Convexity theorems for semisimple symmetric spaces

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Abstract

We prove a convexity theorem for semisimple symmetric spaces G/H which generalizes an earlier theorem of the second named author to a setting without restrictions on the minimal parabolic subgroup involved. The new more general result specializes to Kostant's non-linear convexity theorem for a real semisimple Lie group G in two ways, firstly by taking H maximal compact and secondly by viewing G as a symmetric space for $G \times G$.

1 Introduction

In this paper we prove a generalization of the convexity theorem in [4] for a symmetric space G/H. Here G is a connected semisimple Lie group with finite center, σ an involution of G and H an open subgroup of the group G^{σ} of fixed points for σ. The generalization involves Iwasawa decompositions related to minimal parabolic subgroups of G of arbitrary type instead of the particular type of parabolic subgroup considered in [4].

From now on we assume more generally that G is a real reductive group of the Harish-Chandra class; this will allow an inductive argument relative to the real rank of G. Let $\theta : G \to G$ be a Cartan involution of G that commutes with σ; for its existence, see [21, Thm. 6.16]. The associated group $K := G^{\theta}$ of fixed points is a maximal compact subgroup of G. For the infinitesimal involutions determined by σ and θ we use the same symbols: $\theta, \sigma : \mathfrak{g} \to \mathfrak{g}$; here $\mathfrak{g}$ denotes the Lie algebra of G. With respect to the infinitesimal involutions, $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q},$$

where $\mathfrak{k}$ and $\mathfrak{p}$ are the +1 and -1 eigenspaces for θ and likewise, $\mathfrak{h}$ and $\mathfrak{q}$ are the +1 and -1 eigenspaces for σ. Note that $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{h}$ is the Lie algebra of H.

Since σ and θ commute, their composition $\sigma\theta$ is again an involution of $\mathfrak{g}$. With respect to the latter involution, $\mathfrak{g}$ decomposes into eigenspaces

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-.$$

Observe that the +1 eigenspace $\mathfrak{g}_+$ equals $\mathfrak{k} \cap \mathfrak{h} \oplus \mathfrak{p} \cap \mathfrak{q}$, while the -1 eigenspace $\mathfrak{g}_-$ equals $\mathfrak{k} \cap \mathfrak{q} \oplus \mathfrak{p} \cap \mathfrak{h}$.

We fix a maximal abelian subspace $\mathfrak{a}_{\mathrm{q}}$ of $\mathfrak{p} \cap \mathfrak{q}$, and a a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ that contains $\mathfrak{a}_{\mathrm{q}}$. Then $\mathfrak{a}$ is σ-stable and decomposes as:

$$\mathfrak{a} = \mathfrak{a}_{\mathrm{h}} \oplus \mathfrak{a}_{\mathrm{q}}, \tag{1}$$

where $\mathfrak{a}_{\mathrm{h}} := \mathfrak{a} \cap \mathfrak{h}$. The associated projection onto $\mathfrak{a}_{\mathrm{q}}$ will be denoted by $\mathrm{pr}_{\mathrm{q}} : \mathfrak{a} \to \mathfrak{a}_{\mathrm{q}}$.

Let $\Sigma(\mathfrak{g}, \mathfrak{a})$ be the set of roots of $\mathfrak{a}$ in $\mathfrak{g}$ and $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$ the set of roots of $\mathfrak{a}_{\mathrm{q}}$ in $\mathfrak{g}$. Both of these sets form root systems, possibly non-reduced, see e.g. [26] or [21]. The associated Weyl groups are given by

$$W(\mathfrak{a}) = N_K(\mathfrak{a})/Z_K(\mathfrak{a}) \qquad \text{and} \quad W(\mathfrak{a}_{\mathrm{q}}) = N_K(\mathfrak{a}_{\mathrm{q}})/Z_K(\mathfrak{a}_{\mathrm{q}}). \tag{2}$$

Let $A = \exp \mathfrak{a}$ and let $\mathscr{P}(A)$ be the set of minimal parabolic subgroups of G containing A. If $P \in \mathscr{P}(A)$, then P has a unique Langlands decomposition given by

$$P = MAN_P, \tag{3}$$

where $M := Z_K(\mathfrak{a})$, $N_P = \exp \mathfrak{n}_P$ and $\mathfrak{n}_P$ is the sum of the root spaces corresponding to a uniquely determined positive system of $\Sigma(\mathfrak{g}, \mathfrak{a})$. We denote this positive system by $\Sigma(P)$. The map given by

$$P \mapsto \Sigma(P), \quad P \in \mathscr{P}(A),$$

defines a bijection between $\mathscr{P}(A)$ and the set of positive systems of $\Sigma(\mathfrak{g}, \mathfrak{a})$.

Let

$$\Sigma(P, \sigma\theta) := \{\alpha \in \Sigma(P) : \ \sigma\theta\alpha \in \Sigma(P)\}$$

and

$$\Sigma(P)_- := \{\alpha \in \Sigma(P, \sigma\theta) : \ \sigma\theta\alpha = \alpha \implies \sigma\theta|_{\mathfrak{g}_\alpha} \neq \mathrm{id}_{\mathfrak{g}_\alpha}\}. \tag{4}$$

Any parabolic subgroup $P \in \mathscr{P}(A)$ induces an Iwasawa decomposition

$$G \simeq K \times A \times N_P$$

and the associated infinitesimal decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}_P$. The *Iwasawa projection* corresponding to P is defined as the real analytic map

$$\mathfrak{H}_P : G \longrightarrow \mathfrak{a}, \quad \text{determined by} \quad g \in K \exp \mathfrak{H}_P(g) N_P, \quad (g \in G). \tag{5}$$

The main result of [22], known as 'Kostant's (nonlinear) convexity theorem' characterizes the image under $\mathfrak{H}_P$ of the set aK, for $a \in A$, as follows:

$$\mathfrak{H}_P(aK) = \mathrm{conv}(W(\mathfrak{a}) \cdot \log a).$$

Here 'conv' indicates that the convex hull in $\mathfrak{a}$ is taken.

In our setting, it is natural to study the more general question of convexity of the set $\mathfrak{H}_P(aH)$, for $a \in A$. The first answer to this question was provided in [4, Thm. 1.1] under the assumption that $P \in \mathscr{P}(A)$ satisfies $\Sigma(P, \sigma\theta) = \Sigma(P) \setminus \mathfrak{a}_{\mathrm{h}}^*$; here $\mathfrak{a}_{\mathrm{h}}^*$ and $\mathfrak{a}_{\mathrm{q}}^*$ are viewed as subspaces of $\mathfrak{a}^*$ in accordance with the decomposition (1).

In the present paper we generalize [4, Thm 1.1] to any parabolic subgroup $P \in \mathscr{P}(A)$. To prepare for our main result, we need a few remarks and definitions as well as new notation.

Remark 1.1. Since $\exp : \mathfrak{a} \to A$ is a diffeomorphism, it follows that $A \simeq A_{\mathrm{q}} \times A_{\mathrm{h}}$ where $A_{\mathrm{q}} := \exp(\mathfrak{a}_{\mathrm{q}})$ and $A_{\mathrm{h}} := \exp(\mathfrak{a}_{\mathrm{h}}) = A \cap H$. Thus, we just need to consider $a \in A_{\mathrm{q}}$.

Remark 1.2. Since $\mathfrak{H}_P(aH) = \mathfrak{H}_P(aHA_{\mathrm{h}}) = \mathfrak{H}_P(aH) + \mathfrak{a}_{\mathrm{h}}$, it suffices to consider the image of aH, for $a \in A_{\mathrm{q}}$, under the map

$$\mathfrak{H}_{P,\mathrm{q}} := \mathrm{pr}_{\mathrm{q}} \circ \mathfrak{H}_P : G \to \mathfrak{a}_{\mathrm{q}}.$$

We recall from [4, Eqn. (1.2)] that the subgroup H is said to be *essentially connected* if

$$H = Z_{K\cap H}(\mathfrak{a}_{\mathrm{q}})H^{\circ}, \tag{6}$$

where H° denotes the identity component of H.

Let B be an extension of the Killing form from $[\mathfrak{g},\mathfrak{g}]$ to a bilinear form on the entire algebra $\mathfrak{g}$, such that B is $\mathrm{Ad}(G)$-invariant, invariant under both θ and σ, negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. Then B is non-degenerate.

We define a positive definite inner product on $\mathfrak{g}$ by

$$\langle U,V\rangle := -B(U,\theta V), \qquad (U,V \in \mathfrak{g}). \tag{7}$$

Note that the root space decomposition and the eigenspace decompositions (with respect to θ and σ) are orthogonal with respect to this inner product. Moreover, the extended Killing form and the inner product coincide if either U or V belongs to $\mathfrak{p}$.

Definition 1.3. *The Weyl group $W_{K\cap H}$ is defined as*

$$W_{K\cap H} := N_{K\cap H}(\mathfrak{a}_{\mathrm{q}})/Z_{K\cap H}(\mathfrak{a}_{\mathrm{q}}).$$

Note that $W_{K\cap H}$ may be viewed as a subgroup of $W(\mathfrak{a}_{\mathrm{q}})$. If α is a root in $\Sigma(\mathfrak{g},\mathfrak{a})$ we denote by H_α the element of $\mathfrak{a}$ perpendicular to $\ker\alpha$ with respect to $\langle\cdot,\cdot\rangle$, and normalized by $\alpha(H_\alpha) = 2$.

Definition 1.4. *Let P be a minimal parabolic subgroup of G containing A. Then we define the finitely generated polyhedral cone $\Gamma(P)$ in $\mathfrak{a}_{\mathrm{q}}$ by*

$$\Gamma(P) := \sum_{\alpha\in\Sigma(P)_-} \mathbb{R}_{\geq 0}\, \mathrm{pr}_{\mathrm{q}}(H_\alpha). \tag{8}$$

Main Theorem (Theorem 10.1) *Let H be an essentially connected open subgroup of G^σ, see (6). Let P be any minimal parabolic subgroup of G containing A and let $a \in A_{\mathrm{q}}$. Then*

$$\mathfrak{H}_{P,\mathrm{q}}(aH) = \mathrm{conv}(W_{K\cap H}\cdot\log a) + \Gamma(P).$$

If the two involutions σ and θ are equal, then $K = H$ and $\Sigma(P,\sigma\theta) = \Sigma(P)$. This implies that $W(\mathfrak{a}) = W_{K\cap H}$ and that $\Sigma(P)_- = \emptyset$. Thus, we obtain that $\Gamma(P) = 0$ and hence, in this case our main theorem coincides with the original non-linear convexity theorem of Kostant [22].

For P satisfying $\Sigma(P,\sigma\theta) = \Sigma(P)\setminus\mathfrak{a}_{\mathrm{h}}^*$ the above result coincides with [4, Thm 1.1]. This will be explained in detail in Section 2.2.

The proof of the main theorem follows the line of argument described below, which is a considerable extension of the argumentation of [4], which in turn was inspired by [16].

We first prove the theorem for a regular element $a \in A_{\mathrm{q}}$. Since the map $\mathfrak{H}_P : G \to \mathfrak{a}$ is right $H \cap P$-invariant, see Lemma 4.1, the map

$$F_a : H \to \mathfrak{a}_{\mathrm{q}}, \ \ h \mapsto \mathfrak{H}_{P,\mathrm{q}}(ah)$$

factors through a map $\bar{F}_a : H/H \cap P \to \mathfrak{a}_{\mathrm{q}}$. In order for the idea of the proof in [4] to work in the present situation, one needs to establish properness of the map $\bar{F}_a$. This is done in Section 4 by reducing the problem to the case of a suitable σ-stable parabolic subgroup R combined with application of results of [4]. The established properness implies that the image $F_a(H)$ is closed in $\mathfrak{a}_{\mathrm{q}}$.

The considerations of Section 4 also lead to the constraint on the image $F_a(H)$ that it does not contain any line of $\mathfrak{a}_{\mathrm{q}}$, see Corollary 4.15.

In Section 5 we introduce the functions $F_{a,X} : H \to \mathbb{R}$, for $X \in \mathfrak{a}_{\mathrm{q}}$, defined by

$$F_{a,X}(h) = \langle X, F_a(h) \rangle = B(X, F_a(h)).$$

Geometrically, these functions test the Iwasawa projection by linear forms on $\mathfrak{a}_{\mathrm{q}}$, and give us constraints on the image of H under F_a. For a more detailed exposition on $F_{a,X}$ we refer the reader to [9]. Our own study of this function follows ideas in [4] and [9].

In Section 6 we calculate the critical set $\mathscr{C}_{a,X}$ of the function $F_{a,X}$ explicitly, for $a \in A_{\mathrm{q}}^{\mathrm{reg}}$ and $X \in \mathfrak{a}_{\mathrm{q}}$. In particular, we show that this set is the union of a finite collection $\mathscr{M}_{a,X}$ of injectively immersed connected submanifolds of H. If $\mathscr{C}_{a,X} \subsetneq H$, then all submanifolds in $\mathscr{M}_{a,X}$ are lower dimensional, so that $\mathscr{C}_{a,X}$ is thin in the sense of the Baire theorem, i.e. its closure has empty interior. These considerations allow us to show that in case $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$ spans $\mathfrak{a}_{\mathrm{q}}^*$, the set $\mathscr{C}_a$ of points in H where F_a is not submersive, is closed and thin, see Proposition 6.7. In particular, we then have that

$$F_a(\mathscr{C}_a) \subsetneq F_a(H). \tag{9}$$

In Sections 7 and 8 we calculate the Hessians of $F_{a,X}$ and their transversal signatures along all manifolds from $\mathscr{M}_{a,X}$. These calculations, which are extensive, in particular allow us to determine all points where the transversal signatures are definite. This in turn gives us all points where $F_{a,X}$ attains local maxima and minima. A main result of Section 8 is Lemma 8.14 which asserts that for every local minimum m of the function $F_{a,X}$ we have that $\langle X, \cdot \rangle \geq m$ on the set

$$\Omega := \mathrm{conv}(W_{K\cap H} \cdot \log a) + \Gamma(P). \tag{10}$$

In Section 9 we prepare for the proof of the main theorem by using a limit argument to reduce to the case of a regular element $a \in A_{\mathrm{q}}$.

