid for z is some neighbourhood of zero. The functions $b_{s,n}(\lambda) = [(d/dz)^n\Psi_s(\lambda,z)]_{z=0}$ depend holomorphically on λ. It follows that

$$\phi_\lambda(a) = \sum_{s=I,w} a^{is\lambda-\rho} \sum_{n=0}^{\infty} d(\lambda)^{-1}b_{s,n}(\lambda)a^{-n\alpha}$$

for $\lambda \in \mathfrak{a}_c^*$, $\lambda(H_0) \notin \mathbb{Z}i$. Obviously this series is an asymptotic expansion for $\phi_\lambda(a)$ $(a^\alpha \to \infty)$, and it necessarily corresponds to Harish-Chandra's asymptotic expansion for ϕ_λ (cf. [2], see also Section 4.4 for a more detailed discussion of this expansion). Following Harish-Chandra, we define the c-function to be the coefficient of the principal power $a^{i\lambda-\rho}$. Thus

$$c(\lambda) = d(\lambda)^{-1}\Psi_I(\lambda,0),$$

and we obtain the following theorem.

Theorem 3.12. For all $\lambda \in \mathfrak{a}_c^*$ with $\lambda(H_0) \notin \mathbb{Z}i$ we have

$$c(\lambda) = d(\lambda)^{-1} \int_{\gamma_I} e^{-(i\lambda+\rho)H_I(\bar{n})} \Omega(\bar{n}) \tag{26}$$

It is now possible to derive the usual integral formula for the c-function (cf. Harish-Chandra [2, Theorem 4, p. 291]). This is the subject of the following lemma.

Lemma 3.13. Let $\lambda \in \mathfrak{a}_c^*$, $\mathrm{Im}(\lambda(H_0)) < 0$. Then:

$$\int_{\gamma_I} e^{-(i\lambda+\rho)H_I(\bar{n})} \Omega(\bar{n}) = d(\lambda) \int_{\bar{N}} e^{-(i\lambda+\rho)H(\bar{n})} d\bar{n} \tag{27}$$

where $d\bar{n}$ denotes the Haar measure of $\bar{N}$ normalized by $\int_{\bar{N}} \exp(-2\rho H(\bar{n}))d\bar{n} = 1$. The integral on the right hand side of (27) is absolutely convergent and by (26) it equals $c(\lambda)$.

Proof. The argument follows the familiar pattern of estimation of contour integrals in the elementary theory of functions of one complex variable.

Define the C^∞ map $\Phi: [0,\infty]\times\bar{N}_c \to \bar{N}_c$ by

$$\Phi(t,\bar{n}) = Ad(a(-t),\bar{n}).$$

Consider the homotopy $\Phi_t \circ \gamma_I$ (here $\Phi_t(.) = \Phi(t,.)$). First observe that the image of $\Phi_t \circ \gamma_I$ is disjoint from S for $t \geq 0$ (this follows from the proof of Proposition 3.8). It follows that the branch H_I extends to a branch over $\Phi_t \circ \gamma_I$; we denote it by H_I as well. As in the proof of Theorem 3.10 the value of the integral

$$\int_{\Phi_t \circ \gamma_I} e^{-(i\lambda+\rho)H_I(\bar{n})} \Omega(\bar{n}) \tag{28}$$

is independent of t.

Now consider the integral

$$I(t) = \int_{\Phi_t \circ \gamma_I|([0,\pi]\times\partial B_I)} e^{-(i\lambda+\rho)H_I(\bar{n})}\,\Omega(\bar{n}).$$

Since $Ad(a(-t))^*\Omega = \exp(2\rho(H_0)t)\Omega$, the m-dimensional Euclidean measure of $\Phi_t \circ \gamma_I([0,\pi]\times\partial B_I)$ is $O(\exp 2\rho(H_0)t)$ for $t \to +\infty$. On the other hand, writing $\bar{n} = \bar{n}(t,\tau,X,Y)$ ($\tau \in [0,\pi]$, $(X,Y) \in \partial B_I$), we have

$$\begin{aligned}|e^{-(i\lambda+\rho)H_I(\bar{n})}| &= e^{(\mathrm{Re}\lambda)(\mathrm{Im}H_I(\bar{n}))}\, e^{(\mathrm{Im}\lambda-\rho)(\mathrm{Re}H_I(\bar{n}))}\\ &\leqslant e^{2\pi|\mathrm{Re}\lambda(H_0)|}\, e^{(\mathrm{Im}\lambda-\rho)(\mathrm{Re}H_I(\bar{n}))}\end{aligned}$$

Moreover, writing $a = c(X,X)$, $b = 4c(Y,Y)$, we have

$$\begin{aligned}\alpha(\mathrm{Re}H_I(\bar{n})) &= \tfrac{1}{2}\log|(1 + ae^{2t-2i\tau})^2 + be^{4t-4i\tau}|\\ &= 2t + O(1) \qquad (t \to \infty),\end{aligned}$$

uniformly in $(\tau,(X,Y)) \in [0,\pi]\times\partial B_I$. Hence

$$|I(t)| = O(e^{2\rho(H_0)t}\, e^{2(\mathrm{Im}\lambda-\rho)(H_0)t}),$$

showing that:

$$\lim_{t\to+\infty} I(t) = 0. \tag{29}$$

By the above estimates it also follows that the integral on the right hand side of (27) converges absolutely if $\mathrm{Im}\,\lambda(H_0) < 0$, and therefore, by dominated convergence,

$$\lim_{t\to+\infty} \int_{\Phi_t(B_I)} e^{-(i\lambda+\rho)H(\bar{n})}\,d\bar{n} = \int_{\bar{N}} e^{-(i\lambda+\rho)H(\bar{n})}\,d\bar{n}. \tag{30}$$

Moreover by an argument similar to the one in the proof of Theorem 3.5 it follows that:

$$\int_{\Gamma_t \circ \gamma_I | \mathcal{B}} e^{(i\lambda-\rho)H_I(\bar{n})} \Omega(\bar{n}) = d(\lambda). \int_{\Phi_t(B_I)} e^{-(i\lambda+\rho)H(\bar{n})} d\bar{n} \tag{31}$$

where $\mathcal{B} = (\{0\}\times B_I) \cup (\{\pi\}\times B_I)$. Hence using that the integral (28) is independent of t and decomposing it in a sum of I(t) and the integral on the left hand side of (31), we obtain (27) by application of (29) and (31).

By formula (26) it follows that the c-function is meromorphic; its poles are contained in the set $\{\lambda \in \mathfrak{a}_c^*;\ \lambda(H_0) \in \mathbb{Z}i\}$ and are all at most of first order. This agrees with the formula (cf. Harish-Chandra [2], p.303)

$$c(\lambda) = \frac{c_0 . \Gamma(i\lambda_0)}{(\frac{1}{2}(\frac{1}{2}m(\alpha) + 1 + i\lambda_0))\Gamma(\frac{1}{2}(\frac{1}{2}m(\alpha) + m(2\alpha) + i\lambda_0))}$$

where we have written $\lambda_0 = \lambda(H_0)$, where Γ denotes the classical Gamma function having no zeros and having poles at 0, -1, -2, .., and where c_0 is some non-zero constant.

Remark. By an explicit computation formula (32) can be deduced from (3) and the integral on the right hand side of (27). This has been done by Helgason (cf. [2]) and Schiffmann (cf. [1]).

Chapter 4
The general case

4.1 Introduction

In this chapter we shall study the asymptotic behaviour of $\phi_\lambda(a)$, when $a^\alpha \to +\infty$ for all $\alpha \in \Delta^{++}$ (recall that $a^\alpha =$ $= \exp(\alpha \log a)$). We shall express the latter condition more briefly as: "$a \to \infty$ in A^+".

Fix $C_0 > 0$, and put

$$A(C_0) = \{a \in A;\ a^\alpha > C_0 \text{ for all } \alpha \in \Delta^{++}\}. \tag{1}$$

As we will show, if C_0 is big enough, then there exist smooth dim(K)-cycles $\Gamma_w(a)$ $(w \in W)$, depending smoothly on $a \in A(C_0)$, such that the following holds. The map H has a branch $H_{0,w}$ over the map $(a,y) \to a\Gamma_w(a)(y)$, $A(C_0)\times Y(w) \to G_c\backslash S$, and the functions $\phi_{w,\lambda}\colon A(C_0) \to \mathbb{C}$ defined by

$$\phi_{w,\lambda}(a) = \int_{\Gamma_w(a)} e^{(i\lambda-\rho)H_{0,w}(ak)}\,\omega(k) \tag{2}$$

form a basis of the space $\mathcal{E}_\lambda(A(C_0))$ (cf. Theorem 2.8), whenever λ lies in the complement $\mathfrak{a}^*_{c,0}$ of a certain locally finite union of hyperplanes. For this result, see Theorems 4.21 and 4.23. Moreover, as we will show in Section 4.4, the function $\phi_{w,\lambda}$ has an asymptotic expansion for $a \to \infty$ in A^+ (see Theorem 4.20). In Section 4.5, the principal term of this expansion will appear to be given by

$$\phi_{w,\lambda}(a) \sim d(w\lambda)c(w\lambda)a^{iw\lambda-\rho}. \tag{3}$$

Here d is the holomorphic function $\mathfrak{a}_c^* \to \mathbb{C}$ given by

$$d(\lambda) = \prod_{\alpha\in\Delta^{++}} \{\exp(2\pi(\lambda,\alpha)(\alpha,\alpha)^{-1})-1\}$$

(where (,) denotes the dual of the bilinear form (,) on $\mathfrak{a}_c$), and c denotes the c-function of Harish-Chandra. Let us recall that c is a meromorphic function $\mathfrak{a}_c^* \to \mathbb{C}$. For $\lambda \in \mathfrak{a}_c^*$ with $\mathrm{Im}(\lambda,\alpha) < 0$ $(\alpha \in \Delta^{++})$ it is given by the absolutely convergent integral

$$c(\lambda) = \int_{\overline{N}} e^{-(i\lambda+\rho)H(\overline{n})}d\overline{n}. \qquad (4)$$

Here $d\overline{n}$ is the Haar measure of $\overline{N}$ normalized by $\int_{\overline{N}} \exp(-2\rho H(\overline{n}))d\overline{n} = 1$. Since for $\lambda \in \mathfrak{a}_{c,0}^*$, the functions $\phi_{w,\lambda}$ $(w \in W)$ span $\mathcal{E}_\lambda(A(C_0))$, ϕ_λ must be a linear combination of the $\phi_{w,\lambda}$ with coefficients depending holomorphically on $\lambda \in \mathfrak{a}_{c,0}^*$. By (2) and a result of Harish-Chandra concerning the behaviour of $\phi_\lambda(a)$ when $a \to \infty$ in A^+ (see the proof of Lemma 4.25) these coefficients must be equal to $d(w\lambda)^{-1}$, and we obtain:

$$\phi_\lambda = \sum_{w \in W} d(w\lambda)^{-1}\phi_{w,\lambda}. \qquad (5)$$

For this result, see Section 4.5, where also a detailed comparison with Harish-Chandra's asymptotic expansion for ϕ_λ will be made.

The construction of the cycles $\Gamma_w(a)$ will be carried out in Section 4.4. It is based on the following idea. Fix $w \in W$, and consider the real analytic map $M \times \overline{N} \to K$, $(m,\overline{n}) \to \overline{w}m\kappa(\overline{n})$ (recall that $\overline{w}$ is a fixed representative for w). The pull back of $k \to H(ak)$ under this map is equal to

$$(m,\overline{n}) \to H(Ad(w^{-1}(a),\overline{n})) - H(\overline{n}) + w^{-1}(\log a), \qquad (6)$$

where $w^{-1}(a) = \bar{w}^{-1}a\bar{w}$, and where $Ad(b,\bar{n}) = Ad(b)(\bar{n}) = b\bar{n}b^{-1}$ ($b \in A$, $\bar{n} \in \bar{N}$). If $s \in W$, we set $\bar{N}_s = \bar{N} \cap \bar{s}^{-1}N\bar{s}$, $\Delta^{++}(s) = \{\alpha \in \Delta^{++};\ s(\alpha) \in -\Delta^{++}\}$. Writing $\bar{\mathfrak{n}}_s$ for the Lie algebra of $\bar{N}_s$, and $\mathfrak{n}_\alpha$ for $\mathfrak{g}_{-\alpha} + \mathfrak{g}_{-2\alpha}$ ($\alpha \in \Delta^{++}$), we have:

$$\bar{\mathfrak{n}}_s = \sum_{\alpha\in\Delta^{++}(s)} \bar{\mathfrak{n}}_\alpha,$$

and so $Ad(w^{-1}))|\bar{\mathfrak{n}}_w$ diagonalizes with eigenvalues $a^{-w\alpha}$ ($\alpha \in \Delta^{++}(w)$) tending to $+\infty$ if $a \to \infty$ in A^+. Let w' be the unique element of W with $(w')^{-1}(\mathfrak{a}^+) = -w^{-1}(\mathfrak{a}^+)$. Then Δ^{++} is the disjoint union of $\Delta^{++}(w)$ and $\Delta^{++}(w')$, and so $Ad(w^{-1}(a))|\bar{\mathfrak{n}}_{w'}$ diagonalizes with eigenvalues $a^{-w\alpha}$ ($\alpha \in \Delta^{++}(w')$) tending to 0 if $a \to \infty$ in A^+. Moreover, the map $\bar{N}_w \times \bar{N}_{w'} \to \bar{N}$, $(\bar{n},\bar{n}') \to \bar{n}\,\bar{n}'$ is a real analytic diffeomorphism. To get (6) under control, we apply the transform $Ad(w^{-1}(a))^{-1}$ to $\bar{N}_w$. More precisely, let ξ be the real analytic map $M \times \bar{N}_w \times \bar{N}_{w'} \to K$, $(m,\bar{n},\bar{n}') \to \bar{w}m\kappa(\bar{n}\,\bar{n}')$ and let for $a \in A(C_0)$ the map η_a: $M \times \bar{N}_w \times \bar{N}_{w'} \to M \times \bar{N}_w \times \bar{N}_{w'}$ be given by $\eta_a(m,\bar{n},\bar{n}') = (m, Ad(w^{-1}(a^{-1}),\bar{n}),\bar{n}')$. The pull back of $k \to H(ak) - w^{-1}(\log a)$ under $\xi\circ\eta_a$ is equal to:

$$(m,\bar{n},\bar{n}') \to H(\bar{n}Ad(w^{-1}(a),\bar{n}')) - H(Ad(w^{-1}(a^{-1}),\bar{n})\bar{n}'). \qquad (7)$$

Now if $(\bar{n},\bar{n}')$ varies in a compact subset of $\bar{N}_{w,c} \times \bar{N}_{w',c}$ then $\bar{n}Ad(w^{-1}(a),\bar{n}') \to \bar{n}$ and $Ad(w^{-1}(a^{-1}),\bar{n})\bar{n}' \to \bar{n}'$ when $a \to \infty$ in A^+, uniformly with respect to $(\bar{n},\bar{n}')$. So the multi-valued analytic extension of (7) can be seen as a perturbation of the multi-valued analytic map

$$(m,\bar{n},\bar{n}') \to H(\bar{n}) - H(\bar{n}').$$

We have now come to the fundamental idea behind the construction of $\Gamma_w(a)$. If γ_s $(s = w,w')$ are smooth $\dim(\overline{N}_s)$-cycles in $\overline{N}_{s,c}\backslash S$ such that H has a branch H_s over each γ_s $(s = w,w')$, then the following is true. If C_0 is chosen big enough, and if a varies in $A(C_0)$, then the branch

$$H_w(\overline{n}Ad(w^{-1}(a),\overline{n}')) - H_{w'}(Ad(w^{-1}(a^{-1}),\overline{n})\overline{n}')$$

of (7) is well defined over $id(M)\times\gamma_w\times\gamma_{w'}$. Moreover, the multi-valued analytic extension of $\xi\circ\eta_a$ has a corresponding branch $(\xi\circ\eta_a)_0$ over $id(M)\times\gamma_w\times\gamma_{w'}$ and H has a corresponding branch $H_{0,w}$ over the map $A(C_0)\times Y(w)\to G_c\backslash S$, $(a,y)\to a\Gamma_w(a)(y)$, where $Y(w) = M\times Y_w\times Y_{w'}$, and where $\Gamma_w(a) = (\xi\circ\eta_a)_0 \circ (id(M)\times\gamma_w\times\gamma_{w'})$. Also, the function $\phi_{w,\lambda}$ defined by (2) satisfies

$$\phi_{w,\lambda}(a) = a^{iw\lambda-\rho}\Phi_w'(\lambda,(a^{-\alpha})_{\alpha\in\Delta++}),$$

where Φ_w' is a complex valued function, holomorphic in the first variable, and holomorphic at 0 in the second variable. This provides us with an asymptotic expansion for $\phi_{w,\lambda}(a)$, the principal term being given by

$$\Phi_{w,\lambda}(a) \sim a^{iw\lambda-\rho}\Phi_w'(\lambda,0),$$

with

$$\Phi_w'(\lambda,0) = K_w \int_{\gamma_w\times\gamma_{n'}} e^{(i\lambda-\rho)H(\overline{n})} e^{-(i\lambda+\rho)H(\overline{n}')} \Omega_{w,0}\wedge\Omega_{w',0}.$$

Here $\Omega_{s,0}$ $(s = w,w')$ are certain invariant differential $\dim(\overline{N}_s)$-forms on $\overline{N}_{s,c}$, and K_w is some non-zero constant depending on their normalizations. For the above result, see Theorem 4.20.

As we saw above, the cycles γ_w and $\gamma_{w'}$ should be constructed so that $\Phi_w'(\lambda,0) = d(w\lambda)c(w\lambda)$. The second idea upon which our construction hinges is to adapt the procedure of Gindikin and Karpelevič to achieve this. Originally Gindikin and Karpelevič used this procedure to obtain a product formula for the c-function (cf. [1]). We have extended this procedure to a multi-valued analytic situation, and use it to construct γ_w (and $\gamma_{w'}$) recurrently as a product of cycles corresponding to real rank 1 subgroups. Let us now briefly discuss the idea that is worked out in detail in Section 4.3. If $s \in W$, we set $n(s) = \#\Delta^{++}(s)$. We construct γ_w by induction on $n(w)$. Let $\alpha \in \Delta^{++}(w)$ be such that $-w(\alpha)$ is a simple root, and put $v = w \circ s_\alpha$ (here s_α denotes the orthogonal reflection in the root plane $\ker\alpha$ in $\mathfrak{a}$). Now we have that $\Delta^{++}(w) = \Delta^{++}(v) \cup \{\alpha\}$ (disjoint union) and $n(w) = n(v) + 1$. Write $\overline{N}_\alpha = \exp(\mathfrak{n}_\alpha)$. Then there exists a real analytic diffeomorphism $\Psi_{w,v}$: $\overline{N}_\alpha \times \overline{N}_v \to \overline{N}_w$ such that:

$$H(\Psi_{w,v}(\overline{n}_\alpha, \overline{n}_v)) = H(\overline{n}_\alpha) + H(\overline{n}_v),$$

for $\overline{n}_\alpha \in \overline{N}_\alpha$, $\overline{n}_v \in \overline{N}_v$. Moreover, $\Psi_{w,v}(\overline{n}_\alpha, \overline{n}_v)$ is a holomorphic expression in $\overline{n}_\alpha, \overline{n}_v$, $\nu(\overline{n}_\alpha)$, $h(\overline{n}_\alpha)$ and therefore has a multi-valued analytic extension to $(\overline{N}_{\alpha,c} \setminus S) \times \overline{N}_{v,c}$. Also, if γ_α: $Y_\alpha \to \overline{N}_{\alpha,c} \setminus S$ and γ_v: $Y_v \to \overline{N}_{v,c} \setminus S$ are smooth cycles such that H has branches H_α, H_v over γ_α and γ_v respectively, then $\Psi_{w,v}$ has a branch $\Psi_{w,v,0}$ over $\gamma_\alpha \times \gamma_v$, and H has a branch H_w over $\gamma_w = \Psi_{w,v,0} \circ (\gamma_\alpha \times \gamma_v)$ such that

$$H_w \circ \gamma_w(y_\alpha, y_v) = H_\alpha \circ \gamma_\alpha(y_\alpha) + H_v \circ \gamma_v(y_v).$$

However, from Chapter 3 it follows readily that for each $\beta \in \Delta^{++}$ we may select a smooth $\dim(\overline{N}_\beta)$-cycle γ_β: $Y_\beta \to \overline{N}_{\beta,c} \setminus S$ and a branch H_β of H over γ_β, such that

$$\int_{\gamma_\beta} e^{-(i\lambda+\rho_\beta)H_\beta(\bar{n})}\,\Omega_\beta(\bar{n}) = d_\beta(\lambda)c_\beta(\lambda_\beta). \qquad (8)$$

Here $\rho_\beta = (\frac{1}{2}m(\beta) + m(2\beta))\beta$, $d_\beta(\lambda) = \exp[2\pi(\lambda,\beta)(\beta,\beta)^{-1}] - 1$, Ω_β is a certain invariant $\dim(\bar{N}_\beta)$-form on $\bar{N}_{\beta,c}$, c_β is the c-function associated with a certain real rank 1 group G^β, and finally λ_β denotes the restriction of λ to the $\mathfrak{a}$-part $\mathfrak{a}^\beta$ of $\mathfrak{g}^\beta$, the Lie algebra of G^β (see also Section 4.3). Constructing γ_w recurrently as outlined above, and computing Jacobians in the usual way (the computations do not go beyond holomorphic extensions of the standard computations) we end up with a formula expressing the integral

$$\tilde{I}_w(\lambda) = \int_{\gamma_w} e^{-(i\lambda+\rho)H_w(\bar{n})}\,\Omega_{w,0}(\bar{n})$$

up to a non-zero factor as a product of the integrals in (8), the product being taken over $\Delta^{++}(w)$. With this construction we indeed find that

$$\Phi_w'(\lambda,0) = K_w\tilde{I}_w(-\lambda)\tilde{I}_{w'}(\lambda) = K''d(w\lambda)c(w\lambda),$$

for some $K'' \in \mathbb{C}\setminus\{0\}$. In fact, as one would expect from formal computations, the constant K'' turns out to be 1. We had to keep track of the various multiplicative factors coming from the different normalizations of forms in the course of the construction to establish this.

Resuming, we have organized Chapter 4 as follows. In Section 4.2 we give an exposition of some techniqual prerequisites. In the next section we describe the Gindikin-Karpelevič procedure and the construction of the cycles γ_w and in

Section 4.4 we present the construction of the cycles $\Gamma_w(a)$. Finally, in Section 4.5 we derive formula (5) for ϕ_λ and we compare it with Harish-Chandra's formula.

4.2. Preparations

In this section we discuss some preliminaries on the structure of the algebra $\mathfrak{n}_c$. We start with the following well known lemma.

Lemma 4.1. Let L be a (real or complex) Lie group with a nilpotent Lie algebra $\mathfrak{l}$, such that exp: $\mathfrak{l} \to L$ is a diffeomorphism, and suppose that $\mathfrak{l} = \mathfrak{l}_0 \supsetneq \cdots \supsetneq \mathfrak{l}_m = 0$ is a strictly descending chain of subspaces with $[\mathfrak{l}, \mathfrak{l}_i] \subset \mathfrak{l}_{i+1}$ $(0 \leqslant i < m)$. Moreover, let $\mathfrak{b}$ and $\mathfrak{c}$ be subalgebras of $\mathfrak{l}$ such that $\mathfrak{l}_i = (\mathfrak{l}_i \cap \mathfrak{c}) \oplus (\mathfrak{l}_i \cap \mathfrak{c})$ (direct sum of linear subspaces, $0 \leqslant i \leqslant m$) and set $B = \exp \mathfrak{b}$, $C = \exp \mathfrak{c}$. Then B,C are closed subgroups of L and the map $B \times C \to L$, $(b,c) \to bc$ is a diffeomorphism.

Proof. If $\mathfrak{d}$ is any subalgebra of $\mathfrak{l}$, then $\exp \mathfrak{d} \exp \mathfrak{d} \subset \exp \mathfrak{d}$ by the Baker-Campbell-Hausdorff (BCH) formula for the nilpotent group L, and so $\exp \mathfrak{d}$ is a closed subgroup of L. Write $L_i = \exp \mathfrak{l}_i$, $\mathfrak{b}_i = \mathfrak{b} \cap \mathfrak{l}_i$, $\mathfrak{c}_i = \mathfrak{c} \cap \mathfrak{l}_i$, $B_i = \exp \mathfrak{b}_i$, $C_i = \exp \mathfrak{c}_i$ $(0 \leqslant i \leqslant m)$. Then L_i, B_i, C_i are closed subgroups of L. We shall prove inductively that $L_i = B_i C_i$. The assertion is trivial for $i = m$. Now fix $0 \leqslant j < m$ and assume that the assertion has been proved already for $i = j + 1$. It now suffices to show that $C_j B_j \subset B_j C_j$, for then $B_j C_j$ is an analytic subgroup of L_j with Lie algebra $\mathfrak{b}_j + \mathfrak{c}_j = \mathfrak{l}_j$. Since L_j is connected this implies that $B_jC_j = L_j$. So fix $b \in B_j$, $C \in C_j$. Again by the BCH formula we have $b^{-1}cbc^{-1} \in L_{j+1}$. By the induction hypothesis it follows that there exist $b' \in B_{j+1}$ and $C' \in C_{j+1}$ such that $b^{-1}cbc^{-1} =$ $= b'c'$. Consequently $cb = bb'c'c \in B_jC_j$.

Now let L,B,C be as in Lemma 4.1. Since $(b,c) \to bc$, $B\times C \to L$ is a diffeomorphism, the restriction η of the canonical map $L \to L/C$ to B is a diffeomorphism $B \to L/C$. Consider the action λ of L on L/C given by $\lambda_x(yC) = xyC$ $(x,y \in L)$ and let τ be the action of L on B defined by:

$$\eta\circ\tau(x) = \lambda_x\circ\eta \qquad (x \in L). \tag{9}$$

Note that $\tau(b) = \lambda_b$ if $b \in B$. Since L,C are nilpotent, we have $\det(\mathrm{Ad}_L(c)) = \det(\mathrm{Ad}_C(c)) = 1$, so by Lemma 1.15, there exists a non zero L-invariant $\dim(\mathfrak{l}/\mathfrak{c})$-form $\Omega_{L/C}$ on L/C. Now

$$\Omega'_B = \eta^*(\Omega_{L/C})$$

is a non-zero differential form of degree $\dim(\mathfrak{l}/\mathfrak{c}) = \dim(\mathfrak{b})$ on B. By (9) and the L-invariance of $\Omega_{L/C}$ the form Ω'_B is $\tau(L)$-invariant. In particular it is left B-invariant, and hence every invariant $\dim(\mathfrak{b})$-form on B is a complex scalar times Ω'_B. Thus we obtain:

<u>Lemma 4.2</u>. Let Ω_B be any invariant $\dim(\mathfrak{b})$-form on B. Then for every $x \in L$ we have

$$\tau(x)^*\Omega_B = \Omega_B.$$

Let us now return to the Lie algebra $\mathfrak{n}_C$ and fix an element $w \in W$. Recall that $\overline{w}$ is a representative for w.

Lemma 4.3. Let $\mathfrak{l}$, $\mathfrak{b}$, $\mathfrak{c}$ be subalgebras of $Ad(\bar{w})\mathfrak{n}$ (or $Ad(\bar{w})\mathfrak{n}_c$) such that:

(i) each of $\mathfrak{l}$, $\mathfrak{b}$, $\mathfrak{c}$ is a sum of rootspaces $\mathfrak{g}_\alpha$ ($\mathfrak{g}_{\alpha,c}$);

(ii) $\mathfrak{l} = \mathfrak{b} \oplus \mathfrak{c}$.

Write $L = \exp \mathfrak{l}$, $B = \exp \mathfrak{b}$, $C = \exp \mathfrak{c}$. Then L, B, C are closed subgroups of $\exp(Ad(\bar{w})\mathfrak{n})$ ($\exp(Ad(\bar{w})\mathfrak{n}_c)$), and the map $B\times C \to L$, $(b,c) \to bc$ is a diffeomorphism. Now let τ be the action of L on B defined by

$$\tau(x)(b) \equiv xb \pmod C \qquad (x \in L,\ b \in B). \tag{10}$$

If Ω_B is any invariant $\dim(\mathfrak{b})$-form on B, then for every $x \in L$ we have:

$$\tau(x)^*\Omega_B = \Omega_B. \tag{11}$$

Proof. Of course it suffices to prove the lemma for $w = I$. Now select a system of linear coordinates for $\mathfrak{a}^*$ such that the corresponding lexicographic ordering $<$ satisfies $\alpha \in \Delta^+ \Rightarrow 0 < \alpha$. Define $\Sigma = \{\alpha \in \Delta^+;\ \mathfrak{g}_\alpha \subset \mathfrak{l}\}$ and let $\alpha(1) < \dots < \alpha(p)$ be the numbering of Σ induced by $<$. Put

$$\mathfrak{l}_j = \Sigma_{i>j} (\mathfrak{g}_{\alpha(j),c} \cap \mathfrak{l}) \qquad (0 \leqslant j \leqslant p).$$

Since $[\mathfrak{g}_\alpha,\mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ whereas $\alpha + \beta > \beta$ $(\alpha,\beta \in \Delta^+)$ the descending chain $\mathfrak{l} = \mathfrak{l}_0 \supsetneq \mathfrak{l}_1 \supsetneq \dots \supsetneq \mathfrak{l}_p = 0$ satisfies the condition $[\mathfrak{l},\mathfrak{l}_i] \subset \mathfrak{l}_{i+1}$. Moreover, $\mathfrak{b}$, $\mathfrak{c}$ are direct sums of rootspaces, and $\mathfrak{b} \oplus \mathfrak{c} = \mathfrak{l}$, whence $\mathfrak{l}_j = (\mathfrak{b} \cap \mathfrak{l}_j) \oplus (\mathfrak{c} \cap \mathfrak{l}_j)$ $(0 \leqslant j \leqslant p)$. The lemma now follows by application of Lemmas 4.1 and 4.2.

4.3. An analytic extension of the Gindikin-Karpelevič procedure

Recall that Δ^{++} denotes the set of $\alpha \in \Delta^+$ with $\frac{1}{2}\alpha \notin \Delta^+$. If

$w \in W$ we write $\bar{\mathfrak{n}}_w$ for the subalgebra $\bar{\mathfrak{n}} \cap \mathrm{Ad}(\bar{w}^{-1})\mathfrak{n}$ of $\mathfrak{g}$ (it is independent of the particular choice of the representative $\bar{w}$ of w), we write $\Delta^+(w)$ for the set of $\alpha \in \Delta^+$ with $w(\alpha) \in -\Delta^+$, and $\Delta^{++}(w)$ for $\Delta^{++} \cap \Delta^+(w)$ (these are Schiffmann's notations, cf. [1]). Writing $\mathfrak{n}_\alpha = \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$ and $\bar{\mathfrak{n}}_\alpha = \mathfrak{g}_{-\alpha} + \mathfrak{g}_{-2\alpha}$ if $\alpha \in \Delta^{++}$, we have

$$\bar{\mathfrak{n}}_w = \bar{\mathfrak{n}} \cap \mathrm{Ad}(\bar{w}^{-1})\mathfrak{n} = \sum_{\alpha \in \Delta^{++}(w)} \bar{\mathfrak{n}}_\alpha .$$

Put $\bar{N}_\alpha = \exp(\bar{\mathfrak{n}}_\alpha)$ and $\bar{N}_w = \exp(\bar{\mathfrak{n}}_w)$. Then $\bar{N}_w = \bar{N} \cap \bar{w}^{-1}N\bar{w}$.