The proof of the main theorem is finally given in Section 10. It proceeds by induction over the rank of the root system $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$. More precisely, for $a \in A_{\mathrm{q}}^{\mathrm{reg}}$ the set $\mathscr{C}_{a,X}$ depends on $X \in \mathfrak{a}_{\mathrm{q}}$ through the centralizer $\mathfrak{g}_X$ of X in $\mathfrak{g}$. It is shown that $\mathscr{C}_{a,X} \subsetneq H$ implies that $\mathrm{rk}\,\Sigma(\mathfrak{g}_X, \mathfrak{a}_{\mathrm{q}}) < \mathrm{rk}\,\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$ so that the induction hypothesis holds for the centralizer G_X of X in G. This allows us to determine the image $F_a(C_{a,X})$ for such X. In particular, this leads to a precise description of the image $F_a(\mathscr{C}_a)$ from which it is seen that the latter image contains the boundary of the set Ω.

In the proof we use this observation, together with the earlier obtained constraint that the image $F_a(H)$ does not contain a line, to conclude that $F_a(H)$ is contained in Ω. In particular, this implies that, for each $X \in \mathfrak{a}_{\mathrm{q}}$, every local minimum of $F_{a,X}$ is global.

For the converse inclusion, we first show that the image of $H \setminus \mathscr{C}_a$ under the map F_a is a union of connected components of $\Omega \setminus F_a(\mathscr{C}_a)$. The established fact that every local minimum of $F_{a,X}$ is global then allows us to show that all connected components appear in the image, thereby completing the proof.

We conclude the paper with two appendices, A and B. In Appendix A we give the proof of Lemma 2.11 concerning the decomposition of nilpotent groups in terms of subgroups generated by roots, and in B we discuss the convexity theorem for the case of the group viewed as a symmetric space.

Both the linear and the nonlinear convexity theorems of Kostant, see [22], have been extensively studied. Heckman proved the linear theorem in [16] by means of techniques as above and obtained the non-linear theorem from the linear one by a homotopy argument. Inspired by this, Duistermaat [8] obtained a remarkable universal homotopy containing Heckman's homotopy for all $a \in A$ at once.

Both convexity theorems of Kostant have been explained in the framework of symplectic geometry: see [2], [13], [7], [14], [20] for the linear convexity theorem and [24], [19] for the nonlinear one.

The convexity theorem of [4], which generalizes Kostant's nonlinear convexity theorem, has been given a symplectic interpretation in [11]. This leads us to suspect that such an interpretation should be possible in the present case as well; we intend to investigate this in the future.

Finally, we wish to mention that many of our calculations have been inspired by [16] and [9].

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2 Some structure theory for parabolic subgroups

In this section we will construct a (minimal) parabolic subgroup in $\mathscr{P}(A)$, see the text preceding (3), which has a special position relative to the involution σ; it will play an important role in Section 4. We will also discuss some structure theory of parabolic subgroups from $\mathscr{P}(A)$ and derive a useful decomposition for their unipotent radicals.

We recall that every parabolic subgroup P from $\mathscr{P}(A)$ has a Langlands decomposition of the form (3). Thus, its (θ-stable) Levi component L_P is given by

$$L_P = L = MA$$

and the multiplication map $L \times N_P \to P$ is a diffeomorphism. The opposite parabolic subgroup $\bar{P}$ is defined to be the unique parabolic subgroup from $\mathscr{P}(A)$ with $\Sigma(\bar{P}) = -\Sigma(P)$. It equals $\theta(P)$.

2.1 Extremal minimal parabolic subgroups

If τ is any involution of G which leaves A invariant, then its infinitesimal version $\tau : \mathfrak{g} \to \mathfrak{g}$ leaves $\mathfrak{a}$ invariant, and we put

$$\Sigma(P,\tau) := \{\alpha \in \Sigma(P) : \ \tau\alpha \in \Sigma(P)\}. \tag{11}$$

Observe that $\Sigma(P,\tau) = \Sigma(P) \cap \tau\Sigma(P)$.

Definition 2.1. A minimal parabolic subgroup $Q \in \mathscr{P}(A)$ is said to be $\mathfrak{h}$-extreme if

$$\Sigma(Q,\sigma) = \Sigma(Q) \setminus \mathfrak{a}_{\mathrm{q}}^*. \tag{12}$$

Starting with any minimal parabolic subgroup $P \in \mathscr{P}(A)$, we can obtain an $\mathfrak{h}$-extreme minimal parabolic subgroup by changing one simple root at a time. This process is described in Lemma 2.6 below.

Lemma 2.2. *Let $P \in \mathscr{P}(A)$. Then*

$$\Sigma(P) = \Sigma(P,\sigma) \sqcup \Sigma(P,\sigma\theta) \qquad \text{(disjoint union)}.$$

Proof. Let $\alpha \in \Sigma(P)$. From the fact that $\sigma\theta\alpha = -\sigma\alpha$, the result follows easily. □

Lemma 2.3. *Let $P \in \mathscr{P}(A)$ and assume that*

$$\Sigma(P,\sigma) \subsetneq \Sigma(P) \setminus \mathfrak{a}_{\mathrm{q}}^*. \tag{13}$$

Then there exists a P-simple root $\alpha \in \Sigma(P,\sigma\theta)$ with $\alpha \notin \mathfrak{a}_{\mathrm{q}}^*$.

Remark 2.4. A root $\alpha \in \Sigma(\mathfrak{g},\mathfrak{a})$ is said to be P-simple if it is simple in the positive system $\Sigma(P)$.

Proof. Assume the contrary. Then each P-simple root $\beta \in \Sigma(P,\sigma\theta)$ satisfies $\sigma\theta\beta = \beta$. In view of Lemma 2.2 it follows that for every simple root $\beta \in \Sigma(P)$ we have either $\sigma\beta \in \Sigma(P)$ or $\sigma\beta = \theta\beta = -\beta$.

The set $\Sigma(P)$ is a positive system for the root system $\Sigma(\mathfrak{g},\mathfrak{a})$. Hence, there exists an element $X \in \mathfrak{a}$ such that $\alpha(X) > 0$ for all $\alpha \in \Sigma(P)$. Put $X_{\mathrm{h}} := \frac{1}{2}(X + \sigma(X))$. Then for every simple root β in $\Sigma(P)$ we have either $\sigma\beta = -\beta$, in which case $\beta(X_{\mathrm{h}}) = 0$, or $\sigma\beta \in \Sigma(P)$, in which case $\beta(X_{\mathrm{h}}) > 0$. In any case, for each simple $\beta \in \Sigma(P)$, the value $\beta(X_{\mathrm{h}})$ is a nonnegative real number. Moreover, the number is zero if and only if $\sigma\beta = -\beta$. It follows that for all $\alpha \in \Sigma(P)$ the number $\alpha(X_{\mathrm{h}})$ is nonnegative and furthermore, that it is zero if and only if $\alpha \in \mathfrak{a}_{\mathrm{q}}^*$. Since $\sigma\theta(X_{\mathrm{h}}) = -X_{\mathrm{h}}$ we now infer that $\Sigma(P) \setminus \mathfrak{a}_{\mathrm{q}}^* \cap \Sigma(P,\sigma\theta) = \emptyset$, hence $\Sigma(P) \setminus \mathfrak{a}_{\mathrm{q}}^* \subseteq \Sigma(P,\sigma)$, contradicting (13). □

For a root $\alpha \in \Sigma(\mathfrak{g},\mathfrak{a})$, the associated reflection is denoted by $s_\alpha : \mathfrak{a} \to \mathfrak{a}$.

Corollary 2.5. *If P and α are as in Lemma 2.3, then $P' := s_\alpha(P)$ has the following properties:*

(a) $\Sigma(P) \cap \mathfrak{a}_{\mathrm{q}}^* = \Sigma(P') \cap \mathfrak{a}_{\mathrm{q}}^*$,

(b) $\Sigma(P,\sigma) \subsetneq \Sigma(P',\sigma)$.

In the proof of the above corollary, we will follow the convention to write

$$R_\circ := \{\alpha \in R : \tfrac{1}{2}\alpha \notin R\}$$

for any possibly non-reduced root system R. The elements of $R_\circ$ are called the indivisible roots in R. Furthermore, if $S \subseteq R$ is any subset, we will write $S_\circ := S \cap R_\circ$. Finally, we agree to write $\Sigma_\circ(P)$ for $\Sigma(P)_\circ$.

Proof. It suffices to prove (a) and (b) with everywhere Σ replaced by $\Sigma_\circ$. Since $P' := s_\alpha(P)$ with α simple in $\Sigma(P)$, we have

$$\Sigma_\circ(P') = (\Sigma_\circ(P) \setminus \{\alpha\}) \cup \{-\alpha\},$$

which implies (a).

Let $\beta \in \Sigma_\circ(P) \cap \sigma\Sigma_\circ(P)$. Then $\beta \neq \alpha$ and $\sigma\beta \neq \sigma\alpha$ and we infer that β and $\sigma\beta$ both belong to $\Sigma_\circ(P')$. It follows that $\beta \in \Sigma_\circ(P') \cap \sigma\Sigma_\circ(P')$. This proves the inclusion in (b). We still need to show that equality cannot hold. This follows from the fact that $\theta\alpha = -\alpha \in \Sigma(P', \sigma) \setminus \Sigma(P)$. □

Lemma 2.6. *Let $P \in \mathscr{P}(A)$. Then there exists a minimal parabolic subgroup $Q_\mathrm{h} \in \mathscr{P}(A)$ such that the following conditions hold:*

(a) $\Sigma(Q_\mathrm{h}) \cap \mathfrak{a}_\mathrm{q}^* = \Sigma(P) \cap \mathfrak{a}_\mathrm{q}^*$,

(b) $\Sigma(Q_\mathrm{h}) \cap \mathfrak{a}_\mathrm{h}^* = \Sigma(P) \cap \mathfrak{a}_\mathrm{h}^*$,

(c) $\Sigma(P, \sigma) \subseteq \Sigma(Q_\mathrm{h}, \sigma)$,

(d) Q_h *is $\mathfrak{h}$-extreme, see* (12).

Proof. If $\alpha \in \Sigma(P) \cap \mathfrak{a}_\mathrm{q}^*$, then $\sigma\alpha = -\alpha \notin \Sigma(P)$. Hence

$$\Sigma(P, \sigma) = \Sigma(P) \cap \sigma\Sigma(P) \subseteq \Sigma(P) \setminus \mathfrak{a}_\mathrm{q}^*. \tag{14}$$

If the above inclusion is an equality, the result holds with $Q_\mathrm{h} := P$. If not, then the inclusion in (14) is proper and Lemma 2.3 guarantees the existence of a simple root $\alpha \in \Sigma(P) \setminus \mathfrak{a}_\mathrm{q}^*$ such that $\sigma\theta\alpha \in \Sigma(P)$. By applying Corollary 2.5 we see that the minimal parabolic subgroup $P' := s_\alpha(P)$ satisfies the above conditions (a) and (b), and

$$\Sigma(P, \sigma) \subsetneq \Sigma(P', \sigma). \tag{15}$$

Put $P_0 = P$ and $P_1 = P'$. By applying the above process repeatedly, we obtain a sequence of parabolic subgroups $P = P_0, P_1, \dots, P_k$ satisfying

(a) $\Sigma(P_i) \cap \mathfrak{a}_\mathrm{q}^* = \Sigma(P_{i+1}) \cap \mathfrak{a}_\mathrm{q}^*$,

(b) $\Sigma(P_i) \cap \mathfrak{a}_\mathrm{h}^* = \Sigma(P_{i+1}) \cap \mathfrak{a}_\mathrm{h}^*$,

(c) $\Sigma(P_i, \sigma) \subsetneq \Sigma(P_{i+1}, \sigma)$, for $0 \leq i < k$. The process ends when for some $k > 0$ the condition $\Sigma(P_k) \cap \sigma\Sigma(P_k) = \Sigma(P_k) \setminus \mathfrak{a}_{\mathrm{q}}^*$ is satisfied. The parabolic subgroup $Q_{\mathrm{h}} = P_k$ satisfies all assertions of the lemma. $\square$

Remark 2.7. In analogy with Definition 2.1, a parabolic subgroup $Q \in \mathscr{P}(A)$ is said to be $\mathfrak{q}$-extreme if $\Sigma(Q, \sigma\theta) = \Sigma(Q) \setminus \mathfrak{a}_{\mathrm{h}}^*$. With obvious modifications in the proof, Lemma 2.6 is valid with everywhere $\sigma\theta$ in place of σ and with $\mathfrak{q}$-extreme in place of $\mathfrak{h}$-extreme. However, we will not need this result in the present paper.

2.2 The convexity theorem for a q-extreme parabolic subgroup

We shall now explain why the result of [4] is a special case of the Main Theorem. We keep the notation as above and impose that $P \in \mathscr{P}(A)$ is $\mathfrak{q}$-extreme, see Remark 2.7. Then $\Sigma(P, \sigma\theta) = \Sigma(P) \setminus \mathfrak{a}_{\mathrm{h}}^*$, so that

$$\Delta^+ := \Sigma(P, \sigma\theta)|_{\mathfrak{a}_{\mathrm{q}}}$$

is a positive system for $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$. For $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$, the root space $\mathfrak{g}_\alpha$ is $\sigma\theta$-invariant; we write $\mathfrak{g}_{\alpha,\pm}$ for the ± 1 eigenspaces of $\sigma\theta|_{\mathfrak{g}_\alpha}$. Put

$$\Delta_-^+ = \{\alpha \in \Delta^+ : \ \mathfrak{g}_{\alpha,-} \neq 0\}.$$

Then [4, Thm 1.1] asserts that

$$\mathfrak{H}_{P,\mathrm{q}}(aH) = \mathrm{conv}(W_{K\cap H} \cdot \log a) + \Upsilon(P),$$

where $\Upsilon(P)$ is the finitely generated polyhedral cone in $\mathfrak{a}_{\mathrm{q}}$ defined by

$$\Upsilon(P) = \sum_{\alpha \in \Delta_-^+} \mathbb{R}_{\geq 0} H_\alpha;$$

here H_α denotes the element of $\mathfrak{a}_{\mathrm{q}}$ with $H_\alpha \perp \ker\alpha$ and $\alpha(H_\alpha) = 2$.

Thus, our main theorem coincides with [4, Thm. 1.1] provided that $\Gamma(P) = \Upsilon(P)$. The latter is asserted by the following lemma.

Lemma 2.8. *Let $P \in \mathscr{P}(A)$ be $\mathfrak{q}$-extreme. Then $\Upsilon(P) = \Gamma(P)$.*

Proof. For a root $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ we denote by $H_\alpha^\vee \in \mathfrak{a}$ the element determined by

$$\langle H_\alpha^\vee, X\rangle = \alpha(X) \tag{16}$$

for all $X \in \mathfrak{a}$. Then it is readily verified that

$$H_\alpha^\vee = 2H_\alpha / \langle H_\alpha, H_\alpha\rangle. \tag{17}$$

Similarly, for $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$ we define $H_\alpha^\vee$ to be the element of $\mathfrak{a}_{\mathrm{q}}$ determined by (16) for all $X \in \mathfrak{a}_{\mathrm{q}}$. For this element we also have (17), but now as an identity of elements of $\mathfrak{a}_{\mathrm{q}}$. If $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ has non-zero restriction to $\mathfrak{a}_{\mathrm{q}}$, then $\alpha|_{\mathfrak{a}_{\mathrm{q}}} \in \Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$ and for natural reasons we have

$$\mathrm{pr}_{\mathrm{q}}(H_\alpha^\vee) = H_{\alpha|_{\mathfrak{a}_{\mathrm{q}}}}^\vee.$$

From this we conclude that

$$\mathrm{pr}_{\mathrm{q}}(H_\alpha) = c_\alpha H_{\alpha|_{\mathfrak{a}_{\mathrm{q}}}}, \tag{18}$$

with $c_\alpha = \|H_\alpha\|^2 \|H_{\alpha|_{\mathfrak{a}_{\mathrm{q}}}}\|^{-2} > 0$.

After these preliminary remarks we will now complete the proof. Let $\alpha \in \Sigma(P, \sigma\theta)$. Then $\alpha|_{\mathfrak{a}_{\mathrm{q}}}$ is non-zero, hence a root in $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$, and (18) is valid. As $\sigma\theta$ restricts to the identity on $\mathfrak{a}_{\mathrm{q}}$, the $\mathfrak{a}$-roots α and $\sigma\theta\alpha$ have the same restriction to $\mathfrak{a}_{\mathrm{q}}$ giving the root $\alpha|_{\mathfrak{a}_{\mathrm{q}}}$ of Δ^+. If the given $\mathfrak{a}$-roots are different, then the sum $\mathfrak{g}_\alpha + \sigma\theta(\mathfrak{g}_\alpha)$ is direct and contained in $\mathfrak{g}_{\alpha|_{\mathfrak{a}_{\mathrm{q}}}}$ and we see that $\mathfrak{g}_{\alpha|_{\mathfrak{a}_{\mathrm{q}}},-} \neq 0$, so that $\alpha \in \Sigma(P)_-$ and $\alpha|_{\mathfrak{a}_{\mathrm{q}}} \in \Delta^+_-$. On the other hand, if $\alpha = \sigma\theta\alpha$, then $\mathfrak{g}_\alpha = \mathfrak{g}_{\alpha|_{\mathfrak{a}_{\mathrm{q}}}}$ and we see that $\alpha \in \Sigma(P)_-$ if and only if $\alpha|_{\mathfrak{a}_{\mathrm{q}}} \in \Delta^+_-$. It follows from this argument that $\Sigma(P)_-|_{\mathfrak{a}_{\mathrm{q}}} = \Delta^+_-$. Using (18) we now see that

$$\Gamma(P) = \sum_{\alpha \in \Sigma(P)_-} \mathbb{R}_{\geq 0} H_{\alpha|_{\mathfrak{a}_{\mathrm{q}}}} = \sum_{\alpha \in \Delta^+_-} \mathbb{R}_{\geq 0} H_\alpha = \Upsilon(P).$$

□

2.3 Decompositions of nilpotent Lie groups

In this section we give a brief survey of a number of useful results on decompositions of nilpotent Lie groups that will be needed in this paper.

We start by recalling the following standard result.

Lemma 2.9. *Let N be a connected and simply connected Lie group with nilpotent Lie algebra $\mathfrak{n}$. If $\mathfrak{n}_0$ is a subalgebra of $\mathfrak{n}$, then the exponential map maps $\mathfrak{n}_0$ diffeomorphically onto a closed subgroup of N.*

Lemma 2.10 ([18, Lemma IV.6.8])**.** *Let N be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. Let $(\mathfrak{n}_i)_{0\leq i \leq k}$ be a strictly decreasing sequence of ideals of $\mathfrak{n}$ such that $\mathfrak{n}_0 = \mathfrak{n}$, $\mathfrak{n}_k = 0$ and*

$$[\mathfrak{n}, \mathfrak{n}_i] \subseteq \mathfrak{n}_{i+1} \qquad \text{for all} \quad 0 \leq i < k.$$

Let $\mathfrak{b}_1$ and $\mathfrak{b}_2$ be two mutually complementary subspaces of $\mathfrak{n}$ such that $\mathfrak{n}_i = \mathfrak{b}_1 \cap \mathfrak{n}_i + \mathfrak{b}_2 \cap \mathfrak{n}_i$, for all $0 \leq i \leq k$. Then the mapping

$$\varphi : (X, Y) \to \exp X \exp Y$$

is an analytic diffeomorphism from $\mathfrak{b}_1 \times \mathfrak{b}_2$ onto N.

Lemma 2.11. *Let N_P be the nilpotent radical of a minimal parabolic subgroup $P \in \mathscr{P}(A)$, let $\mathfrak{n}_P$ be its Lie algebra and let $\mathfrak{n}_1, \ldots, \mathfrak{n}_k \subset \mathfrak{n}_P$ be linearly independent subalgebras of $\mathfrak{n}_P$ that are direct sums of $\mathfrak{a}$-root spaces. Assume that $\mathfrak{n} = \mathfrak{n}_1 \oplus \ldots \oplus \mathfrak{n}_k$ is a subalgebra of $\mathfrak{n}_P$. Denote by $N := \exp \mathfrak{n}$ and by $N_i := \exp \mathfrak{n}_i$, $i \in \{1, \ldots, k\}$, the corresponding closed subgroups of N_P. Then the multiplication map*

$$\mu : N_1 \times \ldots \times N_k \to N$$

is a diffeomorphism.

This result is stated in [9, Lemma 2.3] for $\mathfrak{n} = \mathfrak{n}_P$, with reference to [23]. We need the present slightly more general version with $\mathfrak{n}$ a subalgebra of $\mathfrak{n}_P$. A proof of this result can be found in Appendix A.

2.4 Fixed points for the involution in minimal parabolic subgroups

Let $P \in \mathscr{P}(A)$. The decomposition $P = LN_P$ induces a similar decomposition for the intersection $P \cap H$. In the present subsection we present a proof for this fact, see the lemma below.

Lemma 2.12. $P \cap H \simeq (L \cap H) \times (N_P \cap H)$

Proof. Let p be an element in $P \cap H$. According to the decomposition $P = LN_P$, we write $p = ln$. Then, $ln = \sigma(ln) = \sigma(l)\sigma(n)$ and we obtain that $\sigma(n)n^{-1} = \sigma(l)^{-1}l \in L$. Since $\sigma(N_P)$ is the nilpotent radical of the parabolic subgroup $\sigma(P) \in \mathscr{P}(A)$, it follows from Lemma 2.11 with $k = 2$ that the multiplication map

$$(\sigma(N_P) \cap \bar{N}_P) \times (\sigma(N_P) \cap N_P) \to \sigma(N_P)$$

induces a diffeomorphism. We thus see that $\sigma(n)n^{-1} \in \bar{N}_P N_P$. Now, by [21, Lemma 7.64] it follows that $\bar{N}_P N_P \cap L = e$ and thus $\sigma(n) = n$ and $\sigma(l) = l$. □

2.5 Decomposition of nilpotent radicals induced by the involution

In this subsection, we assume that $P \in \mathscr{P}(A)$. We will show that the unipotent radical N_P decomposes as the product of $N_P \cap H$ and a suitable closed subgroup $N_{P,+}$ of N_P. To describe this subgroup, we need the existence of suitable elements of $\mathfrak{a}_{\mathrm{q}}$. As usual, an element $X \in \mathfrak{a}_{\mathrm{q}}$ is said to be regular for the root system $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$ if no root of this system vanishes on it. The set of such regular elements is denoted by $\mathfrak{a}_{\mathrm{q}}^{\mathrm{reg}}$. We observe that in terms of the system $\Sigma(\mathfrak{g}, \mathfrak{a})$ this set may be described as

$$\mathfrak{a}_{\mathrm{q}}^{\mathrm{reg}} = \{X \in \mathfrak{a}_{\mathrm{q}} : \quad \forall \alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}) : \; \alpha(X) = 0 \Rightarrow \alpha|_{\mathfrak{a}_{\mathrm{q}}} = 0\}. \tag{19}$$

Lemma 2.13.

(a) *There exists an element $Z_{\mathrm{q}} \in \mathfrak{a}_{\mathrm{q}}^{\mathrm{reg}}$ such that $\alpha(Z_{\mathrm{q}}) > 0$ for all $\alpha \in \Sigma(P, \sigma\theta)$.*

(b) *There exists an element $Z_{\mathrm{h}} \in \mathfrak{a}_{\mathrm{h}}$ such that $\alpha(Z_{\mathrm{h}}) > 0$ for all $\alpha \in \Sigma(P, \sigma)$.*

Proof. The set

$$\mathfrak{a}' := \{X \in \mathfrak{a} : \quad \forall \alpha, \beta \in \Sigma(\mathfrak{g}, \mathfrak{a}) : \; \alpha(X) = \beta(X) \Rightarrow \alpha = \beta\}$$

is the complement of finitely many hyperplanes in $\mathfrak{a}$, hence open and dense. Let $\mathfrak{a}^+(P)$ denote the positive chamber associated with the positive system $\Sigma(P)$ for $\Sigma(\mathfrak{g}, \mathfrak{a})$. Fix $Z_P \in \mathfrak{a}^+(P) \cap \mathfrak{a}'$. Then it is readily verified that $Z_{\mathrm{q}} := Z_P + \sigma\theta(Z_P)$ satisfies the requirements of (a). Likewise, the element $Z_{\mathrm{h}} = Z_P + \sigma(Z_P)$ satisfies the requirements of (b). □

Given $Z_{\mathrm{q}} \in \mathfrak{a}_{\mathrm{q}}^{\mathrm{reg}}$ we put $\Sigma(P,+) := \{\alpha \in \Sigma(P) : \alpha(Z_{\mathrm{q}}) > 0\}$. Then

$$\mathfrak{n}_{P,+} := \bigoplus_{\alpha \in \Sigma(P,+)} \mathfrak{g}_\alpha$$

is a subalgebra of $\mathfrak{n}_P$. Let $N_{P,+} := \exp \mathfrak{n}_+$ be the corresponding closed subgroup of N_P, see Lemma 2.9. Define

$$\mathfrak{n}_{P,\sigma} := \bigoplus_{\alpha \in \Sigma(P,\sigma)} \mathfrak{g}_\alpha$$

and $N_{P,\sigma}$ as the corresponding closed subgroup.

Proposition 2.14. *Let $Z_{\mathrm{q}} \in \mathfrak{a}_{\mathrm{q}}^{\mathrm{reg}}$ be as in Lemma 2.13 (a) and let $N_{P,+}$ be defined as above. Then the multiplication map*

$$N_{P,+} \times (N_P \cap H) \to N_P$$

is a diffeomorphism.

The proof of this result relies on the following lemma.

Lemma 2.15. *Let $P \in \mathscr{P}(A)$ and let $Z_{\mathrm{q}} \in \mathfrak{a}_{\mathrm{q}}^{\mathrm{reg}}$ be as in Lemma 2.13 (a). Put*

$$\Sigma(P,\sigma,+) := \{\alpha \in \Sigma(P,\sigma) : \ \alpha(Z_{\mathrm{q}}) > 0\}.$$

Then the following statements hold:

(a) $\mathfrak{n}_{P,\sigma,+} := \oplus_{\alpha \in \Sigma(P,\sigma,+)} \mathfrak{g}_\alpha$ *is a subalgebra of* $\mathfrak{n}_{P,\sigma}$,

(b) $N_{P,\sigma,+} := \exp \mathfrak{n}_{P,\sigma,+}$ *is a closed subgroup of* $N_{P,\sigma}$,

(c) $\mathfrak{n}_{P,\sigma} = \mathfrak{n}_{P,\sigma,+} \oplus (\mathfrak{n}_P \cap \mathfrak{h})$,

(d) *the multiplication map*

$$\mu : N_{P,\sigma,+} \times (N_P \cap H) \to N_{P,\sigma}$$

is a diffeomorphism.

Proof. (a): Assume that $\alpha, \beta \in \Sigma(P,\sigma,+)$ and $\alpha+\beta \in \Sigma(\mathfrak{g},\mathfrak{a})$. Then $\alpha+\beta \in \Sigma(P,\sigma)$ and $(\alpha+\beta)(Z_{\mathrm{q}}) > 0$ so that $\alpha+\beta \in \Sigma(P,\sigma,+)$. This implies (a).

Assertion (b) follows from (a) by application of Lemma 2.9.

Next, we prove (c). If $\alpha \in \Sigma(P,\sigma,+)$ then $\sigma\alpha(Z_{\mathrm{q}}) < 0$, which implies $\sigma\alpha \notin \Sigma(P,\sigma,+)$. Hence, $\mathfrak{n}_{P,\sigma,+} \cap \mathfrak{h} = \{0\}$. It follows that

$$\mathfrak{n}_{P,\sigma,+} \cap (\mathfrak{n}_P \cap \mathfrak{h}) = \{0\}.$$

It remains to be shown that any $X \in \mathfrak{n}_{P,\sigma}$ can be written as

$$X = X_+ + X_{\mathrm{h}},$$

with $X_+ \in \mathfrak{n}_{P,\sigma,+}$ and $X_\mathrm{h} \in \mathfrak{n}_P \cap \mathfrak{h}$. It suffices to prove this for $X \in \mathfrak{g}_\alpha \subset \mathfrak{n}_{P,\sigma}$. If $\alpha(Z_\mathrm{q}) > 0$, then $X \in \mathfrak{n}_{P,\sigma,+}$ by definition. On the other hand if $\alpha(Z_\mathrm{q}) = 0$, then by regularity of Z_q we have that $\alpha \in \mathfrak{a}_\mathrm{h}^*$ and thus $\mathfrak{g}_\alpha \in \mathfrak{h}$, which implies that $X \in \mathfrak{n}_P \cap \mathfrak{h}$. Finally, if $\alpha(Z_\mathrm{q}) < 0$, then

$$X = (X + \sigma(X)) - \sigma(X)$$

with $X + \sigma(X) \in \mathfrak{n}_P \cap \mathfrak{h}$ and $-\sigma(X) \in \mathfrak{n}_{P,\sigma,+}$, and we are done.