Let $w,v \in W$. We call w,v adjacent (cf. also Harish-Chandra [4, p. 120]) iff the Weyl chambers $w^{-1}(\mathfrak{a}^+)$ and $v^{-1}(\mathfrak{a}^+)$ are adjacent, i.e. they have a wall $\ker(\alpha)$ ($\alpha \in \Delta^{++}$) of maximal dimension in common. Now observe that s_α, the orthogonal reflection in $\ker(\alpha)$, maps $v^{-1}(\mathfrak{a}^+)$ onto $w^{-1}(\mathfrak{a}^+)$. Consequently $w = v \circ s_\alpha$. Interchanging the roles of v,w if necessary we may assume that α is strictly positive on $v^{-1}(\mathfrak{a}^+)$, and then it is strictly negative on $w^{-1}(\mathfrak{a}^+)$. For a general $s \in W$ we have

$$\Delta^{++}(s) = \{\beta \in \Delta^{++};\ \beta < 0 \text{ on } s^{-1}(\mathfrak{a}^+)\},$$

and therefore:

$$\Delta^{++}(w) = \Delta^{++}(v) \cup \{\alpha\} \qquad \text{(disjoint union).}$$

So writing $n(s)$ for $\#\Delta^{++}(s)$ ($s \in W$), we have in particular that $n(w) = n(v) + 1$.

Before proceeding let us briefly introduce some notations. If L is a connected subgroup of G with Liealgebra $\mathfrak{l}$, we write $\mathfrak{l}_c$ for the complexification of $\mathfrak{l}$ in $\mathfrak{g}_c$ and L_c for the connected subgroup of G_c with Lie algebra $\mathfrak{l}_c$. If $s \in W$ (or $s \in \Delta^{++}$), and if $\bar{N}_s$ is provided with some orientation, we denote the invariant

$\dim(\overline{N}_s)$-form on $\overline{N}_s$ associated with that orientation and the inner product (,) on $\overline{\mathfrak{n}}_s$ by $\Omega_{s,0}$ (so $\Omega_{s,0}$ is determined by the condition that $(\Omega_{s,0})_e\ (\xi_1,\ldots,\xi_d) = 1$ for an oriented orthonormal basis $\xi_1,\ldots\xi_d$ of $\overline{\mathfrak{n}}_s$).

Now let w,v be two adjacent elements of W, and suppose that $n(w) = n(v) + 1$. Let α be the unique element of $\Delta^{++}(w)\backslash\Delta^{++}(v)$. We have a direct sum of vector spaces

$$\overline{\mathfrak{n}}_w = \overline{\mathfrak{n}}_\alpha + \overline{\mathfrak{n}}_v,$$

so, by Lemma 4.4, the map $(\overline{n},\overline{n}') \to \overline{n}\ \overline{n}'$, $\overline{N}_\alpha\times\overline{N}_v \to \overline{N}_w$ is a diffeomorphism. Assume that $\overline{N}_\alpha$, $\overline{N}_v$, $\overline{N}_w$ have orientations such that this map is orientation preserving.

Proposition 4.4. The map $(\overline{n},\overline{n}') \to \overline{n}\ \overline{n}'$, $\overline{N}_{\alpha,c}\times\overline{N}_{v,c} \to \overline{N}_{w,c}$ is a holomorphic diffeomorphism; the form $\Omega_{\alpha,0}\wedge\Omega_{v,0}$ corresponds to $\Omega_{w,0}$ under this diffeomorphism.

Notation. In the assertion of Proposition 4.4 we have used the following notation. Let X_i $(1 \leqslant i \leqslant t)$ be C^∞ manifolds and let $X = X_1\times\ldots\times X_t$. Denote the natural projection $X \to X_i$ by π_i. If ξ_i are differential forms on X_i respectively $(1 \leqslant i \leqslant t)$, we write $\xi_1\wedge\ldots\wedge\xi_t$ for the form $\pi_1^*(\xi_1)\wedge\ldots\wedge\pi_t^*(\xi_t)$ on X, and we call it the product of $\xi_1,\ldots,\xi_t$.

Proof of Proposition 4.4. By Lemma 4.3 the map ϕ: $(\overline{n},\overline{n}') \to \overline{n}\ \overline{n}'$, $\overline{N}_{\alpha,c}\times\overline{N}_{v,c} \to \overline{N}_{w,c}$ is a holomorphic diffeomorphism. Clearly $(\phi^{-1})^*(\Omega_{\alpha,0}\wedge\Omega_{v,0})$ is a holomorphic left $\overline{N}_{\alpha,c}$-invariant and right $\overline{N}_{v,c}$-invariant form of dimension $\dim(\overline{N}_w)$ on $\overline{N}_{w,c}$. By nilpotence of $\overline{N}_{w,c}$ the form $\Omega_{w,0}$ is bi-invariant. Consequently

$(\phi^{-1})^*(\Omega_{\alpha,0}\wedge\Omega_{V,0}) = C.\Omega_{W,0}$ for some $C \in \mathbb{C}\setminus\{0\}$. Now the derivative $d\phi(e,e)\colon \bar{\mathfrak{n}}_{\alpha,c}\times\bar{\mathfrak{n}}_{V,c} \to \bar{\mathfrak{n}}_{W,c}$ of ϕ at (e,e) is given by $(\xi,\eta) \to \xi + \eta$ and we see that $C = 1$.

Consider the nilpotent algebra $\mathfrak{l} = \mathrm{Ad}(\bar{v}^{-1})\mathfrak{n}_c$ with its subalgebras $\mathfrak{b} = \bar{\mathfrak{n}}_{V,c} = \bar{\mathfrak{n}}_c \cap \mathfrak{l}$ and $\mathfrak{c} = \mathfrak{n}_c \cap \mathfrak{l}$. These algebras satisfy all conditions of Lemma 4.3. Now let τ_V be the action of $L = \bar{v}^{-1}N_c\bar{v}$ on $B = \bar{N}_{V,c}$ defined by

$$\tau_V(x)(\bar{n}) \equiv x\bar{n} \bmod C, \tag{12}$$

where $C = \exp \mathfrak{c} = N_c \cap \bar{v}^{-1}N_c\bar{v}$. Applying Lemma 4.3 we obtain:

Lemma 4.5. If $x \in \bar{v}^{-1}N_c\bar{v}$, then

$$\tau_V(x)^*(\Omega_{V,0}) = \Omega_{V,0}. \tag{13}$$

In particular this holds for $x \in N_{\alpha,c} \subset \bar{v}^{-1}N_c\bar{v}$.

Let $\mathfrak{g}^\alpha$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{n}_\alpha$, $\bar{\mathfrak{n}}_\alpha$. It is a semisimple algebra, invariant under the Cartan involution. Moreover $\mathfrak{k}^\alpha = \mathfrak{k} \cap \mathfrak{g}^\alpha$ is a maximal compact subalgebra of $\mathfrak{g}^\alpha$. Now let H_α be the element of $\mathfrak{a}$ such that $(H_\alpha,H) = \alpha(H)$ for all $H \in \mathfrak{a}$. Then $\mathfrak{a} \cap \mathfrak{g}^\alpha = [\![H_\alpha]\!]$ is maximal abelian in $\mathfrak{s} \cap \mathfrak{g}^\alpha = \mathfrak{s}^\alpha$. Finally $\mathfrak{n} \cap \mathfrak{g}^\alpha = \mathfrak{n}_\alpha$ and $\bar{\mathfrak{n}} \cap \mathfrak{g}^\alpha = \bar{\mathfrak{n}}_\alpha$. It follows that $\mathfrak{g}^\alpha$ is a real-rank 1 semisimple Lie algebra with Iwasawa decomposition $\mathfrak{g}^\alpha = \mathfrak{k}^\alpha + \mathfrak{a}^\alpha + \mathfrak{n}_\alpha$. G has a closed connected subgroup G^α with Lie algebra $\mathfrak{g}^\alpha$. The Iwasawa decomposition of G induces a decomposition of G^α by

$$G^\alpha = K^\alpha A^\alpha N_\alpha, \quad K^\alpha = K \cap G^\alpha, \; A^\alpha = A \cap G^\alpha.$$

K^α, A^α, N^α are connected closed subgroups of G^α with Lie algebras $\mathfrak{k}^\alpha$, $\mathfrak{a}^\alpha$, $\mathfrak{n}^\alpha$, so $G^\alpha = K^\alpha A^\alpha N_\alpha$ is an Iwasawa decomposition of the semisimple group G^α. It follows that the natural maps $G^\alpha \to K^\alpha$, $G^\alpha \to A^\alpha$, $G^\alpha \to N_\alpha$ associated with this Iwasawa decomposition are the restrictions to G^α of the maps κ, h, ν respectively.

By the discussion above, ν maps $\overline{N}_\alpha$ into N_α. Therefore we may define the map $\Psi_{w,v}\colon \overline{N}_\alpha \times \overline{N}_v \to \overline{N}_w$ by

$$\Psi_{w,v}(\overline{n},\overline{n}') = \overline{n}\ \tau_v(\nu(\overline{n})^{-1})Ad(h(\overline{n})^{-1})(\overline{n}') \tag{14}$$

<u>Lemma 4.6</u>. The map $\Psi_{w,v}\colon \overline{N}_\alpha \times \overline{N}_v \to \overline{N}_w$ is a real analytic diffeomorphism. If $(\overline{n},\overline{n}') \in \overline{N}_\alpha \times \overline{N}_v$, we have:

$$H(\Psi_{w,v}(\overline{n},\overline{n}')) = H(\overline{n}) + H(\overline{n}'). \tag{15}$$

Moreover, writing $\rho_\alpha = (\frac{1}{2}m(\alpha) + m(2\alpha))\alpha$, we have

$$(\Psi_{w,v})^*(e^{-\rho H(.)}\Omega_{w,0})(\overline{n},\overline{n}') = e^{-\rho_\alpha H(\overline{n})}e^{-\rho H(\overline{n}')}(\Omega_{\alpha,0}\wedge\Omega_{v,0})(\overline{n},\overline{n}').$$

<u>Proof</u>. Denote the map $\overline{N}_\alpha \times \overline{N}_v \to \overline{N}_w$, $(\overline{n},\overline{n}') \to \overline{n}\ \overline{n}'$ by ϕ. From Proposition 4.4 it follows readily that ϕ is a real analytic diffeomorphism with $\phi^*(\Omega_{w,0}) = \Omega_{\alpha,0}\wedge\Omega_{v,0}$. Let the map $\phi_1\colon \overline{N}_\alpha \times \overline{N}_v \to \overline{N}_\alpha \times \overline{N}_v$ be defined by

$$\phi_1(\overline{n},\overline{n}') = (\overline{n},\ \tau_v(\nu(\overline{n})^{-1})Ad(h(\overline{n})^{-1})(\overline{n}')).$$

Then ϕ_1 is bijective, its inverse is given by the formula $\phi_1^{-1}(\overline{n},\overline{n}') = (\overline{n}, Ad(h(\overline{n}))\ \tau_v(\nu(\overline{n}))(\overline{n}'))$. This shows that ϕ_1 is a real analytic diffeomorphism, and so $\Psi_{w,v} = \phi\circ\phi_1$ is a real analytic diffeomorphism.

Let $\overline{n} \in \overline{N}_\alpha$, $\overline{n}' \in \overline{N}_v$. By (12) and (14) we have

$$\Psi_{w,v}(\bar{n},\bar{n}') \equiv \bar{n}\nu(\bar{n})^{-1}h(\bar{n})^{-1}\bar{n}'h(\bar{n}) \bmod (\bar{N} \cap \bar{v}^{-1}N\bar{v}).$$

Since $\bar{n}\nu(\bar{n})^{-1}h(\bar{n})^{-1} = \kappa(\bar{n})$, it follows that $H(\Psi_{w,v}(\bar{n},\bar{n}')) =$ $= H(\bar{n}'h(\bar{n})) = H(h(\bar{n}')h(\bar{n})h(\bar{n})^{-1}\nu(\bar{n}')h(\bar{n}))$, hence (15).

Finally, let us write

$$\rho_v = \sum_{\alpha\in\Delta^{++}(v)} \rho_\alpha \cdot \tag{16}$$

Then $Ad(h(\bar{n})^{-1})^*(\Omega_{v,0}) = \exp(2\rho_v H(\bar{n}))\Omega_{v,0}$. By (13) it follows that

$$\phi_1^*(\Omega_{\alpha,0}\wedge\Omega_{v,0}) = e^{2\rho_v H(n)}\,\Omega_{\alpha,0}\wedge\Omega_{v,0}$$

hence the last statement of Lemma 4.6 follows from $\Psi^*(\Omega_{w,0}) =$ $= \phi_1^*(\Omega_{\alpha,0}\wedge\Omega_{v,0})$ and Proposition 4.7 below (observe that $H(\bar{n}) \in \mathfrak{a}^\alpha$).

Proposition 4.7. Let ρ_v be as in formula (16). Then

$$2\rho_v - \rho = -\rho_\alpha \quad \text{on } \mathfrak{a}^\alpha.$$

Proof. Let H_α be the element of $\mathfrak{a}$ with $(H_\alpha,H) = \alpha(H)$ for all $H \in \mathfrak{a}$. Then $H_\alpha \perp \ker\alpha$ (with respect to (,)), hence $s_\alpha(H_\alpha) = -H_\alpha$. Since α is a simple root for the Weyl chamber $v^{-1}(\mathfrak{a}^+)$, s_α leaves the set of roots $R = \{\beta \in \Delta^{++} \cup (-\Delta^{++});$ $\beta \neq \alpha,\ \beta < 0$ on $v^{-1}(\mathfrak{a}^+)$ invariant. Since

$$R = [\Delta^{++}(v) \cup -(\Delta^{++}\setminus\Delta^{++}(v))]\setminus\{-\alpha\} \tag{17}$$

and

$$2\rho_v - \rho = \sum_{\beta\in R} \rho_\beta - \rho_\alpha$$

we see that $s_\alpha(2\rho_v - \rho) = 2\rho_v - \rho + 2\rho_\alpha$. Hence $-(2\rho_v - \rho)(H_\alpha) =$ $= s_\alpha(2\rho_v - \rho)(H_\alpha) = (2\rho_v - \rho)(H_\alpha) + 2\rho_\alpha(H_\alpha)$. The assertion now follows from the fact that $\mathfrak{a}^{\alpha} = [\![H_{\alpha}]\!]$.

The procedure of Gindikin and Karpelevič is based on Lemma 4.6. Our first objective is to derive a suitable holomorphic extension of this lemma.

Let $s \in W$ (or $s \in \Delta^{++}$). By Lemma 1.8, $\overline{N}_{s,c} \setminus S$ is the biggest connected open subset of $\overline{N}_{s,c}$ such that the map $H|\overline{N}_s$ has a multi-valued analytic extension to it. We denote this extension by H^s. H^s is the restriction (in the sense of the appendix to Chapter 1) of the multi-valued analytic map H to $\overline{N}_{s,c}\setminus S$. Similarly, we let κ^s, h^s, ν^s denote the restrictions of the multi-valued analytic maps κ, h, ν to $\overline{N}_{s,c}\setminus S$. These maps are the multi-valued analytic extensions to $\overline{N}_{s,c}\setminus S$ of $\kappa|\overline{N}_s$, $h|\overline{N}_s$, $\nu|\overline{N}_s$ respectively. By an argument similar to the one used in the proof of Theorem 1.5, we obtain

<u>Proposition 4.8</u>. Let $\beta \in \Delta^{++}$. The multi-valued analytic maps κ^{β}, h^{β}, ν^{β} map $\overline{N}_{\beta,c}\setminus S$ into K_c^{β}, A_c^{β}, $N_{\beta,c}$ respectively.

If $s \in W$ (or $s \in \Delta^{++}$) we denote the covering with base points associated with the multi-valued analytic map $H^s\colon \overline{N}_{s,c}\setminus S \to \mathfrak{a}_c$ by

$$\pi_s\colon (\tilde{\overline{N}}_s, \varepsilon_s) \to (\overline{N}_{s,c}\setminus S, e).$$

Recall that this covering is the analogon of the Riemann surface, cf. the appendix to Chapter 1. The lifting of H^s to a holomorphic map $\tilde{\overline{N}}_s \to \mathfrak{a}_c$ is denoted by $\tilde{H}^s$, and we write $\tilde{h}^s = \exp \tilde{H}^s$, $\tilde{\nu}^s = \tilde{\nu}^s \circ \pi_s$.

Let us now return to the adjacent Weyl group elements w and v, and recall the definition of $\Psi_{w,v}$ (cf. (14)). Since ν^{α} maps

$\overline{N}_{\alpha,c}\backslash S$ into $N_{\alpha,c}$ we may define the multi-valued analytic map $\Psi_{w,v,c}\colon (\overline{N}_{\alpha,c}\backslash S)\times\overline{N}_{v,c} \to \overline{N}_{w,c}$ by

$$\Psi_{w,v,c}(\overline{n},\overline{n}') = \overline{n}\tau_v(\nu^{\alpha}(\overline{n})^{-1})Ad(h^{\alpha}(\overline{n})^{-1})(\overline{n}').$$

Thus $\Psi_{w,v,c}$ is the multi-valued analytic extension of the map $\Psi_{w,v}$ with respect to the base point (e,e).

Proposition 4.9. Each branch $\Psi_{w,v,U}$ of $\Psi_{w,v,c}$ over a simply connected open subset U of $(\overline{N}_{\alpha,c}\backslash S)\times\overline{N}_{v,c}$ is a local diffeomorphism.

Proof. Straightforward, by first showing that the proposition holds for the multi-valued analytic map $\phi_{1,c}\colon (\overline{N}_{\alpha,c}\backslash S)\times\overline{N}_{v,c} \to$ $\to (\overline{N}_{\alpha,c}\backslash S)\times\overline{N}_{v,c}$ defined by $\phi_{1,c}(\overline{n},\overline{n}') = (\overline{n},\tau_v(\nu^{\alpha}(\overline{n})^{-1})Ad(h^{\alpha}(\overline{n})^{-1})(\overline{n}'))$. The map $\phi_c\colon \overline{N}_{\alpha,c}\times\overline{N}_{v,c} \to \overline{N}_{w,c}$, $(\overline{n},\overline{n}') \to \overline{n}\,\overline{n}'$ is a holomorphic diffeomorphism, and $\Psi_{w,v,c} = \phi_c\circ\phi_{1,c}$.

Lemma 4.10. The restriction $\Psi_{w,v}$ of $\Psi_{w,v,c}$ to $(\overline{N}_{\alpha,c}\backslash S)\times(\overline{N}_{v,c}\backslash S)$ is a multi-valued analytic map of $(\overline{N}_{\alpha,c}\backslash S)\times(\overline{N}_{v,c}\backslash S)$ into $\overline{N}_{w,v}\backslash S$. Moreover, $\Psi_{w,v}$ has a unique lifting to a holomorphic map

$$\tilde{\Psi}_{w,v}\colon\ \overline{\mathcal{N}}_{\alpha}\times\overline{\mathcal{N}}_{v} \to \overline{\mathcal{N}}_{w}$$

with

$$\tilde{\Psi}_{w,v}(\varepsilon_{\alpha},\varepsilon_{v}) = \varepsilon_{w}.$$

Proof. From the definition of $\Psi_{w,v,c}$ it follows that $\Psi_{w,v}$ has a unique lifting to a holomorphic map

$$\overline{\Psi}_{w,v}\colon \overline{\mathcal{N}}_{\alpha}\times\overline{\mathcal{N}}_{v} \to \overline{N}_{w,c}$$

with $\overline{\Psi}_{w,v}(\varepsilon_\alpha,\varepsilon_v) = e$. In fact, $\overline{\Psi}_{w,v}$ is defined by

$$\overline{\Psi}_{w,v}(x,y) = \pi_\alpha(x) \,.\, \tau_v(\nu^\alpha(x)^{-1})Ad(h^\alpha(x)^{-1})(\pi_v(y)).$$

By Proposition 4.9 $\overline{\Psi}_{w,v}$ is a local diffeomorphism, hence its image is an open subset U of $\overline{N}_{w,c}$. If $z \in U$ then there exists a continuous curve $\sigma\colon [0,1] \to \overline{N}_\alpha \times \overline{N}_v$ with $\sigma(0) = (\varepsilon_\alpha,\varepsilon_v)$, $\overline{\Psi}_{w,v}(\sigma(1)) = z$. Let $\sigma'\colon [0,1] \to \overline{N}_\alpha$ and $\sigma''\colon [0,1] \to \overline{N}_v$ be the curves with $\sigma = (\sigma',\sigma'')$. From (15) it follows that the germ of the real branch of $H^w\colon (\overline{N}_{w,c}\setminus S) \to \mathfrak{a}_c$ at e is given by

$$(H^w)_e \circ (\overline{\Psi}_{w,c})_{(\varepsilon_\alpha,\varepsilon_v)}(x,y) = \widetilde{H}^\alpha(x) + \widetilde{H}^v(y).$$

This shows that H^w can be analytically continuated along the curve $\overline{\Psi}_{w,v}\circ\sigma$, the branch over $\overline{\Psi}_{w,v}\circ\sigma$ being given by

$$H^w \circ \overline{\Psi}_{w,v} \circ \sigma = \widetilde{H}^\alpha \circ \sigma' + \widetilde{H}^v \circ \sigma''.$$

This implies that $U \subset \overline{N}_{w,c}\setminus S$. Hence $\mathrm{im}(\Psi_{w,v})$ is an open subset of $\overline{N}_{w,c}\setminus S$. Moreover, the germ of H^w at z obtained by continuation along $\overline{\Psi}_{w,v}\circ\sigma$ is given by

$$(H^w)_z \circ (\overline{\Psi}_{w,v})_{\sigma(1)}\,(x,y) = \widetilde{H}^\alpha_{\sigma'(1)}(x) + \widetilde{H}^v_{\sigma''(1)}\,(y), \tag{18}$$

so it depends on $\sigma(1)$ only. It follows that $\overline{\Psi}_{w,v}$ has a unique lifting to a map $\widetilde{\Psi}_{w,v}\colon \overline{N}_\alpha \times \overline{N}_v \to \overline{N}_w$ such that $\widetilde{\Psi}_{w,v}(\varepsilon_\alpha,\varepsilon_v) = \varepsilon_w$.

We now come to the holomorphic extension of Lemma 4.6. If $s \in W$ (or $s \in \Delta^{++}$) we write $\widetilde{\Omega}_{s,0}$ for the pull back of $\Omega_{s,0}$ to $\overline{N}_s$.

Lemma 4.11. The map $\widetilde{\Psi}_{w,v}\colon \overline{N}_\alpha \times \overline{N}_v \to \overline{N}_w$ is a holomorphic local diffeomorphism, and if $x \in \overline{N}_\alpha$, $y \in \overline{N}_v$, then:

$$\tilde{H}^w(\tilde{\Psi}_{w,v}(x,y)) = \tilde{H}^\alpha(x) + \tilde{H}^v(y).$$

Moreover, we have

$$(\tilde{\Psi}_{w,v})^*(e^{-\rho\tilde{H}^w}\tilde{\Omega}_{w,0})(x,y) = e^{-\rho_\alpha\tilde{H}^\alpha(x)}\, e^{-\rho\tilde{H}^v(y)}(\tilde{\Omega}_{\alpha,0}\wedge\tilde{\Omega}_{v,0})(x,y).$$

Proof. The first two statements follow from the proof of Lemma 4.10 (cf. formula (18)). In view of Lemma 4.7 the last statement holds locally at $(\varepsilon_\alpha,\varepsilon_v)$. By analytic continuation this completes the proof.

Now let $w \in W$ be fixed in the remainder of this section, and let $q = n(w)$. Select a sequence of Weyl group elements $w(j)$ $(0 \leqslant j \leqslant q)$ such that $w(0) = I$ and $w(q) = w$, and such that $w(j)$ and $w(j+1)$ are adjacent and $n(w(j+1)) = n(w(j)) + 1$ for $0 \leqslant j \leqslant 1$. (It is easy to see how such a sequence might be defined recurrently, starting with $w(q) = w$). If $0 \leqslant j < q$ let $\alpha(j+1)$ be the unique element of $\Delta^{++}(w(j+1))\setminus\Delta^{++}(w(j))$. Then $w(j+1) = w(j)\circ s_{\alpha(j+1)}$ $(0 \leqslant j < q)$ and by induction we obtain

$$\Delta^{++}(w(j)) = \{\alpha(1),\dots,\alpha(j)\}, \qquad w(j) = s_{\alpha(1)}\circ\cdots\circ s_{\alpha(j)}.$$

for $1 \leqslant j \leqslant q$. Select orientations for the spaces $\bar{N}_{\alpha(j)}$, $\bar{N}_{w(j)}$ $(1 \leqslant j \leqslant q)$ such that the maps $\bar{N}_{\alpha(j+1)}\times\bar{N}_{w(j)} \to \bar{N}_{w(j+1)}$, $(\bar{n},\bar{n}') \to \bar{n}\,\bar{n}'$ $(1 \leqslant j < q)$ become orientation preserving.

Define maps $\Psi_j\colon \bar{N}_{\alpha(j)}\times\dots\times\bar{N}_{\alpha(1)} \to \bar{N}_{w(j)}$ $(1 \leqslant j \leqslant q)$ recurrently as follows:

$$\Psi_1 = \mathrm{id}(\bar{N}_{\alpha(1)}),$$

$$\Psi_{j+1} = \Psi_{w(j+1),w(j)}\circ(\mathrm{id}(\bar{N}_{\alpha(j+1)})\times\Psi_j) \qquad (1 \leqslant j < q);$$

and let $\Psi_w = \Psi_q$.

Theorem 4.12. The map Ψ_w: $\overline{N}_{\alpha(q)} \times \ldots \times \overline{N}_{\alpha(1)} \to \overline{N}_w$ is a real analytic diffeomorphism. If $\overline{n}_j \in \overline{N}_{\alpha(j)}$ $(1 \leqslant j \leqslant q)$, then

$$H(\Psi_w(\overline{n}_q,\ldots,\overline{n}_1)) = H(\overline{n}_q) + \ldots + H(\overline{n}_1).$$

Moreover,

$$(\Psi_w)^*(e^{-\rho H(.)}\Omega_{w,0}) = \prod_{j=1}^{q} e^{-\rho_{\alpha(j)}H(\overline{n}_j)}\Omega_{\alpha(q),0}\wedge\ldots\wedge\Omega_{\alpha(1),0}$$

Proof. Straightforward by using Lemma 4.6 and induction on the definition of Ψ_w.

Corollary 4.13. (Gindikin-Karpelevič) Let $\lambda \in \mathfrak{a}_c$. Then the integral

$$I_w(\lambda) = \int_{\overline{N}_w} e^{-(i\lambda+\rho)H(\overline{n})}\Omega_{w,0}(\overline{n})$$

converges absolutely iff for every $\alpha \in \Delta^{++}(w)$ the integral

$$I_\alpha(\lambda) = \int_{\overline{N}_\alpha} e^{-(i\lambda+\rho_\alpha)H(\overline{n})}\Omega_{\alpha,0}(\overline{n})$$

converges absolutely. If these conditions are fulfilled, then

$$I_w(\lambda) = \prod_{\alpha\in\Delta^{++}(w)} I_\alpha(\lambda). \tag{19}$$

By similar estimates as in the proof of Lemma 3.13 it can be shown that for $\alpha \in \Delta^{++}$ the integral $I_\alpha(\lambda)$ converges absolutely iff $\mathrm{Im}(\lambda,\alpha) < 0$ (cf. also Helgason [2] and Schiffmann [1]). Hence $I_w(\lambda)$ converges absolutely iff $\mathrm{Im}(\lambda,\alpha) < 0$ for all $\alpha \in \Delta^{++}(w)$. Let, for $\alpha \in \Delta^{++}$, Ω_α be the invariant $\dim(\overline{N}_\alpha)$-form on $\overline{N}_{\alpha,c}$ normalized by

$$\int_{\bar{N}_\alpha} \exp(-2\rho H(\bar{n}))\Omega_\alpha(\bar{n}) = 1.$$

The form Ω_α is the analogon of Ω for the real rank 1 group $G^\alpha = K^\alpha A^\alpha N_\alpha$. Now let K_α be the positive real number with $\Omega_{\alpha,0} = K_\alpha \Omega_\alpha$. In view of Theorem 3.12 and Lemma 3.13 we obtain that

$$I_\alpha(\lambda) = K_\alpha c_\alpha(\lambda_\alpha), \tag{20}$$

if $\alpha \in \Delta^{++}$, and if $\lambda \in \mathfrak{a}_c^*$ satisfies $\text{Im}(\lambda,\alpha) < 0$. Here c_α denotes the c-funtion associated with G^α, and λ_α denotes the restriction of λ to $\mathfrak{a}_c^\alpha$. Similarly, let K be the positive real number with

$$\Omega_0 = K\Omega. \tag{21}$$

If we write u for the Coxeter element of W (i.e. the element $s \in W$ with $s(\mathfrak{a}^+) = -\mathfrak{a}^+$), then

$$I_u(\lambda) = Kc(\lambda), \tag{22}$$

if $\lambda \in \mathfrak{a}_c^*$ is such that $\text{Im}(\lambda,\alpha) < 0$ for all $\alpha \in \Delta^{++}$. By (22), (19) and (21) it follows that for $\lambda \in \mathfrak{a}_c^*$ with $\text{Im}(\lambda,\alpha) < 0$ (all $\alpha \in \Delta^{++}$) we have

$$c(\lambda) = K_0 \prod_{\alpha \in \Delta^{++}} c_\alpha(\lambda_\alpha),$$

with

$$K_0 = K^{-1} \prod_{\alpha \in \Delta^{++}} K_\alpha. \tag{23}$$

Remark. The constant K_0 has been computed explicitly by Duistermaat, Kolk, Varadarajan [1, §3.8].

We now come to the holomorphic version of the Gindikin-

Karpelevič procedure. Define maps $\tilde{\Psi}_j: \bar{N}_{\alpha(j)}\times\dots\times\bar{N}_{\alpha(1)} \to \bar{N}_{w(j)}$ $(1 \leq j \leq q)$ recurrently by:

$$\tilde{\Psi}_1 = \mathrm{id}(\bar{N}_{\alpha(1)}),$$

$$\tilde{\Psi}_{j+1} = \tilde{\Psi}_{w(j+1),w(j)}\circ(\mathrm{id}(\bar{N}_{\alpha(j+1)})\times\tilde{\Psi}_j) \qquad (1 \leq j < q);$$

and let $\tilde{\Psi}_w = \tilde{\Psi}_q$.

Theorem 4.14. The map $\tilde{\Psi}_w: \bar{N}_{\alpha(q)}\times\dots\times\bar{N}_{\alpha(1)} \to \bar{N}_w$ is a holomorphic local diffeomorphism. If $x_j \in \bar{N}_{\alpha(j)}$ $(1 \leq j \leq q)$, then

$$\tilde{H}^w(\tilde{\Psi}_w(x_q,\dots,x_1)) = \tilde{H}^{\alpha(q)}(x_q) + \dots + \tilde{H}^{\alpha(1)}(x_1).$$

Moreover,

$$(\tilde{\Psi}_w)^*(e^{-\rho\tilde{H}^w}\tilde{\Omega}_{w,0}) = (\prod_{j=1}^{q} e^{-\rho_{\alpha(j)}\tilde{H}^{\alpha(j)}(x_j)})\tilde{\Omega}_{\alpha(q),0}\wedge\dots\wedge\tilde{\Omega}_{\alpha(1),0}.$$

Proof. Straightforward by application of Lemma 4.11 and induction on the definition of $\tilde{\Psi}_w$.