For (d) fix $Z_\mathfrak{h}$ as in Lemma 2.13 (b). Then for all $\alpha \in \Sigma(P,\sigma)$ we have that $v_\alpha := \alpha(Z_\mathfrak{h}) > 0$. Let the set of positive real numbers thus obtained be ordered by $v_{\alpha_1} < v_{\alpha_2} < \cdots < v_{\alpha_m}$. We define $\mathfrak{n}_0 = \mathfrak{n}_{P,\sigma}$, $\mathfrak{n}_m = 0$, and for $1 \le i < m$,

$$\mathfrak{n}_i := \bigoplus_{\substack{\alpha\in\Sigma(P,\sigma)\\ \alpha(Z_\mathrm{h})>v_{\alpha_i}}} \mathfrak{g}_\alpha.$$

Then $\mathfrak{n}_1,\dots,\mathfrak{n}_m$ is a strictly decreasing sequence of ideals in $\mathfrak{n}_{P,\sigma}$ with $[\mathfrak{n},\mathfrak{n}_i] \subseteq \mathfrak{n}_{i+1}$ for $0 \le i < m$. We note that each $\mathfrak{n}_i$ is invariant under σ. Hence by the same argument as used in the proof of (c) above it follows that

$$\mathfrak{n}_i = (\mathfrak{n}_i \cap \mathfrak{n}_{P,\sigma,+}) \oplus (\mathfrak{n}_i \cap (\mathfrak{n}_P \cap \mathfrak{h}))$$

for all $0 \le i \le m$. Thus, we may apply Lemma 2.10 to conclude that

$$N_{P,\sigma} \simeq N_{P,\sigma,+} \times \exp(\mathfrak{n}_P \cap \mathfrak{h}).$$

It remains to show that $\exp(\mathfrak{n}_P \cap \mathfrak{h}) = N_P \cap H$. This follows from

$$N_P \cap H \subseteq \{n \in N_P \,:\, \sigma(n) = n\} = \{\exp X \,:\, X \in \mathfrak{n}_P \cap \mathfrak{h}\} \subseteq N_P \cap H.$$

The proof is complete. □

Proof of Prop. 2.14. Let

$$\mathfrak{n}_{P,\sigma\theta} := \sum_{\alpha\in\Sigma(P,\sigma\theta)} \mathfrak{g}_\alpha$$

and let $N_{P,\sigma\theta}$ be the corresponding closed subgroup of N_P. Then $\mathfrak{n}_P = \mathfrak{n}_{P,\sigma\theta} \oplus \mathfrak{n}_{P,\sigma}$ and by Lemma 2.11 we obtain that

$$N_P \simeq N_{P,\sigma\theta} \times N_{P,\sigma}. \tag{20}$$

Applying Lemma 2.15 to the second component we obtain that

$$N_P \simeq N_{P,\sigma\theta} \times N_{P,\sigma,+} \times (N_P \cap H).$$

On the other hand, $\mathfrak{n}_{P,+} = \mathfrak{n}_{P,\sigma,+} \oplus \mathfrak{n}_{P,\sigma\theta}$. From this we infer by application of Lemma 2.11 that

$$N_{P,\sigma\theta} \times N_{P,\sigma,+} \simeq N_{P,+}$$

The result follows. □

Remark 2.16. For the case of an $\mathfrak{h}$-extreme parabolic subgroup, Proposition 2.14 is due to [1], where, for this special case, a different proof of the result is given.

3 Auxiliary results in convex linear algebra

In this section we present a few results in convex linear algebra which will be used in Section 4.

Lemma 3.1. *Let V be a finite dimensional real linear space and $B \subseteq V$ a closed subset, star-shaped about the origin. If B is non-compact, then there exists a $v \in V \setminus \{0\}$ such that $\mathbb{R}_{\geq 0} v \subseteq B$.*

Proof. Since B is star-shaped, we have $sB = t(s/t)B \subseteq tB$ for all $0 < s < t$. Fix a positive definite inner product on V and let S be the associated unit sphere centered at the origin. For $s > 0$ we define the compact set $C_s := s^{-1}B \cap S$. Then $s < t \implies C_s \supseteq C_t$. As B is unbounded and starshaped, each of the sets C_s is non-empty. It follows that the intersection

$$C := \cap_{s>0} C_s$$

is non-empty. Let v be a point in this intersection. Then $v \neq 0$ and for all $s > 0$ we have $sv \in sC_s \subseteq B$. Hence, $\mathbb{R}_{\geq 0} v \subseteq B$. $\square$

Lemma 3.2. *Let V and W be two finite dimensional real linear spaces, $p : V \to W$ a linear map and $\Gamma \subseteq V$ a closed convex cone. Then the following assertions are equivalent.*

(a) *$p|_\Gamma$ is a proper map.*

(b) $\ker p \cap \Gamma = \{0\}$.

Proof. First we prove that (a) implies (b). Assume (b) doesn't hold, i.e. there exists $v \in \ker p \cap \Gamma$, $v \neq 0$. Then $\mathbb{R}_{\geq 0} v \subseteq \ker p \cap \Gamma = (p|_\Gamma)^{-1}(0)$ and we obtain that $(p|_\Gamma)^{-1}(0)$ is not compact and hence $p|_\Gamma$ is not a proper map.

For the converse implication, assume that (a) does not hold. Then there exists a compact set $K \subseteq W$, such that the set $p^{-1}(K) \cap \Gamma$ is not compact. As the latter set is closed, it is unbounded in V. Let $\bar{K}$ be the convex hull of $K \cup \{0\}$. Then $\bar{K}$ is compact and $p^{-1}(\bar{K}) \cap \Gamma$ is convex, contains 0 and is unbounded in V, hence not compact. We apply Lemma 3.1 and obtain that there exists $v \neq 0$ such that $\forall t \geq 0 : tv \in p^{-1}(\bar{K}) \cap \Gamma$. Hence, $t \cdot p(v) \in \bar{K}$ for every $t \geq 0$. Since $\bar{K}$ is compact, it follows that $p(v) = 0$ and $v \in \ker p \cap \Gamma$, which implies that (b) cannot hold. $\square$

Lemma 3.3. *Let V be a finite dimensional real linear space, and Γ a closed convex cone in V such that there exists a linear functional $\xi \in V^*$ with $\xi > 0$ on $\Gamma \setminus \{0\}$. Then the following holds.*

(a) *For every $R > 0$ the set $\{x \in \Gamma : \ \xi(x) \leq R\}$ is compact.*

(b) *The addition map $a : (x,y) \mapsto x + y$, $\Gamma \times \Gamma \to V$, is proper.*

Proof. Let $R > 0$. The set $\Gamma_R := \{x \in \Gamma : \ \xi(x) \leq R\}$ is closed and convex and it contains the origin. If $v \in \Gamma_R \setminus \{0\}$ then the half line $\mathbb{R}_{\geq 0} v$ is not contained in Γ_R. By Lemma 3.1 we infer that Γ_R is compact, hence (a).

We turn to (b). Assume $\mathscr{K} \subseteq V$ is compact. Then there exist an $R > 0$ such that $\xi \leq R$ on $\mathscr{K}$. Let $(x,y) \in a^{-1}(\mathscr{K})$. Then it follows that $\xi(x+y) \leq R$, hence $\xi(x) \leq R$ and $\xi(y) \leq R$, so that (x,y) belongs to the compact set $\Gamma_R \times \Gamma_R$. We conclude that $a^{-1}(\mathscr{K})$ is a closed subset of $\Gamma_R \times \Gamma_R$, hence compact. $\square$

If S is a subset of $\Sigma(\mathfrak{g}, \mathfrak{a})$ then the convex cone

$$\Gamma_{\mathfrak{a}}(S) := \sum_{\alpha \in S} \mathbb{R}_{\geq 0} H_\alpha.$$

is finitely generated, hence closed in $\mathfrak{a}$. Likewise,

$$\Gamma_{\mathfrak{a}_\mathfrak{q}}(S) := \mathrm{pr}_\mathfrak{q} \Gamma_{\mathfrak{a}}(S) = \sum_{\alpha \in S} \mathbb{R}_{\geq 0} \, \mathrm{pr}_\mathfrak{q}(H_\alpha)$$

is a closed and convex cone in $\mathfrak{a}_\mathfrak{q}$.

Corollary 3.4. *Let $P \in \mathscr{P}(A)$. Then the following assertions are valid.*

(a) *The map $\mathrm{pr}_\mathfrak{q} : \Gamma_{\mathfrak{a}}(\Sigma(P, \sigma\theta)) \to \mathfrak{a}_\mathfrak{q}$ is proper.*

(b) *The addition map $a : \Gamma_{\mathfrak{a}_\mathfrak{q}}(\Sigma(P, \sigma\theta)) \times \Gamma_{\mathfrak{a}_\mathfrak{q}}(\Sigma(P, \sigma\theta)) \to \mathfrak{a}_\mathfrak{q}$ is proper.*

Proof. We start with (a). In view of Lemma 3.2 it suffices to establish the claim that $\Gamma_{\mathfrak{a}}(\Sigma(P, \sigma)) \cap \mathfrak{a}_\mathfrak{h} = 0$. This can be done as follows. There exists a $Y \in \mathfrak{a}$ such that $\alpha(Y) > 0$ for all $\alpha \in \Sigma(P)$. Put $X := Y + \sigma\theta Y = Y - \sigma(Y)$, then $X \in \mathfrak{a}_\mathfrak{q}$ and $\langle X, H_\alpha \rangle = \langle H_\alpha, H_\alpha \rangle \alpha(X)/2 = \langle H_\alpha, H_\alpha \rangle (\alpha + \sigma\theta\alpha)(Y)/2 > 0$ for all $\alpha \in \Sigma(P, \sigma\theta)$. It follows that the linear functional $\xi = \langle X, \cdot \rangle \in \mathfrak{a}^*$ has strictly positive values on $\Gamma_{\mathfrak{a}}(\Sigma(P, \sigma\theta)) \setminus \{0\}$. Now $\xi = 0$ on $\mathfrak{a}_\mathfrak{h}$ and we see that the claim is valid. Hence, (a).

For (b) we proceed as follows. Let ξ be as above, then $\ker \mathrm{pr}_\mathfrak{q} \subseteq \ker \xi$ and we see that $\xi > 0$ on $\Gamma_{\mathfrak{a}_\mathfrak{q}}(\Sigma(P, \sigma\theta)) \setminus \{0\}$. Now use Lemma 3.3. $\square$

4 Properness of the Iwasawa projection

Let $P \in \mathscr{P}(A)$ and let $\mathfrak{H}_P : G \to \mathfrak{a}$ be the Iwasawa projection defined by (5). Let $\mathfrak{H}_{P,\mathfrak{q}} : G \to \mathfrak{a}_\mathfrak{q}$ be defined as in Remark 1.2. The purpose of this section is to prove that the restriction of $\mathfrak{H}_{P,\mathfrak{q}}$ to H factors through a proper map $H/H \cap P \to \mathfrak{a}_\mathfrak{q}$.

We start with a simple lemma.

Lemma 4.1. *The map $\mathfrak{H}_{P,\mathfrak{q}}|_H : H \to \mathfrak{a}_\mathfrak{q}$ is left $K \cap H$- and right $(P \cap H)$-invariant.*

Proof. Let $h \in H$, $k_H \in K \cap H$ and $p \in P \cap H$. By the Iwasawa decomposition, the element h may be decomposed as $h = kan$, with $k \in K$, $a \in A$ and $n \in N_P$. In view of Lemma 2.12 we may decompose $p = mbn'$, with $m \in M \cap H$, $b \in A \cap H$ and $n' \in N_P \cap H$. Since MA normalizes N_P and centralizes A we find

$$k_H hp = k_H kanmbn' = (k_H km)ab((mb)^{-1} n (mb))n' \in KabN_P.$$

From this we deduce that

$$\mathfrak{H}_{P,\mathfrak{q}}(k_H hp) = \mathrm{pr}_\mathfrak{q}(\log a + \log b) = \mathrm{pr}_\mathfrak{q} \log a = \mathfrak{H}_{P,\mathfrak{q}}(h). \qquad \square$$

It follows from the above lemma that the restriction of $\mathfrak{H}_{P,\mathrm{q}}$ to H induces a smooth map

$$\overline{\mathfrak{H}}_{P,\mathrm{q}} : H/H\cap P \to \mathfrak{a}_{\mathrm{q}}. \tag{21}$$

The following proposition is the main result of this section.

Proposition 4.2. *The induced map (21) is proper.*

In order to prove the proposition, we will reduce to another result, Prop. 4.7, establishing some useful lemmas along the way.

We fix Q_{h} in $\mathfrak{h}$-extreme position and related to P as in Lemma 2.6. Let $Z_G(\mathfrak{a}_{\mathrm{h}})$ denote the centralizer of $\mathfrak{a}_{\mathrm{h}}$ in G and define the parabolic subgroup

$$R := Z_G(\mathfrak{a}_{\mathrm{h}})N_{Q_{\mathrm{h}}}. \tag{22}$$

Let $\mathfrak{n}_R$ be the sum of the root spaces $\mathfrak{g}_\alpha$ for $\alpha \in \Sigma(Q_{\mathrm{h}},\sigma) = \Sigma(Q_{\mathrm{h}}) \setminus \mathfrak{a}_{\mathrm{q}}^*$ and put $N_R := \exp(\mathfrak{n}_R)$. Then N_R is σ-stable. It is readily seen that R has the Levi decomposition $R = L_R N_R$ where $L_R = Z_G(\mathfrak{a}_{\mathrm{h}})$ is σ-stable. Hence, R is σ-stable. Let $\Sigma(R)$ denote the set of $\mathfrak{a}$-roots that appear in $\mathfrak{n}_R$.

Lemma 4.3. $\Sigma(P)\cap\Sigma(\bar{R}) \subseteq \Sigma(P,\sigma\theta)$.

Proof. Let $\alpha \in \Sigma(P)\cap\Sigma(\bar{R})$. Then $\alpha \in \Sigma(\bar{Q}_{\mathrm{h}}) \setminus \mathfrak{a}_{\mathrm{q}}^* = -\Sigma(Q_{\mathrm{h}},\sigma)$, hence $\alpha \notin \Sigma(P,\sigma)$, see Lemma 2.6 (b). This implies that $\alpha \in \Sigma(P,\sigma\theta)$. □

Let $R = M_R A_R N_R$ be the Langlands decomposition of R. Then $L_R = M_R A_R$.