Remark. Let $\alpha \in \Delta^{++}$. By formula (3.3) it follows that the monodromy group of the multi-valued map $H^\alpha: \bar{N}_{\alpha,c}\backslash S \to \mathfrak{a}_c$ at e is isomorphic to the lattice $\Lambda(M^\alpha)$ generated over $\mathbb{Z}$ by the vector $2\pi i H_{\alpha,0}$. Now let $w \in W$. Then from Theorem 4.14 it follows that the monodromy group of $H^w: \bar{N}_{w,c}\backslash S \to \mathfrak{a}_c$ at e is isomorphic to the lattice

$$\Lambda(M^w) = \sum_{\alpha\in\Delta^{++}(w)} \Lambda(M^\alpha).$$

In particular it follows that the monodromy group of $H: G_c\backslash S \to \mathfrak{a}_c$ at e is the lattice $\Lambda(M) = \{X \in \mathfrak{a}_c;\ \exp X \in M\}$ (see also the remark following the proof of Theorem 1.6).

Select for each $\alpha \in \Delta^{++}$ a cylinder $m(\alpha)$-cycle $\gamma_{I,\alpha}: \partial([0,\pi]\times B_{I,\alpha}) \to \overline{N}_{\alpha,c}\backslash S$ as in Chapter 3. This can be done, for $\overline{N}_{\alpha,c}\backslash S$ is the biggest connected open subset of $\overline{N}_{\alpha,c}\backslash S$, containing $\overline{N}_\alpha$, such that $H|\overline{N}_\alpha$ (the Iwasawa projection associated with $G^\alpha = K^\alpha A^\alpha N_\alpha$) has a multi-valued analytic extension to it; so the set $\overline{N}_{\alpha,c}\backslash S$ corresponds to the set $\overline{N}_c\backslash S$ of Chapter 3. The multi-valued analytic map $H^\alpha: \overline{N}_{\alpha,c}\backslash S \to \mathfrak{a}_c^\alpha$ has a branch $H_{I,\alpha}^\alpha$ as in Chapter 3 over $\gamma_{I,\alpha}$. Let $\tilde{\gamma}_{I,\alpha}$ be the corresponding lifting of $\gamma_{I,\alpha}$ to $\overline{\mathcal{N}}$. Let $H_{\alpha,0}$ denote the element of $\mathfrak{a}^\alpha$ with $\alpha(H_{\alpha,0}) = 1$, and define the function $d_\alpha: \mathfrak{a}_c^* \to \mathbb{C}$ by:

$$d_\alpha(\lambda) = e^{2\pi\lambda(H_{\alpha,0})} - 1.$$

Then if $\lambda \in \mathfrak{a}_c^*$, $\mathrm{Im}(\lambda,\alpha) < 0$ we have:

$$\int_{\tilde{\gamma}_{I,\alpha}} e^{-(i\lambda+\rho_\alpha)\tilde{H}^\alpha(x)} \tilde{\Omega}_{\alpha,0}(x) = d_\alpha(\lambda) I_\alpha(\lambda) \tag{24}$$

(cf. Chapter 3 and formula (14)).

By an easy application of Stokes' theorem we see that we may replace the cycle $\tilde{\gamma}_{I,\alpha}$ by a smooth $\dim(\overline{N}_\alpha)$-cycle $\tilde{\gamma}_\alpha: Y_\alpha \to \overline{\mathcal{N}}_\alpha$ (where Y_α is diffeomorphic to a $\dim(\overline{N}_\alpha)$-sphere), such that:

$$\int_{\tilde{\gamma}_{I,\alpha}} \xi = \int_{\tilde{\gamma}_\alpha} \xi$$

for any closed $\dim(\overline{N}_\alpha)$-form ξ on $\overline{\mathcal{N}}_\alpha$. In particular the value of the integral in (24) does not change if we replace $\tilde{\gamma}_{I,\alpha}$ by $\tilde{\gamma}_\alpha$.

Define $Y_w = Y_{\alpha(q)}\times\ldots\times Y_{\alpha(1)}$. It is a connected compact C^∞ manifold of dimension $\dim(\overline{N}_w)$; we provide it with the product orientation. Define the smooth cycle $\tilde{\gamma}_{\alpha(q)}\times\ldots\times\tilde{\gamma}_{\alpha(1)}: Y_w \to$ $\to \overline{\mathcal{N}}_{\alpha(q)}\times\ldots\times\overline{\mathcal{N}}_{\alpha(1)}$ by

$$(t_q,\dots t_1) \to (\tilde{\gamma}_{\alpha(q)}(t_q),\dots,\tilde{\gamma}_{\alpha(1)}(t_1)).$$

Theorem 4.15. Denote the smooth $\dim(\bar{N}_w)$-cycle $\tilde{\Psi}_w \circ (\tilde{\gamma}_{\alpha(q)} \times \dots \times \tilde{\gamma}_{\alpha(1)})$ by $\tilde{\gamma}_w$. If $\lambda \in \mathfrak{a}_c^*$, we have

$$\int_{\tilde{\gamma}_w} e^{-(i\lambda+\rho)\tilde{H}^w} \tilde{\Omega}_{w,0} = \prod_{\alpha \in \Delta^{++}(w)} \int_{\tilde{\gamma}_\alpha} e^{-(i\lambda+\rho_\alpha)\tilde{H}^\alpha} \tilde{\Omega}_{\alpha,0} \qquad (25)$$

Proof. The integral on the left hand side of (25) is equal to

$$\int_{\tilde{\gamma}_{\alpha(q)} \times \dots \times \tilde{\gamma}_{\alpha(1)}} (\tilde{\Psi}_w)^* (e^{-(i\lambda+\rho)\tilde{H}^w} \tilde{\Omega}_{w,0}).$$

Application of Theorem 4.14 yields (25).

If $s = w$ of $s \in \Delta^{++}$ we denote the smooth $\dim(\bar{N}_s)$-cycle $\pi_s \circ \tilde{\gamma}_s : Y_s \to \bar{N}_{s,c} \setminus S$ by γ_s. Moreover, we denote the branch of H^s over γ_s corresponding to $\tilde{H}^s$ by H_s. Thus, writing

$$\tilde{I}_s(\lambda) = \int_{\gamma_s} e^{-(i\lambda+\rho)H_s(\bar{n})} \Omega_{s,0}(n) \qquad (26)$$

for $\lambda \in \mathfrak{a}_c^*$, $s = w$ or $s \in \Delta^{++}$, we may rewrite (25) as

$$\tilde{I}_w(\lambda) = \prod_{\alpha \in \Delta^{++}(w)} \tilde{I}_\alpha(\lambda). \qquad (27)$$

Moreover, (24) can be rewritten as

$$\tilde{I}_\alpha(\lambda) = d_\alpha(\lambda) I_\alpha(\lambda)$$

if $\lambda \in \mathfrak{a}_c^*$ satisfies $\operatorname{Im}(\lambda,\alpha) < 0$. By (19) we obtain the following lemma.

Lemma 4.16. Let $\lambda \in \mathfrak{a}_c^*$ be such that $\operatorname{Im}(\lambda,\alpha) < 0$ for all $\alpha \in \Delta^{++}(w)$. Then

$$\tilde{I}_w(\lambda) = \{ \prod_{\alpha \in \Delta^{++}(w)} d_\alpha(\lambda)\} I_w(\lambda). \tag{28}$$

If u is the Coxeter element of W, we write $\tilde{I}(.)$ for $\tilde{I}_u(.)$. Obviously $\tilde{I}$ is a holomorphic function $\mathfrak{a}_c^* \to \mathbb{C}$. Now let d be the holomorphic function $\mathfrak{a}_c^* \to \mathbb{C}$ defined by

$$d(\lambda) = \prod_{\alpha \in \Delta^{++}} d_\alpha(\lambda). \tag{29}$$

Then we have the following corollary of Lemma 4.16.

Corollary 4.17. If $\lambda \in \mathfrak{a}_c^*$ is such that $\mathrm{Im}(\lambda,\alpha) < 0$ for all $\alpha \in \Delta^{++}$, then

$$\tilde{I}(\lambda) = K d(\lambda) c(\lambda). \tag{30}$$

Consequently, the c-function extends meromorphically to $\mathfrak{a}_c^*$.

4.4 Construction of the cycles Γ_w

Let $w \in W$ be fixed throughout this section. Recall that the map $M \times \bar{N} \to K$, $(m,\bar{n}) \to m\kappa(\bar{n})$ is a diffeomorphism onto a dense open subset of K. Fix an orientation of M such that this map becomes orientation preserving, and let ω_M be the invariant dim(M)-form on M such that

$$\int_M \omega_M = 1.$$

Let Ω be the invariant dim($\bar{N}$)-form on $\bar{N}$ such that

$$\int_{\bar{N}} e^{-2\rho H(\bar{n})} \Omega(\bar{n}) = 1 \tag{31}$$

(cf. Section 1.5). The map $M \times \bar{N} \to K$, $(m,\bar{n}) \to \bar{w} m \kappa(\bar{n})$ is an orientation preserving diffeomorphism onto a dense open subset of K, and the pull back of ω under this map is equal to

$$e^{-2\rho H(\bar{n})}\omega_M \wedge \Omega$$

(this follows readily from formula (20), Section 1.5).

We denote the element of W sending $\mathfrak{a}^+$ onto $-w(\mathfrak{a}^+)$ by w'. We obviously have $w^{-1}(\mathfrak{a}^+) = -(w')^{-1}(\mathfrak{a}^+)$, hence

$$\Delta^{++} = \Delta^{++}(w) \cup \Delta^{++}(w') \qquad \text{(disjoint union)},$$

and

$$\bar{\mathfrak{n}}_c = \bar{\mathfrak{n}}_{w,c} \oplus \bar{\mathfrak{n}}_{w',c}.$$

by Lemma 4.3 the map $\bar{N}_w \times \bar{N}_{w'} \to \bar{N}$ $(\bar{n},\bar{n}') \to \bar{n}\,\bar{n}'$ is a real analytic diffeomorphism. Assume that orientations of $\bar{N}_w$ and $\bar{N}_{w'}$ are fixed such that this map is orientation preserving.

<u>Proposition 4.18</u>. The map $\bar{N}_{w,c} \times \bar{N}_{w',c} \to \bar{N}_c$, $(\bar{n},\bar{n}') \to \bar{n}\,\bar{n}'$ is a holomorphic diffeomorphism, and the holomorphic forms $\Omega_{w,0} \wedge \Omega_{w',0}$ and Ω_0 correspond under this map.

<u>Proof</u>. Just as the proof of Proposition 4.3.

<u>Proposition 4.19</u>. The map ξ: $M \times \bar{N}_w \times \bar{N}_{w'} \to K$, $(m,\bar{n},\bar{n}') \to \bar{w} m \kappa(\bar{n}\,\bar{n}')$ is a diffeomorphism onto a dense open subset of K. Morever,

$$\xi^*(\omega) = K^{-1} e^{-2\rho H(\bar{n}\,\bar{n}')} \omega_M \wedge \Omega_{w,0} \wedge \Omega_{w',0}, \tag{32}$$

and if $(m,\bar{n},\bar{n}') \in M \times \bar{N}_w \times \bar{N}_{w'}$, writing $\bar{w}^{-1} a \bar{w} = \bar{w}^{-1}(a)$, we have:

$$H(a\xi(m,\bar{n},\bar{n}')) = H(Ad(w^{-1}(a))(\bar{n}\,\bar{n}')) - H(\bar{n}\,\bar{n}') + w^{-1}(\log a). \tag{33}$$

<u>Proof</u>. Since $\Omega = K^{-1}\Omega_0$ (see formula (21)) the first statement and formula (32) follow from the preceding discussion. Now let $(m,\bar{n},\bar{n}') \in M \times \bar{N}_w \times \bar{N}_{w'}$, $a \in A$. Then

$$H(a\bar{w}m\kappa(\bar{n}\,\bar{n}')) = H(w^{-1}(a)\kappa(\bar{n}\,\bar{n}'))$$

$$= H(w^{-1}(a)\bar{n}\,\bar{n}') - H(\bar{n}\,\bar{n}'),$$

whence (33).

We now apply a coordinate transform to put $H(a\xi(m,\bar{n},\bar{n}'))$ in a form we can handle. If $a \in A$, define the map $\eta_a: M\times N_w\times N_{w'} \to M\times N_w\times N_{w'}$ by

$$\eta_a(m,\bar{n},\bar{n}') = (m,Ad(w^{-1}(a^{-1}),\bar{n}),\bar{n}')$$

where we have written $Ad(w^{-1}(a^{-1}),\bar{n})$ for $Ad(w^{-1}(a^{-1}))(\bar{n})$. η_a is a real analytic diffeomorphism, and

$$H(a(\xi\circ\eta_a)(m,\bar{n},\bar{n}')) =$$

$$= H(\bar{n}Ad(w^{-1}(a),\bar{n}')) - H(Ad(w^{-1}(a^{-1}),\bar{n})\bar{n}') + w^{-1}(\log a). \tag{34}$$

Moreover, since obviously

$$\eta_a^*(\omega_M\wedge\Omega_{w,0}\wedge\Omega_{w',0}) = a^{2w\rho_w}\omega_M\wedge\Omega_{w,0}\wedge\Omega_{w',0},$$

where $\rho_w = \Sigma_\alpha \rho_\alpha$ (summation over $\Delta^{++}(w)$), we have

$$(\xi\circ\eta_a)^*(\omega) =$$

$$= K^{-1}a^{2w\rho_w}e^{-2\rho H(Ad(w^{-1}(a^{-1}),n)n')}\omega_M\wedge\Omega_{w,0}\wedge\Omega_{w',0}. \tag{35}$$

Let $\mathbb{C}(\Delta^{++})$ denote the set $\mathbb{C}^{\Delta^{++}}$ of functions $\Delta^{++} \to \mathbb{C}$. If $z \in \mathbb{C}(\Delta^{++})$, $\bar{n} \in \bar{N}_c$, $\bar{n} = \exp(\Sigma_\alpha(X_\alpha+Y_\alpha))$ (summation over Δ^{++}; $X_\alpha \in \mathfrak{g}_{-\alpha,c}$, $Y_\alpha \in \mathfrak{g}_{-2\alpha,c}$), we define

$$z.\bar{n} = \exp\left(\sum_{\alpha\in\Delta^{++}}[z_\alpha X_\alpha + (z_\alpha)^2 Y_\alpha]\right).$$

Obviously the map $\mathbb{C}(\Delta^{++})\times\bar{N}_c \to \bar{N}_c$, $(z,\bar{n}) \to z.\bar{n}$ is holomorphic. We define the map $z_w\colon A \to \mathbb{C}(\Delta^{++})$ as follows:

$$(z_w(a))_{\alpha} = a^{w(\alpha)} \quad \text{if} \quad \alpha \in \Delta^{++}(w),$$
$$= a^{-w(\alpha)} \quad \text{if} \quad \alpha \in \Delta^{++}(w') = \Delta^{++}\setminus\Delta^{++}(w). \tag{36}$$

Observe that $z_w(a) \to 0$ if $a^{\alpha} \to +\infty$ for all $\alpha \in \Delta^{++}$.

With these notations we have

$$Ad(w^{-1}(a^{-1}),\bar{n}) = z_w(a).\bar{n},$$
$$Ad(w^{-1}(a),\ \bar{n}') = z_w(a).\bar{n}',$$

for $a \in A$, $\bar{n} \in \bar{N}_{w,c}$, $\bar{n}' \in \bar{N}_{w',c}$. Hence, writing $z_w = z_w(a)$, formula (34) becomes:

$$H(a(\xi\eta_a)(m,\bar{n},\bar{n}')) = H(\bar{n}(z_w.\bar{n}')) - H((z_w.\bar{n})\bar{n}') + w^{-1}(\log a) \tag{37}$$

Let $\gamma_s\colon Y_s \to \bar{N}_{s,c}\setminus S$ $(s = w,w')$ be cycles as constructed in the previous section, and let H_s $(s = w,w')$ be the corresponding branches of H. Since $\mathrm{im}(\gamma_s)$ $(s = w,w')$ are compact subsets of $\bar{N}_s\setminus S$ we may select a simply connected open neighbourhood U_w of 0 in $\mathbb{C}(\Delta^{++})$ such that for $z \in U_w$, $\bar{n} \in \mathrm{im}(\gamma_w)$, $\bar{n}' \in \mathrm{im}(\gamma_{w'})$ we have:

$$\bar{n}(z.\bar{n}') \notin S \ \ \& \ \ (z.\bar{n})\bar{n}' \notin S. \tag{38}$$

It follows that the multi-valued analytic maps $H(\bar{n}(z.\bar{n}'))$ and $H((z.\bar{n})\bar{n}')$ have branches over $U_w\times\gamma_w\times\gamma_{w'}$ that restrict to $H_w(\bar{n})$ and $H_{w'}(\bar{n}')$ over $\{0\}\times\gamma_w\times\gamma_{w'}$ respectively. We denote these branches by $H_w(\bar{n}(z.\bar{n}'))$ and $H_{w'}((z.\bar{n})\bar{n}')$ respectively.

Now consider the map $\xi\circ\eta_a$. If $a \in A$, $(m,\bar{n},\bar{n}') \in M\times\bar{N}_w\times\bar{N}_{w'}$, then

$$(\xi \circ \eta_a)(m,\bar{n},\bar{n}') = \bar{w}m\kappa((z_w(a).\bar{n})\bar{n}').$$

Let $U_{w,r} = \{z \in U_w;\ z_\alpha \in \mathbb{R}$ for all $\alpha \in \Delta^{++}\}$, and define the map $\psi: U_{w,r}\times M\times\bar{N}_w\times\bar{N}_{w'} \to K$ by

$$\psi(z,m,\bar{n},\bar{n}') = \bar{w}m\kappa((z.\bar{n})\bar{n}').$$

Then for $a \in A$ we have

$$\psi(z_w(a),.) = \xi\circ\eta_a. \tag{39}$$

By the conditions on U_w, the multi-valued extension ψ_c of ψ has a branch ψ_w over $U_w\times M\times\gamma_w\times\gamma_{w'}$, that restricts to

$$\bar{w}m\kappa_{w'}(\bar{n}')$$

over $\{0\}\times M\times\gamma_w\times\gamma_{w'}$. Here $\kappa_{w'}$ denotes the branch of κ over $\gamma_{w'}$ corresponding to the branch $h_{w'} = \exp\circ H_{w'}$ of h over $\gamma_{w'}$ (recall that the map κh is single valued, cf. Theorem 1.5).

Select a constant $C_w > 0$ such that $z_w(a) \in U_w$ for $a \in A(C_w)$ (cf. formula (1)), and define $\Gamma_w: A(C_w)\times Y_w\times Y_{w'} \to K_c$ by

$$\Gamma_w(a,.) = \psi_w(z_w(a),.)\circ(\mathrm{id}(M)\times\gamma_w\times\gamma_{w'}). \tag{40}$$

Writing $\Gamma_w(a) = \Gamma_w(a,.)$, Γ_w is a smooth homotopy of smooth $\dim(K)$-cycles $\Gamma_w(a): Y(w) = M\times Y_w\times Y_{w'} \to K_c$. By formulas (40), (39) and (37), the multi-valued map H has a unique branch $H_{0,w}$ over Γ_w with

$$\psi_w(z_w(a),.)^*(H_{0,w}(a.)) =$$

$$= H_w(\bar{n}(z_w.\bar{n}')) - H_{w'}((z_w.\bar{n})\bar{n}') + w^{-1}(\log a). \tag{41}$$

If $a \in A(C_w)$, we obtain by analytic continuation of (35) that

$$\psi_w(z_w(a),.)^*(\omega) = \kappa^{-1}a^{2w\rho_w}e^{-2\rho H_{w'}((z_w.\bar{n})\bar{n}')}\omega_M\wedge\Omega_{w,0}\wedge\Omega_{w',0}$$

Hence

$$\psi_w(z_w,.)^*(e^{(i\lambda-\rho)H_{0,w}(a,.)}\omega) =$$

$$= a^{w(i\lambda-\rho+w\rho_w)}\tilde{\Phi}_w(\lambda,z_w,\bar{n},\bar{n})\omega_M\wedge\Omega_{w,0}\wedge\Omega_{w',0},$$

where:

$$\tilde{\Phi}_w(\lambda,z,\bar{n},\bar{n}') = K^{-1}e^{(i\lambda-\rho)H_w(\bar{n}(z.\bar{n}'))-(i\lambda+\rho)H_{w'}((z.\bar{n})\bar{n}')}. \tag{42}$$

Define the map Φ_w: $\mathfrak{a}_c^*\times U_w \to \mathbb{C}$ by

$$\Phi_w(\lambda,z) = \int_{Y_w\times Y_{w'}} \tilde{\Phi}_w(\lambda,z,\bar{n},\bar{n}')\Omega_{w,0}\wedge\Omega_{w',0}. \tag{43}$$

Since $\tilde{\Phi}_w$ is holomorphic in $(\lambda,z) \in \mathfrak{a}_c^*\times U_w$, Φ_w is a holomorphic map. It follows that

$$\int_{\Gamma_w(a)} e^{(i\lambda-\rho)H_{0,w}(ak)}\omega(k) =$$

$$= \int_{M\times Y_w\times Y_{w'}} \psi_w(z_w(a),.)^*(\omega)$$

$$= a^{w(i\lambda-\rho+w\rho_w)}\Phi_w(\lambda,z_w(a))$$

From (42) and (43) it follows that

$$\Phi_w(\lambda,0) = K^{-1}\int_{Y_w\times Y_{w'}} e^{(i\lambda-\rho)H_w(\bar{n})}e^{-(i\lambda+\rho)H_{w'}(\bar{n}')}\Omega_{w,0}\wedge\Omega_{w',0}.$$

So, using the notation (26) of the previous section, $\Phi_w(\lambda,0) = K^{-1}\tilde{I}_w(-\lambda)\tilde{I}_{w'}(\lambda)$. Finally, observing that $w(\rho-2\rho_w) = w(\rho_{w'}-\rho_w) = \rho$, we have proved the following theorem.

<u>Theorem 4.20</u>. There exists a constant $C_w > 0$ and a C^∞ map Γ_w: $A(C_w)\times M\times Y_w\times Y_{w'} \to K_c$ together with a branch $H_{0,w}$ of H over Γ_w such that the following holds. For every $a \in A(C_w)$, $\lambda \in \mathfrak{a}_c^*$ we have that

$$\int_{\Gamma_w(a)} e^{(i\lambda-\rho)H_{0,w}(ak)}\,\omega(k) = a^{iw\lambda-\rho}\Phi_w(\lambda, z_w(a)).$$

Here Φ_w is the holomorphic function $\mathfrak{a}_c^* \times U_w \to \mathbb{C}$ defined by (43), U_w is an open neighbourhood of 0 in $\mathbb{C}(\Delta^{++})$, and z_w is the map $A(C_w) \to U_w$ defined by (36). Finally, employing the notation (26), we have

$$\Phi_w(\lambda, 0) = \kappa^{-1}\tilde{I}_w(-\lambda)\tilde{I}_{w'}(\lambda) \qquad (44)$$

Observe that in view of the above theorem the function $\phi_{w,\lambda}\colon A(C_w) \to \mathbb{C}$ defined by

$$\phi_{w,\lambda}(a) = \int_{\Gamma_w(a)} e^{(i\lambda-\rho)H_{0,w}(ak)}\,\omega(k) \qquad (45)$$

has an asymptotic expansion for $a \to \infty$ in A^+ (cf. also the discussions in Section 3.4 and in Section 4.5). The principal term of this expansion is given by

$$\phi_{w,\lambda}(a) \sim a^{iw\lambda-\rho}\Phi_w(\lambda, 0). \qquad (46)$$

Theorem 4.21. The function $\phi_{w,\lambda}\colon A(C_w) \to \mathbb{C}$ defined by (45) is a solution of the system of radial differential equations

$$\Delta(D)\phi = \gamma(D, i\lambda)\phi \qquad (D \in U(\mathfrak{g}_c)^K). \qquad (47)$$

Proof. Fix $a_0 \in A(C_w)$. Since the cycles $\Gamma_w(a)$ are smoothly homotopic to $\Gamma_w(a_0)$, there exists an open neighbourhood $\mathcal{A}$ of a_0 in $A(C_w)$ such that

$$\phi_{w,\lambda}(a) = \int_{\Gamma_w(a_0)} e^{(i\lambda-\rho)H_{0,w}(ak)}\,\omega(k)$$

for $a \in \mathcal{A}$. By Theorem 2.9 it follows that $\phi_{w,\lambda}$ satisfies the system (47) in $\mathcal{A}$.

4.5 Harish-Chandra's formula

Let $\mathfrak{a}^*_{c,0}$ be the set of $\lambda \in \mathfrak{a}^*_c$ with $(\lambda,\alpha) \neq 0$ and $\Phi_w(\lambda,0) \neq 0$ for all $\alpha \in \Delta^{++}$, $w \in W$. In view of (44), (28), (19), (20) and (23) we have that

$$\Phi_w(\lambda,0) = K_0 \prod_{\alpha\in\Delta^{++}(w)} d_\alpha(-\lambda)c_\alpha(\lambda_\alpha) \prod_{\alpha\in\Delta^{++}(w')} d_\alpha(\lambda)c_\alpha(\lambda_\alpha),$$

and so by formula (3.32) for the real rank 1 c-function we easily obtain the following proposition.

Proposition 4.22. The set $\mathfrak{a}^*_{c,0}$ is the complement of a Weyl group invariant, locally finite union of hyperplanes in $\mathfrak{a}^*_c$. In particular, $\mathfrak{a}^*_{c,0}$ is a connected dense open subset of $\mathfrak{a}^*_c$.

Now fix a constant $C_0 > 0$ with $C_0 > C_w$ for all $w \in W$. If $\lambda \in \mathfrak{a}^*_c$, and $(\lambda,\alpha) \neq 0$ for all $\alpha \in \Delta^{++}$, then the functions

$$A(C_0) \to \mathbb{C},\ a \to a^{iw\lambda-\rho} \qquad (w \in W)$$

are linearly independent. Consequently, by (46) we obtain

Theorem 4.23. If $\lambda \in \mathfrak{a}^*_{c,0}$ then the functions $\phi_{w,\lambda}: A(C_0) \to \mathbb{C}$ ($w \in W$) are linearly independent elements of $\mathcal{E}_\lambda(A(C_0))$ (for this notation see Theorem 2.8). Since $\dim \mathcal{E}_\lambda(A(C_0)) \leq \#W$ it follows that $\dim \mathcal{E}_\lambda(A(C_0)) = \#W$, and $(\phi_{w,\lambda})_{w\in W}$ is a basis for $\mathcal{E}_\lambda(A(C_0))$.

Lemma 4.24. There exist holomorphic functions $e_w: \mathfrak{a}^*_{c,0} \to \mathbb{C}$ ($w \in W$) such that for $\lambda \in \mathfrak{a}^*_{c,0}$:

$$\phi_\lambda = \sum_{w \in W} e_w(\lambda)\phi_{w,\lambda}. \tag{48}$$

Proof. Fix $a_0 \in A(C_0)$, and let $P(a_0)\colon \mathcal{E}_\lambda(A(C_0)) \to \mathbb{C}^W$ be the linear map defined in Theorem 2.8. It is injective, hence $(P(a_0)\phi_{w,\lambda})_{w\in W}$ is a basis for $\mathbb{C}^W$ whose elements depend holomorphically on $\lambda \in \mathfrak{a}^*_{c,0}$. The function $\lambda \to P(a_0)\phi_\lambda$, $\mathfrak{a}^*_c \to \mathbb{C}^W$ is holomorphic, and therefore there exist holomorphic functions $e_w\colon \mathfrak{a}^*_{c,0} \to \mathbb{C}$ such that $P(a_0)\phi_\lambda = \Sigma_{w\in W}\ e_w(\lambda)P(a_0)\phi_{w,\lambda}$. Since $P(a_0)$ is an injective linear map this implies (48).

Lemma 4.25. Let $\lambda \in \mathfrak{a}^*_{c,0}$, $w \in W$. Then $e_w(\lambda) = e_I(w\lambda)$ $(w \in W)$. Moreover,

$$e_I(\lambda) = \prod_{\alpha \in \Delta^{++}} d_\alpha(\lambda)^{-1} = d(\lambda)^{-1}. \tag{49}$$

Proof. First we establish (49) by a technique due to Harish-Chandra. Let us fix $\lambda \in \mathfrak{a}^*_{c,0}$ such that $\mathrm{Im}(\lambda,\alpha) < 0$ for all $\alpha \in \Delta^{++}$. Select $H \in \mathfrak{a}^+$ such that $\exp H \in A(C_0)$, and write $a(t) = \exp(tH)$ $(t \geqslant 1)$. If $w \in W$, $w \neq I$ then $\mathrm{Im}(w\lambda-\lambda) > 0$ on $\mathfrak{a}^+$, hence by (46) it follows that $\exp(t(\rho-i\lambda)(H))\phi_{w,\lambda}(a(t)) \to 0$ $(t \to +\infty)$. By (46), (48) we obtain that

$$\lim_{t\to+\infty} a(t)^{\rho-i\lambda}\ \phi_\lambda(a(t)) = e_I(\lambda)\Phi_I(\lambda,0). \tag{50}$$

On the other hand

$$a(t)^{\rho-i\lambda}\ \phi_\lambda(a(t)) = \int_{\bar N} e^{(i\lambda-\rho)\{H(Ad(a(t),\bar n))-H(\bar n)\}} e^{-2\rho H(n)} d\bar n. \tag{51}$$

In his paper [2] Harish-Chandra showed that the integral on the right hand side of (51) converges to the absolutely convergent integral

$$c(\lambda) = \int_{\bar{N}} e^{-(i\lambda+\rho)H(\bar{n})} d\bar{n} \tag{52}$$

(see Harisch-Chandra [2, p. 287, p. 291]). Consequently $c(\lambda) = e_I(\lambda)\Phi_I(\lambda,0)$, and in view of (36), (30) this proves (49) for λ with $\mathrm{Im}(\lambda,\alpha) < 0$ (all $\alpha \in \Delta^{++}$), and hence by analytic continuation for all $\lambda \in \mathfrak{a}^*_{c,0}$.

By the same argument as above, for $\lambda \in \mathfrak{a}^*_c$ with $\mathrm{Im}(w\lambda,\alpha) < 0$ we have

$$\lim_{t\to+\infty} a(t)^{\rho-iw\lambda} \phi_\lambda(a(t)) = e_w(\lambda)\Phi_w(\lambda,0),$$

and since $\phi_\lambda = \phi_{w\lambda}$ we obtain that

$$e_w(\lambda)\Phi_w(\lambda,0) = e_I(w\lambda)\Phi_I(w\lambda,0). \tag{53}$$

By analytic continuation (53) holds for all $\lambda \in \mathfrak{a}^*_{c,0}$ and in view of the proposition below this completes the proof.