Lemma 4.4. *The multiplication map*

$$\mu : (K\cap H)\times(M_R\cap H)\times(N_R\cap H)/(N_R\cap H\cap P) \longrightarrow H/H\cap P,$$

given by $(k,m,[n]) \mapsto km[n]$ *is surjective.*

Proof. The map $K\times(\mathfrak{l}_R\cap\mathfrak{p})\times N_R \to G$ given by $(k,X,n)\mapsto k\exp Xn$ is a diffeomorphism. Since K, $\mathfrak{l}_R\cap\mathfrak{p}$ and N_R are σ-stable, whereas $N_R^\sigma = N_R\cap H$, it follows that

$$H = (K\cap H)(L_R\cap H)(N_R\cap H). \tag{23}$$

Now $L_R = M_R A_R$ with M_R and A_R both σ-stable. Since $A_R\cap H$ normalizes $N_R\cap H$, we have that

$$\begin{aligned} H &= (K\cap H)(M_R\cap H)(A_R\cap H)(N_R\cap H) \\ &= (K\cap H)(M_R\cap H)(N_R\cap H)(A_R\cap H). \end{aligned}$$

This implies the result. □

We equip $M_R \cap H$ with the natural right-action of the closed subgroup $M_R \cap H \cap P$. The latter group acts on $N_R \cap H$ by conjugation. Moreover, since M_R normalizes N_R and P normalizes N_P, the conjugation action leaves the closed subgroup $N_R \cap H \cap P$ invariant. Accordingly, we have an induced right-action of $M_R \cap H \cap P$ on $(N_R \cap H)/(N_R \cap H \cap P)$ given by

$$[n] \cdot m := [m^{-1}nm], \qquad (m \in M_R \cap H \cap P, n \in N_R \cap H).$$

We equip $(M_R \cap H)$ with the usual right-action by $M_R \cap H \cap P$, and $(N_R \cap H)/(N_R \cap H \cap P)$ with the product action. The latter action is proper and free, so that the associated quotient space $(M_R \cap H) \times_{M_R \cap H \cap P} (N_R \cap H)/(N_R \cap H \cap P)$ is a smooth manifold.

Lemma 4.5. *The multiplication map of Lemma 4.4 induces a surjective smooth map*

$$\bar{\mu} : (K \cap H) \times (M_R \cap H) \times_{M_R \cap H \cap P} (N_R \cap H)/(N_R \cap H \cap P) \to H/H \cap P.$$

Proof. Let $k \in K \cap H$, $m \in M_R \cap H$ and $n \in N_R \cap H$. Then for $p \in M_R \cap H \cap P$ we have

$$\mu(k, (m, [n]) \cdot p) = \mu(k, mp, [p^{-1}np]) = kmp(p^{-1}np)[e] = kmn[e] = \mu(k, m, [n]).$$

This implies that μ induces a smooth map $\bar{\mu}$ as described. The surjectivity of $\bar{\mu}$ follows from the surjectivity of μ. □

Proposition 4.2 will follow from the result that the composition $\mathfrak{H}_{P,\mathrm{q}} \circ \bar{\mu}$ is proper. The latter map is left-invariant under the left action of $K \cap H$ on the first component. Thus, Proposition 4.2 will already follow from the following result.

Lemma 4.6. *The map $(m, n) \mapsto \mathfrak{H}_{P,\mathrm{q}}(mn)$ induces a smooth map*

$$\varphi : \ (M_R \cap H) \times_{M_R \cap H \cap P} (N_R \cap H)/(N_R \cap H \cap P) \to \mathfrak{a}_{\mathrm{q}}$$

which is proper.

The inclusion map $N_R \cap H \to N_R$ induces an embedding of $(N_R \cap H)/(N_R \cap H \cap P)$ onto a closed submanifold of $N_R/N_R \cap P$. This embedding is equivariant for the conjugation action of $M_R \cap H \cap P$. Accordingly, we may view

$$(M_R \cap H) \times_{M_R \cap H \cap P} (N_R \cap H)/(N_R \cap H \cap P)$$

as a closed submanifold of

$$(M_R \cap H) \times_{M_R \cap H \cap P} N_R/(N_R \cap P).$$

Thus, for the proof of Lemma 4.6 it suffices to establish the following result.

Proposition 4.7. *The map $\psi : (m, n) \mapsto \mathfrak{H}_{P,\mathrm{q}}(mn)$ induces a smooth map*

$$\bar{\psi} : (M_R \cap H) \times_{M_R \cap H \cap P} N_R/(N_R \cap P) \to \mathfrak{a}_{\mathrm{q}}. \tag{24}$$

This map is proper.

Before we proceed with the proof of Proposition 4.7 we will first study the maps $M_R \cap H/M_R \cap H \cap P \to \mathfrak{a}_{\mathrm{q}}$ and $N_R/(N_R \cap P) \to \mathfrak{a}_{\mathrm{q}}$ induced by $\mathfrak{H}_{P,\mathrm{q}}$.

Lemma 4.8. *The map $\mathfrak{H}_{P,\mathrm{q}}^R := \mathfrak{H}_{P,\mathrm{q}}|_{M_R \cap H}$ induces a smooth map $\bar{\mathfrak{H}}_{P,\mathrm{q}}^R : (M_R \cap H)/(M_R \cap H \cap P) \to \mathfrak{a}_{\mathrm{q}}$ which is proper and has image equal to the cone $\Gamma_{\mathfrak{a}_{\mathrm{q}}}(\Sigma_-^R)$, where*

$$\Sigma_-^R = \{\alpha \in \Sigma(P) \cap \mathfrak{a}_{\mathrm{q}}^* : \ \mathfrak{g}_\alpha \not\subseteq \ker(\sigma\theta - I)\}.$$

In particular, the image is contained in the cone $\Gamma_{\mathfrak{a}_{\mathrm{q}}}(\Sigma(P, \sigma\theta))$.

Proof. We start by noting that $(M_R, M_R \cap H)$ is a reductive symmetric pair of the Harish-Chandra class, which is invariant under the Cartan involution θ. Furthermore, ${}^*\mathfrak{a}_R := \mathfrak{m}_R \cap \mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{m}_R \cap \mathfrak{p}$ (contained in $\mathfrak{a}_{\mathrm{q}}$) and $M_R \cap P$ is a minimal parabolic subgroup of M_R containing ${}^*A_R := \exp {}^*\mathfrak{a}_R$. Accordingly, by restriction the Iwasawa projection map $\mathfrak{H}_{P,\mathrm{q}} : H \to \mathfrak{a}_{\mathrm{q}}$ induces the similar projection map $\mathfrak{H}_{P,\mathrm{q}}^R : M_R \cap H \to \mathfrak{a}_{\mathrm{q}}$ which is the analogue of $\mathfrak{H}_{P,\mathrm{q}}$ defined relative to the data $M_R, M_R \cap K, P \cap M_R, H \cap M_R$, in place of G, K, P, H.

The ${}^*\mathfrak{a}_R$-roots in $N_P \cap M_R$ are precisely the restrictions of the roots from $\Sigma(P) \cap \mathfrak{a}_{\mathrm{q}}^*$. From this we see that the minimal parabolic subgroup $P \cap M_R$ of M_R is $\sigma\theta$-stable. Hence, in view of [4, Theorem 1.1, Lemma 3.3], the map $\bar{\mathfrak{H}}_{P,\mathrm{q}}^R$ is proper and has image equal to the cone $\Gamma_{\mathfrak{a}_{\mathrm{q}}}(\Sigma_-^R)$ given above. The final assertion now follows from the observation that $\Sigma(P) \cap \mathfrak{a}_{\mathrm{q}}^* \subseteq \Sigma(P, \sigma\theta)$. $\square$

The following lemma is well known. For completeness of the exposition, we provide the proof.

Lemma 4.9. *The Iwasawa map $\mathfrak{H}_P|_{\bar{N}_P} : \bar{N}_P \to \mathfrak{a}$ is proper. If $Q \in \mathscr{P}(A)$, then*

$$\mathfrak{H}_P(N_Q \cap \bar{N}_P) = \Gamma_{\mathfrak{a}}(\Sigma(P) \cap \Sigma(\bar{Q})).$$

Proof. For the first assertion, let $(\bar{n}_j)$ be sequence in $\bar{N}_P$ such that $\mathfrak{H}_P(\bar{n}_j)$ converges. Then $\bar{n}_j = k_j a_j n_j$, with $k_j \in K$, $a_j = \exp \mathfrak{H}_P(\bar{n}_j)$ and $n_j \in N_P$. By passing to a converging subsequence, we may arrange that in addition the sequence (k_j) converges in K. It follows that $\bar{n}_j n_j^{-1} = k_j a_j$ converges in G. By [15, Lemma 39], the sequence $(\bar{n}_j)$ converges.

For the second assertion, we may assume $\Sigma(\bar{Q}) \cap \Sigma(P) \neq \emptyset$ and use the idea due to S. Gindikin and F. Karpelevic [12], to decompose $N_Q \cap \bar{N}_P$ by using a P-simple root in $\Sigma(\bar{Q}) \cap \Sigma(P)$. Let α be such a root. Let $\mathfrak{n}_\alpha = \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$ and $N_\alpha = \exp \mathfrak{n}_\alpha$. Put $Q' = s_\alpha Q s_\alpha$. Then, with the notation of Subsection 2.1,

$$\Sigma_\circ(\bar{Q}) \cap \Sigma_\circ(P) = \{\alpha\} \sqcup (\Sigma_\circ(\bar{Q}') \cap \Sigma_\circ(P)),$$

so that

$$N_Q \cap \bar{N}_P = \bar{N}_\alpha(N_{Q'} \cap \bar{N}_P) \simeq \bar{N}_\alpha \times (N_{Q'} \cap \bar{N}_P),$$

in view of Lemma 2.11. Let $\bar{n} \in N_Q \cap \bar{N}_P$. Then according to the above decomposition we may write $\bar{n} = \bar{n}_\alpha \bar{n}'$, where $\bar{n}_\alpha \in \bar{N}_\alpha$ and $\bar{n}' \in N_{Q'} \cap N_P$. Let $\mathfrak{g}(\alpha)$ be the semisimple subalgebra generated by $\mathfrak{n}_\alpha$ and $\bar{\mathfrak{n}}_\alpha$, and let $G(\alpha)$ be the corresponding analytic subgroup of G. By the Iwasawa decomposition of $G(\alpha)$ for the minimal parabolic subgroup $P \cap G(\alpha)$ we may write $\bar{n}_\alpha = k_\alpha a_\alpha n_\alpha$

with $k_\alpha \in G(\alpha) \cap K$, $a_\alpha \in \exp(\mathbb{R}H_\alpha)$ and $n_\alpha \in N_\alpha$. From application of Lemma 2.11 we find that
$$N_{Q'} \cap \bar{N}_P \simeq N_{Q'}/(N_{Q'} \cap N_P)$$
and we see that there exists a diffeomorphism τ_{n_α} from $N_{Q'} \cap \bar{N}_P$ onto itself, such that
$$n_\alpha \bar{n}' \in \tau_{n_\alpha}(\bar{n}')N_P, \quad \text{for all } \bar{n}' \in N_{Q'} \cap \bar{N}_P.$$
This implies that
$$\mathfrak{H}_P(\bar{n}_\alpha n') = \mathfrak{H}_P(a_\alpha \tau_{n_\alpha}(\bar{n}')a_\alpha^{-1}) + \log a_\alpha,$$
and we see that
$$\mathfrak{H}_P(\bar{N}_\alpha(N'_Q \cap \bar{N}_P)) = \mathfrak{H}_P(N'_Q \cap \bar{N}_P) + \mathfrak{H}_P(\bar{N}_\alpha).$$
Now $\mathfrak{H}_P(\bar{N}_\alpha)$ equals the image of $\bar{N}_\alpha$ under the Iwasawa projection $\mathfrak{H}_\alpha$ for the split rank 1 group $G(\alpha)$ and the minimal parabolic subgroup $P \cap G(\alpha)$. By [17, Thm. IX.3.8], which is based on $\mathrm{SU}(2,1)$-reduction, we see that $\mathfrak{H}_\alpha(\bar{N}_\alpha) = \mathbb{R}_{\geq 0}H_\alpha$. It follows that
$$\mathfrak{H}_P(\bar{N}_\alpha(N'_Q \cap \bar{N}_P)) = \mathfrak{H}_P(N'_Q \cap \bar{N}_P) + \mathbb{R}_{\geq 0}H_\alpha.$$
The proof is completed by induction on the number of elements in $\Sigma_\circ(\bar{Q}) \cap \Sigma_\circ(P)$. □

The following lemma is the second ingredient for the proof of Proposition 4.7.

Lemma 4.10. *The Iwasawa map $\mathfrak{H}_{P,\mathfrak{q}}|_{N_R} : N_R \to \mathfrak{a}_\mathfrak{q}$ factors through a proper map $N_R/N_R \cap N_P \to \mathfrak{a}_\mathfrak{q}$ with image equal to the cone*
$$\Gamma_{\mathfrak{a}_\mathfrak{q}}(\Sigma(P) \cap \Sigma(\bar{R})). \tag{25}$$
In particular, the image is contained in the cone $\Gamma_{\mathfrak{a}_\mathfrak{q}}(\Sigma(P, \sigma\theta))$.

Proof. We denote the induced map by $\mathfrak{H}$. It follows by application of Lemma 2.11 that the multiplication map $(N_R \cap \bar{N}_P) \times (N_R \cap N_P) \to N_R$ is a diffeomorphism. Let $\nu : N_R \cap \bar{N}_P \to N_R/N_R \cap N_P$ denote the induced diffeomorphism. Then $\mathfrak{H} \circ \nu$ equals $\mathrm{pr}_\mathfrak{q} \circ \mathfrak{H}_{P,R}$, where $\mathfrak{H}_{P,R}$ denotes the restriction of $\mathfrak{H}_P$ to $N_R \cap \bar{N}_P$. This restriction is proper with image $\Gamma_\mathfrak{a}(\Sigma(P) \cap \Sigma(\bar{R}))$, by Lemma 4.9 above. In particular, the image is contained in the cone $\Gamma_\mathfrak{a}(\Sigma(P, \sigma\theta))$, by Lemma 4.3. In view of Corollary 3.4 (a) it now follows that $\mathfrak{H} \circ \nu = \mathrm{pr}_\mathfrak{q} \circ \mathfrak{H}_{P,R}$ is proper with image equal to (25). This implies the result. □

We proceed with a final lemma needed for the proof of Proposition 4.7.

Lemma 4.11. *Let $\bar{\psi}$ be as in (24) and let*
$$\bar{\mathrm{pr}}_1 : (M_R \cap H) \times_{M_R \cap H \cap P} N_R/(N_R \cap P) \to (M_R \cap H)/(M_R \cap H \cap P)$$
denote the map induced by projection onto the first component.