<u>Proposition 4.26</u>. For all $\lambda \in \mathfrak{a}^*_c$ we have

$$\Phi_w(\lambda,0) = \Phi_I(w\lambda,0). \tag{54}$$

<u>Proof</u>. We start with the observation that for $\alpha \in \Delta^{++}\setminus\Delta^{++}(w)$ we have $\tilde{I}_\alpha(\lambda) = \tilde{I}_{w\alpha}(w\lambda)$ whereas for $\alpha \in \Delta^{++}(w)$ we have $\tilde{I}_\alpha(\lambda) = \tilde{I}_{-w\alpha}(-w\lambda)$ (this can be proved by showing that similar statements hold for $d_\alpha(\lambda)$, $I_\alpha(\lambda)$). It follows that

$$\tilde{I}_w(-\lambda)\tilde{I}_{w'}(\lambda) =$$

$$= \prod_{\alpha\in\Delta^{++}(w)} \tilde{I}_\alpha(-\lambda) \prod_{\alpha\in\Delta^{++}(w')} \tilde{I}_\alpha(\lambda) =$$

$$= \prod_{\alpha\in\Delta^{++}(w)} \tilde{I}_{-w\alpha}(w\lambda) \prod_{\alpha\in\Delta^{++}(w')} \tilde{I}_{w\alpha}(w\lambda) =$$

$$= \prod_{\alpha \in \Delta^{++}} \tilde{I}_\alpha(w\lambda) = \tilde{I}_{I'}(w\lambda),$$

where we have used that $\Delta^{++} = w(-\Delta^{++}(w) \cup \Delta^{++}(w'))$. By Theorem 4.20 this proves (54).

Now define the subset $'\mathfrak{a}_c^*$ of $\mathfrak{a}_c^*$ by

$$'\mathfrak{a}_c^* = \{\lambda \in \mathfrak{a}_c^*;\ \lambda(H_{\alpha,0}) \notin i\mathbb{Z} \text{ for all } \alpha \in \Delta^{++}\}. \tag{55}$$

Clearly $'\mathfrak{a}_c^*$ is the complement of a Weyl group invariant locally finite union of hyperplanes in $\mathfrak{a}_c^*$, and so it is a connected dense open subset of $\mathfrak{a}_c^*$. Obviously if $w \in W$, then

$$'\mathfrak{a}_c^* = \{\lambda \in \mathfrak{a}_c^*;\ d(w\lambda) \neq 0\}.$$

Finally from Lemmas 4.24 and 4.25 we obtain:

<u>Theorem 4.27</u>. If $\lambda \in {}'\mathfrak{a}_c^*$, $a \in A(C_0)$ then:

$$\phi_\lambda(a) = \sum_{w \in W} a^{iw\lambda-\rho}\, d(w\lambda)^{-1} \Phi_w(\lambda, z_w(a)). \tag{56}$$

Formula (56) corresponds to Harish-Chandra's asymptotic expansion for $\phi_\lambda(a)$, when $a \to \infty$ in A^+. In fact, let S be the set of simple roots corresponding to the choice of positive roots Δ^+, and let L denote the set of all sums $\Sigma_{\alpha \in S}\, n_\alpha \alpha$ (n_α nonnegative integers). The function $\Phi_I\colon \mathfrak{a}_c^* \times U_W \to \mathbb{C}$ is holomorphic, so using the power series expansion of $\Phi_I(\lambda, .)$ at 0 we see that there exist holomorphic functions $\Gamma'_\mu\colon \mathfrak{a}_c^* \to \mathbb{C}$ such that

$$\Phi_I(\lambda, z_I(a)) = \sum_{\mu \in L} \Gamma'_\mu(\lambda) e^{-\mu(\log a)}. \tag{57}$$

Obviously this series is an asymptotic expansion. Moreover, it converges uniformly absolutely in all derivatives on $A(C_0)$. Now let ω be the Casimir operator of G. Since $\gamma(\omega,i\lambda) = -(\lambda,\lambda) - (\rho,\rho)$ (cf. Harish-Chandra [2, p. 271]) the function $\phi_{I,\lambda}(a) = \exp((i\lambda-\rho)\log a)\Phi_I(\lambda,z_I(a))$ satisfies the differential equation

$$\Delta(\omega)\phi_{I,\lambda} = (-(\lambda,\lambda) - (\rho,\rho))\phi_{I,\lambda}. \tag{58}$$

The radial part of ω is given by

$$\Delta(\omega) = L_A + \sum_{\alpha\in\Delta^+} m(\alpha)\ (\coth\alpha)(.)H_\alpha, \tag{59}$$

where L_A denotes the Laplacian with respect to the inner product (,) on $\mathfrak{a}$, and where $H_\alpha \in \mathfrak{a}$ is viewed as a first order invariant differential operator on A. (cf. Harish-Chandra [2, p. 270]). Substituting (57) and (59) in (58) and using the powerseries expansion $\coth\alpha = 1 + 2\Sigma_{k\geqslant 1}\exp(-2k\alpha)$ we obtain the following recurrence relations for $\Gamma'_\mu(\lambda)$:

$$\Gamma'_0(\lambda) = \Phi_I(\lambda,0),$$

$$\{(\mu,\mu) - 2i(\mu,\lambda)\}\Gamma'_\mu(\lambda) =$$

$$= 2\sum_{\alpha\in\Delta^+} m(\alpha) \sum_{k\geqslant 1} \Gamma'_{\mu-2k\alpha}(\lambda)\{(\mu+\rho-2k\alpha,\alpha) - (\alpha,\lambda)\}. \tag{60}$$

Now let Γ_μ be the functions determined by the recurrence relations (60) and the condition $\Gamma_0 \equiv 1$. The Γ_μ are rational functions on $\mathfrak{a}_c^*$, and we have

$$\Gamma'_\mu(\lambda) = \Phi_I(\lambda,0)\Gamma_\mu(\lambda).$$

Since $\phi_{w\lambda} = \phi_\lambda$ for all $w \in W$, it follows from Lemma 4.24 and Lemma 4.25 that $\phi_{w,\lambda} = \phi_{I,w\lambda}$ for all $\lambda \in \mathfrak{a}_c^*$, $w \in W$. Also $\Phi_I(\lambda,0) = K^{-1}\tilde{I}(\lambda) = d(\lambda)c(\lambda)$ and hence we have proved:

Theorem 4.28. Let Γ_μ $(\mu \in L)$ be the rational functions on $\mathfrak{a}_c^*$ defined by the recurrence relations (60) and $\Gamma_0 \equiv 1$. If $\lambda \in \mathfrak{a}_{c,0}^*$, $a \in A(C_0)$ we have Harish-Chandra's formula (cf. [2]):

$$\phi_\lambda(a) = \sum_{w\in W} c(w\lambda)e^{(iw\lambda-\rho)\log a} \sum_{\mu\in L} \Gamma_\mu(w\lambda)e^{-\mu(\log a)}.$$

Chapter 5

Asymptotics along the walls of $\overline{A^+}$

5.1 Introduction

In the previous chapter we derived the formula

$$\phi_\lambda(a) = \sum_{w \in W} d(w\lambda)^{-1} \int_{\Gamma_w(a)} e^{(i\lambda-\rho)H_{0,w}(ak)} \omega(k), \qquad (1)$$

valid for $a \in A(C_0)$, $\lambda \in {}'\mathfrak{a}_c^*$. It does not give us any information when a varies in a neighbourhood of a wall of $\overline{A^+}$. In fact by a careful analysis of the radial differential equations one can show that the functions $\phi_{w,\lambda}: A(C_0) \to \mathbb{C}$ defined by (4.2) extend real analytically to the whole of A^+ but become singular at the walls. A good example of what might happen to the cycles $\Gamma_w(a)$ if a tends to a wall is provided by the case of $SL(2,\mathbb{R})$. In Figure 3.1 we see that the singularities of the function $\bar{n} \to H(a\bar{n}a^{-1}) - H(\bar{n})$ correspond to the points i, -i, $a^\alpha i$, $-a^\alpha i$. Thus if $a^\alpha \downarrow 1$ then the cycle γ_I is pinched by i, $a^\alpha i$ and by -i, $-a^\alpha i$.

In this chapter we will derive formulas like (1), expressing ϕ_λ as a sum of integrals, valid in certain neighbourhoods of walls of $\overline{A^+}$. To explain the main results, let us first introduce some notations. F will be a fixed subset of S, the collection of simple roots of Δ^{++}. We set

$$\mathfrak{a}_F = \bigcap_{\alpha \in F} \ker \alpha, \; A_F = \exp \mathfrak{a}_F,$$

and we will be concerned with the wall $A_F \cap \overline{A^+}$. Note that $\mathfrak{a}_F$ is the split component of the standard parabolic subalgebra $\mathfrak{p}_F$ determined by F (for standard facts concerning parabolic subalgebras and subgroups, we refer the reader to Varadarajan [2]). Moreover, we write

$$^*\mathfrak{a} = (\mathfrak{a}_F)^{\perp} \cap \mathfrak{a}, \qquad ^*A = \exp(^*\mathfrak{a}),$$

and if $a_0 \in A$ we set:

$$a_0 = {}^*aa \qquad (^*a \in {}^*A,\ a \in A_F).$$

Now let $\Delta_F = \Delta \cap \mathbb{Z}.F$, $\Delta_F^+ = \Delta_F \cap \Delta^+$ and $\Delta_F^{++} = \Delta_F \cap \Delta^{++}$. If $^*C > 0$, $C_F > 0$ we put

$$A(F,{}^*C,C_F) =$$

$$= \{a_0 \in A;\ |\alpha(\log{}^*a)| < {}^*C,\ a^{\beta} > C_F \text{ for } \alpha \in \Delta^{++},$$

$$\beta \in \Delta^{++}\backslash\Delta\ \}.$$

A basic result of this chapter is the following (see Theorem 5.5, Lemma 5.9). If $^*C > 0$ is arbitrary then there exists a $C_F > 0$, cycles $\Gamma_{F,v}(a_0)$ $(v \in W)$ depending smoothly on $a_0 \in A(F,{}^*C,C_F)$ and branches $H_{F,v}$ of H such that the functions

$$\phi_{F,v,\lambda}(a_0) = \int_{\Gamma_{F,v}(a_0)} e^{(i\lambda-\rho)H_{F,v}(a_0k)}\,\omega(k)$$

satisfy the radial differential equations on $A^+ \cap A(F,{}^*C,C_F)$. Now let the function $d_F\colon \mathfrak{a}_c^* \to \mathbb{C}$ be defined by

$$d_F(\lambda) = \prod_{\alpha \in \Delta^{++}\backslash\Delta_F} d_{\alpha}(\lambda),$$

and let $d_{F,v}: \mathfrak{a}_c^* \to \mathbb{C}$, $\lambda \to d_F(v\lambda)$ $(v \in W)$. Then the functions $\phi_{F,v,\lambda}$ and $d_{F,v}$ depend on the coset $W_F v$ of v in $W_F\backslash W$ only (here W_F denotes the centralizer of $\mathfrak{a}_F$ in W). If $\sigma = W_F v$ we shall also write $\phi_{F,\sigma,\lambda}$ and $d_{F,\sigma}$ for those functions. With these notations we have:

$$\phi_\lambda(a_0) = \sum_{\sigma \in W_F\backslash W} d_{F,\sigma}(\lambda)^{-1} \phi_{F,\sigma,\lambda}(a_0), \tag{2}$$

for $a_0 \in A(F,{}^*C,C_F)$, $\lambda \in {}'\mathfrak{a}_c^*$ (see Theorem 5.14). We will use this formula to study the asymptotic behaviour of $\phi_\lambda(a_0)$ when $a_0 \to \infty$ in $A(F,{}^*C,C_F)$. By the latter statement we mean that $a_0 \in A(F,{}^*C,C_F)$ and $a^\alpha \to +\infty$ for every $\alpha \in \Delta^{++}\backslash\Delta_F$. Observe that since F and ${}^*C$ are allowed to be chosen freely the set $\overline{A^+}$ can be covered by a finite number of sets $A(F_j,{}^*C_j,C_{F,j})$, and thus in principle the asymptotic behaviour of ϕ_λ is determined by the asymptotic behaviour of the $\phi_{F_j,v,\lambda}(a_0)$ when $a_0 \to \infty$ in $A(F_j,{}^*C_j,C_{F,j})$.

As we will show in Section 5.3 there exist functions $\Phi'_{F,v}$, holomorphic in the first and last and real analytic in the second variable, such that for $\lambda \in \mathfrak{a}_c^*$, $a_0 \in A(F,{}^*C,C_F)$ we have:

$$\phi_{F,v,\lambda}(a_0) = a^{iv\lambda-\rho}.\Phi'_{F,v}\left(\lambda,{}^*a,(a^{-\alpha})_{\alpha \in \Delta^{++}\backslash\Delta_F}\right) \tag{3}$$

Hence $\phi_{F,v,\lambda}(a_0)$ has a series expansion which is asymptotic for $a_0 \to \infty$ in $A(F,{}^*C,C_F)$. The principal term is given by

$$\phi_{F,v,\lambda}(a_0) \sim$$

$$\sim a^{iv\lambda-\rho}.K_F.\phi_{v^{-1}(F)}(\lambda : v^{-1}({}^*a)).\tilde{I}_w(-\lambda).\tilde{I}_{w'}(\lambda). \tag{4}$$

Here K_F is a positive constant, w,w' are the Weyl group elements determined by

$$\Delta^{++}(w) = \Delta^{++}(v)\backslash v^{-1}(\Delta_F),$$

$$\Delta^{++}(w') = \Delta^{++}\backslash\{\Delta^{++}(v) \cup v^{-1}(\Delta_F)\},$$

and $\phi_{v^{-1}(F)}(\mu:.)$ denotes the elementary spherical function associated with the reductive pair $(G_{v^{-1}(F)}, K_{v^{-1}(F)})$ and the linear functional $\mu \in \mathfrak{a}_c^*$ (here we have written $G_{v^{-1}(F)}$ for the centralizer of $v^{-1}(\mathfrak{a}_F)$ in G, and $K_{v^{-1}(F)}$ for $G_{v^{-1}(F)} \cap K$). For this result, we refer the reader to Theorem 5.7.

Let us now expose the structure of this chapter. In Section 5.1 we discuss some preliminaries. In Section 5.3 the cycles $\Gamma_{F,v}(a_0)$ are constructed. As in Chapter 4 this is achieved by studying a pull back of the map $k \to H(a_0k)$, but now under ξ: $K_{v^{-1}(F)} \times \overline{N}_w \times \overline{N}_{w'} \to K$, $(k,\overline{n},\overline{n}') \to \overline{v}k\kappa(\overline{n}\,\overline{n}')$. This pull back is equal to

$$H({}^*\tilde{a}kAd(\tilde{a},\overline{n}\,\overline{n}')) - H(\overline{n}\,\overline{n}') + v^{-1}(\log a),$$

where we have written ${}^*\tilde{a} = v^{-1}({}^*a)$, $\tilde{a} = v^{-1}(a)$. By a further pull back under a suitable coordinate transformation $\eta(a_0)$ in $K_{v^{-1}(F)} \times \overline{N}_w \times \overline{N}_{w'}$, we bring $H({}^*\tilde{a}kAd(\tilde{a},\overline{n}\,\overline{n}')) - H(\overline{n}\,\overline{n}')$ in a form that is a perturbation of

$$H({}^*\tilde{a}k) + H(\overline{n}) - H(\overline{n}'). \tag{5}$$

The cycle $\Gamma_{F,v}(a_0)$ is then obtained by transportation under $\xi \circ \eta(a_0)$ of the cycle $id(K_{v^{-1}(F)}) \times \gamma_w \times \gamma_{w'}$ over which (5) has a branch. We end Section 5.3 with a proof of (3) and (4).

In Section 5.4 formula (2) for ϕ_λ is derived. The idea is as follows. The functions $\phi_{F,v,\lambda}$ satisfy the radial differential equations on $A = A(F,{}^*C,C_F) \cap A(C_0)$, and hence are linear combinations of the $\phi_{u,\lambda}$ ($u \in W$) with coefficients depending holomorphically on $\lambda \in \mathfrak{a}^*_{c,0}$. Comparing the principal terms of asymptotic expansions these coefficients can be identified and we obtain

$$d_{F,v}(\lambda)^{-1}\phi_{F,v,\lambda}(a_0) = \sum_{u\in W_F} d(uv\lambda)^{-1}\phi_{uv,\lambda}(a_0).$$

By (4.5) this proves (2) on A, hence by analytic continuation on $A(F,{}^*C,C_F)$ as well.

Formulas (2) and (3) lead to a converging series expansion for $\phi_\lambda(a_0)$ which is asymptotic when $a_0 \to \infty$ in $A(F,{}^*C,C_F)$. This is the asymptotic expansion Trombi and Varadarajan obtained in [1, §2.11] using a certain system of radial differential equations. Thus we see that the expansion they gave actually converges. In the last sections 5.5 and 5.6 we examine this expansion and give estimates for the remainder terms, with certain properties of uniformity with respect to real values of λ. If λ remains bounded we obtain the same result as Trombi and Varadarajan (see Theorem 5.21). If λ is allowed to vary freely we have not been able to obtain their result. However, if $\|\lambda\|^{-1}\lambda$ varies in a compact subset of the set $\mathfrak{a}^{*\prime}$ of regular points we have obtained results that are sharper. We hope that a synthesis of our techniques with those of Duistermaat - Kolk - Varadarajan [2] will eventually lead to a full understanding of the asymptotic behaviour of $\phi_\lambda(a)$ in a and λ simultaneously.

Remark: In their research for [2], DKV independently used similar "rescaling arguments" as we did by our definition of

η_a, p. 4 - 26.

5.2 Preparations

In this section Y is a fixed element of $\mathfrak{a}$. If L is any subgroup of G, L_Y denotes the centralizer of Y in L. Similarly we write $\mathfrak{l}_Y$ for the centralizer of Y in a Lie subalgebra $\mathfrak{l}$ of $\mathfrak{g}$. Thus G_Y, K_Y, N_Y, $\overline{N}_Y$ are closed subgroups of G; their Lie algebras are $\mathfrak{g}_Y$, $\mathfrak{k}_Y$, $\mathfrak{n}_Y$ and $\overline{\mathfrak{n}}_Y$ respectively. Writing $\Delta_Y^{++} =$ $= \{\alpha \in \Delta^{++};\ \alpha(Y) = 0\}$, we have:

$$\mathfrak{n}_Y = \sum_{\alpha \in \Delta_Y^{++}} \mathfrak{n}_\alpha, \qquad \overline{\mathfrak{n}}_Y = \sum_{\alpha \in \Delta_Y^{++}} \overline{\mathfrak{n}}\ .$$

Moreover, we have the direct sum of vector spaces

$$\overline{\mathfrak{n}} = \overline{\mathfrak{n}}_Y \oplus \sum_{\substack{\alpha \in \Delta^{++} \\ \alpha(Y) < 0}} \overline{\mathfrak{n}}_\alpha \oplus \sum_{\substack{\alpha \in \Delta^{++} \\ \alpha(Y) > 0}} \overline{\mathfrak{n}}_\alpha; \tag{6}$$

the three summands are subalgebras of $\overline{\mathfrak{n}}$.

Proposition 5.1. Let $Z \in \mathfrak{a}$. Then there exists a unique $w \in W$ such that

$$\{\alpha \in \Delta^{++};\ \alpha(Z) < 0\} = \Delta^{++}(w).$$

Proof. The uniqueness of w is obvious. For the existence, consider the root systems $\Phi = \Delta^{++} \cup (-\Delta^{++})$ and $\Phi_Z = \{\alpha \in \Phi;$ $\alpha(Z) = 0\}$. Φ_Z is a subsystem of Φ, and Δ_Z^{++} is a choice of positive roots for Φ_Z. Let Σ be the set of simple roots of Δ_Z^{++}. Its elements are linearly independent, hence there exists a $Z_1 \in \mathfrak{a}$ such that $\alpha(Z_1) > 0$ for every $\alpha \in \Sigma$. Since $\Delta_Z^{++} \subset \mathbb{N}.\Sigma$ it follows that $\alpha(Z_1) > 0$ for every $\alpha \in \Delta_Z^{++}$. Now write $Z(t) = Z + tZ_1$ $(t > 0)$. If t is sufficiently close to 0 we have:

$$\alpha(Z(t)) < 0 \Leftrightarrow \alpha(Z) < 0,$$

$$\alpha(Z(t)) > 0 \Leftrightarrow \alpha(Z) \geqslant 0,$$

for any $\alpha \in \Delta^{++}$. In particular, $Z(t)$ is regular and contained in some Weyl chamber $w^{-1}(\mathfrak{a}^{+})$ $(w \in W)$ of $\mathfrak{a}$ for t sufficiently small. Consequently $\Delta^{++}(w) = \{\alpha \in \Delta^{++};\ \alpha(Z) < 0\}$. Observe that $Z \in w^{-1}(\overline{\mathfrak{a}^{+}})$.

By the above proposition there exist unique elements $w, w' \in$ $\in W$ such that for every $\alpha \in \Delta^{++}$ we have:

$$\begin{aligned} &\alpha(Y) < 0 \Leftrightarrow w(\alpha) \in -\Delta^{++}, \\ &\alpha(Y) > 0 \Leftrightarrow w'(\alpha) \in -\Delta^{++}. \end{aligned} \tag{7}$$

So, (6) can be rewritten as:

$$\bar{\mathfrak{n}} = \bar{\mathfrak{n}}_Y \oplus \bar{\mathfrak{n}}_W \oplus \bar{\mathfrak{n}}_{W'}.$$

Now $\bar{\mathfrak{n}}_Y \oplus \bar{\mathfrak{n}}_W$ is a subalgebra of $\bar{\mathfrak{n}}$, and so, applying Lemma 4.3 twice, we see that the map $\bar{N}_Y \times \bar{N}_W \times \bar{N}_{W'} \to \bar{N}$, $(\bar{n}_Y, \bar{n}', \bar{n}'') \to \bar{n}'\bar{n}_Y\bar{n}''$ is a diffeomorphism. Let $\bar{N}_Y, \bar{N}_W, \bar{N}_{W'}$ be oriented so that this map is orientation preserving, and let $\Omega_{Y,0}, \Omega_{W,0}, \Omega_{W',0}$ be the Riemannian volume forms associated with the invariant oriented Riemannian structures on $\bar{N}_Y, \bar{N}_W, \bar{N}_{W'}$ induced by the inner products $(\ ,\ )$ on $\bar{\mathfrak{n}}_Y, \bar{\mathfrak{n}}_W, \bar{\mathfrak{n}}_{W'}$. As usual we denote the holomorphic extensions of these forms by the same symbols.

<u>Lemma 5.2</u>. The map $\bar{N}_Y \times \bar{N}_W \times \bar{N}_{W'} \to \bar{N}$, $(\bar{n}_Y, \bar{n}', \bar{n}'') \to \bar{n}'\bar{n}_Y\bar{n}''$ is a diffeomorphism. The forms $\Omega_{Y,0} \wedge \Omega_{W,0} \wedge \Omega_{W',0}$ and Ω_0 correspond under under this diffeomorphism.

Proof. Denote the above map by ε. By the preceding discussion it is an orientation preserving diffeomorphism and so there exists a smooth function $f\colon \bar{N}_Y\times\bar{N}_W\times\bar{N}_{W'} \to \mathbb{R}^+$ such that

$$\varepsilon^*(\Omega_0) = f(\bar{n}_Y,\bar{n}',\bar{n}'')\Omega_{Y,0}\wedge\Omega_{W,0}\wedge\Omega_{W',0}.$$

If $\bar{n}' \in \bar{N}_W$, $\bar{n}'' \in \bar{N}_{W'}$, we have

$$\lambda(\bar{n}')\circ\rho(\bar{n}'')\circ\varepsilon = \varepsilon\circ(\mathrm{id}\times\lambda(\bar{n}')\times\rho(\bar{n}''))$$

and since Ω_0 is bi-invariant, whereas $\Omega_{Y,0}\wedge\Omega_{W,0}\wedge\Omega_{W',0}$ is invariant under $\mathrm{id}\times\lambda(\bar{n}')\times\rho(\bar{n}'')$ it follows that $f(\bar{n}_Y,\bar{n}',\bar{n}'') = f(\bar{n}_Y,e,e)$. Now G normalizes the subalgebra

$$\sum_{\alpha\in\Delta,\alpha(Y)>0} \mathfrak{g}_\alpha$$

of $\mathfrak{g}$, hence $\bar{N}_Y$ normalizes the intersection $\bar{\mathfrak{n}}_W$ of this subalgebra with $\bar{\mathfrak{n}}$. Similarly $\bar{N}_Y$ normalizes $\bar{\mathfrak{n}}_{W'}$. It follows that $\bar{N}_Y$ normalizes $\bar{N}_W$, and by nilpotence of $\bar{N}$, we have $Ad(\bar{n}_Y)^*(\Omega_{W,0}) = \Omega_{W,0}$ for $\bar{n}_y \in \bar{N}_Y$. If $\bar{n}_Y \in \bar{N}_Y$, then

$$\lambda(\bar{n}_y)\circ\varepsilon = \varepsilon\circ(\lambda(\bar{n}_y)\times Ad(\bar{n}_y)\times\mathrm{id}),$$

and Ω_0 is invariant under $\lambda(\bar{n}_y)$ whereas $\Omega_{y,0}\wedge\Omega_{W,0}\wedge\Omega_{W',0}$ is invariant under $\lambda(\bar{n}_y)\times Ad(\bar{n}_y)\times\mathrm{id}$. Therefore $f(\bar{n}_y,e,e) = f(e,e,e)$, showing that f is constant. The derivative $d\varepsilon(e,e,e)\colon \bar{\mathfrak{n}}_Y\times\bar{\mathfrak{n}}_W\times\bar{\mathfrak{n}}_{W'} \to \bar{\mathfrak{n}}$ of ε at (e,e,e) is given by $(U_Y,U',U'') \to U' + U_Y + U''$. Therefore $f \equiv 1$ and the proof is complete.

G_Y is a reductive subgroup of G. It has an Iwasawa decomposition $G_Y = K_YAN_Y$, and the corresponding map $G_Y \to K_Y$ is equal to the restriction of κ to G_Y. Hence the map $\phi_0\colon M\times\bar{N}_Y \to K_Y$, $(\bar{m},\bar{n}_Y) \to m\kappa(\bar{n}_Y)$ is a diffeomorphism onto a dense open subset of

K_Y. We give K_Y the orientation that makes this map orientation preserving. Now let ω_Y be the invariant $\dim(K_Y)$-form on K_Y with

$$\int_{K_Y} \omega_Y = 1.$$

As in Section 4.4 we have that

$$(\phi_0)^*(\omega_Y) = \mathcal{K}_Y^{-1} e^{-2\rho_Y H(\overline{n}_Y)} \omega_M \wedge \Omega_{Y,0},$$

where $\rho_Y = \Sigma_\alpha \rho_\alpha$ (summation over $\Delta^{++}(Y)$), and where $\mathcal{K}_Y$ is the positive real number given by the absolutely convergent integral

$$\mathcal{K}_Y = \int_{\overline{N}_Y} e^{-2\rho_Y H(\overline{n}_Y)} \Omega_{Y,0}.$$

<u>Lemma 5.3</u>. Let $\mathcal{O}_Y$ be the image of the map $\phi_0 \colon M \times \overline{N}_Y \to m\kappa(\overline{n}_Y)$. Then the map $\eta \colon \mathcal{O}_Y \times \overline{N}_W \times \overline{N}_{W'} \to K$, given by

$$(k_Y, \overline{n}', \overline{n}'') = \kappa(\overline{n}' k_Y \overline{n}'')$$

is an orientation preserving diffeomorphism onto a dense open subset of K. Moreover, if $(m, \overline{n}_Y, \overline{n}', \overline{n}'') \in M \times \overline{N}_Y \times \overline{N}_W \times \overline{N}_{W'}$, $k_Y = m\kappa(\overline{n}_Y)$, we have

$$\eta^*(\omega) = \mathcal{K}_Y \mathcal{K}^{-1} e^{-2\rho H(\overline{n}' k_Y \overline{n}'') - 2\rho_W H(\overline{n}_Y)} \omega_Y^*. \qquad (8)$$

Here we have written ω_Y^* for $\omega_Y \wedge \Omega_{W,0} \wedge \Omega_{W',0}$.

<u>Proof</u>. Consider the diagram:

$$\begin{array}{ccc}
\mathcal{O}_Y \times \overline{N}_W \times \overline{N}_{W'} & \xrightarrow{\ \eta\ } & K \\
\phi \uparrow & & \\
M \times \overline{N}_Y \times \overline{N}_W \times \overline{N}_{W'} & & \uparrow \delta \\
\psi \uparrow & & \\
M \times \overline{N}_Y \times \overline{N}_W \times \overline{N}_{W'} & \xrightarrow{\ \varepsilon\ } & M \times \overline{N}
\end{array}$$

where ϕ, ψ, ε, δ are the maps defined by:

$$\phi(m,\bar{n}_Y,\bar{n}',\bar{n}'') = (m\kappa(\bar{n}_Y),\bar{n}',\bar{n}''),$$

$$\psi(m,\bar{n}_Y,\bar{n}',\bar{n}'') = (m,\bar{n}_Y,m\bar{n}'m^{-1},Ad(h(\bar{n}_Y))\tau_{w'}(\nu(\bar{n}_Y))\bar{n}''),$$

$$\varepsilon(m,\bar{n}_Y,\bar{n}',\bar{n}'') = (m,\bar{n}'n_Y\bar{n}'')$$

$$\delta(m,\bar{n}) = m\kappa(\bar{n}).$$

Here $\tau_{w'}$ is the map $\bar{w}'^{-1}N\bar{w}'\times\bar{N}_{w'} \to \bar{N}_{w'}$ defined by formula (4.12). Observe that ν maps G_Y into $N_Y \subset \bar{w}'^{-1}N\bar{w}'$. Now $\delta\varepsilon(m,\bar{n}_Y,\bar{n}',\bar{n}'') =$ $= m\kappa(\bar{n}'\bar{n}_Y\bar{n}'')$, whereas

$$\eta\phi\psi(m,\bar{n}_Y,\bar{n}',\bar{n}'') =$$

$$= \kappa\,[m\bar{n}'m^{-1}m\kappa(\bar{n}_Y)h(\bar{n}_Y)\tau_{w'}(\nu(\bar{n}_Y),\bar{n}'')h(\bar{n}_Y)^{-1}],$$

and since $\tau_{w'}(\nu(\bar{n}_Y),\bar{n}'') \equiv \nu(\bar{n}_Y)\bar{n}''$ mod N it follows that $\delta\varepsilon = \eta\phi\psi$. Hence the diagram commutes, and since ϕ, ψ, ε are orientation preserving diffeomorphisms, whereas δ is an orientation preserving diffeomorphism onto a dense open subset of K, this proves the first assertion of the lemma.

Writing $\Omega^* = \omega_M\wedge\Omega_{Y,0}\wedge\Omega_{w,0}\wedge\Omega_{w',0}$, we have that

$$\phi^*(\omega_Y^*) = K_Y^{-1}e^{-2\rho_Y H(\bar{n}_Y)}\Omega^*,$$

$$\psi^*(\Omega^*) = e^{-2\rho_{w'}H(\bar{n}_Y)}\Omega^*,$$

$$\varepsilon^*(\omega_M\wedge\Omega_0) = \Omega^*,$$

$$\delta^*(\omega) = K^{-1}e^{-2\rho H(\bar{n})}\omega_M\wedge\Omega_0.$$

Now let $g\colon M\times\bar{N}_Y\times\bar{N}_w\times\bar{N}_{w'} \to \mathbb{R}^+$ be the smooth function determined by

$$\xi^*(\omega)(m\kappa(\bar{n}_Y),\bar{n}',\bar{n}'') = g(m,\bar{n}_Y,\bar{n}',\bar{n}'')\omega_Y^*.$$

Then we obtain that

$$\phi^*\eta^*(\omega) = K_Y^{-1}e^{-2\rho_Y H(\bar{n}_Y)}g(m,\bar{n}_Y,\bar{n}',\bar{n}'')\Omega^*,$$

hence

$$\psi^*\phi^*\eta^*(\omega) = K_Y^{-1}e^{-2\rho_Y H(\bar{n}_Y)-2\rho_{W'}H(\bar{n}_Y)}g(\psi(m,\bar{n}_Y,\bar{n}',\bar{n}''))\Omega^*.$$

On the other hand, we have that

$$\varepsilon^*\delta^*(\omega) = K^{-1}e^{-2\rho H(\bar{n}'\bar{n}_Y\bar{n}'')}\Omega^*.$$

Therefore, writing $\nu = \nu(\bar{n}_Y)$, $h = h(\bar{n}_Y)$, and using that

$$\psi^{-1}(m,\bar{n}_Y,\bar{n}',\bar{n}'') = (m,\bar{n}_Y,m^{-1}\bar{n}'m\tau_{W'}(\nu^{-1})Ad(h^{-1})\bar{n}''),$$

we obtain that

$$KK_Y^{-1}\exp[(-2\rho_Y-2\rho_{W'})H(\bar{n}_Y)]g(m,\bar{n}_Y,\bar{n}',\bar{n}'') =$$

$$= \exp[-2\rho H(m^{-1}\bar{n}'m\bar{n}_Y\tau_{W'}(\nu^{-1})Ad(h^{-1})\bar{n}'']$$

$$= \exp[-2\rho\{H(\bar{n}'m\bar{n}_Y\nu^{-1}h^{-1}\bar{n}'') + H(\bar{n}_Y)\}]$$

$$= \exp[-2\rho H(\bar{n}'m\kappa(\bar{n}_Y)\bar{n}'') - 2\rho H(\bar{n}_Y)].$$

Since $\rho = \rho_Y + \rho_W + \rho_{W'}$, this proves

$$g(m,\bar{n}_Y,\bar{n}',\bar{n}'') = K_Y K^{-1}\exp[-2\rho H(\bar{n}'k_Y\bar{n}'') - 2\rho_W H(\bar{n}_Y)]$$

with $k_Y = m\kappa(\bar{n}_Y)$. Hence (8).

Lemma 5.4. Let ξ: $K_Y\times\bar{N}_W\times\bar{N}_{W'} \to K$ be defined by

$$\xi(k_Y,\bar{n}',\bar{n}'') = k_Y\kappa(\bar{n}'\bar{n}'').$$

Then there exists a real analytic function $J: \overline{N}_w \times \overline{N}_{w'} \to \mathbb{R}$ such that

$$\xi^*(\omega) = J(\overline{n}',\overline{n}")\omega_Y \wedge \Omega_{w,0} \wedge \Omega_{w',0}. \tag{9}$$

Moreover, if $\overline{n}" \in \overline{N}_{w'}$, we have

$$J(e,\overline{n}") = \kappa_Y \kappa^{-1} e^{-2\rho H(\overline{n}")}. \tag{10}$$

Proof. There exists a real analytic funtion $f: K_Y \times \overline{N}_w \times \overline{N}_{w'} \to \mathbb{R}$ such that

$$\xi^*(\omega) = f(k_Y,\overline{n}',\overline{n}")\omega_Y \wedge \Omega_{w,0} \wedge \Omega_{w',0}.$$

Now $\lambda_k \circ \xi = \xi \circ (\lambda_k \times id \times id)$ for $k \in K_Y$, and since ω is left K-invariant, whereas $\omega_Y \wedge \Omega_{w,0} \wedge \Omega_{w',0}$ is invariant under $\lambda_k \times id \times id$ for $k \in K_Y$, it follows that $f(k_Y,\overline{n}',\overline{n}") = f(e,\overline{n}',\overline{n}")$. This proves (9) with $J = f(e,.,.)$.

For (10), observe that $J(e,\overline{n}")$ is determined by

$$d\xi(e,e,\overline{n}")^*(\omega_{\kappa(\overline{n}")}) = J(e,\overline{n}")(\omega_Y)_{(e,e,\overline{n}")}.$$

Here the derivative $d\xi(e,e,\overline{n}")$ of ξ at $(e,e,\overline{n}")$ is the map $\mathfrak{k}_Y \times \overline{\mathfrak{n}}_w \times T_{\overline{n}"}(\overline{n}_{w'}) \to T_{\kappa(\overline{n}")}K$ given by

$$(\delta k_Y,\delta\overline{n}',\delta\overline{n}") \to d\kappa(\overline{n}")\{d\rho_{\overline{n}"}(e)(\delta k_Y + \delta\overline{n}') + \delta\overline{n}"\}$$

and this map is also the derivative of η at $(e,e,\overline{n}")$. But by Lemma 5.3 we have that

$$d\eta(e,e,\overline{n}")^*(\omega_{\kappa(\overline{n}")}) = \kappa_Y \kappa^{-1} e^{-2\rho H(\overline{n}")}(\omega_Y^*)_{(e,e,\overline{n}")}.$$

This proves (10).

5.3 Construction of the cycles $\Gamma_{F,v}$

Recall the subset F of S. For technical reasons we fix an element $X \in \mathfrak{a}_F$ such that $\beta \in \Delta \backslash \Delta_F \Rightarrow \beta(X) \neq 0$. Thus

$$\Delta_F^{++} = \Delta_X^{++}.$$

In our constructions and definitions we will often refer to the element X. However the constructed or defined objects will never depend on the particular choice of X. We will often indicate them by the subscript F. For instance the centralizer of X in G depends on F only, and is also denoted by G_F.

In this section v will be a fixed element of W, and we write

$$Y = v^{-1}(X).$$

We will use the results of Section 5.2 for this element $Y \in \mathfrak{a}$. In particular, let w,w' be the elements of W determined by (7). Fix ${}^*C > 0$ arbitrarily. C_F will stand for a big enough positive real number, depending on F, v, ${}^*C$. In the course of the construction we will encounter conditions on its magnitude.

We now start the construction of the cycles $\Gamma_{F,v}(a_0)$ for $a_0 \in A(F,{}^*C,C_F)$. Observe that

$$K_Y = \overline{v}^{-1} K_X \overline{v},$$

and consider the map $\xi = \xi_{F,v}\colon K_Y \times \overline{N}_w \times \overline{N}_{w'} \to K$ given by

$$\xi(k,\overline{n},\overline{n}') = \overline{v} k \kappa(\overline{n}\ \overline{n}').$$

By Lemma 5.4 there exists a real analytic function $J = J_{F,v}\colon \overline{N}_w \times \overline{N}_{w'} \to K$, such that

$$\xi^*(\omega) = J(\overline{n},\overline{n}')\omega_{F,v}, \tag{11}$$

where $\omega_{F,v} = \omega_Y \wedge \Omega_{w,0} \wedge \Omega_{w',0}$. Moreover,

$$J(e,\bar{n}') = K_Y K^{-1} e^{-2\rho H(\bar{n}')} \qquad (\bar{n}' \in \bar{N}_{w'})$$

Now let $a_0 \in A(F, {}^*C, C_F)$. Then

$$\xi(k,\bar{n},\bar{n}') = H({}^*a a\bar{v}k\bar{n}\,\bar{n}') - H(\bar{n}\,\bar{n}')$$

$$= H({}^*\tilde{a}k\tilde{a}\bar{n}\,\bar{n}'\tilde{a}^{-1}) - H(\bar{n}\,\bar{n}') + \log \tilde{a}$$

where we have written ${}^*\tilde{a} = v^{-1}({}^*a) = \bar{v}^{-1*}a\bar{v}$ and $\tilde{a} = v^{-1}(a)$. As in Section 4.4 we shall first apply coordinate transformations to bring $H(a_0\,\xi(k,\bar{n},\bar{n}'))$ in suitable form. First, if ${}^*a \in {}^*A$, $k \in K_Y$, we define the map $\sigma({}^*a,k)\colon \bar{N}_w \to \bar{N}_w$ by

$$\sigma({}^*a,k) = \tau_w(\nu({}^*\tilde{a}k)^{-1}) \circ Ad(h({}^*\tilde{a}k)^{-1}).$$

Here it should be observed that ν maps ${}^*\tilde{a}k \in G_Y$ into $N_Y \subset \bar{w}^{-1}N\bar{w}$. Obviously $\sigma({}^*a,k)$ is a diffeomorphism, and we have:

$$\sigma({}^*a,k)^*(\Omega_{w,0}) = e^{2\rho_w H({}^*\tilde{a}k)}\,\Omega_{w,0}.$$

Moreover, we have

$$H({}^*\tilde{a}k\sigma({}^*a,k)(\bar{n})) =$$

$$= H({}^*\tilde{a}k\tau_w(\nu({}^*\tilde{a}k)^{-1})[h({}^*\tilde{a}k)^{-1}\bar{n}]) + H({}^*\tilde{a}k)$$

$$= H({}^*\tilde{a}k\,\nu(\tilde{a}k)^{-1}h({}^*\tilde{a}k)^{-1}\bar{n}) + H({}^*\tilde{a}k)$$

$$= H(\bar{n}) + H({}^*\tilde{a}k). \qquad (12)$$

If $a_0 \in A$, define $\eta(a_0)\colon K_Y \times \bar{N}_w \times \bar{N}_{w'} \to K_Y \times \bar{N}_w \times \bar{N}_{w'}$ by

$$\eta(a_0)(k,\bar{n},\bar{n}') = (k, \sigma({}^*a,k)\circ Ad(\tilde{a}^{-1})(\bar{n}), \bar{n}').$$

Obviously $\eta(a_0)$ is a diffeomorphism, and we have that

$$\eta(a_0)^*(\omega_{F,v}) = e^{2\rho_w H(^*\tilde{a}k)} e^{2\rho_w \log \tilde{a}} \omega_{F,v} \tag{13}$$

By (11) and (13) it follows that

$$(\xi \circ \eta(a_0))^*(\omega) =$$

$$= J(\sigma(^*a,k)Ad(\tilde{a}^{-1},\bar{n}),\bar{n}')e^{2\rho_w H(^*\tilde{a}k)} \tilde{a}^{2\rho_w} \omega_{F,v}. \tag{14}$$

Moreover, writing $H(a_0;.)$ for the map $k \to H(a_0 k)$, $K \to \mathfrak{a}$, we have

$$H(a_0;\xi\eta(a_0)(k,\bar{n},\bar{n}')) =$$

$$= H(^*\tilde{a}k\sigma(^*a,k)(\bar{n})Ad(\tilde{a},\bar{n}')) +$$

$$- H(\sigma(^*a,k)(Ad(\tilde{a}^{-1},\bar{n}))\bar{n}') + v^{-1}(\log a). \tag{15}$$

Now if $^*a$ varies in a compact subset of $^*A$, and if $(\bar{n},\bar{n}')$ varies in a compact subset of $\bar{N}_{w,c} \times \bar{N}_{w',c}$, then

$$\sigma(^*a,k)(Ad(\tilde{a}^{-1},\bar{n})) \to e,$$

$$Ad(\tilde{a},\bar{n}') \to e,$$

uniformly with respect to $(^*a,\bar{n},\bar{n}')$, when $a^\alpha \to +\infty$ for every $\alpha \in \Delta^{++} \setminus \Delta_F$. Consequently the multi-valued analytic extension of $(\bar{n},\bar{n}') \to H(a_0;\xi\eta(a_0)(k,\bar{n},\bar{n}')) - v^{-1}(\log a)$ is a perturbation of

$$H(^*\tilde{a}k\sigma(^*a,k)(\bar{n})) - H(\bar{n}'),$$

which by (12) is equal to

$$H(^{*}ak) + H(\bar{n}) - H(\bar{n}').$$

At this stage, let us introduce some notations. We write $\mathbb{C}(\Delta^{++}\backslash\Delta_Y)$ for the space of functions $\Delta^{++}\backslash\Delta_Y \to \mathbb{C}$. If $z \in \mathbb{C}(\Delta^{++}\backslash\Delta_Y)$, $\bar{n} = \exp(\Sigma_\alpha(U_\alpha + U_{2\alpha}))$ (summation over $\Delta^{++}(s)$, $s = w$ or $s = w'$), with $U_\alpha \in \mathfrak{g}_{-\alpha,c}$, $U_{2\alpha} \in \mathfrak{g}_{-2\alpha,c}$ we set

$$z.\bar{n} = \exp[\Sigma_\alpha\ z_\alpha U_\alpha + z^2 U_{2\alpha}].$$

Obviously the maps $(z,\bar{n}) \to z.\bar{n}$, $\mathbb{C}(\Delta^{++}\backslash\Delta_Y)\times\bar{N}_{s,c} \to \bar{N}_{s,c}$ $(s = w,w')$ are holomorphic. Now define the map $z_{F,v}\colon \mathfrak{a}_F \to \mathbb{C}(\Delta^{++}\backslash\Delta_Y)$ by

$$\begin{aligned}(z_{F,v}(a))_\alpha &= a^{v(\alpha)} && \text{if } \alpha \in \Delta^{++}(w),\\ &= a^{-v(\alpha)} && \text{if } \alpha \in \Delta^{++}(w'). \qquad (16)\end{aligned}$$

With these notations, and writing $\tilde{a} = v^{-1}(a) = \bar{v}^{-1}a\bar{v}$, we have that:

$$Ad(\tilde{a}^{-1},\bar{n}) = z_{F,v}(a).\bar{n} \qquad (n \in \bar{N}_{w,c})$$

$$Ad(\tilde{a},\bar{n}) = z_{F,v}(a).\bar{n} \qquad (n \in \bar{N}_{w',c})$$

Recalling that $Y = v^{-1}X$ and that (7), we see that

$$z_{F,v}(a) \to 0 \text{ if } a_0 \to \infty \text{ in } A(F,{}^{*}C,C_F).$$

If we write $z = z_{F,v}(a)$ formulas (15) and (14) become:

$$H(a_0;\xi\eta(a_0)(k,\bar{n},\bar{n}')) = \tag{17}$$

$$= H({}^*\tilde{a}k\sigma({}^*a,k)(\bar{n})(z.\bar{n})) - H(\sigma({}^*a,k)(z.\bar{n})\bar{n}') - v^{-1}(\log a),$$

and

$$(\xi\circ\eta(a_0))^*(\omega) = J(\sigma({}^*a,k)(z.\bar{n}),\bar{n}')e^{2\rho_W H({}^*ak)}a^{2\rho_W}\omega_{F,v}.$$

Moreover, observe that

$$\xi\circ\eta(a_0)(k,\bar{n},\bar{n}') = \bar{v}k\kappa(\sigma({}^*a,k)(z.\bar{n})\bar{n}').$$

The set ${}^*A({}^*C) = \{{}^*a \in {}^*A;\ |\alpha(\log{}^*a)| < {}^*C \text{ for } \alpha \in \Delta_F^{++}\}$ is relatively compact in ${}^*A$. Therefore $\sigma({}^*a,k)(\bar{n})$ varies within a compact subset of $\bar{N}_{w,c}$ as long as $\bar{n}$ does. Let $\gamma_s\colon Y_s \to \bar{N}_{s,c}\backslash S$ $(s = w,w')$ be the smooth cycles constructed in Section 4.3. We denote the branch of κ over γ_s $(s = w,w')$ corresponding to the branch H_s by κ_s. Now select an open neighbourhood $U_{F,v}$ of 0 in $\mathbb{C}(\Delta^{++}\backslash\Delta_Y)$ such that the following condition holds:

For all ${}^*a \in {}^*A({}^*C)$, $k \in K_Y$, $\bar{n} \in \mathrm{im}\gamma_w$, $\bar{n}' \in \mathrm{im}\gamma_{w'}$, $z \in U_{F,v}$ we have:

$$\sigma({}^*a,k)(z.\bar{n})\bar{n}' \notin S, \tag{18}$$

$$({}^*\tilde{a}k)\sigma({}^*a,k)(\bar{n})(z.\bar{n}') \notin S. \tag{19}$$

For (19), note that $({}^*\tilde{a}k)\sigma({}^*a,k)(\bar{n}) \equiv \kappa({}^*\tilde{a}k)\bar{n} \bmod N_c$, and so, by left K_c- and right N_c-invariance of S, we see that $({}^*\tilde{a}k)\sigma({}^*a,k)(\bar{n}) \notin S$ if $\bar{n} \in \mathrm{im}\gamma_w$.

Now write $U_{F,v,r}$ for $U_{F,v} \cap \mathbb{R}(\Delta^{++}\backslash\Delta_Y)$ (here $\mathbb{R}(\Delta^{++}\backslash\Delta_Y)$ denotes the space of functions $\Delta^{++}\backslash\Delta_Y \to \mathbb{R}$), and define ψ: $U_{F,v,r}\times{}^*A({}^*C)\times K_Y\times\bar{N}_w\times\bar{N}_{w'} \to K$ by

$$\psi(z,{}^*a,k,\bar{n},\bar{n}') = \bar{v}k\kappa(\sigma({}^*a,k)(z.\bar{n})\bar{n}').$$

Then we have

$$\xi\eta(a_0)(k,\bar{n},\bar{n}') = \psi(z_{F,v}(a),{}^*a,k,\bar{n},\bar{n}').$$

From the conditions on $U_{F,v}$ it follows that the multi-valued analytic extension of ψ has a branch ψ_v over $U_{F,v}\times{}^*A({}^*C)\times K_Y\times\bar{N}_w\times\bar{N}_{w'}$ that restricts to

$$\bar{v}k\kappa_{w'}(\bar{n}')$$

over $\{0\}\times{}^*A({}^*C)\times K_Y\times\gamma_w\times\gamma_{w'}$. Now assume that C_F is such that for any $a \in A_F$ with $a^\alpha > C_F$ for all $\alpha \in \Delta^{++}\backslash\Delta_F$ we have $z_{F,v}(a) \in U_{F,v}$. Then it follows that the map $\Gamma_{F,v}$: $A(F,{}^*C,C_F)\times K_Y\times Y_w\times Y_{w'} \to K_c$ defined by

$$\Gamma_{F,v}(a_0,.) = \psi_v(z_{F,v}(a),{}^*a)\circ(id(K_Y)\times\gamma_w\times\gamma_{w'})$$

is a smooth cycle in K_c (recall that Y_s = domain(γ_s), $s = w,w'$). Moreover, the multi-valued analytic maps

$$H({}^*\tilde{a}k\sigma({}^*a,k)(\bar{n})(z.\bar{n})), \qquad H(\sigma({}^*a,k)(z.\bar{n})\bar{n}')$$

have branches over $U_{F,v}\times{}^*A({}^*C)\times K_Y\times\gamma_w\times\gamma_{w'}$ that restrict to $H_w(\bar{n}) + H({}^*\tilde{a}k)$ and $H_{w'}(\bar{n}')$ over $\{0\}\times{}^*A({}^*C)\times K_Y\times\gamma_w\times\gamma_{w'}$. We denote these branches by the subscripts w and w' respectively. In view of (17) the multi-valued analytic continuation of

$H(a_0;.)$ has a unique branch $H_{F,v}(a_0;.)$ over $\Gamma_{F,v}(a_0) = = \Gamma_{F,v}(a_0,.)$ such that (with $z = z_{F,v}(a)$)

$$\psi_v(z_{F,v}(a),{}^*a)^*H_{F,v}(a_0;.) =$$

$$= H_w({}^*\tilde{a}k\sigma({}^*a,k)(\bar{n})(z.\bar{n})) - H_{w'}(\sigma({}^*a,k)(z.\bar{n})\bar{n}') +$$

$$+ v^{-1}(\log a).$$

Finally, the function J defined by (11) has a multi-valued analytic extension that has a branch $J_0 = J_{F,v,0}$ over $\Gamma_{F,v}$, extending the real branch, and we have:

$$\psi_v(z_{F,v}(a),{}^*a)^*(\omega) =$$

$$= \tilde{a}^{2\rho_w} e^{2\rho_w H({}^*\tilde{a}k)} J_0(\sigma({}^*a,k)(z.\bar{n}),\bar{n}')\omega_{F,v}.$$

Now let

$$\tilde{\Phi}_{F,v}(\lambda,{}^*a,z,k,\bar{n},\bar{n}') =$$

$$= \exp[(i\lambda-\rho)\{H_w({}^*\tilde{a}k\sigma({}^*a,k)(\bar{n})(z.\bar{n}')) - H_{w'}(\sigma({}^*a,k)(z.\bar{n})\bar{n}')\}]\times$$

$$\times \exp[2\rho_w H({}^*\tilde{a}k)]\, J_0(\sigma({}^*a,k)(z.\bar{n}),\bar{n}'),$$

and define the map $\Phi_{F,v}$: $\mathfrak{a}_c^*\times{}^*A({}^*C)\times U_{F,v} \to \mathbb{C}$ by

$$\Phi_{F,v}(\lambda,{}^*a,z) = \int_{\gamma_{F,v}} \tilde{\Phi}_{F,v}(\lambda,{}^*a,z,k,\bar{n},\bar{n}')\omega_{F,v} \qquad (20)$$

(where $\gamma_{F,v} = id(K_Y)\times\gamma_w\times\gamma_{w'}$). Then the map $\Phi_{F,v}$ is holomorphic in $(\lambda,z) \in \mathfrak{a}_c^*\times U_{F,v}$ and real analytic in ${}^*a \in {}^*A({}^*C)$, and it follows that

$$\int_{\Gamma_{F,v}(a_0)} e^{(i\lambda-\rho)H_{F,v}(a_0k)}\omega(k) =$$

$$= \int_{\gamma_{F,v}} \psi_v(z_{F,v}(a), {}^*a)^*(e^{(i\lambda-\rho)H(a_0;.)}\omega) =$$

$$= a^{v(i\lambda-\rho+2\rho_w)}\Phi_{F,v}(\lambda, {}^*a, z_{F,v}(a)).$$

Observing that $\Delta^{++}(w) \subset \Delta^{++}(v) \subset \Delta^{++}(w) \cup \Delta_Y^{++}$ we see that $\rho_w = \rho_v$ on $v^{-1}(\mathfrak{a}_F)$. Since $v(\rho-2\rho_v) = \rho$ (cf. the proof of Theorem 4.20) we have proved the following theorem.

Theorem 5.5. Let ${}^*C > 0$ be arbitrary. Then there exists a constant $C_F > 0$ and a family of smooth dim(K)-cycles $\Gamma_{F,v}(a_0)$: $Y(F,v) \to K_c$ depending smoothly on $a_0 \in A(F, {}^*C, C_F)$, such that the following holds. If $a_0 \in A(F, {}^*C, C_F)$, then $\lambda(a_0)$ (im $\Gamma_{F,v}(a_0)) \cap S = \emptyset$. Moreover H has a branch $H_{F,v}$ over the map $A(F, {}^*C, C_F) \times Y(F,v) \to G_c \backslash S$, $(a_0, y) \to a_0\Gamma_{F,v}(a_0)(y)$ such that

$$\phi_{F,v,\lambda}(a_0) = \int_{\Gamma_{F,v}(a_0)} e^{(i\lambda-\rho)H_{F,v}(a_0k)}\omega(k) =$$

$$= a^{iv\lambda-\rho}\Phi_{F,v}(\lambda, {}^*a, z_{F,v}(a)). \tag{21}$$

Here $z_{F,v}(a)$ is defined by (16), and $\Phi_{F,v}$ is the function ${}^*\mathfrak{a}_c^* \times {}^*A({}^*C) \times U_{F,v} \to \mathbb{C}$, holomorphic in the first and last and real analytic in the second variable, defined by (20).

Using the Taylor expansion for $z \to \Phi_{F,v}(\lambda, {}^*a, z)$ at 0 we see that the function $\phi_{F,v,\lambda}$ has an asymptotic expansion for $a_0 \to \infty$ in $A(F, {}^*C, C_F)$. The principal term of this expansion is given by

$$\phi_{F,v,\lambda}(a_0) \sim a^{iv\lambda-\rho}\Phi_{F,v}(\lambda, {}^*a, 0). \tag{22}$$

For the computation of $\Phi_{F,v}(\lambda, {}^*a, 0)$ we need the following proposition. Recall that W_F denotes the centralizer of $\mathfrak{a}_F$ in W, and write

$$W^F = \{s \in W;\ \Delta^{++}(s^{-1}) \cap \Delta_F = \emptyset\}.$$

Proposition 5.6. Each $v \in W$ decomposes uniquely as $v = uw$ with $u \in W_F$, $w \in w^F$. Moreover, we have $\Delta^{++}(w) = \{\alpha \in \Delta^{++};\ \alpha(v^{-1}X) < 0\}$.

Proof. The first assertion is a consequence of Proposition 3.9 in Borel-Tits [1]. As for the second assertion, fix $\alpha \in \Delta^{++}$. If $\alpha(v^{-1}X) < 0$ then $(v\alpha)(X) < 0$ and since $X \in \overline{\mathfrak{a}^+}$ this implies that $v\alpha \in -\Delta^{++}$. Conversely let $v\alpha \in -\Delta^{++}$, and write $\beta = -v\alpha$. Then $\beta \in \Delta^{++}(v^{-1})$ and so $\beta(X) \neq 0$. Consequently $\alpha(v^{-1}X) = -\beta(X) < 0$.

If Σ is any subset of Δ we write G_Σ for the centralizer of $\cap_{\alpha \in \Sigma} \ker \sigma$ in G. Moreover we put $K_\Sigma = K \cap G_\Sigma$. Now G_Σ is a reductive closed subgroup of G with maximal compact subgroup K_Σ; it admits the Iwasawa decomposition $G_\Sigma = K_\Sigma A(N \cap G_\Sigma)$. If $\mu \in \mathfrak{a}_c^*$ we denote the elementary spherical function of (G_Σ, K_Σ) determined by μ by $\phi_\Sigma(\mu:.)$. With these notations we have the following theorem.

Theorem 5.7. Let all notations be as in Theorem 5.5 and put $K_F = K_X K^{-1}$. Then for every $\lambda \in \mathfrak{a}_c^*$, ${}^*a \in {}^*A({}^*C)$ we have

$$\Phi_{F,v}(\lambda,{}^*a,0) =$$

$$= K_F.\phi_{w^{-1}(F)}(\lambda:w^{-1}({}^*a)).\tilde{I}_w(-\lambda).\tilde{I}_{w'}(\lambda). \tag{23}$$

Remark. From this theorem it follows that the right hand side of (22) depends on the coset W_Fv of v in $W_F\backslash W$ only. As we will see in the next section this even implies that the function $\phi_{F,v,\lambda}$ depends on the coset W_Fv only.

Proof of Theorem 5.7. By analytic continuation we have

$$J_{F,v,0}(e,\bar{n}') = K_Y K^{-1}.\exp(-2\rho_{w'}H(\bar{n}'))$$

over $\gamma_{w'}$. Substituting this in (20) and taking into account that $K_X = K_Y$ and that

$$H_w({}^*\tilde{a}k\sigma({}^*a,k)(\bar{n})) = H_w(\bar{n}) + H({}^*\tilde{a}k),$$

we obtain that

$$\Phi_{F,v}(\lambda,{}^*a,0) = K_F.\phi({}^*a).\tilde{I}_w(-\lambda).\tilde{I}_{w'}(\lambda), \tag{24}$$

$$\phi({}^*a) = \int_{K_Y} e^{(i\lambda-\rho+2\rho_w)H({}^*\tilde{a}k)}\,\omega_Y(k).$$

Now $W_Fv = W_Fw$ and hence $Y = v^{-1}X = w^{-1}X$ and $v^{-1}(F) = w^{-1}(F)$. Moreover, $w^{-1}(\Delta_F)$ is the rootsystem of the pair $(\mathfrak{g}_Y,\mathfrak{a}) = (\mathfrak{g}_{w^{-1}(F)},\mathfrak{a})$, and $w^{-1}(\Delta_F^+)$ is the choice of positive roots corresponding to $N \cap G_Y$. Consequently, if we write

$$\rho_F = \tfrac{1}{2}\sum_{\alpha\in\Delta_F^+} m(\alpha)\alpha,$$

then $w^{-1}(\rho_F)$ is the ρ corresponding to the choice $w^{-1}(\Delta_F^+)$ of positive roots. Now consider the semisimple algebra $\mathfrak{g}_{Y1} = [\mathfrak{g}_Y, \mathfrak{g}_Y]$. Then $\mathfrak{g}_{Y1} \cap \mathfrak{a}$ is the orthocomplement of $\cap_{\alpha(Y)=0}(\ker \alpha)$ in $\mathfrak{a}$. Hence $\mathfrak{g}_{Y1} \cap \mathfrak{a} = w^{-1}({}^*\mathfrak{a})$. It follows that $\mathfrak{g}_{Y1}$ admits the Iwasawa decomposition

$$\mathfrak{g}_{Y1} = \mathfrak{k}_Y + w^{-1}({}^*\mathfrak{a}) + \mathfrak{n}_Y.$$

Consider the closed subset $G_{Y1} = K_Y w^{-1}({}^*A)N_Y$ of G_Y. It is a subgroup with Lie algebra $\mathfrak{g}_{Y1}$. By the given Iwasawa decomposition it follows that H maps G_{Y1} into $w^{-1}({}^*\mathfrak{a})$. Hence we see that $H({}^*\tilde{a}k) \in w^{-1}({}^*\mathfrak{a})$ for ${}^*a \in {}^*A$, $k \in K_Y$. By Proposition 5.8 below, ρ_F and ρ have the same restriction to ${}^*\mathfrak{a}$. Hence on $w^{-1}({}^*\mathfrak{a})$ we have:

$$\rho - 2\rho_w = w^{-1}(\rho) = w^{-1}(\rho_F).$$

Consequently $\phi({}^*a) = \phi_{w^{-1}(F)}(\lambda : {}^*\tilde{a})$. Now let u be the element of W_F such that $v = uw$. Then ${}^*\tilde{a} = v^{-1}({}^*a) = w^{-1}u^{-1}({}^*a) = w^{-1}u^{-1}w[w^{-1}({}^*a)]$ and since $\phi_{w^{-1}(F)}(\lambda : .)$ is invariant under the Weyl group $w^{-1}W_F w$ of the pair $(\mathfrak{g}_{Y1}, w^{-1}({}^*\mathfrak{a}))$ it follows that $\phi({}^*a) = \phi_{w^{-1}(F)}(\lambda : w^{-1}({}^*a))$.

<u>Proposition 5.8</u>. $\rho_F|{}^*\mathfrak{a} = \rho|{}^*\mathfrak{a}$.

<u>Proof</u>. Since ${}^*\mathfrak{a} = (\cap_{\alpha\in F} \ker \alpha)^\perp = \sum_{\alpha\in F} (\ker \alpha)^\perp$ it suffices to prove that $\rho_F(H_\alpha) = \rho(H_\alpha)$ for each $\alpha \in F$. So fix $\alpha \in F$. Since $s_\alpha(X) = X$, s_a permutes Δ_F. Now α is simple and hence s_α permutes $\Delta^{++}\setminus\{\alpha\}$ as well. Consequently s_α permutes $\Delta_F^{++}\setminus\{\alpha\}$ and

so $-\rho_F(H_\alpha) = (s_\alpha \rho_F)(H_\alpha) = (\rho_F - 2\rho_\alpha)(H_\alpha)$, showing that $\rho_F(H_\alpha) =$ $= \rho_\alpha(H_\alpha) = \rho(H_\alpha)$.

5.4 A formula for ϕ_λ

Let C_0 be the positive constant fixed in Section 4.5, and select a constant ${}^*C > 0$ such that ${}^*A({}^*C)A_F \cap A(C_0) \neq \emptyset$. Moreover, fix a constant $C_F > 0$ such that the results of the previous section hold for each $v \in W$. Put

$$\mathcal{A} = A(F, {}^*C, C_F) \cap A(C_0).$$

Then $\mathcal{A}$ is a nonempty connected open subset of A^+.