Let $C \subseteq \mathfrak{a}_\mathfrak{q}$ be a compact set. Then the set $\bar{\mathrm{pr}}_1(\bar{\psi}^{-1}(C))$ is relatively compact in $(M_R \cap H)/(M_R \cap H \cap P)$.

Proof. Let $m \mapsto [m]$ denote the canonical projection $M_R \cap H \to (M_R \cap H)/(M_R \cap H \cap P)$. Let (m_j) and (n_j) be sequences in $M \cap H$ and N_R, respectively, such that $\mathfrak{H}_P(m_j n_j) \in C$ for all j. Then it suffices to show that the sequence $([m_j])$ in $(M_R \cap H)/(M_R \cap H \cap P)$ has a converging subsequence.

In accordance with the Iwasawa decomposition $M_R = (M_R \cap K)(M_R \cap A)(M_R \cap N_P)$, we may decompose $m_j = k_j a_j \nu_j$. Since $\mathfrak{a}_{\mathrm{h}} \subseteq \mathfrak{a}_R = \mathrm{center}(\mathfrak{l}_R) \cap \mathfrak{p}$, we have $\mathfrak{m}_R \cap \mathfrak{a} = \mathfrak{a}_R^{\perp} \cap \mathfrak{a} \subseteq \mathfrak{a}_{\mathrm{q}}$, so that $\log a_j = \mathfrak{H}^R_{P,\mathrm{q}}(m_j)$.

The element $t_j = a_j \nu_j$ belongs to M_R, hence $n'_j := t_j n_j t_j^{-1} \in N_R$, for all j. From $m_j n_j = k_j n'_j a_j \nu_j$ it follows that

$$\mathfrak{H}_{P,\mathrm{q}}(m_j n_j) = \mathfrak{H}_{P,\mathrm{q}}(k_j n'_j) + \log a_j = \mathfrak{H}_{P,\mathrm{q}}(n'_j) + \mathfrak{H}^R_{P,\mathrm{q}}(m_j).$$

We now note that both $\mathfrak{H}_{P,\mathrm{q}}(n'_j)$ and $\mathfrak{H}_{P,\mathrm{q}}(m_j)$ belong to $\Gamma_{\mathfrak{a}_{\mathrm{q}}}(P, \sigma\theta)$ by Lemmas 4.10 and 4.8. By application of Corollary 3.4 we infer that the sequence $\mathfrak{H}_{P,\mathrm{q}}(m_j)$ is contained in a relatively compact subset of $\mathfrak{a}_{\mathrm{q}}$. By application of Lemma 4.8 it now follows that $([m_j])$ is contained in a relatively compact subset of $(M_R \cap H)/(M_R \cap H \cap P)$, hence contains a convergent subsequence. □

Completion of the proof of Proposition 4.7. Let C be a compact subset of $\mathfrak{a}_{\mathrm{q}}$ and let (m_j) be a sequence in $M_R \cap H$ and (n_j) a sequence in N_R such that $\bar{\psi}([(m_j, n_j)]) \in C$ for all j. Then it suffices to show that the sequence of points

$$[(m_j, n_j)] \in (M_R \cap H) \times_{M_R \cap H \cap P} N_R/(N_R \cap N_P)$$

has a converging subsequence.

In view of Lemma 4.11 we may pass to a subsequence of indices and assume that the sequence $([m_j])$ in $D := (M_R \cap H)/(M_R \cap H \cap P)$ converges. Since the canonical projection $M_R \cap H \to D$ determines a principal fiber bundle, we may invoke a local trivialization to obtain a converging sequence $({}^{\backprime}m_j)$ in $M_R \cap H$ such that ${}^{\backprime}m_j \in m_j(M_R \cap H \cap P)$ for all j. Let $p_j \in M_R \cap H \cap P$ be such that $m_j = {}^{\backprime}m_j p_j$ for all j. Then

$$[(m_j, n_j)] = [({}^{\backprime}m_j, {}^{\backprime}n_j)],$$

with ${}^{\backprime}n_j = p_j n_j p_j^{-1} \in N_R$.

Replacing the original sequence of points (m_j, n_j) in this fashion if necessary, we may as well assume that the original sequence (m_j) converges in $M_R \cap H$. Let $m \in M_R \cap H$ be the limit of this sequence. As in the proof of Lemma 4.11 we may decompose $m_j = k_j a_j \nu_j$ and $m = ka\nu$ in accordance with the Iwasawa decomposition $M_R = (M_R \cap K)(M_R \cap A)(M_R \cap N_P)$. Then $k_j \to k$, $a_j \to a$ and $\nu_j \to \nu$, for $j \to \infty$. Put $t_j = a_j \nu_j$ and $n'_j = t_j n_j t_j^{-1}$. As in the proof of Lemma 4.11 it follows that

$$\bar{\psi}([m_j, n_j]) = \log a_j + \mathfrak{H}_{P,\mathrm{q}}(n'_j).$$

Since (a_j) converges, it follows that the sequence $\mathfrak{H}_{P,\mathrm{q}}(n'_j)$ is contained in a compact subset $C' \subseteq \mathfrak{a}_{\mathrm{q}}$. By Lemma 4.10 it follows that the sequence $([n'_j])$ in $N_R/N_R \cap N_P$ is contained in a compact subset. Passing to a suitable subsequence of indices we may as well assume that the sequence $([n_j'])$ converges to a point $[n]$, for some $n \in N_R$. It follows that

$$[n_j] = [t_j^{-1} n_j' t_j] = t_j^{-1} \cdot [n_j'] \to t^{-1} \cdot [n] = [t^{-1} n t], \quad (j \to \infty),$$

where $t = a\nu$. We conclude that the sequence $[(m_j, n_j)]$ converges with limit equal to $[(m, t^{-1} n t)]$. $\square$

We finish this section with a number of results that will be needed in Section 10.

Corollary 4.12. *Let $\mathscr{A}$ be a compact subset of A_{q}. Then*

(a) $\mathfrak{H}_{P,\mathrm{q}}(ah) \in \mathfrak{H}_{P,\mathrm{q}}(\mathscr{A}K) + \mathfrak{H}_{P,\mathrm{q}}(h)$, *for all* $(a,h) \in \mathscr{A} \times H$;

(b) *the map* $(a,h) \mapsto \mathfrak{H}_{P,\mathrm{q}}(ah)$ *induces a proper map* $\mathscr{A} \times H/H \cap P \to \mathfrak{a}_{\mathrm{q}}$.

Proof. We first prove (a). Let $a \in \mathscr{A}$ and $h \in H$. We may decompose $h = kbn$ with $k \in K$, $b \in A$ and $n \in N_P$. Furthermore, $ak = k'a'n'$ with $k' \in K$, $n' \in N_P$ and $\log a' \in \mathfrak{H}_P(\mathscr{A}K)$. Now

$$ah = akbn = k'a'n'bn = k'a'bn''$$

with $n'' = b^{-1} n' bn \in N_P$. It follows that

$$\mathfrak{H}_{P,\mathrm{q}}(ah) = \mathrm{pr}_{\mathrm{q}}(\log a' + \log b) \in \mathfrak{H}_{P,\mathrm{q}}(\mathscr{A}K) + \mathfrak{H}_{P,\mathrm{q}}(h).$$

This establishes (a).

Since $\mathfrak{H}_{P,\mathrm{q}}(\mathscr{A}K)$ is compact, (b) follows from combining (a) with Proposition 4.7. $\square$

Lemma 4.13. *Let $P \in \mathscr{P}(A)$. Then $\mathfrak{H}_{P,\mathrm{q}}(H) \subseteq \Gamma_{\mathfrak{a}_{\mathrm{q}}}(\Sigma(P, \sigma\theta))$.*

Proof. By (23) we have

$$H = (H \cap K)(H \cap N_R)(H \cap L_R) \subseteq K N_R (H \cap L_R).$$

Fix $h \in H$, then we may write $h = k n_R h_L$ with $k \in K$, $n_R \in N_R$ and $h_L \in (H \cap L_R)$. The group $P \cap L_R$ is a minimal parabolic subgroup of L_R, containing A. In accordance with the associated Iwasawa decomposition for L_R, we may write $h_L = k_L a_L n_L$ with $k_L \in K \cap L_R$, $a_L \in A$ and $n_L \in N_P \cap L_R$. Since L_R normalizes N_R, it follows that

$$h = k n_R k_L a_L n_L \in K n_R' a_L n_L$$

with $n_R' \in N_R$. We now observe that $n_R' \in K b N_P$ with $b = \exp \mathfrak{H}_P(n_R')$. Thus, $h \in K b a_L N_P$. It follows that

$$\mathfrak{H}_{P,\mathrm{q}}(h) = \mathrm{pr}_{\mathrm{q}}(\log b + \log a_L) \in \mathfrak{H}_{P,\mathrm{q}}(N_R) + \mathfrak{H}_{P,\mathrm{q}}(H \cap L_R). \tag{26}$$

Since $L_R \cap H = (M_R \cap H)(A \cap H)$, we have $\mathfrak{H}_{P,\mathrm{q}}(H \cap L_R) = \mathfrak{H}_{P,\mathrm{q}}(H \cap M_R)$. The result now follows from (26) by applying Lemmas 4.8 and 4.10. $\square$

Lemma 4.14. *Let V be a finite dimensional real vector space (or more generally a real locally convex Hausdorff space), Γ_1 a convex cone in V, Γ_2 a closed convex cones in V and $B \subseteq V$ a bounded subset. If $\Gamma_1 \subseteq B + \Gamma_2$ then $\Gamma_1 \subseteq \Gamma_2$.*

Proof. Let $\gamma \in \Gamma_1$. Then for any positive integer $n \geq 1$ we have that $n\gamma \in \Gamma_1 \subseteq B + \Gamma_2$, hence

$$\gamma = b_n/n + \gamma_n,$$

with $b_n \in B$ and $\gamma_n \in \Gamma_2$. As B is bounded, $b_n/n \to 0$ and we conclude that $\gamma_n \to \gamma$, for $n \to \infty$. Since Γ_2 is closed, it follows that $\gamma \in \Gamma_2$. □

Corollary 4.15. *Let $P \in \mathscr{P}(A)$. Then for each $a \in A_{\mathrm{q}}$, the set $\mathfrak{H}_{P,\mathrm{q}}(aH)$ does not contain any line of $\mathfrak{a}_{\mathrm{q}}$.*

Proof. From Corollary 4.12 (a) combined with Lemma 4.13 we see that

$$\mathfrak{H}_{P,\mathrm{q}}(aH) \subseteq \mathfrak{H}_{P,\mathrm{q}}(aK) + \Gamma_{\mathfrak{a}_{\mathrm{q}}}(\Sigma(P,\sigma\theta)). \tag{27}$$

Arguing by contradiction, assume that $\mathfrak{H}_{P,\mathrm{q}}(aH)$ contains a line of the form $Z + \mathbb{R}Y$, with $Y, Z \in \mathfrak{a}_{\mathrm{q}}$, $Y \neq 0$. Then $\mathbb{R}Y \subseteq (-Z) + \mathfrak{H}_{P,\mathrm{q}}(aK) + \Gamma(\Sigma(P,\sigma,\theta))$, and by Lemma 4.14 we conclude that $\mathbb{R}Y \in \Gamma(\Sigma(P,\sigma\theta))$. This implies that

$$Y \in \Gamma(\Sigma(P,\sigma\theta)) \cap -\Gamma(\Sigma(P,\sigma\theta)) = \{0\},$$

contradiction. □

5 Critical points of components of the Iwasawa map

In this section we assume that $P \in \mathscr{P}(A)$ is a fixed minimal parabolic subgroup and that a is a fixed element of A_{q}. We will investigate the critical sets of vector components of the map $h \mapsto \mathfrak{H}_{P,\mathrm{q}}(ah)$, $H \to \mathfrak{a}_{\mathrm{q}}$. For this, let $X \in \mathfrak{a}_{\mathrm{q}}$, and consider the function $F_{a,X} : H \to \mathbb{R}$ defined by

$$F_{a,X}(h) = \langle X, \mathfrak{H}_P(ah) \rangle = \langle X, \mathfrak{H}_{P,\mathrm{q}}(ah) \rangle = B(X, \mathfrak{H}_{P,\mathrm{q}}(ah)). \tag{28}$$

The second equality is valid because $\mathfrak{a}_{\mathrm{h}}$ and $\mathfrak{a}_{\mathrm{q}}$ are perpendicular with respect to the inner product $\langle \cdot, \cdot \rangle$, while the third holds because $\mathfrak{H}_{P,\mathrm{q}}(ah) \in \mathfrak{a}_{\mathrm{q}} \subset \mathfrak{p}$. We start with a result on derivatives of the function

$$F_X : G \to \mathbb{R}, \quad g \mapsto \langle X, \mathfrak{H}_P(g) \rangle. \tag{29}$$

In order to formulate it, we need a bit of additional notation. If $F \in C^\infty(G)$ and $U \in \mathfrak{g}$, we define:

$$F(g;U) = R_U F(g) := \left.\frac{d}{dt}\right|_{t=0} F(g\exp(tU)).$$

The following result and its proof can be found in [9, Cor. 5.2]. See also [4, Cor. 4.2].

Lemma 5.1. *Let $g \in G$ and $U \in \mathfrak{g}$. Then*

$$F_X(g;U) = B(\mathrm{Ad}(\tau(g))U, X) = B(U, \mathrm{Ad}(\nu(g)^{-1})X),$$

where we have used the decompositions $g = k(g)\tau(g)$ and $\tau(g) = a(g)\nu(g)$, according to the Iwasawa decomposition $G = KAN_P$.