Lemma 5.9. Let $\lambda \in \mathfrak{a}_c^*$, $v \in W$. The function $\phi_{F,v,\lambda}$ defined by (21) satisfies the radial differential equations (4.47) on $\mathcal{A}$.

Proof. Just as the proof of Theorem 4.21.

Lemma 5.10. Let $v \in W$. Then there exist holomorphic functions $e_{F,v,u}\colon \mathfrak{a}_{c,0}^* \to \mathbb{C}$ $(u \in W_F)$ such that for every $\lambda \in \mathfrak{a}_{c,0}^*$ we have:

$$\phi_{F,v,\lambda} = \sum_{u \in W_F} e_{F,v,u}(\lambda)\phi_{uv,\lambda} \tag{25}$$

on $\mathcal{A}$.

Proof. Just as in the proof of Lemma 4.24 it follows that there exist holomorphic functions $e_s = e_{F,v,s}\colon \mathfrak{a}_{c,0}^* \to \mathbb{C}$ $(s \in W)$ such that for every $\lambda \in \mathfrak{a}_{c,0}^*$ we have:

$$\phi_{F,v,\lambda} = \sum_{s \in W} e_s(\lambda)\phi_{s,\lambda}$$

on A. The asymptotic behaviour of the functions $\phi_{F,v,\lambda}$, $\phi_{s,\lambda}$ in the variable a (recall the notation $a_0 = {}^*aa$) is given by the formulas

$$\phi_{S,\lambda}(a_0) = a^{is\lambda-\rho}\,({}^*a)^{is\lambda-\rho}\,\Phi_s(\lambda, z_s(a_0)), \tag{26}$$

$$\phi_{s,v,\lambda}(a_0) = a^{iv\lambda-\rho}\,\Phi_{F,v}(\lambda, {}^*a, z_{F,v}(a)). \tag{27}$$

From this it follows immediately that $e_s \equiv 0$ unless $s^{-1} = v^{-1}$ on $\mathfrak{a}_F$. The latter condition is equivalent to: $s = uv$ for some $u \in W_F$. Therefore we obtain (25) with $e_{F,v,u} = e_{uv}$ $(u \in W_F)$.

<u>Theorem 5.11</u>. Let $v \in W$ and let $w, w' \in W$ be the elements determined by $\Delta^{++}(w) = \{\alpha \in \Delta^{++};\ \alpha(v^{-1}X) < 0$, $\Delta^{++}(w') = \{\alpha \in \Delta^{++};\ \alpha(v^{-1}X) > 0\}$. If $\lambda \in \mathfrak{a}_c^*$, then for all $a_0 \in A$ we have

$$d_{F,w}(\lambda)^{-1}\phi_{F,v,\lambda}(a_0) = \sum_{u \in W_F} d(uw\lambda)^{-1}\phi_{uw,\lambda}(a_0),$$

where (for $s \in W$):

$$d_{F,s}(\lambda) = \prod_{\substack{\alpha \in \Delta^{++} \\ \alpha(s^{-1}X) < 0}} d_\alpha(-\lambda) \prod_{\substack{\alpha \in \Delta^{++} \\ \alpha(s^{-1}X) > 0}} d_\alpha(\lambda) .$$

<u>Proof</u>. Consider formula (25) in Lemma 5.10. Since $W_F v = W_F w$ it can be rewritten as

$$\phi_{F,v,\lambda}(a_0) = \sum_{u \in W_F} e_{uw}(\lambda)\phi_{uw,\lambda}(a_0) \tag{28}$$

where $e_{uv} = e_{F,v,u}$ $(u \in W_F)$. If we divide (28) by $a^{iw\lambda-\rho}$ and take the limit for $a^{\alpha} \to +\infty$ $(\alpha \in \Delta^{++}\setminus\Delta_F)$, then in view of (26), (27) we obtain that:

$$\Phi_{F,v}(\lambda, {}^*a, 0) =$$

$$= \sum_{u \in W_F} ({}^*a)^{iuw\lambda-\rho}\, e_{uw}(\lambda)\Phi_{uw}(\lambda, z^F_{uw}({}^*a)) =$$

$$= \sum_{u \in w^{-1}W_F w} ({}^*a)^{iwu\lambda-\rho} e_{wu}(\lambda)\Phi_{wu}(\lambda, z^F_{wu}({}^*a)). \qquad (29)$$

Here z^F_s $(s \in W)$ is the function ${}^*A \to \mathbb{C}(\Delta^{++})$ given by:

$$(z^F_s({}^*a))_{\alpha} = 0 \qquad \text{if } \alpha(s^{-1}X) = 0,$$

$$= ({}^*a)^{s\alpha} \qquad \text{if } \alpha(s^{-1}X) < 0,$$

$$= ({}^*a)^{-s\alpha} \qquad \text{if } \alpha(s^{-1}X) > 0.$$

Since the constant ${}^*C$ could have been fixed arbitrarily big, the identity (29) is actually valid on the whole of ${}^*A \cap A(C_0)$. We shall now compare the asymptotic behaviour of both sides of the identity (29) when $({}^*a)^{\alpha} \to +\infty$ for every $\alpha \in \Delta^{++}_F$. First consider the formula (23) for $\Phi_{F,v}(\lambda, a, 0)$. If $({}^*a)^{\alpha} \to +\infty$ for every $\alpha \in \Delta^{++}_F$, then $w^{-1}({}^*a)$ tends to infinity in the positive Weyl chamber of $w^{-1}({}^*\mathfrak{a})$. Therefore $\phi_{w^{-1}(F)}(\lambda : w^{-1}(a))$ has an asymptotic expansion as in Secton 4.5. Since $w^{-1}W_F w$ is the Weyl group of the pair $(\mathfrak{g}_{w^{-1}(F)1}, w^{-1}({}^*\mathfrak{a}))$, the principal terms of this expansion are given by

$$\phi_{w^{-1}(F)}(\lambda : w^{-1}({}^*a)) \sim \sum_{u \in w^{-1}W_F} (w^{-1}({}^*a))^{iu\lambda - w^{-1}\rho_F} . c_{w^{-1}(F)}(u\lambda)$$

where $c_{w^{-1}(F)}$ denotes the c-function of the pair $(G_{w^{-1}(F)}, K_{w^{-1}(F)})$. By Proposition 5.8 we have $\rho_F|{}^*\mathfrak{a} = \rho|{}^*\mathfrak{a}$, and so, comparing the principal terms of the asymptotic expansion of $\Phi_{F,v}(\lambda, {}^*a, 0)$ thus obtained with those of the right hand side of (29), we obtain

$$K_F\, c_{w^{-1}(F)}(u\lambda).\tilde{I}_w(-\lambda)\tilde{I}_{w'}(\lambda) = e_{wu}(\lambda)\Phi_{wu}(\lambda,0), \tag{30}$$

for every $u \in w^{-1}W_F w$. Before proceeding with this proof, we first prove the following proposition.

Proposition 5.12. Let $w \in W^F$, and let $w' \in W$ be the element determined by $\Delta^{++}(w') = \{\alpha \in \Delta^{++};\ \alpha(w^{-1}x) > 0\}$. If $u \in w^{-1}W_F w$, we have the following identity of meromorphic functions:

$$K_F\, c_{w^{-1}(F)}(u\lambda)I_w(-\lambda)I_{w'}(\lambda) = d(wu\lambda)^{-1}\Phi_{wu}(\lambda,0). \tag{31}$$

Proof. As we proved in Sections 4.3 - 4.5 the functions $K_X^{-1}\, c_{w^{-1}(F)}(u\lambda)$, $I_w(-\lambda)I_{w'}(\lambda)$ and $K^{-1}d(wu\lambda)\Phi_{wu}(\lambda,0)$ can be written as products

$$\prod_{\alpha\in\Sigma_i^-} I_\alpha(-\lambda) \prod_{\alpha\in\Sigma_i^+} I_\alpha(\lambda) \qquad (i = 0,\ 1,\ 2)$$

respectively, where $\Sigma_i^\pm$ are the subsets of Δ^{++} defined by:

$$\Sigma_0^\pm = \{\alpha \in \Delta^{++};\ \alpha(w^{-1}X) = 0\ \&\ u\alpha \in \pm\Delta^{++}\},$$

$$\Sigma_1^\pm = \{\alpha \in \Delta^{++};\ \pm\alpha(w^{-1}X) > 0\},$$

$$\Sigma_2^\pm = \{\alpha \in \Delta^{++};\ wu\alpha \in \pm\Delta^{++}\}.$$

Consequently (31) is equivalent to

$$\Sigma_0^{\pm} \cup \Sigma_1^{\pm} = \Sigma_2^{\pm} \text{ (disjoint unions).} \tag{32}$$

Now this is established as follows. First let $\alpha \in \Delta^{++}$, $\alpha(w^{-1}X) = 0$. Then $u\alpha(w^{-1}X) = 0$ and so $wu\alpha$ has the same sign on $\mathfrak{a}^+$ as $w^{-1}(wu\alpha) = u\alpha$. Consequently $\Sigma_0^{\pm} = \Sigma_2^{\pm} \cap \Delta_Y^{\pm}$. Next, let $\alpha \in \Delta^{++}$, $\alpha(w^{-1}X) \neq 0$. Then $wu\alpha(X) \neq 0$, and so $\pm\alpha(w^{-1}X) > 0 \Leftrightarrow$ $\Leftrightarrow \pm u\alpha(w^{-1}X) > 0 \Leftrightarrow \pm wu\alpha \in \Delta^{++}$. Therefore $\Sigma_1^{\pm} = \Sigma_2^{\pm}\setminus\Delta_Y^{++}$ and hence indeed (32).

<u>End of proof of Th. 5.11</u>. By (4.27) we have that $d_{F,w}(\lambda)I_w(-\lambda)I_{w'}(\lambda) = \tilde{I}_w(-\lambda)\tilde{I}_{w'}(\lambda)$, and hence:

$$e_{wu}(\lambda) = d_{F,w}(\lambda)d(wu\lambda)^{-1}$$

for every $u \in w^{-1}W_F w$. This completes the proof.

<u>Corollary 5.13</u>. Let $\lambda \in \mathfrak{a}_c^*$. If $v(1), v(2) \in W$ are such that $W_F v(1) = W_F v(2)$, then

$$\phi_{F,v(1),\lambda} = \phi_{F,v(2),\lambda}.$$

<u>Notations</u>. If σ belongs to the coset space $W_F\backslash W$, we write $\phi_{F,\sigma,\lambda}$ instead of $\phi_{F,v,\lambda}$ $(v \in \sigma)$ and $d_{F,\sigma}$ instead of $d_{F,v}$ $(v \in \sigma)$. With these notations we have the following theorem.

<u>Theorem 5.14</u>. Let $\lambda \in {}'\mathfrak{a}_c^*$, $a_0 \in A(F, {}^*C, C_F)$. Then

$$\phi_\lambda(a_0) = \sum_{\sigma \in W_F\backslash W} d_{F,\sigma}(\lambda)^{-1}\phi_{F,\sigma,\lambda}(a_0). \tag{33}$$

Moreover, if v is any representative of σ in W, then

$$\phi_{F,\sigma,\lambda}(a_0) = \int_{\Gamma_{F,v}(a_0)} e^{(i\lambda-\rho)H_{F,v}(a_0 k)}\omega(k)$$

$$= a^{iv\lambda-\rho}\Phi_{F,v}(\lambda, {}^*a, z_{F,v}(a)).$$

<u>Proof</u>. By Theorem 5.11 and Lemmas 4.24, 4.25 formula (33) holds for $a_0 \in A = A(C_0) \cap A(F, {}^*C, C_F)$. By analytic continuation it is valid on the whole of $A(F, {}^*C, C_F)$. The second assertion is a consequence of Theorem 5.5 and Corollary 5.13.

<u>Remark</u>. If $v \in W^F$, $\lambda \in \mathfrak{a}_c^*$, then the function $z \to \Phi_{F,v}(\lambda,z)$ has a converging power series expansion at z = 0. This yields a series expansion

$$\phi_\lambda(a_0) = \sum_{v\in W^F} a^{iv\lambda-\rho} \sum_{\mu\in L_0} \Gamma'_{F,v,\mu}(\lambda, {}^*a)a^{-\mu}. \qquad (34)$$

Here L_0 denotes the set $\mathbb{N}.(S\backslash F)$, the $\Gamma'_{F,v,\mu}$ are functions ${}'\mathfrak{a}_c^* \times ({}^*\mathfrak{a}) \to \mathbb{C}$, holomorphic in the first and real analytic in the second variable. The series (34) converges absolutely in all derivatives for $\lambda \in {}'\mathfrak{a}_c^*$, $a_0 \in A(F, {}^*C, C_F)$, locally uniformly with respect to λ and a_0. Using standard estimates for the remainder terms of the power series expansion of $\Phi_{F,v}(\lambda,z)$ we see that (34) is an asymptotic expansion for $\phi_\lambda(a_0)$, when $a_0 \to \infty$ in $A(F, {}^*C, C_F)$. It is locally uniform with respect to $(\lambda, {}^*a) \in {}'\mathfrak{a}_c^* \times ({}^*\mathfrak{a})$. Moreover, by (23) and since

$$d_{F,v}(\lambda)^{-1}\, \tilde{I}_v(-\lambda).\tilde{I}_{v'}(\lambda) = I_v(-\lambda).I_{v'}(\lambda),$$

the principal terms of this expansion are given by

$$\phi_\lambda(a_0) \sim \tag{35}$$

$$\sim \sum_{v\in W^F} a^{iv\lambda-\rho}.K_F.\phi_{v^{-1}(F)}(\lambda:v^{-1}(*a)).I_v(-\lambda).I_{v'}(\lambda).$$

5.5 Asymptotics along A_F, for λ bounded

In this section we will study the asymptotic expansion (34) for $\phi_\lambda(a_0)$ when $a_0 \to \infty$ in $A(F,*C,C_F)$ in more detail.

Let L be the set $\mathbb{N}.S$. Each element $\mu \in L$ can be written uniquely as $\mu = \Sigma_{\alpha\in S}\,\mu(\alpha)\alpha$ with $\mu(\alpha)$ nonnegative integral. The number

$$|\mu| = \sum_{\alpha\in S} \mu(\alpha)$$

is called the order of μ. Let L_0 denote the subset $\mathbb{N}.(S\setminus F)$ of L. Fix a numbering $\{\alpha_1,\dots,\alpha_p\}$ of $S\setminus F$ and identify $\mathbb{N}^p$ with L_0 via the map $(\mu(i)) \to \Sigma\mu(i)\alpha_i$. We thus have $|(\mu(i))| = \Sigma\mu(i)$. We will focus our attention on the problem of estimating the remainder terms:

$$R_{F,k}(\lambda,a_0) =$$

$$= \phi_\lambda(a_0) - \sum_{v\in W^F} a^{iv\lambda-\rho} \sum_{\substack{\mu\in L_0\\ |\mu|\leq k}} \Gamma'_{F,v,\mu}(\lambda,*a)a^{-\mu} \tag{36}$$

when λ varies in the set

$$\mathfrak{a}^{*\prime} = \{\xi \in \mathfrak{a}^*;\ (\xi,\alpha) \neq 0 \text{ for every } \alpha \in \Delta^{++}\}.$$

If λ is kept in a bounded subset of $\mathfrak{a}^{*\prime}$, then the remainder terms $R_{F,k}(\lambda,a_0)$ can be estimated uniformly with respect to λ, even though at first sight (36) becomes singular when $(\lambda,\alpha) = 0$ for some $\alpha \in \Delta^{++}$ (cf. Theorem 5.22). This main result of Section 5.5 is the local version (with respect to λ) of a result of Trombi-Varadarajan (cf. [1, §2.11]). In the next section we will discuss the problem of estimating $R_{F,k}(\lambda,a_0)$ when λ is allowed to tend to ∞.

First we show that the estimation of $R_{F,k}(\lambda,a_0)$ can be reduced to estimation of the remainder terms of the asymptotic expansion for $\Phi_{F,I}(\lambda,{}^*a,z_{F,I}(a))$. The following lemma is the initial step in this reduction.

<u>Lemma 5.15</u>. If $w \in W$, $\lambda \in \mathfrak{a}_c^*$, then

$$d_{F,w}(\lambda) = d_{F,I}(w\lambda), \quad \phi_{F,w,\lambda} = \phi_{F,I,w\lambda}. \tag{37}$$

<u>Proof</u>. By Corollary 5.13 and the fact that $d_{F,w}$ depends on the coset $W_F w$ only it suffices to prove this for $w \in W^F$. If $u \in W_F$ then $\phi_{uw,\lambda} = \phi_{I,uw\lambda} = \phi_{u,w\lambda}$. Hence by Theorem 5.11 we obtain

$$d_{F,w}(\lambda)^{-1}\phi_{F,w,\lambda} = d_{F,I}(w\lambda)^{-1}\phi_{F,I,w\lambda}.$$

Therefore we may restrict ourselves to the first identity of (37). If $\alpha \in \Delta^{++}\setminus\Delta^{++}(w)$ then $d_\alpha(\lambda) = d_{w\alpha}(w\lambda)$ and if $\alpha \in \Delta^{++}(w)$ then $d_\alpha(\lambda) = d_{-w\alpha}(-w\lambda)$. Now

$$w(\{\alpha \in \Delta^{++};\ \alpha(w^{-1}X) < 0\}) = -(\Delta^{++}(w^{-1})\setminus\Delta_F),$$

$$w(\{\alpha \in \Delta^{++};\ \alpha(w^{-1}X) > 0\}) = \Delta^{++}\setminus(\Delta^{++}(w^{-1}) \cup \Delta_F).$$

Hence substituting $\beta = -w\alpha$ in the first product and $\beta = w\alpha$ in the second product on the right hand side of the identity for $d_{F,w}(\lambda)$ (cf. Theorem 5.11) we obtain that

$$d_{F,w}(\lambda) = \prod_{\beta \in \Delta^{++} \setminus \Delta_F} d_\beta(w\lambda) = d_{F,I}(w\lambda).$$

From now on we shall also write d_F for $d_{F,I}$ and $\phi_{F,\lambda}$ for $\phi_{F,I,\lambda}$. With this notation we have the following corollary.

Corollary 5.16. Let $\lambda \in {}'\mathfrak{a}_c^*$, $a_0 \in A(F, {}^*C, C_F)$. Then

$$\phi_\lambda(a_0) = \sum_{w \in W^F} d_F(w\lambda)^{-1} \phi_{F,w\lambda}(a_0) =$$

$$= \frac{1}{|W_F|} \sum_{w \in W} d_F(w\lambda)^{-1} \phi_{F,w\lambda}(a_0).$$

Now let $\pi \colon \mathfrak{a}_c^* \to \mathbb{C}$ be the function defined by

$$\pi(\lambda) = \prod_{\alpha \in \Delta^{++}} (\alpha, \lambda).$$

The following lemma, combined with the observation that the function πd_F^{-1} is real analytic on $\mathfrak{a}^*$ is the next step in our reduction.

Lemma 5.17. Let $D \in U(\mathfrak{a}_c^*)$. Then there exist finitely many $D_i \in U(\mathfrak{a}_c^*)$ $(1 \leq i \leq l)$ of order $\leq$ order $(D) + \#\Delta^{++}$, such that the following holds. If $f \colon \mathfrak{a}^* \to \mathbb{C}$ is any C^∞ (real analytic) function, then the function $\mathfrak{a}^{*\prime} \to \mathbb{C}$,

$$\lambda \to \sum_{w \in W} \pi(w\lambda)^{-1} f(w\lambda) \tag{38}$$

extends to a C^∞ (real analytic) function $F: \mathfrak{a}^* \to \mathbb{C}$. Moreover, if $\lambda_0 \in \mathfrak{a}^*$, then:

$$|DF(\lambda_0)| \leqslant \max_{1 \leqslant i \leqslant l} \max_{\lambda \in Ch(\lambda_0)} |D_i f(\lambda)|, \tag{39}$$

where $Ch(\lambda_0)$ denotes the convex hull of the set $\{w\lambda_0 : w \in W\}$.

Proof. We postpone the proof to the appendix at the end of this chapter. It is based on a formula of Demazure, expressing F as an iterated difference quotient (cf. [1]).

Let us now concentrate on the function

$$\phi_{F,\lambda}(a_0) = a^{i\lambda-\rho}\Phi_F(\lambda, {}^*a, z_F(a)),$$

where we have written Φ_F for $\Phi_{F,I}$ and z_F for $z_{F,I}$. Consider again the construction of $\Gamma_F = \Gamma_{F,I}(a_0)$. In this case we have $v = I$, $Y = X$, and by (7) $w = I$. Set

$$u = w'.$$

Since $\bar{N}_{w,c} = \{e\}$, $J_{F,I}$ corresponds to the single valued analytic function $J_F: \bar{N}_{u,c} \backslash S \to \mathbb{C}$ given by

$$J_F(\bar{n}) = K_F\, e^{-2\rho H(\bar{n})}$$

(cf. (10) and Lemma 1.19). Moreover, by (20) we have

$$\Phi_F(\lambda, {}^*a, z) = \tag{40}$$

$$= K_F \int_{K_X \times \Upsilon_u} e^{(i\lambda-\rho)H_0({}^*ak(z.\bar{n}))}\, e^{-(i\lambda+\rho)H_u(\bar{n})}\, \omega_X \wedge \Omega_{u,0},$$

where H_0 denotes the real branch of H at e. Now G_F normalizes the Lie algebra

$$\overline{\mathfrak{n}}_{u,c} = \sum_{\alpha\in\Delta,\alpha(X)<0} \mathfrak{g}_{\alpha,c},$$

and so it normalizes $\overline{N}_{u,c}$. Hence for $a_0 \in A(F,{}^*C,C_F)$ we can rewrite the integrand of (40) with $z = z_F(a)$ as:

$$e^{(i\lambda-\rho)H({}^*ak)}\, e^{(i\lambda-\rho)H_0[Ad(a{}^*ak,\overline{n})\kappa({}^*ak)]}\, e^{-(i\lambda+\rho)H_u(\overline{n})}, \qquad (41)$$

where H_0 now denotes the real branch of H in some neighbourhood of K_X in $G_c\backslash S$. Now put

$${}^*_+A({}^*C) = \{{}^*a \in {}^*A;\ \alpha(\log{}^*a) > -C^*,\ \alpha \in \Delta^{++}\backslash\Delta_F\}.$$

If $a^\alpha \to +\infty$ for all $\alpha \in \Delta^{++}\backslash\Delta_F$, then $(a{}^*a)^{-\alpha} \to 0$, uniformly with respect to ${}^*a \in {}^*_+A({}^*C)$, and so (41) can be seen as a perturbation of

$$e^{(i\lambda-\rho)H({}^*ak)}\, e^{-(i\lambda+\rho)H_u(\overline{n})}.$$

Keeping this in mind, we introduce the map $\zeta_F\colon A_F \to \mathbb{C}^P$. First, if $\zeta \in \mathbb{C}^P$ we define $z(\zeta) \in \mathbb{C}(\Delta^{++}\backslash\Delta_F)$ as follows. If $\alpha \in \Delta^{++}\backslash\Delta_F$, $\alpha \equiv \Sigma_{i=1}^{p} k(i)\alpha_i \bmod \mathbb{N}.F$, then

$$z(\zeta)_\alpha = \zeta_1^{k(1)}\cdots\zeta_p^{k(p)}.$$

We shall also write $\zeta.\overline{n}$ for $z(\zeta).\overline{n}$ ($\zeta \in \mathbb{C}^p$, $\overline{n} \in \overline{N}_{u,c}$). Now let the map $\zeta_F\colon A_F \to \mathbb{C}^p$ be defined by:

$$\zeta_F(a)_j = a^{-\alpha_j} \qquad (1 \leqslant j \leqslant p).$$

Then obviously, if $a \in A_F$, $\bar{n} \in \bar{N}_{u,c}$, we have:

$$\zeta_F(a).\bar{n} = z_F(a).\bar{n} = Ad(a,\bar{n}).$$

Let V be an open neighbourhood of K_X in $G_c\backslash S$ such that the real branch H_0 of H exists in V. If $\varepsilon > 0$, we set

$$D(0;\varepsilon) = \{\zeta \in \mathbb{C}^p;\ |\zeta_j| < \varepsilon,\ \ 1 \leqslant j \leqslant p\}.$$

We now have the following easy proposition.

Proposition 5.18. There exists a $\varepsilon > 0$ such that for every ${}^*a \in {}^*_+A({}^*C)$, $k \in K_X$, $\bar{n} \in \mathrm{im}\gamma_u$, $\zeta \in D(0;\varepsilon)$ we have

$$[\zeta.Ad({}^*ak,\bar{n})]\kappa({}^*ak) \in V.$$

By Proposition 5.18 we may define the map Ψ_F: $\mathfrak{a}^*_c \times [{}^*_+A({}^*C)] \times D(0;\varepsilon) \to \mathbb{C}$ by

$$\Psi_F(\lambda,{}^*a,\zeta) = \int_{K_X\times\gamma_u} \tilde{\Psi}_F.\omega_X\wedge\Omega_{u,0}, \tag{42}$$

$$K_F^{-1}\tilde{\Psi}_F = K_F^{-1}\tilde{\Psi}_F(\lambda,{}^*a,k,\zeta,\bar{n}) =$$

$$= e^{(i\lambda-\rho)H({}^*ak)}\, e^{(i\lambda-\rho)H_0([\zeta.Ad({}^*ak,\bar{n})]\kappa)}\, e^{-(i\lambda+\rho)H_u(\bar{n})}.$$

Here we have written $\kappa = \kappa({}^*ak)$.

If $C' > 0$ we set $A_F(C') = \{a \in A_F;\ \ a^\alpha > C',\ \alpha \in S\backslash F\}$ and:

$${}_+A(F,{}^*C,C') = {}^*_+A({}^*C).A_F(C'). \tag{43}$$

With these notations we have:

Lemma 5.19. The map $\Psi_F\colon \mathfrak{a}^*_C\times[{}_+^*A({}^*C)]\times D(0;\varepsilon) \to \mathbb{C}$ is holomorphic in the first and last variable, and real analytic in the second variable. Moreover, if $a_0 \in {}_+A(F,{}^*C,\varepsilon^{-1})$, then:

$$\phi_{F,\lambda}(a_0) = a^{i\lambda-\rho}\Psi_F(\lambda,{}^*a,\zeta_F(a)). \tag{44}$$

Proof. The first assertion is obvious. Formula (44) is valid for $a_0 \in A(F,{}^*C,C_F) \cap {}_+A(F,{}^*C,\varepsilon^{-1})$ and hence by analytic continuation it is valid on ${}_+A(F,{}^*C,\varepsilon^{-1})$.

The function $\Psi_F(\lambda,{}^*a,.)$ has a power series expansion

$$\Psi_F(\lambda,{}^*a,\zeta) = \sum_{\mu\in L_0} \theta_{F,\mu}(\lambda,{}^*a)\zeta^{\mu}, \tag{45}$$

where

$$\theta_{F,\mu}(\lambda,{}^*a) = \frac{1}{\mu!}\left(\frac{\partial}{\partial\zeta}\right)^{\mu}\Psi_F(\lambda,{}^*a,\zeta)\Big|_{\zeta=0}. \tag{46}$$

Here we have identified $\mathbb{N}^p$ with L_0 under $(\mu(i)) \to \Sigma\mu(i)\alpha_i$ and we have used the multi-index notations:

$$\mu! = \mu(1)!\cdot\ldots\cdot\mu(p)!,\ \zeta^{\mu} = \zeta_1^{\mu(1)}\ldots\zeta_p^{\mu(p)},$$

$$\left(\frac{\partial}{\partial\zeta}\right)^{\mu} = \left(\frac{\partial}{\partial\zeta_1}\right)^{\mu(1)}\ldots\left(\frac{\partial}{\partial\zeta_p}\right)^{\mu(p)}.$$

Observe that with these notations we have $\zeta_F(a)^{\mu} = a^{-\mu}$ for $a \in A_F$. Hence (45) leads to the converging series expansion

$$\phi_{F,\lambda}(a_0) = a^{i\lambda-\rho}\sum_{\mu\in L_0} \theta_{F,\mu}(\lambda,{}^*a)a^{-\mu}. \tag{47}$$

Thus, in view of Corollary 5.16 we obtain (34) with

$$\Gamma'_{F,v,\mu}(\lambda,{}^*a) = d_F(v\lambda)^{-1}\theta_{F,\mu}(v\lambda,{}^*a). \tag{48}$$

Now let

$$R_{F,k}(\lambda,{}^*a,\zeta) =$$

$$= \Psi_F(\lambda,{}^*a,\zeta) - \sum_{\substack{\mu\in L_0 \\ |\mu|\leqslant k}} \theta_{F,\mu}(\lambda,{}^*a)\zeta^{\mu}. \tag{49}$$

Then the following lemma follows immediately.

Lemma 5.20. If $\lambda \in {}'\mathfrak{a}_c^*$, $a_0 \in {}_+A(F,{}^*C,\varepsilon^{-1})$, then

$$R_{F,k}(\lambda,a_0) =$$

$$= \frac{1}{|W_F|} \sum_{w\in W} \frac{1}{d_F(w\lambda)} a^{iw\lambda-\rho} R_{F,k}(w\lambda,{}^*a,\zeta_F(a)). \tag{50}$$

Before proceeding, let us introduce some notations. We write $\|\cdot\|$ for the norms determined by (,) on $\mathfrak{a}$ and $\mathfrak{a}^*$. If $\zeta \in \mathbb{C}^p$ we put $\|\zeta\| = \max\{|\zeta_1|,\ldots,|\zeta_p|\}$. Finally let Ξ_F be the spherical function of (G_F,K_F) given by

$$\Xi_F(x) = \int_{K_F} e^{-\rho H(xk)}dk_F, \tag{51}$$

where dk_F denotes the normalized Haar measure of $K_F = K_X$ (observe that by Prop. 5.8 and the proof of Th. 5.7 we have $\Xi_F = \phi_F(0:.)$)

Lemma 5.21. Let $D_\lambda \in U(\mathfrak{a}_c^*)$ be of order $\leqslant d$, and let $R > 0$, $k \in \mathbb{N}$. Then there exists a constant $A > 0$ such that

$$|D_\lambda R_{F,k}(\lambda,{}^*a,\zeta)| \leqslant A.(1+\|\log{}^*a\|)^d\,\Xi_F({}^*a)\|\zeta\|^{k+1},$$

for $\lambda \in \mathfrak{a}^*$ with $\|\lambda\| \leqslant R$ and for ${}^*a \in {}^*_+A({}^*C)$, $\zeta \in D(0;\varepsilon)$.

Before proving Lemma 5.21 we present the principal result of this section. Let $\beta \in \mathfrak{a}^*$ be defined by

$$\beta = \min_{\alpha \in S\setminus F} \alpha$$

Then obviously $a^{-\beta} = \|\zeta_F(a)\|$ for $a \in A_F$.

Theorem 5.22. Let $D_\lambda \in U(\mathfrak{a}_c^*)$ be of order $\leqslant d$, and let $R > 0$, $k \in \mathbb{N}$. Then there exists a constant $B > 0$ such that

$$|D_\lambda R_{F,k}(\lambda,a_0)| \leqslant \tag{52}$$

$$\leqslant B \;.\; [(1 + \|\log a\|)(1 + \|\log{}^*a\|)]^{d+N} \Xi_F({}^*a)a^{-(k+1)\beta-\rho}$$

for $\lambda \in \mathfrak{a}^{*\prime}$ with $\|\lambda\| \leqslant R$ and for $a_0 \in {}_+A(F,{}^*C,\varepsilon^{-1})$. Here $N = \#\Delta^{++}$.

Proof. In view of (50) this follows from Lemma 5.21 by application of Lemma 5.16 to the functions

$$f_{a_0} : \lambda \to \frac{\pi(\lambda)}{d_F(\lambda)}\, a^{i\lambda-\rho} R_{F,k}(\lambda,{}^*a,\zeta_F(a)).$$

Proof of Lemma 5.21. Consider the Taylor expansion for the function $t \to D_\lambda \Psi_F(\lambda,{}^*a,t\zeta)$ around $t = 0$. At $t = 1$ its k-th order remainder term is just equal to $D_\lambda R_{F,k}(\lambda,{}^*a,\zeta)$, and so

$$\left|D_\lambda R_{F,k}(\lambda,{}^*a,\zeta)\right| \leqslant \max_{0\leqslant t\leqslant 1} \frac{1}{(k+1)!}\left|\left(\frac{d}{dt}\right)^{k+1} D_\lambda \Psi_F(\lambda,{}^*a,t\zeta)\right|.$$

Now let P: $\mathfrak{a}_c \to \mathbb{C}$ be the polynomial function defined by P(H) = $= D_\lambda(\exp i\lambda(H))|_{\lambda=0}$. Then differentiation of (42) under the integral sign yields

$$D_\lambda \Psi_F(\lambda, {}^*a, \zeta) = \int_{K_X \times Y_u} \tilde{P}.\tilde{\Psi}_F.\omega_X \wedge \Omega_{u,0}$$

where $\tilde{P} = \tilde{P}({}^*a,k,\zeta,\bar{n}) = P[H({}^*ak) + H_0([\zeta.Ad({}^*ak,\bar{n})]\kappa({}^*ak)) - H_u(\bar{n})]$. By Kostant's convexity theorem we have $\|H({}^*ak)\| \leqslant \leqslant \|\log {}^*a\|$ (for a recent elegant proof of this theorem, see Heckman's thesis [1, Theorem 1]). Now Lemma 5.21 follows from a straightforward estimation of

$$\left(\frac{d}{dt}\right)^{k+1}[\tilde{P}({}^*a,k,t\zeta,\bar{n})\tilde{\Psi}_F(\lambda,{}^*a,k,t\zeta,\bar{n})].$$

5.6 Asymptotics along A_F, for λ unbounded

In [1, §2.11], Trombi and Varadarajan obtained estimates for $R_{F,k}(\lambda, a_0)$, uniform with respect to $\lambda \in \mathfrak{a}^{*\prime}$. These estimates are like (52), with an additional power of $(1 + \|\lambda\|)$ on the right hand side of the inequality. We have not managed yet to obtain these results with our techniques. However if Λ_1 is a compact subset of $\{\xi \in \mathfrak{a}^{*\prime};\ \|\xi\| = 1\}$, and if λ varies in the set

$$\Lambda = \{\tau\xi;\ \xi \in \Lambda_1,\ \tau \geqslant 1\} \tag{53}$$

then we are able to obtain estimates which are sharp in λ. Of course here the problem is to suitably estimate the function

$$-\lambda\ \mathrm{Im}(H_0[\{\zeta.Ad({}^*ak,\gamma_u(y))\}\kappa({}^*ak)] - H_u[\gamma_u(y)]) \tag{54}$$

which occurs in the exponent of the integral for $R_{F,k}(\lambda,a_0)$. For $\lambda \in \Lambda$ this can be achieved by deforming the cycle γ_u so that a multi-variable version of the method of steepest descent can be applied (for a description of the single variable method, see Erdélyi [1, §2.5]). After this deformation, the main contribution to the integral comes from a certain non degenerate stationary point $\sigma(\sigma) \in Y_u$ = domain (γ_u) of the function (54) of y. This point is independent of the value of the parameters $(\lambda, {}^*a, k, \zeta)$ if ζ is sufficiently close to 0. Here the allowed magnitude of ζ depends locally uniformly on $\lambda \in \mathfrak{a}^{*\prime}$. On the other hand if $(\lambda,\alpha) = 0$ for some $\alpha \in \Delta^{++} \setminus \Delta_F$ then the set of stationary points of (54) is completely different, and therefore the obtained results are not uniform if λ becomes singular.

The idea to analyse the set of stationary points of the function (54) in order to obtain asymptotic expansions is due to Duistermaat, Kolk and Varadarajan (DKV). In DKV [2] they use the method of stationary phase to obtain asymptotic expansions for integrals of the form

$$\int_K e^{i\tau\lambda H(ak)} g(a,k)\,dk \qquad (g \in C^\infty(A\times K)) \tag{55}$$

when $\lambda \in \mathfrak{a}^*$, $\tau \to +\infty$, with a kept in a compact subset of A. By an ingenious method they are able to obtain uniform estimates for (55) even when λ varies in neighbourhoods of singular points in $\mathfrak{a}^*$. We hope that a synthesis of their techniques with those presented here will eventually lead to a complete understanding of the asymptotic behaviour of $\phi_\lambda(a)$ with respect to λ and a simultaneously.

We now come to the main result of this section.

Theorem 5.23. Let Λ be a set as in (53). Then there exists a constant $\varepsilon > 0$ such that the following holds. If $k \in \mathbb{N}$ then there exists a $A_{\Lambda,k} > 0$ such that

$$|R_{F,k}(\lambda,a_0)| \leqslant A_{\Lambda,k}.(1 + \|\lambda\|)^{m(F,k)}.\Xi_F(^*a).a^{-(k+1)\beta-\rho}$$

for every $a_0 \in {}_+A(F,^*C,\varepsilon^{-1})$, $\lambda \in \Lambda$. Here $m(F,k) = k + 1 - \frac{1}{2}\dim(\bar{\mathfrak{n}}_u)$.

Proof. This theorem follows immediately from (50) and the lemma below.

Lemma 5.24. Let Λ be a set as in (53). Then there exists a constant $\varepsilon > 0$ such that the following holds. If $k \in \mathbb{N}$ then there exists a $B_{\Lambda,k} > 0$ such that

$$|d_F(\lambda)^{-1}R_{F,k}(\lambda,^*a,\zeta)| \leqslant$$

$$\leqslant B_{\Lambda,k}.(1 + \|\lambda\|)^{m(F,k)}.\Xi_F(^*a).\|\zeta\|^{k+1} \qquad (56)$$

for every $\lambda \in \Lambda$, $^*a \in {}^*_+A(^*C)$, $\zeta \in D(0;\varepsilon)$.

Proof. As in the proof of Lemma 5.21 the left hand side of (56) can be estimated on

$$\max_{0\leqslant t\leqslant 1} \frac{|d_F(\lambda)|^{-1}}{(k+1)!} \cdot \left|\left(\frac{d}{dt}\right)^{k+1}\Psi_F(\lambda,^*a,t\zeta)\right|. \qquad (57)$$

Now $(d/dt)^{k+1}\Psi_F(\lambda,^*a,t\zeta)$ is a finite sum of terms of the form

$$P(\lambda)Q(\zeta)\int_{K_X\times Y_u} e^{(i\lambda-\rho)H(^*ak)}.e^{\tau f}.g.\omega_X\wedge\Omega_{u,0}$$

where P and Q are polynomials of degree $\leqslant k+1$, where $g = g(\lambda, {}^*a, k, t\zeta, \bar{n})$ is a uniformly bounded function, and where finally we have written

$$f = f(\xi, {}^*a, k, t\zeta, \bar{n}) =$$

$$= -\xi \operatorname{Im}\{H_0[(t\zeta).Ad({}^*ak, \bar{n})\kappa({}^*ak)] - H_u(\bar{n})\},$$

$$\tau = \|\lambda\|, \ \xi = \|\lambda\|^{-1}\lambda.$$

The proof is completed by application of the lemma below.

Lemma 5.25. The cycle γ_u can be chosen so that for every compact subset Λ_1 of $\{\xi \in \mathfrak{a}^{*\prime}; \|\xi\| = 1\}$ there exist constants $\varepsilon > 0$, $A' > 0$ with the following property. If $G: Y_u \to \mathbb{C}$ is any continuous function, then

$$\left|\int_{Y_u} e^{\tau f(\xi, {}^*a, k, \zeta, \gamma_u(y))} G(y)\gamma_u^*(\Omega_{u,0})\right| \leqslant$$

$$\leqslant A'. \max_{y \in Y_u} |G(y)|.d_F(\tau\xi).\tau^{-\frac{1}{2}d}$$

for ${}^*a \in {}^*_+A({}^*C)$, $k \in K_X$, $\zeta \in D(0;\varepsilon)$, $\xi \in \Lambda_1$, $\tau \geqslant 1$. Here we have written $d = \dim(\bar{\mathfrak{n}}_u) = \dim(Y_u)$.

Proof. We will give a sketch of the proof, and leave the details to the reader. It will occupy the remainder of this section, with at some places interruptions by propositions.

It will be convenient to work with a parameter $\kappa \in K_X$, and with the parameter $\eta = (\xi, k, \kappa)$ varying in the compact set $E = \Lambda_1 \times K_X \times K_X$. Moreover, we shall work with a parameter $z \in \mathbb{C}(\Delta^{++} \setminus \Delta_F)$, varying in a sufficiently small polydisc

$$D'(0;\varepsilon) = \{z;\ |z_\alpha| < \varepsilon' \text{ for all } \alpha \in \Delta^{++}\backslash\Delta_F\}.$$

Now let the function $F\colon E\times D'(0;\varepsilon)\times Y_u \to \mathbb{R}$ be given by:

$$F(\eta,z,y) = \tag{59}$$

$$= -\xi\ \mathrm{Im}\left\{H_0\left[\{z.Ad(k,\gamma_u(y))\}\kappa\right] - H_u[\gamma_u(y)]\right\}$$

Then with $\eta = (\xi,k,\kappa(^*ak))$, $z_\alpha = z(\zeta)_\alpha.(^*a)^{-\alpha}$ for $\alpha \in \Delta^{++}\backslash\Delta_F$, we have

$$F(\eta,z,y) = f(\xi,{}^*a,k,\zeta,\gamma_u(y)).$$

Hence we are done if we can choose γ_u, ε' so that (58) holds with f replaced by F, for all $\eta \in E$, $\tau \geqslant 1$, $z \in D'(0;\varepsilon')$.

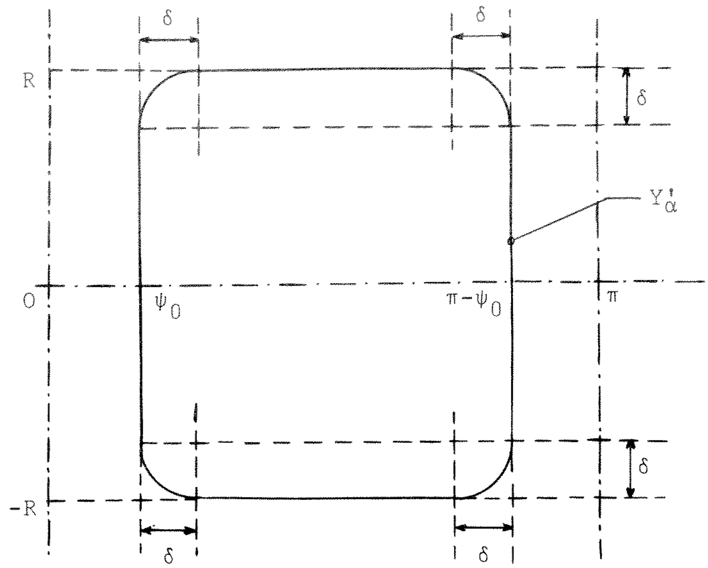
We will view $F(\eta,z,y)$ as a perturbation of

$$F(\eta,0,y) = \xi\ \mathrm{Im}\ \{H_u\circ\gamma_u(y)\} =$$

$$= \sum_{\alpha\in\Delta^{++}} \xi\ \mathrm{Im}\ \{H_\alpha\circ\gamma_\alpha(y_\alpha)\}.$$

Here we have used the notations of Secton 4.3 and we have written y_α for the Y_α-coordinate of $y \in Y_u$. We will now describe how the $\gamma_\alpha\colon Y_\alpha \to \overline{N}_{\alpha,c}\backslash S$ must be chosen. Let us fix $\alpha \in \Delta^{++}(u)$, and use the notations of Chapter 3 for the corresponding real rank 1 group G^α. Fix a number $0 < \psi_0 < \frac{1}{4}\pi$, a sufficiently small number $\delta > 0$, and a 1-dimensional C^∞ submanifold Y'_α of $\mathbb{R}\times\mathbb{R}$ as drawn in Figure (5.1).

Figure 5.1.

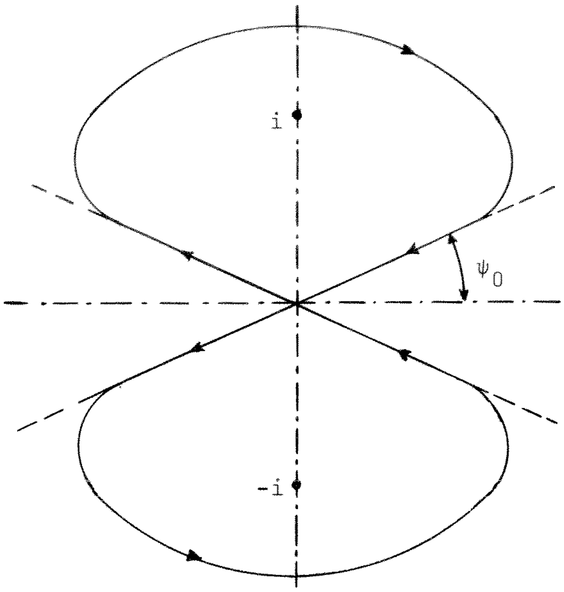


Moreover, let Y_α be the set of $(t,(X,Y)) \in [0,\pi]\times B_I$ with $(t,\|(X,Y)\|) \in Y'_\alpha$. Thus Y_α is a compact C^∞ manifold of dimension $\dim(\overline{N}_\alpha)$. It tends to $\partial([\psi_0,\pi-\psi_0]\times B_I)$ if $\delta \downarrow 0$. Now let $\gamma_\alpha\colon Y_\alpha \to \overline{N}_{\alpha,c}\backslash S$ be defined by

$$\gamma_\alpha(t,(X,Y)) = a(e^{-it})E(X,Y)a(e^{it})$$

(cf. also the formula above (3.8)). If δ is sufficiently small, then γ_α is homotopic to γ_I, and so we may use it in the constructions of Chapter 4. In Figure 5.2 we have drawn the image of γ_α in the case of $SL(2,\mathbb{R})$ using the same coordinates as in Figure 3.1.

Figure 5.2.



Now let H_α be the branch of H over γ_α corresponding to H_I, an let $\sigma(\alpha,+1) = (\psi_0,(0,0)) \in Y_\alpha$, $\sigma(\alpha,-1) = (\pi-\psi_0,(0,0)) \in Y_\alpha$. Then we have the following easy proposition.

Proposition 5.26. If $y \in Y_\alpha \setminus \{\sigma(\alpha,+1),\sigma(\alpha,-1)\}$, then

$$0 < \alpha \, \mathrm{Im} \, \{H_\alpha \circ \gamma_\alpha(y)\} < 2\pi,$$

and $\alpha H_\alpha \circ \gamma_\alpha(\sigma(\alpha,+1)) = 0$, $\alpha H_\alpha \circ \gamma_\alpha(\sigma(\alpha,-1)) = 2\pi i$. The function $y \to \alpha \, \mathrm{Im} \, \{H_\alpha \circ \gamma_\alpha(y)\}$ has non degenerate stationary points at $\sigma(\alpha,+1)$, $\sigma(\alpha,-1)$. Moreover, the second order total derivative $d^2(\alpha \, \mathrm{Im} \, (H_\alpha \circ \gamma_\alpha))$ is positive definite at $\sigma(\alpha,+1)$ and negative definite at $(\alpha,-1)$.

Now let $\mathfrak{a}^{*+}$ be the positive Weyl chamber of $\mathfrak{a}^*$ corresponding to the choice Δ^+ of positive roots. Writing $\Lambda_{1\sigma} = \Lambda_1 \cap \sigma^{-1}(\mathfrak{a}^{*+}) = \{\xi \in \Lambda_1;\ (\xi,\sigma\alpha) > 0 \text{ for } \alpha \in \Delta^{++}\}$ we have

$$\Lambda_1 = \bigcup_{\sigma \in W} \Lambda_{1\sigma} \qquad\qquad \text{(disjoint union)},$$

each $\Lambda_{1\sigma}$ being a compact subset of $\mathfrak{a}^{*\prime}$. Clearly it suffices to prove Lemma 5.25 for each of the $\Lambda_{1\sigma}$. So fix $\sigma \in W$, and suppose that $\Lambda_1 = \Lambda_{1\sigma}$. Now define the point $o(\sigma) \in Y_u$ as follows:

$$o(\sigma)_\alpha = o(\alpha,-1) \qquad \text{if } \alpha > 0 \text{ on } \sigma^{-1}(\mathfrak{a}^{*+}),$$

$$o(\sigma)_\alpha = o(\alpha,+1) \qquad \text{if } \alpha < 0 \text{ on } \sigma^{-1}(\mathfrak{a}^{*+}).$$

Next, define the function $\Sigma^\sigma : \mathfrak{a}_c^* \to \mathbb{C}$ by

$$\Sigma^\sigma(\lambda) = 2\pi. \sum_{\substack{\alpha \in \Delta^{++}\setminus\Delta_F \\ \sigma(\alpha) \in \Delta^{++}}} \lambda(H_{\alpha,0}). \tag{60}$$

Since for every $y \in Y_u$ we have

$$H_u \circ \gamma_u(y) = \sum_{\alpha \in \Delta^{++}\setminus\Delta_F} H_\alpha \circ \gamma_\alpha(y_\alpha),$$

the following proposition is an easy corollary of Proposition 5.26.

<u>Proposition 5.27</u>. If $y \in Y_\alpha \setminus \{o(\sigma)\}$, $\xi \in \Lambda_{1\sigma}$, then

$$\xi \operatorname{Im} \{H_u \circ \gamma_u(y)\} < \Sigma^\sigma(\xi),$$

and $\xi \operatorname{Im} \{H_u \circ \gamma_u(o(\sigma))\} = \Sigma^\sigma(\xi)$. The function $y \to \xi \operatorname{Im} \{H_u \circ \gamma_u(y)\}$

has a non degenerate stationary point at $o(\sigma)$. Moreover, the second order total derivative $d^2(\xi \text{ Im } \{H_u \circ \gamma_u\})(o(\sigma))$ is negative definite.

Now recall the definition (59) of F, and observe that $d_y^2 F(\eta,z,o(\sigma))$ (the subscript y means that the total derivative with respect to the variable y is taken) is a perturbation of $d_y^2 F(\eta,0,o(\sigma)) = d^2(\xi \text{ Im } \{H_u \circ \gamma_u\})(o(\sigma))$.

Proposition 5.28. There exists a constant $\varepsilon' > 0$ such that for all $\eta \in E$, $z \in D'(0;\varepsilon')$ we have

$$F(\eta,z,o(\sigma)) = \Sigma^{\sigma}(\xi),$$

$$d_y F(\eta,z,o(\sigma)) = 0$$

$$d_y^2 F(\eta,z,o(\sigma)) \quad \text{is negative definite.}$$

Let us now finish the proof of Lemma 5.25. By the Morse Lemma with parameters (cf. Hörmander [1, Lemma 3.23]) there exist a neighbourhood U of $o(\sigma)$ in Y_u, and a system of coordinates $x = x(\eta,z,.): U \to \mathbb{R}^d$, depending C^∞ on η and z such that

$$F(\eta,z,y) = Q_{\eta,z}(x(\eta,z,y),x(\eta,z,y)) + \Sigma^{\sigma}(\xi)$$

for some negative definite quadratic form $Q_{\eta,z}$ depending smoothly on η,z. Using a C^∞ partition of unity $\{\phi_1,\phi_2\}$ on Y_u, with $\text{supp}(\phi_1) \subset U$, $\phi_2 = 0$ in an open neighbourhood U' of $o(\sigma)$ we may rewrite the integral in (58) with f replaced by F as $I_1(G) + I_2(G)$, with

$$I_j(G) = \int_{Y_u} e^{\tau F(\eta,z,y)} G(y)\phi_j(y)\gamma_u^*(\Omega_{u,0}).$$

In the coordinates $x = x(\eta,z,y)$, we have

$$I_1(G) = e^{\Sigma^\sigma(\tau\xi)} \int_{\mathbb{R}^d} e^{\tau Q_{\eta,z}(x,x)} [\phi_1.G](y(\eta,z,x))J(\eta,z,x)dx$$

where $y(\eta,z,.)$ is the inverse of $x(\eta,z,.)$, and where J is some Jacobian. Applying the substitution $x' = \sqrt{\tau}x$, we see that $\exp(-\Sigma^\sigma(\tau\xi))I_1(G)$ equals

$$\tau^{-\frac{1}{2}d} \int_{\mathbb{R}^d} e^{Q_{\eta,z}(x',x')}.(\phi_1.G)[y(\eta,z,\tfrac{x'}{\sqrt{\tau}})]J(\eta,z,\tfrac{x'}{\sqrt{\tau}})dx'.$$

Now let

$$B' = \max|(\phi_1\circ y).J| \;.\; \max_{\eta,z}\left(\int_{\mathbb{R}^d} e^{Q_{\eta,z}(x,x)} dx\right)$$

(the integral is absolutely convergent and defines a C^∞ function of (η,z)). It follows that

$$|I_1(G)| \leqslant B'.\max|G|.\exp(\Sigma^\sigma(\tau\xi)).\tau^{-\frac{1}{2}} \tag{61}$$

On the other hand by Proposition 5.27 there exists a constant $\varepsilon_0 > 0$ such that

$$F(\eta,z,y) < \Sigma^\sigma(\xi) - \varepsilon_0 \tag{62}$$

for $\eta \in E$, $y \in Y_u\backslash U'$, $z = 0$. If ε' is chosen sufficiently small then (62) still holds for $z \in D'(0;\varepsilon')$. Consequently there exists a constant $B'' > 0$ such that

$$|I_2(G)| \leqslant B''.\max|G|.\exp(\Sigma^\sigma(\tau\xi) - \tau\varepsilon_0) \tag{63}$$

for $\eta \in E$, $z \in D'(0;\varepsilon')$. Now (58) follows from (61), (63) and the observation that

$$\exp(\Sigma^{\sigma}(\tau\xi)) = \mathcal{O}(d_F(\tau\xi)) \qquad (\tau \to +\infty),$$

uniformly with respect to $\xi \in \Lambda_{1\sigma}$.

Remark 1. In the case of $SL(2,\mathbb{R})$ we may use the curve along which $\mathrm{Im}(H(\zeta.\bar{n}) - H(\bar{n}))$ has its steepest descent. By the Cauchy-Riemann equations this is the level curve $\mathrm{Re}(H(\zeta.\bar{n}) - H(\bar{n})) = 0$. In the coordinates used in Figure 3.1 it is given by the equation:

$$|1 + \zeta^2 z^2| = |1 + z^2|.$$

If $\zeta \to 0$, this curve tends to the limit curve $|1 + z^2| = 1$ (see the cover of this thesis). Thus the method of steepest descent can be applied in its purest form here. This leads to an asymptotic expansion of $d(w\lambda)^{-1}\Phi(w\lambda,\zeta)$ $(w \in W)$, when $(\lambda,\alpha) \to +\infty$, locally uniformly with respect to $\zeta \in \mathbb{C}$, $|\zeta| < 1$. The principal term is given by

$$d(w\lambda)^{-1}\Phi(w\lambda,\zeta) \sim \frac{1}{\sqrt{\pi}}\, e^{-\frac{\pi i}{4}\det(w)} . \lambda(H_{\alpha,0})^{-\frac{1}{2}} . \sqrt{1-\zeta^2}^{\,-1},$$

where $\sqrt{1-\zeta^2}$ is the complex root having its argument in the interval $(-\frac{\pi}{2},\frac{\pi}{2})$.

Remark 2. By putting the cycle γ_u ($u \in W$ determined by $u(\Delta^+) = -\Delta^+$) in a position of steepest descent in a neighbourhood of the relevant stationary point, it is possible to obtain asymptotic expansions for $d(\tau\lambda)^{-1}\Phi(\tau\lambda,\zeta)$, when $\tau \to +\infty$, locally uniformly with respect to $\lambda \in \mathfrak{a}^{*\prime}$ and uniformly with respect to ζ varying in a sufficiently small neighbourhood of 0 in $\mathbb{C}(\Delta^{++})$. Substitution of $\zeta = 0$ gives the asymptotic expansion for $c(\tau\lambda)$ ($\tau \to +\infty$, $\lambda \in \mathfrak{a}^{*\prime}$) which Cohn derived by application of the method of stationary phase (cf. [1]).

Remark 3. As I learned recently, curves as the loop in Figure 5.2 are known as contours of Pochhammer type. In his paper [1] Pochhammer introduced such loops in order to represent solutions of ordinary second order differential equations of the regular singular type by integrals over them, the advantage being that convergence is ensured. In [2] he applied this idea to the study of Euler's Bêta function, and to the study of the hypergeometric functions.

Appendix to Chapter 5

Iterated difference quotients

In this appendix we give a proof of Lemma 5.17, based on a lemma of Demazure (cf. [1, Lemma 4]).

We shall consider the space $\mathfrak{a}^*$ together with the root system $R = \Delta^{++} \cup (-\Delta^{++})$ and the choice $R^+ = \Delta^{++}$ of positive roots. The corresponding set of simple roots is denoted by S. The action of the Weyl group W on $\mathfrak{a}^*$ naturally extends to $S(\mathfrak{a}^*)$, the symmetric algebra of $\mathfrak{a}^*$ (over $\mathbb{C}$). We define the linear endomorphism J of $S(\mathfrak{a}^*)$ by

$$J(p) = \sum_{w \in W} \det(w).(wp) \tag{64}$$

and we define the element $\pi \in S(\mathfrak{a}^*)$ by

$$\pi = \prod_{\alpha \in R^+} \alpha .$$

If $\alpha \in R$, $p \in S(\mathfrak{a}^*)$ then it is easily seen that α divides $p - s_\alpha(p)$ in $S(\mathfrak{a}^*)$. We set

$$\Delta_\alpha(p) = \frac{p - s_\alpha(p)}{\alpha}.$$

Thus Δ_α is a linear endomorphism of $S(\mathfrak{a}^*)$ mapping the homogeneous component $S^{m+1}(\mathfrak{a}^*)$ into $S^m(\mathfrak{a}^*)$.

As usual, if $w \in W$, an expression

$$w = s_{\alpha(1)} \circ \dots \circ s_{\alpha(t)}$$

with $s_{\alpha(j)} \in S$ $(1 \leq j \leq t)$ will be called a *reduced expression* for w if it is of minimal length. This minimal length is called the length of w (notation: $\ell(w)$). As is well known, we have $\ell(w) = n(w)$ (cf. Varadarajan [1 , Theorem 4.15.10]).

Lemma 5.29. (Demazure) Let $w_0 \in W$ be the element with $w_0(R^+) = -R^+$, and let $w_0 = s_{\alpha(1)} \circ \cdots \circ s_{\alpha(N)}$ be a reduced expression. Then for every $p \in S(\mathfrak{a}^*)$ we have:

$$\pi . \Delta_{\alpha(1)} \circ \cdots \circ \Delta_{\alpha(N)}(p) = J(p) \tag{65}$$

Proof. Identical to the proof of Lemma 4 in Demazure [1].

We now come to the analogue of Lemma 5.29 for the space $C^\infty(\mathfrak{a}^*)$ of C^∞ functions $\mathfrak{a}^* \to \mathbb{C}$. Let $P(\mathfrak{a}^*)$ denote the subspace of polynomial functions $\mathfrak{a}^* \to \mathbb{C}$. The identification of $\mathfrak{a}^*$ with $\mathfrak{a}^{**}$ by $\alpha \to (\alpha,.)$ induces an algebra isomorphism $S(\mathfrak{a}^*) \to P(\mathfrak{a}^*)$. From now on we identify $S(\mathfrak{a}^*)$ with the subalgebra $P(\mathfrak{a}^*)$ of $C^\infty(\mathfrak{a}^*)$ in this manner. Observe that the action of W on $P(\mathfrak{a}^*)$ is given by

$$(wf)(\lambda) = f(w^{-1}\lambda),$$

for $w \in W$, $f \in P(\mathfrak{a}^*)$, $\lambda \in \mathfrak{a}^*$. We extend it to $C^\infty(\mathfrak{a}^*)$, using the same formula. Moreover, we extend the endomorphism J to an endomorphism of $C^\infty(\mathfrak{a}^*)$ by just using formula (64). If $f \in C^\infty(\mathfrak{a}^*)$, then the function $\lambda \to (\alpha,\lambda)^{-1}[f(\lambda) - f(s_\alpha(\lambda))]$ is C^∞. To see this, note that

$$f(\lambda) - f(s_\alpha(\lambda)) = \int_0^1 \frac{\partial}{\partial t}[f(s_\alpha(t)(\lambda))]dt$$

where $s_\alpha(t)$ is the linear map $\mathfrak{a}^* \to \mathfrak{a}^*$,

$$\lambda \to \lambda - (1 - t)\frac{2(\lambda, \alpha)}{(\alpha, \alpha)}\alpha .$$

Writing $\partial(\alpha)$ for the differential operator defined by $(\partial(\alpha)f)[\lambda] = (d/dt)[f(\lambda + t\alpha)]_{t=0}$, we thus obtain

$$\frac{f(\lambda) - f(s_\alpha(\lambda))}{(\alpha, \lambda)} = \frac{2}{(\alpha, \alpha)} \int_0^1 (\partial(\alpha)f)[s_\alpha(t)(\lambda)]dt. \quad (66)$$

Consequently the endomorphism Δ_α of $P(\mathfrak{a}^*)$ naturally extends to $C^\infty(\mathfrak{a}^*)$.

Lemma 5.30. Let notations be as in Lemma 5.29. Then for every $f \in C^\infty(\mathfrak{a}^*)$ we have

$$\Delta_{\alpha(1)} \circ \dots \circ \Delta_{\alpha(N)} f(\lambda) = \sum_{w \in W} \pi(w\lambda)^{-1} f(w\lambda). \quad (67)$$

Proof. Give $C^\infty(\mathfrak{a}^*)$ the usual topology of locally uniform convergence in all derivatives. Then $S(\mathfrak{a}^*) \cong P(\mathfrak{a}^*)$ is dense in $C^\infty(\mathfrak{a}^*)$, and since the operators Δ_α $(\alpha \in R)$ and J are continuous, this proves (65) for every $p \in C^\infty(\mathfrak{a}^*)$. Since $\pi(w\lambda) = \det(w).\pi(\lambda)$ this proves (67).

Proof of Lemma 5.17. By repeated application of formula (66) it follows that there exist finitely many polynomials $p_j(t) = p_j(t_1, \dots t_N)$ and finitely many differential operators $P_j \in U(\mathfrak{a}_c^*)$ of degree $\leq N = \#\Delta^{++}$, such that for any $f \in C^\infty(\mathfrak{a}^*)$ the left hand side of (67) is equal to

$$\sum_j \int_0^1 .. \int_0^1 p_j(t).(P_j f)[s_N(t_N) \circ \dots \circ s_1(t_1)(\lambda)]dt_1 \dots dt_N$$

(here we have written $s_i = s_{\alpha(i)}$). It is easy now to complete the proof.