We define the set of regular elements in A_q by $A_\mathrm{q}^\mathrm{reg} := \exp(\mathfrak{a}_\mathrm{q}^\mathrm{reg})$, see (19). If $X \in \mathfrak{a}_\mathrm{q}$ we denote by G_X the centralizer of X in G and put

$$N_{P,X} := N_P \cap G_X. \tag{30}$$

Lemma 5.2. *Let $a \in A_\mathrm{q}$ and let $X \in \mathfrak{a}_\mathrm{q}$. The point $h \in H$ is a critical point for the function $F_{a,X}$ if and only if $ah = kbn$ for certain $k \in K$, $b \in A$ and $n \in N_{P,X}(N_P \cap H)$.*

Proof. Let $h \in H$. Then h is a critical point for the function $F_{a,X}$ if and only if

$$\forall U \in \mathfrak{h}: \;\; 0 = F_{a,X}(h;U) = B(U, \mathrm{Ad}(\nu(ah)^{-1})X). \tag{31}$$

Since $\mathfrak{h}$ and $\mathfrak{q}$ are perpendicular with respect to B, see text above (7), the condition (31) is equivalent to the assertion that $\mathrm{Ad}(\nu(ah)^{-1})X \in \mathfrak{q}$. Write $n = \nu(ah)$ and decompose $n = n_+ n_H$ according to the decomposition $N_P = N_{P,+}(N_P \cap H)$ of Proposition 2.14. Since $\mathrm{Ad}(n_H)$ normalizes $\mathfrak{q}$, the above condition is equivalent to $\mathrm{Ad}(n_+)^{-1}X \in \mathfrak{q}$. Now apply the lemma below to see that the latter is equivalent to $n_+ \in N_{P,+} \cap N_{P,X}$. It follows that (31) is equivalent to $n \in N_{P,X}(N_P \cap H)$. □

Lemma 5.3. *Let $n \in N_{P,+}$ (cf. Prop. 2.14) and $X \in \mathfrak{a}_\mathrm{q}$. Then*

$$\mathrm{Ad}(n)X \in \mathfrak{q} \iff \mathrm{Ad}(n)X = X.$$

Proof. The implication '$\Leftarrow$' is obvious. Thus, assume that $\mathrm{Ad}(n)X \in \mathfrak{q}$. We may write $n = \exp(U)$, where $U \in \mathfrak{n}_{P,+}$. Then by nilpotence of $\mathfrak{n}_{P,+}$,

$$\mathrm{Ad}(n)X = e^{\mathrm{ad}(U)}X \in X + \mathfrak{n}_{P,+}$$

By assumption, $\mathrm{Ad}(n)X - X \in \mathfrak{q}$. Since obviously $\sigma(\mathfrak{n}_{P,+}) \cap \mathfrak{n}_{P,+} = 0$, it follows that $\mathfrak{n}_{P,+} \cap \mathfrak{q} = 0$ and we infer that $\mathrm{Ad}(n)X = X$. □

Given $X \in \mathfrak{a}_\mathrm{q}$ we agree to denote by $\mathscr{C}_{a,X}$ the set of critical points for the function $F_{a,X}$. The remainder of this section will be dedicated to proving the following description of this set in case a is regular. We recall the definitions of the Weyl groups $W(\mathfrak{a}_\mathrm{q})$ and $W_{K\cap H}$ from (2) and Definition 1.3.

Remark 5.4. In the following we will use the notation

$$a^w := w^{-1} \cdot a$$

for $a \in A_\mathrm{q}$ and $w \in W(\mathfrak{a}_\mathrm{q})$. This notation has the advantage that $(a^v)^w = a^{vw}$ and $(a^w)^\beta = a^{w\beta}$, for $v, w \in W(\mathfrak{a}_\mathrm{q})$ and $\beta \in \Sigma(\mathfrak{g}, \mathfrak{a}_\mathrm{q})$. In particular, $\mathrm{Ad}(a^w) = a^{w\beta}I$ on $\mathfrak{g}_\beta$.

We will use similar notation for $a \in A$ and $w \in W(\mathfrak{a})$.

Lemma 5.5. *Let $a \in A_{\mathrm{q}}^{\mathrm{reg}}$ and $X \in \mathfrak{a}_{\mathrm{q}}$. Then*
$$\mathscr{C}_{a,X} = \bigcup_{w \in W_{K\cap H}} wH_X(N_P \cap H). \tag{32}$$

Proof. Let x_w be a representative of w in $N_{K\cap H}(\mathfrak{a}_{\mathrm{q}})$, let $h \in H_X$ and $n_P \in N_P \cap H$. Then, with notation as in Lemma 5.1,
$$\nu(ax_w h n_P) = \nu(x_w^{-1} a x_w h n_P) = \nu(a^w h n_P) = \nu(a^w h) n_P.$$
The element $a^w h$ belongs to G_X, and according to [9, Eqn. (2.6)],
$$G_X \simeq K_X A N_{P,X}.$$
Thus, $\nu(a^w h) \in N_{P,X}$ and it follows that $\nu(ax_w h n_P) \in N_{P,X}(N_P \cap H)$. This proves that the set on the right-hand side of (32) is included in the set on the left-hand side. It remains to prove the converse inclusion.

Let $h \in \mathscr{C}_{a,X}$. Then by Lemma 5.2 we may write $ah = kbn_X n_H$ with $k \in K$, $b \in A$, $n_X \in N_{P,X}$ and $n_H \in N_P \cap H$. From this we see that $k^{-1} a h n_H^{-1} = b n_X \in G_X$. The element $h' := h n_H^{-1}$, belongs to H. In view of the Cartan decomposition $H = (K \cap H) \times \exp(\mathfrak{p} \cap \mathfrak{h})$, we may write $h' = h_1 h_2$, where $h_1 \in K \cap H$ and $h_2 \in \exp(\mathfrak{p} \cap \mathfrak{h})$. Then
$$k^{-1} a h_1 h_2 = k^{-1} h_1 (h_1^{-1} a h_1) h_2 \in G_X. \tag{33}$$

By [25], the group G decomposes as
$$G \simeq K \times \exp(\mathfrak{p} \cap \mathfrak{q}) \times \exp(\mathfrak{p} \cap \mathfrak{h}).$$
According to [25, Thm. 5], G_X has a similar decomposition
$$G_X \simeq K_X \times \exp(\mathfrak{p} \cap \mathfrak{q}_X) \times \exp(\mathfrak{p} \cap \mathfrak{h}_X).$$
By the uniqueness properties of the latter decomposition it follows from (33) that $k^{-1} h_1 \in K_X$, $h_1^{-1} a h_1 \in \exp(\mathfrak{p} \cap \mathfrak{q}_X)$ and $h_2 \in \exp(\mathfrak{p} \cap \mathfrak{h}_X)$.

We note that $\sigma\theta$ fixes X hence leaves the centralizer G_X invariant. The fixed point group $G_{X,+}$ of this involution in G_X admits the Cartan decomposition
$$G_{X,+} \simeq (K \cap H_X) \times \exp(\mathfrak{p} \cap \mathfrak{q}_X).$$
Obviously $\mathfrak{a}_{\mathrm{q}}$ is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}_X$. Hence, every element of the latter space is conjugate to an element of $\mathfrak{a}_{\mathrm{q}}$ under the group $(K \cap H_X)^\circ$. We infer that there exists an element $l \in (K \cap H_X)^\circ$ such that
$$l^{-1} h_1^{-1} a h_1 l \in A_{\mathrm{q}}. \tag{34}$$
Since a was assumed to be regular for $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$, it follows that a is regular for $\Sigma(\mathfrak{g}_+, \mathfrak{a}_{\mathrm{q}})$ as well. Hence, (34) implies that the element $h_1 l \in K \cap H$ normalizes $\mathfrak{a}_{\mathrm{q}}$. It follows that $h_1 \in N_{K\cap H}(\mathfrak{a}_{\mathrm{q}})(K \cap H_X)$. Then,
$$h' = h_1 h_2 \in N_{K\cap H}(\mathfrak{a}_{\mathrm{q}})(K \cap H_X)\exp(\mathfrak{p} \cap \mathfrak{h}_X) = N_{K\cap H}(\mathfrak{a}_{\mathrm{q}}) H_X$$
and we conclude that $h n_H^{-1} \in N_{K\cap H}(\mathfrak{a}_{\mathrm{q}}) H_X$. This finally implies that
$$h \in N_{K\cap H}(\mathfrak{a}_{\mathrm{q}}) H_X (N_P \cap H),$$
which concludes the proof. $\square$

6 Properties of the set of critical points

As in the previous section, we assume that $P \in \mathscr{P}(A)$ and that a is a regular point in A_{q}. In the previous section we defined the function $F_{a,X} : H \to \mathbb{R}$, for $X \in \mathfrak{a}_{\mathrm{q}}$, by (28) and we determined its set of critical points $\mathscr{C}_{a,X}$, see (32). The purpose of the present section is to study this set in more detail.

We start with the following lemma.

Lemma 6.1. *The map $\varphi : H_X \times (N_P \cap H) \to H$ given by $(h,n) \mapsto hn$ induces an injective immersion*

$$\bar{\varphi} : H_X \times_{N_P \cap H_X} (N_P \cap H) \to H$$

with image $H_X(N_P \cap H)$.

Proof. The group $H_X \times (N_P \cap H)$ has a natural left action on H given by the formula: $(h,n) \cdot x = hxn^{-1}$. The set $H_X(N_P \cap H)$ is the orbit for this action through the identity element e of H. Let F be the stabilizer of e for this action. Then it follows that the map $(h,n) \mapsto (h,n) \cdot e = hn^{-1}$ factors through an injective immersion $(H_X \times (N_P \cap H))/F \to H$ with image $H_X(N_P \cap H)$. The stabilizer F consists of the elements (h,h) with $h \in H_X \cap N_P$. To complete the proof of the lemma, we note that the map $(h,n) \mapsto (h,n^{-1})$ induces a diffeomorphism $H_X \times_{N_P \cap H_X} (N_P \cap H) \to (H_X \times (N_P \cap H))/F$. □

Lemma 6.2. *Let $X \in \mathfrak{a}_{\mathrm{q}}$. Then the set $\mathscr{C}_{a,X}$ is closed in H. Moreover, the following holds.*

(a) *If $\mathfrak{h}_X + (\mathfrak{n}_P \cap \mathfrak{h}) = \mathfrak{h}$ then $\mathscr{C}_{a,X} = H$.*

(b) *If $\mathfrak{h}_X + (\mathfrak{n}_P \cap \mathfrak{h}) \subsetneq \mathfrak{h}$ then $\mathscr{C}_{a,X}$ is a finite union of lower dimensional injectively immersed submanifolds.*

Proof. Since $\mathscr{C}_{a,X}$ is the set of critical points of the smooth function $F_{a,X}$, it is closed.

From Lemma 5.5 combined with Lemma 6.1 it follows that $\mathscr{C}_{a,X}$ is a finite union of injectively immersed submanifolds of dimension $d_X := \dim(\mathfrak{h}_X + (\mathfrak{n}_P \cap \mathfrak{h}))$. From this, (b) is immediate.

For (a) we assume the hypothesis to be fulfilled, or equivalently, that $d_X = \dim(H)$. Then $\mathscr{C}_{a,X}$ is open in H. Since this set is also closed in H, and contains $H_X(N_P \cap H)$, it follows that $\mathscr{C}_{a,X} \supseteq H^\circ$. From Lemma 5.5 it follows that $\mathscr{C}_{a,X}$ is left $N_{K \cap H}(\mathfrak{a}_{\mathrm{q}})$-invariant, so that $\mathscr{C}_{a,X} \supseteq N_{K \cap H}(\mathfrak{a}_{\mathrm{q}})H^\circ$. Since H is essentially connected, the latter set equals H, see (6). □

Lemma 6.3. *Let $X \in \mathfrak{a}_{\mathrm{q}}$. Then the following assertions are equivalent:*

(a) $\mathfrak{h} = \mathfrak{h}_X + (\mathfrak{n}_P \cap \mathfrak{h})$;

(b) $\forall \alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}) : \alpha(X) = 0$.

Proof. First assume (b). Then $\mathfrak{g}_X = \mathfrak{g}$ and (a) follows. We will prove the converse implication by contraposition. Thus, assume that (b) does not hold. Then there exists a root $\beta \in \Sigma(\mathfrak{g}, \mathfrak{a})$ such that $\beta(X) \neq 0$. By changing sign if necessary, we may in addition arrange that $\beta \in \Sigma(P)$.

Given a subset $\mathscr{O} \subseteq \Sigma(\mathfrak{g},\mathfrak{a}) \cup \{0\}$, we agree to write

$$\mathfrak{g}_{\mathscr{O}} = \oplus_{\alpha \in \mathscr{O}}\, \mathfrak{g}_\alpha. \tag{35}$$

In particular, we see that $\mathfrak{n}_P = \mathfrak{g}_{\Sigma(P)}$. We also agree to write $\mathscr{O}^\sigma := \mathscr{O} \cap \sigma(\mathscr{O})$. Then using $\sigma(\mathfrak{g}_\alpha) = \mathfrak{g}_{\sigma\alpha}$ we readily see that

$$\mathfrak{g}_{\mathscr{O}} \cap \mathfrak{h} = (\mathfrak{g}_{\mathscr{O}^\sigma})^\sigma = \oplus_{\omega \in \mathscr{O}^\sigma/\{1,\sigma\}}\ (\mathfrak{g}_\omega)^\sigma; \tag{36}$$

here $\mathscr{O}^\sigma/\{1,\sigma\}$ denotes the set of orbits for the action on $\mathscr{O}^\sigma$ of the subgroup $\{1,\sigma\}$ of $\mathrm{Aut}(\mathfrak{g})$. If we apply (36) to the set $\mathscr{O}_X := \{\alpha \in \Sigma(\mathfrak{g},\mathfrak{a}) :\ \alpha(X) = 0\} \cup \{0\}$, we find

$$\mathfrak{h}_X = \oplus_{\omega \in \mathscr{O}_X/\{1,\sigma\}}\ (\mathfrak{g}_\omega)^\sigma.$$

We note that $\Sigma(P)^\sigma = \Sigma(P,\sigma)$, so that

$$\mathfrak{n}_P \cap \mathfrak{h} = \mathfrak{g}_{\Sigma(P,\sigma)} \cap \mathfrak{h}.$$

We now consider the set $\mathscr{O}_\beta := \{\beta, \sigma\beta, -\beta, -\sigma\beta\}$. Since $\mathscr{O}_X \cap \mathscr{O}_\beta = \emptyset$, it follows from the above that

$$(\mathfrak{h}_X + (\mathfrak{n}_P \cap \mathfrak{h})) \cap \mathfrak{g}_{\mathscr{O}_\beta} = \mathfrak{n}_P \cap \mathfrak{h} \cap \mathfrak{g}_{\mathscr{O}_\beta} = (\mathfrak{g}_{\Sigma(P,\sigma)\cap\mathscr{O}_\beta})^\sigma. \tag{37}$$

On the other hand,

$$\mathfrak{h} \cap \mathfrak{g}_{\mathscr{O}_\beta} = (\mathfrak{g}_{\mathscr{O}_\beta})^\sigma.$$

From $\beta(X) \neq 0$ it follows that $\beta \notin \mathfrak{a}_\mathrm{h}^*$. If $\beta \in \mathfrak{a}_\mathrm{q}^*$ then $\Sigma(P,\sigma) \cap \mathscr{O}_\beta = \emptyset$ and if $\beta \notin \mathfrak{a}_\mathrm{q}^*$ then $\Sigma(P,\sigma) \cap \mathscr{O}_\beta \subseteq \{\beta, \sigma\beta\}$. In any case, $\Sigma(P,\sigma) \cap \mathscr{O}_\beta$ is a proper σ-invariant subset of $\mathscr{O}_\beta$. By application of (36) it now follows that

$$(\mathfrak{g}_{\Sigma(P,\sigma)\cap\mathscr{O}})^\sigma \subsetneq (\mathfrak{g}_{\mathscr{O}})^\sigma.$$

Using (37) we infer that (a) is not valid. □

We agree to write

$$S := \mathfrak{a}_\mathrm{q} \setminus \cap_{\alpha \in \Sigma(\mathfrak{g},\mathfrak{a}_\mathrm{q})} \ker \alpha. \tag{38}$$

Remark 6.4. If $\Sigma(\mathfrak{g},\mathfrak{a}_\mathrm{q})$ spans $\mathfrak{a}_\mathrm{q}$ then it follows that $S = \mathfrak{a}_\mathrm{q} \setminus \{0\}$.