Chapter 6

Asymptotic behaviour for singular values of λ

6.1 Introduction.

In this chapter we shall study the asymptotic behaviour of $\phi_\lambda(a_0)$, when $a_0 \to \infty$ in $A(F, {}^*C, C_F)$ and when $\lambda \in \mathfrak{a}_c^*$ is not necessarily contained in ${}'\mathfrak{a}_c^*$. For the meaning of notations not specified here, see Chapter 5.

If $\lambda \in {}'\mathfrak{a}_c^*$, we have

$$\phi_\lambda(a_0) = \sum_{v \in W^F} d_F(v\lambda)^{-1} \phi_{F,v,\lambda}(a_0), \tag{1}$$

for $a_0 \in A(F, {}^*C, C_F)$. This formula breaks down at points $\lambda \in \mathfrak{a}_c^*$ with $d_F(v\lambda) = 0$ for some $v \in W^F$. The aim of this chapter is to obtain formulas for ϕ_λ valid at such points.

We will obtain formulas expressing ϕ_λ as a sum of integrals over the cycles $\Gamma_{F,v}(a_0)$, valid when λ is kept equisingular (see Theorem 6.3). We get these formulas by differentiation of (1) with respect to λ; this explains the appearance of the polynomials $Q^\xi_{F,v}(\lambda,.)$ in (3). The formulas thus obtained lead to formulas from which the asymptotic behaviour of $\phi_\lambda(a_0)$ when $a_0 \to \infty$ in $A(F, {}^*C, C_F)$ can be read off (see Theorem 6.4). This behaviour is locally uniform with respect to the parameter λ that is kept equisingular. To obtain results uniform in λ one would have to tackle the same problem as we indicated in Chapter 5, that of bringing the cycles in such a position that the function $k \to \exp[(i\lambda-\rho)H_{F,v}(a_0 k)]$ can be estimated suitably.

At the end of Section 6.2 we will derive Harish-Chandra's well known estimate for $\Xi = \phi_0$ from the principal terms of the asymptotic expansions for Ξ (see Corollary 6.5). Moreover, these principal terms will be computed explicitly (see Theorem 6.6).

6.2 Formulas for singular values of λ

Fix an element $\xi \in \mathfrak{a}_c^*$ and put

$$\Delta^{++}(\xi) = \{\alpha \in \Delta^{++};\ \xi(H_{\alpha,0}) \in i\mathbb{Z}\},$$

$$\mathfrak{a}_c^*(\xi) = \{\alpha \in \mathfrak{a}_c^*\ ;\ \eta(H_{\alpha,0}) \in i\mathbb{Z} \text{ for } \alpha \in \Delta^{++}(\xi)\ .$$

Thus $\mathfrak{a}_c^*(\xi)$ is a locally finite union of mutually disjoint linear varieties in $\mathfrak{a}_c^*$. Therefore $\mathfrak{a}_c^*(\xi)$ is a complex analytic submanifold of $\mathfrak{a}_c^*$. Define

$$'\mathfrak{a}_c^*(\xi) = \{\eta \in \mathfrak{a}_c^*;\ \Delta^{++}(\eta) = \Delta^{++}(\xi)\}.$$

Thus $'\mathfrak{a}_c^*(\xi)$ consists of the $\eta \in \mathfrak{a}_c^*$ equisingular to ξ. Obviously this set is the complement of a locally finite union of lower dimensional linear subvarieties in $\mathfrak{a}_c^*(\xi)$. In particular it is a connected dense open subset of $\mathfrak{a}_c^*(\xi)$, and we may speak of holomorphic functions on $'\mathfrak{a}_c^*(\xi)$.

We define the meromorphic functions d^ξ and $d^\xi_{F,v}$ $(v \in W^F)$ on $\mathfrak{a}_c^*$ by:

$$d^\xi(\lambda) = \prod_{\alpha \in \Delta^{++}(\xi)} d_\alpha(\lambda),$$

$$d^\xi_{F,v}(\lambda) = d^\xi(\lambda) d_{F,v}(\lambda)^{-1} = d^\xi(\lambda) d_F(v\lambda)^{-1}.$$

Thus d^ξ is holomorphic on $\mathfrak{a}_c^*$ and $d^\xi_{F,v}$ is holomorphic at each $\mu \in {}'\mathfrak{a}_c^*(\xi)$. Moreover, on some open neighbourhood of ${}'\mathfrak{a}_c^*(\xi)$ in $\mathfrak{a}_c^*$ we have:

$$d^\xi(\lambda)\phi_\lambda(a_0) = \sum_{v \in W^F} d^\xi_{F,v}(\lambda)\phi_{F,v,\lambda}(a_0) \qquad (2)$$

Let $\mathbb{D}(\mathfrak{a}_c^*)$ denote the algebra of constant coefficient holomorphic differential operators on $\mathfrak{a}_c^*$, and let ∂ be the isomorphism $S(\mathfrak{a}_c^*) \to \mathbb{D}(\mathfrak{a}_c^*)$ determined by

$$[\partial(\eta)f](\lambda) = \left(\frac{d}{dt}\right)_{t=0} f(\lambda + t\eta) = df(\lambda)(\eta),$$

for $f \in C^\infty(\mathfrak{a}_c^*)$, $\lambda \in \mathfrak{a}_c^*$, $\eta \in \mathfrak{a}^*$. If D is a linear holomorphic differential operator defined in a neighbourhood of a point $\mu \in \mathfrak{a}_c^*$, then there exists a unique differential operator $D_\mu \in \mathbb{D}(\mathfrak{a}_c^*)$ such that for any holomorphic function f defined in a neighbourhood of μ we have

$$(Df)(\mu) = (D_\mu f)(\mu).$$

The operator D_μ is called the local expression of D at μ. Now let $P(\mathfrak{a}_c^*)$ denote the algebra of polynomial functions $\mathfrak{a}_c^* \to \mathbb{C}$, and let $\pi^\xi \in P(\mathfrak{a}_c^*)$ and $\tilde{\omega}^\xi \in S(\mathfrak{a}_c^*)$ be defined by:

$$\pi^\xi(\lambda) = \prod_{\alpha \in \Delta^{++}(\xi)} (\alpha,\lambda), \qquad \tilde{\omega}^\xi = \prod_{\alpha \in \Delta^{++}(\xi)} \alpha. \qquad (\lambda \in \mathfrak{a}_c^*)$$

Then we have the following propositions.

<u>Proposition 6.1</u>. $\partial(\tilde{\omega}^\xi)(\pi^\xi)$ is constant and strictly positive.

Proof. It is obvious that $\partial(\omega^{\xi})(\pi^{\xi})$ is constant. As for the second assertion, see e.g. Varadarajan [2 , p. 59, Corollary 7].

Proposition 6.2. Let $M(\xi)$ be the positive real number

$$M(\xi) = (\partial(\tilde{\omega}^{\xi})\pi^{\xi})(0) . \prod_{\alpha \in \Delta^{++}(\xi)} \frac{2\pi}{(\alpha,\alpha)} .$$

Then the holomorphic differential operator

$$D^{\xi} : f \to M(\xi)^{-1} . \partial(\tilde{\omega}^{\xi})(d^{\xi} f)$$

has at each point $\mu \in \mathfrak{a}_c^*(\xi)$ the identity as its local expression.

Proof. Fix $\mu \in \mathfrak{a}_c^*(\xi)$, and let f be a holomorphic function defined in some neighbourhood of μ. Using Leibniz' rule repeatedly and observing that for every $\alpha \in \Delta^{++}(\xi)$ the function $d_{\alpha} : \lambda \to \exp(2\pi\lambda(H_{\alpha,0})) - 1$ vanishes at $\lambda = \mu$, we see that

$$[\partial(\omega^{\xi})(d^{\xi} f)](\mu) = [\partial(\omega^{\xi}) d^{\xi}](\mu) . f(\mu) .$$

Moreover, we obviously have

$$\partial(\omega^{\xi}) d^{\xi} = [\prod_{\alpha \in \Delta^{++}(\xi)} \frac{2\pi}{(\alpha,\alpha)}] . \partial(\omega^{\xi})\pi^{\xi}$$

and this proves the Proposition.

If $\lambda \in {}'\mathfrak{a}_c^*(\xi)$, let $Q^{\xi}_{F,v}(\lambda)$ denote the local expression of $D^{\xi} d^{-1}_{F,v}$ at λ. If $k \in \mathbb{N}$ we write $\mathbb{D}^k(\mathfrak{a}_c^*)$ for the elements of $\mathbb{D}(\mathfrak{a}_c^*)$ of order at most k. Obviously the map

$$\lambda \to Q^{\xi}_{F,v}(\lambda), \ '\mathfrak{a}^*_c(\xi) \to \mathbb{D}^q(\mathfrak{a}^*_c) \qquad (q = \#\Delta^{++}(\xi))$$

is holomorphic. Now consider the algebra isomorphism $D \to P_D$, $\mathbb{D}(\mathfrak{a}^*_c) \to P(\mathfrak{a}^*_c)$ defined by

$$P_D(H) = D(\lambda \to e^{i\lambda(H)})_{\lambda=0}.$$

Writing $Q^{\xi}_{F,v}(\lambda,.)$ for the image of $Q^{\xi}_{F,v}(\lambda)$ in $P(\mathfrak{a}^*_c)$ under this isomorphism, we have the following theorem.

Theorem 6.3. If $a_0 \in A(F,{}^*C,C_F)$, $\lambda \in {}'\mathfrak{a}^*_c(\xi)$, then:

$$\phi_\lambda(a_0) =$$

$$= \sum_{v \in W^F} \int_{\Gamma_{F,v}} Q^{\xi}_{F,v}(\lambda, H_{F,v}(a_0k)) e^{(i\lambda-\rho)H_{F,v}(a_0k)} \omega(k) \tag{3}$$

Proof. By (2), Proposition 6.2 and the definition of the $Q^{\xi}_{F,v}(\lambda)$ it follows that for $a_0 \in A(F,{}^*C,C_F)$, $\mu \in {}'\mathfrak{a}^*_c(\xi)$ we have:

$$\phi_\mu(a_0) = \sum_{v \in W^F} Q^{\xi}_{F,v}(\mu)[\lambda \to \phi_{F,v,\lambda}(a_0)]_{\lambda=\mu},$$

Now

$$Q^{\xi}_{F,v}(\mu)[\lambda \to e^{(i\lambda-\rho)H_{F,v}(a_0k)}] =$$

$$= Q^{\xi}_{F,v}(\mu, H_{F,v}(a_0k)) e^{(i\lambda-\rho)H_{F,v}(a_0k)}$$

and so differentiation under the integral sign completes the proof.

Let L be the set $\mathbb{N}.S$ (S is the collection of simple roots in Δ^{++}). If $\ell \in L$ we write

$$\ell = \sum_{\alpha \in S} \ell_\alpha \alpha \qquad (\ell_\alpha \in \mathbb{N}),$$

and if $H \in \mathfrak{a}$, $\ell \in L$ we define

$$H^\ell = \prod_{\alpha \in S} \alpha(H)^{\ell_\alpha}.$$

Finally if $\ell \in L$, let $|\ell| = \Sigma_{\alpha \in S}\, \ell_\alpha$.

<u>Theorem 6.4</u>. There exist functions $\Phi^\xi_{F,v,\ell}$: ${}'\mathfrak{a}_c(\xi) \times A({}^*C) \times U_{F,v} \to \mathbb{C}$ ($v \in W^F$, $\ell \in L$, $|\ell| \leqslant q = \#\Delta^{++}(\xi)$) depending holomorphically on the first and on the last variable, and real analytically on the second variable, such that for all $a_0 \in A(F, {}^*C, C_F)$, $\lambda \in {}'\mathfrak{a}^*_c(\xi)$ we have:

$$\phi_\lambda(a_0) = \sum_{v \in W^F} a^{iv\lambda-\rho} \sum_{\substack{\ell \in L \\ |\ell| \leqslant q}} (\log a)^\ell . \Phi^\xi_{F,v,\ell}(\lambda, {}^*a, z_{F,v}(a)). \tag{4}$$

<u>Proof</u>. There exist polynomials $Q^\xi_{F,v,\ell}(\lambda,\cdot) \in P(\mathfrak{a}^*_c)$ ($v \in W^F$, $\ell \in L$, $|\ell| \leqslant q$) depending holomorphically on λ, such that for H', $H \in \mathfrak{a}^*_c$ we have

$$Q^\xi_{F,v}(\lambda, H' + v^{-1}(H)) = \sum_{|\ell| \leqslant q} Q^\xi_{F,v,\ell}(\lambda, H')H^\ell.$$

With the notations of Section 5.3 we have

$$\int_{\Gamma_{F,v}(a_0)} Q^\xi_{F,v}(\lambda, H_{F,v}(a_0k))e^{(i\lambda-\rho)H_{F,v}(a_0k)}\,\omega(k) =$$

$$= \int_{\gamma_{F,v}} a^{iv\lambda-\rho} Q^{\xi}_{F,v}(\lambda,\psi_v(z,{}^*a)^* H_{F,v}(a_0;.))\tilde{\Phi}_{F,v}\omega_{F,v} \qquad (5)$$

where $\gamma_{F,v} = K_{v^{-1}(F)} \times \gamma_v \times \gamma_{v'}$, $z = z_{F,v}(a)$, and where $\tilde{\Phi}_{F,v} =$ $= \tilde{\Phi}_{F,v}(\lambda,{}^*a,z_{F,v}(a),k,\bar{n},\bar{n}')$. In view of the formula for $\psi_v(z_{F,v}(a),{}^*a)^* H_{F,v}(a_0;.)$ in Section 5.3 we have

$$Q^{\xi}_{F,v}(\lambda,\psi_v(z_{F,v}(a),{}^*a)^* H_{F,v}(a_0;.)) =$$

$$= \sum_{\substack{\ell \in L \\ |\ell| \leqslant q}} (\log a)^{\ell}.\tilde{Q}^{\xi}_{F,v,\ell}(\lambda,{}^*a,z_{F,v}(a),k,\bar{n},\bar{n}'), \qquad (6)$$

where:

$$\tilde{Q}^{\xi}_{F,v,\ell}(\lambda,{}^*a,z,k,\bar{n},\bar{n}^*) =$$

$$= Q^{\xi}_{F,v,\ell}(\lambda,H_v[{}^*ak\sigma({}^*a,k)(\bar{n})(z.\bar{n}')] - H_{v'}[\sigma({}^*a,k)(z.\bar{n})\bar{n}']).$$

Define $\Phi^{\xi}_{F,v,\ell}: {}'\mathfrak{a}^*_c(\xi)\times A({}^*C)\times U_{F,v} \to \mathbb{C}$ by

$$\Phi^{\xi}_{F,v,\ell}(\lambda,{}^*a,z) =$$

$$= \int_{\gamma_{F,v}} (\tilde{Q}^{\xi}_{F,v,\ell}\tilde{\Phi}_{F,v})(\lambda,{}^*a,z,k,\bar{n},\bar{n}')\omega_{F,v}.$$

Then obviously $\Phi^{\xi}_{F,v,\ell}$ is holomorphic in its first and last variable, and real analytic in its second variable. Moreover, (4) follows from (3) by substitution of (6) in (5).

Corollary 6.5. Let $\lambda \in {}'\mathfrak{a}^*_c(\xi)$, and set $q = \#\Delta^{++}(\xi)$. Then the function

$$a \to a^{\rho}(1 + \|\log a\|)^{-q}\phi_{\lambda}(a)$$

is bounded on A, locally uniformly with respect to λ.

Proof. The assertion holds on all sets of type $A(F, {}^*C, C_F)$ satisfying the conditions of Chapter 5. The constants ${}^*C$ can be chosen arbitrarily big and so $\overline{A^+}$ can be covered with a finite number of such sets. Therefore the assertion holds on $\overline{A^+}$ as well. Now fix $w \in W$. Then we have $w^{-1}(\rho) \leq \rho$ on $\overline{\mathfrak{a}^+}$. Hence if $a \in \overline{A^+}$ then:

$$(wa)^{\rho}(1 + \|\log w(a)\|)^{-q}\phi_{\lambda}(wa) =$$

$$= a^{w^{-1}(\rho)}(1 + \|\log a\|)^{-q}\phi_{\lambda}(a) \leq$$

$$\leq a^{\rho}(1 + \|\log a\|)^{-q}\phi_{\lambda}(a).$$

This shows that the assertion is valid on $w(\overline{A^+})$ for every $w \in W$, and therefore on A.

Remark. Corollary 6.5 with $\xi = 0$, $\lambda = 0$ yields the well known estimate

$$\Xi(a) = \int_K e^{-\rho H(ak)}dk \leq E(1 + \|\log a\|)^N, \tag{7}$$

where $N = \#\Delta^{++}$ and where E is a positive constant (cf. Harish-Chandra [2 , p. 279]). It is sharp, as can be read off from Theorem 6.6 below.

If $F \subset S$, we write G_F for the centralizer of $\mathfrak{a}_F$ in G, and we put $K_F = G_F \cap K$. The analogon of Ξ for the reductive pair (G_F, K_F) is denoted by Ξ_F. By the expression "$a_0 \to \infty$ along A_F"

we mean that there exist constants ${}^*C > 0$, $C_F > 0$ such that $a_0 \to \infty$ in $A(F,{}^*C,C_F)$ (cf. Section 5.1). Finally we define $\tilde{\omega},\tilde{\omega}_F \in S(\mathfrak{a}_c^*)$ and π, $\pi_F \in P(\mathfrak{a}_c^*)$ by

$$\tilde{\omega} = \prod_{\alpha\in\Delta^{++}} \alpha, \quad \tilde{\omega}_F = \prod_{\alpha\in\Delta_F^{++}} \alpha,$$

$$\pi(\lambda) = \prod_{\alpha\in\Delta^{++}} (\alpha,\lambda), \quad \pi_F(\lambda) = \prod_{\alpha\in\Delta_F^{++}} (\alpha,\lambda),$$

and we let c_F denote the c-function associated with the pair (G_F,K_F).

Theorem 6.6. Let $F \subset S$ and let $a_0 \to \infty$ along A_F. Then:

$$\Xi(a_0) \sim E_F\, \Xi_F({}^*a)a^{-\rho} \prod_{\alpha\in\Delta^{++}\setminus\Delta_F} \alpha(i \log a),$$

$$E_F = [W:W_F]\cdot\frac{(\partial(\omega_F)\pi_F)(0)}{(\partial(\omega)\pi)(0)}\cdot\frac{(c\pi)(0)}{(c_F\pi_F)(0)}\ .$$

Proof. Consider the proof of Theorem 6.4 with $\xi = 0$, $\lambda = 0$. The principal term of the right hand side of (6) is equal to

$$Q^0_{F,v}(0,v^{-1}\log a).$$

Hence by (5) and by (5.20), the principal term of the expansion for $\Xi(a_0)$ corresponding to (4) is equal to

$$\sum_{v\in W^F} a^{-\rho}Q^0_{F,v}(0,v^{-1}(\log a))\Phi_{F,v}(0,{}^*a,0). \tag{8}$$

By Theorem 5.7 we have

$$\Phi_{F,v}(0,{}^*a,0) = \Phi_{F,I}(0,{}^*a,0) =$$

$$= \Xi_F(^*a)K_X K^{-1}\,\tilde{I}_u(0) \tag{9}$$

where $u \in W$ is determined by $\Delta^{++}(u) = \Delta^{++}\setminus\Delta_F$. Writing $\tilde{I}_F$ for the analogon of $\tilde{I}$ for G_F we have

$$\tilde{I}_F = \prod_{\alpha\in\Delta_F^{++}} \tilde{I}_\alpha, \qquad \tilde{I}_u = \prod_{\alpha\in\Delta^{++}\setminus\Delta_F} \tilde{I}_\alpha$$

and hence $\tilde{I}_u\tilde{I}_F = \tilde{I}$. Moreover, $K^{-1}\tilde{I} = dc$ and $K_X^{-1}\tilde{I}_u = \delta_F c_F$, where

$$\delta_F = \prod_{\alpha\in\Delta_F^{++}} d_\alpha,$$

and so $K_X K^{-1}\tilde{I}_u(0) = (dc)(0).(\delta_F c_F)(0)^{-1}$. Observe that $(dc)(0) \neq$ $\neq 0$ and $(\delta_F c_F)(0) \neq 0$, by (3.32) and the product formula for the c-function. If $\alpha \in \Delta^{++}$ then $(\lambda,\alpha)^{-1}d_\alpha(\lambda) \to 2\pi(\alpha,\alpha)^{-1}$ if $\lambda \to 0$, and so (9) becomes

$$\Phi_{F,v}(0,\ a,0) = \Xi_F(^*a)\ \frac{(\pi c)(0)}{(\pi_F c_F)(0)} \prod_{\alpha\in\Delta^{++}\setminus\Delta_F} \left[\frac{2\pi}{(\alpha,\alpha)}\right] \tag{10}$$

On the other hand, $Q^0_{F,v}(0)$ is the local expression of the differential operator $D^0\circ d_{F,v}^{-1} = M(0)^{-1}\partial(\tilde{\omega})\circ(dd_{F,v}^{-1})$ at 0. Writing v^* for the pull back of v: $\mathfrak{a}_c^* \to \mathfrak{a}_c^*$ operating on functions we have $\partial(\eta)\circ v^* = v^*\circ\partial(v\eta)$ for $\eta \in \mathfrak{a}_c^*$ and so

$$\begin{aligned} D^0\circ d_{F,v}^{-1} &= M(0)^{-1}.\det(v).\partial(\tilde{\omega})\circ v^*\circ\delta_F\circ(v^{-1})^* \\ &= M(0)^{-1}.\det(v).v^*\circ\partial(v\tilde{\omega})\circ\delta_F\circ(v^{-1})^* \\ &= M(0)^{-1}.v^*\circ\partial(\tilde{\omega})\circ\delta_F\circ(v^{-1})^*. \end{aligned}$$

This shows that for $H \in \mathfrak{a}_c^*$ we have

$$Q^0_{F,v}(0,v^{-1}(H)) = M(0)^{-1}\partial(\tilde{\omega})[\lambda \to \delta_F(\lambda)e^{i\lambda(H)}]_{\lambda=0}.$$

Now $\tilde{\omega} = \tilde{\omega}_F{}^*\tilde{\omega}$, with

$$^*\tilde{\omega} = \prod_{\alpha \in \Delta^{++}\setminus\Delta_F} \alpha.$$

If $H \in \mathfrak{a}_F$, then $\partial(\alpha)[\lambda \to e^{i\lambda(H)}]_{\lambda=0} = 0$, and therefore

$$Q^0_{F,v}(0,v^{-1}(H)) = M(0)^{-1}.(\partial(\omega_F)\delta_F)(0). \prod_{\alpha \in \Delta^{++}\setminus\Delta_F} \alpha(iH).$$

By (8), (10), the formula for M(0) and the fact that

$$\partial(\tilde{\omega}_F)(\delta_F) = \{\prod_{\alpha \in \Delta_F^{++}} \frac{2\pi}{(\alpha,\alpha)}\}\partial(\tilde{\omega}_F)(\pi_F)$$

this proves the theorem.

References

Araki, S.

[1] On root systems and an infinitesimal classification of irreducible symmetric spaces, J. of Math., Osaka City Univ., 13 (1962), 1 - 34.

Bhanu Murti, T.S.

[1] Plancherel's measure for the factor space $SL(n,\mathbb{R})/SO(n,\mathbb{R})$, Doklady Akad. Nauk. SSSR 133 (1960), 503 - 506 = Soviet Math. Dokl. 1 (1960).

[2] The asymptotic behavior of zonal spherical functions on the Siegel upper half-plane, Doklady Akad. Nauk. SSSR 135 (1960), 1027 - 1029 = Soviet Math. Dokl. 1 (1960), 1325 - 1329.

Borel, A. and Tits, J.

[1] Compléments à l'article "Groupes réductifs", Publ. Math. IHES 41 (1972), 253 - 276.

Cohn, L.

[1] Analytic Theory of the Harish-Chandra C-function, Lecture Notes in Math. 429, Springer-Verlag, Berlin, 1974.

Demazure, M.

[1] Invariants symétriques entiers des groupes de Weyl et torsion, Invent. Math. 21 (1973), 287 - 301.

Duistermaat, J.J., Kolk J.A.C. and Varadarajan V.S.

[1] Spectra of compact locally symmetric manifolds of negative curvature, Invent. Math. 52 (1979), 29 - 93.

[2] Functions, flows and oscillatory integrals on flag manifolds and conjugacy classes in real semisimple Lie groups, Preprint No. 200, Univ. of Utrecht, Department of Math., 1981.

Erdélyi, A.

[1] Asymptotic Expansions, Dover Publ.,Inc., New York, 1956.

Gelfand, I.M., Neumark, M.A.

[1] Unitäre Darstellungen der klassischen Gruppen, Akademie-Verlag, Berlin, 1957.

Gindikin, S.G. and Karpelevič, F.I.

[1] Plancherel measure for Riemann symmetric spaces of non-positive curvature, Doklady Akad. Nauk. SSSR, 145 (1962), 252 - 255 = Soviet Math. Dokl. 3 (1962), 962 - 965.

Griffiths, P.A. and Harris, J.
[1] Principles of Algebraic Geometry, Wiley Intersc. Publ., New York, 1978.
Guillemin, V. and Pollack, A.
[1] Differential Topology, Prentice-Hall, Inc., Englewood Cliffs, 1974.
Harish-Chandra
[1] Representations of semisimple Lie groups II, Trans. Amer. Math. Soc. 76 (1954), 26 - 65.
[2] Spherical functions on a semisimple Lie group I, Amer. J. Math. 80 (1958), 241 - 310.
[3] idem II; Amer. J. Math. 80 (1958), 553 - 613.
[4] Harmonic analysis on real reductive groups III. The Maass-Selberg relations and the Plancherel formula, Ann. of Math. 104 (1976), 117 - 201.
Heckman, G.J.
[1] Projections of Orbits and Asymptotic Behaviour of Multiplicities for Compact Lie Groups, Thesis, Rijksuniversiteit Leiden, 1980.
Helgason, S.
[1] Differential Geometry and Symmetric Spaces, Academic Press, New York, 1962.
[2] A duality for symmetric spaces with applications to group representations, Advances in Math. 5 (1970), 1 - 154.
[3] Analysis on Lie Groups and Homogeneous Spaces, Conf. Board of the Math. Sci., Regional Conf. Ser. Math. 14, AMS, Providence, R.I., 1972.
[4] Functions on symmetric spaces, Proc. of Symp. in Pure Math. 26, 101 - 146, AMS, Providence, R.I., 1973.
Hörmander, L.
[1] Fourier integral operators I, Acta Math. 127 (1971), 79 - 183.
Humphreys, J.E.
[1] Linear Algebraic Groups, Springer-Verlag, New York, 1975.

Pochhammer, L.

[1] Ueber ein Integral mit doppeltem Umlauf, Mathematische Annalen 35 (1890), 470 - 494.

[2] Zur Theorie der Euler'schen Integrale, Mathematische Annalen 35 (1890), 495 - 526.

de Rham, G.

[1] Variétés Différentiables, Hermann, Paris, 1954.

Schiffmann, G.

[1] Intégrales d'entrelacement et fonctions de Whittaker, Bull. Soc. Math. France 99 (1971), 3 - 72.

Trombi, P.C. and Varadarajan, V.S.

[1] Spherical transforms on semisimple Lie groups, Ann. of Math. 94 (1971), 246 - 303.

Varadarajan, V.S.

[1] Lie Groups, Lie Algebras, and their Representations, Prentice-Hall, Englewood Cliffs, 1974.

[2] Harmonic Analysis on Real Reductive Groups, Lecture Notes in Math. 576, Springer-Verlag, Berlin, 1977.

Wallach, N.R.

[1] Harmonic Analysis on Homogeneous Spaces, Marcel Dekker, Inc., New York, 1973.

Warner, G.

[1] Harmonic Analysis on Semi-Simple Lie Groups I, Springer-Verlag, Berlin, 1972.

[2] idem, II.

Samenvatting

In dit proefschrift wordt het asymptotisch gedrag bestudeerd van elementaire bolfuncties behorend bij een reële halfenkelvoudige Liegroep G (samenhangend, met eindig centrum) en een maximaal compacte ondergroep K. De harmonisch analyse van bi-K-invariante functies (ook wel bolfuncties genoemd) op G is grotendeels door Harish-Chandra ontwikkeld ([2], [3]). Hierin spelen de elementaire bolfuncties een rol die analoog is aan die van de exponentiële functies in de Fourieranalyse op $\mathbb{R}^n$. Zo bestaan er bijvoorbeeld een inversieformule en een Plancherelformule.

De elementaire bolfuncties kunnen geparametriseerd worden door een eindig dimensionale complex lineaire ruimte $\mathfrak{a}_c^*$ modulo een eindige spiegelingsgroep W, de Weylgroep. Als $\lambda \in \mathfrak{a}_c^*$ zekere regulariteitscondities vervult dan kan de bijbehorende elementaire bolfunctie ϕ_λ volgens een bekende formule van Harish-Chandra geschreven worden als eindige som van functies $\phi_\lambda = \Sigma_{w \in W}\, \psi_{w,\lambda}$. Iedere summand $\psi_{w,\lambda}$ wordt gegeven door een reeksontwikkeling die het asymptotisch gedrag beschrijft indien de groepsvariabele in een bepaalde collectie van richtingen naar oneindig gaat. In dit proefschrift worden de functies $\psi_{w,\lambda}$ op nieuwe wijze voorgesteld, namelijk als integralen over compacte cykels in een complexificatie K_c van de groep K. Deze voorstellingen dienen vervolgens als uitgangspunt voor de studie van het asymptotisch gedrag van de $\psi_{w,\lambda}$.

De ontwikkelde techniek is ook toepasbaar als de groepsvariabele in andere richtingen naar oneindig gaat. Dit werpt nieuw licht op resultaten van Trombi en Varadarajan ([1]).

Tenslotte is het mogelijk formules af te leiden voor waarden van $\lambda \in \mathfrak{a}_c^*$ die niet voldoen aan bovengenoemde regulariteitscondities. Dit leidt tot enige nieuwe asymptotische ontwikkelingen.

Curriculum Vitae

De auteur van dit proefschrift werd op 24 januari 1956 te Eindhoven geboren. Van 1968 tot 1974 bezocht hij het Gemeentelijk Lyceum te Eindhoven. Vervolgens ging hij wiskunde met bijvak natuurkunde studeren aan de Rijksuniversiteit te Utrecht.

Na zijn kandidaatsexamen in 1976 koos hij het bijvak capita selecta van de wiskunde. Hij volgde colleges van o.a. de docenten van der Blij, Duistermaat, Springer en Stegeman. Tijdens zijn afstudeerfase over complexe analyse in meer variabelen maakte hij door een college van zijn afstudeerdocent Prof.dr. J.J. Duistermaat kennis met de theorie van de Liegroepen. Na zijn doctoraalexamen in juni 1978 besloot hij zich in die richting verder te ontwikkelen.

Sinds september 1978 is hij als wetenschappelijk medewerker in dienst van het Mathematisch Instituut van de Rijksuniversiteit Utrecht. Naast zijn onderwijstaken verrichtte hij daar o.l.v. Prof.dr. J.J. Duistermaat het onderzoek dat geleid heeft tot dit proefschrift.

In het academische jaar 1982 - 1983 zal hij als Member verbonden zijn aan het Institute for Advanced Study in Princeton.

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