Corollary 6.5. $S = \{X \in \mathfrak{a}_\mathrm{q} :\ \mathscr{C}_{a,X} \subsetneq H\}$.

Proof. Let $X \in \mathfrak{a}_\mathrm{q}$. In the situation of Lemma 6.2 (b) the set $\mathscr{C}_{a,X}$ is a countable union of lower dimensional submanifolds, hence nowhere dense by the Baire category theorem. Thus, by application of Lemmas 6.2 and 6.3 it follows that $\mathscr{C}_{a,X} \subsetneq H \iff X \in S$. □

For each $Z \in \mathfrak{a}_\mathrm{q}$, let $\Sigma(Z)$ denote the collection of roots in $\Sigma(\mathfrak{g},\mathfrak{a}_\mathrm{q})$ vanishing on Z. We define the equivalence relation $\sim$ on $\mathfrak{a}_\mathrm{q}$ by

$$X \sim Y \iff \Sigma(X) = \Sigma(Y).$$

Then clearly, $\sim$ has finitely many equivalence classes in $\mathfrak{a}_{\mathrm{q}}$ and

$$X \sim Y \iff G_X = G_Y.$$

The class of 0 is given by $[0] = \cap_{\alpha\in\Sigma(\mathfrak{g},\mathfrak{a}_{\mathrm{q}})} \ker\alpha$ and S is the union of the remaining finitely many equivalence classes for $\sim$. Furthermore, the set $\mathscr{C}_{a,X}$ depends on $X \in S$ through the centralizer G_X, hence through the equivalence class $[X]$ for $\sim$. Accordingly, we will also write $\mathscr{C}_{a,[X]}$ for this set.

We define

$$\mathscr{C}_a := \cup_{X\in S}\,\mathscr{C}_{a,X}. \tag{39}$$

Lemma 6.6.

(a) *There exists a finite subset $S_0 \subseteq S$ such that (39) is valid for the union over S_0 in place of S.*

(b) *The set $\mathscr{C}_a$ is closed and a finite union of lower dimensional injectively immersed submanifolds of H.*

(c) *The set $\mathscr{C}_a$ is nowhere dense in H.*

Proof. By the discussion preceding the lemma, $\mathscr{C}_a$ is the union of the sets $\mathscr{C}_{a,[X]}$, for $[X] \in S/\sim$. Since the latter set is finite, assertion (a) follows with S_0 a complete set of representatives for $S/\sim$. Assertion (b) now follows by application of Corollary 6.5 and Lemma 6.2. Assertion (c) follows from (b) by application of the Baire category theorem. □

The following result illustrates the importance of the set $\mathscr{C}_a$.

Proposition 6.7. *The set $H \setminus \mathscr{C}_a$ is open and dense in H. Assume that $\Sigma(\mathfrak{g},\mathfrak{a}_{\mathrm{q}})$ spans $\mathfrak{a}_{\mathrm{q}}^*$. Then the map $F_a : h \mapsto \mathfrak{H}_{P,\mathrm{q}}(ah)$, $H \to \mathfrak{a}_{\mathrm{q}}$ is submersive at all points of $H \setminus \mathscr{C}_a$.*

Proof. The first assertion is a consequence of Lemma 6.6.

Let $h_0 \in H \setminus \mathscr{C}_a$. Then for every $X \in S$ the point h_0 is not critical for the function $F_{a,X}$. As $S = \mathfrak{a}_{\mathrm{q}} \setminus \{0\}$, see Remark 6.4, it follows that $F_a : h \mapsto \mathfrak{H}_{P,\mathrm{q}}(ah)$ is submersive at h_0. □

Lemma 6.8. *Let $P \in \mathscr{P}(A)$ and $a \in A_{\mathrm{q}}^{\mathrm{reg}}$. Then the following assertions are valid.*

(a) *The sets $\mathfrak{H}_{P,\mathrm{q}}(aH)$ and $\mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a)$ are closed in $\mathfrak{a}_{\mathrm{q}}$.*

(b) *If $\Sigma(\mathfrak{g},\mathfrak{a}_{\mathrm{q}})$ spans $\mathfrak{a}_{\mathrm{q}}^*$ then the set $\mathfrak{H}_{P,\mathrm{q}}(aH) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a)$ is open and closed in $\mathfrak{a}_{\mathrm{q}} \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a)$.*

Proof. For $\mathscr{A} \subseteq A$ compact, the map $\mathscr{A} \times H/(H\cap P) \to \mathfrak{a}_{\mathrm{q}}, (b,[h]) \mapsto \mathfrak{H}_{P,\mathrm{q}}(bh)$ is proper, hence closed; see Corollary 4.12. In particular, it follows that $\mathfrak{H}_{P,\mathrm{q}}(aH)$ is closed in $\mathfrak{a}_{\mathrm{q}}$.

It follows from Lemma 6.6 that $\mathscr{C}_a$ is closed in H. Moreover, $\mathscr{C}_a$ is a countable union of lower dimensional submanifolds of H. Thus, by the Baire property, $\mathscr{C}_a$ has empty interior in H. In particular, it is a proper subset of H.

Furthermore, the set $\mathscr{C}_a$ is right $H\cap P$-invariant, hence has closed image in $H/H\cap P$. It follows that $\mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a)$ is closed in $\mathfrak{a}_{\mathrm{q}}$. This establishes (a).

By Proposition 6.7 the map $F_a : h \mapsto \mathfrak{H}_{P,\mathrm{q}}(ah)$ is submersive at the points of $H \setminus \mathscr{C}_a$. Hence $\mathfrak{H}_{P,\mathrm{q}}(a(H \setminus \mathscr{C}_a))$ is open in $\mathfrak{a}_\mathrm{q}$. It follows that

$$\mathfrak{H}_{P,\mathrm{q}}(aH) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a) = \mathfrak{H}_{P,\mathrm{q}}(a(H \setminus \mathscr{C}_a)) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a) \tag{40}$$

is open in $\mathfrak{a}_\mathrm{q}$ hence in $\mathfrak{a}_\mathrm{q} \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a)$. Finally, since $\mathfrak{H}_{P,\mathrm{q}}(aH)$ is closed, the first set in (40) is closed in $\mathfrak{a}_\mathrm{q} \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a)$. We conclude that the set (40) is both open and closed in $\mathfrak{a}_\mathrm{q} \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a)$. □

Lemma 6.9. *Assume that* $\Sigma(\mathfrak{g}, \mathfrak{a}_\mathrm{q})$ *spans* $\mathfrak{a}_\mathrm{q}^*$. *Then* $\mathfrak{H}_{P,\mathrm{q}}(aH) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a) \neq \emptyset$.

Proof. Under the assumption that $\Sigma(\mathfrak{g}, \mathfrak{a}_\mathrm{q})$ spans $\mathfrak{a}_\mathrm{q}^*$, the map $\mathfrak{H}_{P,\mathrm{q}} : aH \to \mathfrak{a}_\mathrm{q}$ is submersive except at points of $\mathscr{C}_a$. The set $H \setminus \mathscr{C}_a$ is open and non-empty. Thus, $\mathfrak{H}_{P,\mathrm{q}}(a(H \setminus \mathscr{C}_a))$ is open and non-empty. By Sard's Theorem, $\mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a)$ has measure zero. This implies that

$$\mathfrak{H}_{P,\mathrm{q}}(a(H \setminus \mathscr{C}_a)) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a) \neq \emptyset,$$

and hence

$$\mathfrak{H}_{P,\mathrm{q}}(aH) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a) \neq \emptyset.$$

□

Remark 6.10. The lemma can readily be extended to the case that $\Sigma(\mathfrak{g}, \mathfrak{a}_\mathrm{q})$ does not span $\mathfrak{a}_\mathrm{q}^*$.

7 The computation of Hessians

We retain the assumption that $P \in \mathscr{P}(A)$. Furthermore, we assume that $a \in A_\mathrm{q}^{\mathrm{reg}}$ and $X \in \mathfrak{a}_\mathrm{q}$. In this section we will compute the Hessian of the function $F_{a,X} : H \to \mathbb{R}$, defined in (28), at all points of its critical locus $\mathscr{C}_{a,X}$.

Given $U \in \mathfrak{h}$, we denote by R_U the associated left-invariant vectorfield on H defined by

$$R_U(h) = dl_h(e)U = \frac{\partial}{\partial t}(h \exp tU)|_{t=0}, \qquad (h \in H).$$

The associated derivation on $C^\infty(H)$ is denoted by the same symbol.

If $f : H \to \mathbb{R}$ is a C^2-function with critical point at h, then its Hessian at h is the symmetric bilinear form $H(f)(h) = H(f)_h$ on T_hH given by

$$\mathrm{H}(f)_h(R_U(h), R_V(h)) := R_U R_V f(h) = \partial_s \partial_t f(h \exp sU \exp tV)|_{s=t=0},$$

for $U, V \in \mathfrak{h}$.

Lemma 7.1. *Let* $a \in A_\mathrm{q}$, $X \in \mathfrak{a}_\mathrm{q}$ *and* $h \in H$. *Then for all* $U, V \in \mathfrak{h}$ *we have:*

$$R_U R_V F_{a,X}(h) = B(U, L_{a,X,h}(V)) = -\langle U, \theta L_{a,X,h}(V) \rangle,$$

where $L_{a,X,h} : \mathfrak{h} \to \mathfrak{h}$ is the linear map given by

$$L_{a,X,h}(V) = -\mathrm{Ad}(h^{-1}) \circ \pi_{\mathfrak{h}} \circ \mathrm{Ad}(a^{-1}) \circ \mathrm{Ad}(k_a(h)) \circ \mathrm{ad}(X) \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(\tau(ah))V. \qquad (41)$$

Here $\pi_{\mathfrak{h}} : \mathfrak{g} \to \mathfrak{h}$ denotes the projection according to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ and $E_{\mathfrak{k}} : \mathfrak{g} \to \mathfrak{k}$ is the projection associated with the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}_P$. The notation $k_a(h)$ is used to express the K-part of the element ah with respect to the Iwasawa decomposition $G = KAN_P$. Finally, $\tau(ah)$ denotes the (AN_P)-part of ah with respect to the same Iwasawa decomposition.

Proof. By [4, Lemma 5.1], see also [9], we obtain that for $x \in G$ and $U, V \in \mathfrak{g}$,

$$R_U R_V F_X(x) = B([\mathrm{Ad}(\tau)U, E_{\mathfrak{k}} \circ \mathrm{Ad}(\tau)V], X),$$

where F_X is the function defined in (29) and where $\tau := \tau(x)$. Therefore,

$$\begin{aligned} R_U R_V F_X(x) &= -B(\mathrm{Ad}(\tau)U, \mathrm{ad}X \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(\tau)V) \\ &= -B(U, \mathrm{Ad}(\tau)^{-1} \circ \mathrm{ad}X \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(\tau)V). \end{aligned}$$

We can restrict now to the case where $x = ah$ and $U, V \in \mathfrak{h}$. Since $F_{a,X}(h) = F_X(ah)$, we obtain

$$R_U R_V F_{a,X}(h) = R_U R_V F_X(ah) = B(U, -\pi_{\mathfrak{h}} \circ \mathrm{Ad}(\tau)^{-1} \circ \mathrm{ad}X \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(\tau)V). \qquad (42)$$

Since $ah = k_a(h)\tau(ah)$, it follows that $\tau^{-1} = \tau(ah)^{-1} = h^{-1}a^{-1}k_a(h)$ and by applying Ad to this equality we obtain

$$\mathrm{Ad}(\tau)^{-1} = \mathrm{Ad}(h^{-1})\mathrm{Ad}(a^{-1})\mathrm{Ad}(k_a(h)).$$

We complete the proof by substituting this equality in (42) and observing that $\pi_{\mathfrak{h}}$ commutes with $\mathrm{Ad}(h^{-1})$. $\square$

8 The transversal signature of the Hessian

In this section we fix $P \in \mathscr{P}(A)$, $a \in A_{\mathrm{q}}^{\mathrm{reg}}$ and $X \in \mathfrak{a}_{\mathrm{q}}$. We will study the behavior of the Hessian $H(F_{a,X})_h$ of the function $F_{a,X} : H \to \mathbb{R}$ defined in (28) at each point h of its critical set $\mathscr{C}_{a,X}$. This Hessian is a symmetric bilinear form on T_hH. Its kernel at h is by definition equal to the following linear subspace of T_hH,

$$\ker(H(F_{a,X})(h)) := \{V \in T_hH : \; H(F_{a,X})(h)(V, \cdot) = 0\}.$$

By symmetry, the Hessian induces a non-degenerate symmetric bilinear form $\bar{H}(F_{a,X})(h)$ on the quotient space $T_hH/\ker(H(F_{a,X})(h))$. For each $w \in W_{K\cap H}$ we select a representative $x_w \in N_{K\cap H}(\mathfrak{a}_{\mathrm{q}})$. The set

$$\mathscr{C}_{a,X,w} := x_w H_X (H \cap N_P)$$

is an injectively immersed submanifold of H, see Lemma 6.1. In particular this set has a well-defined tangent space at each of its points. We will show that the Hessian of $F_{a,X}$ is transversally non-degenerate along $\mathscr{C}_{a,X,w}$.

Lemma 8.1. *Let $w \in W_{K\cap H}$. Then at each point $\bar{h} \in \mathscr{C}_{a,X,w}$ the kernel of the Hessian $H(F_{a,X})(\bar{h})$ equals the tangent space $T_{\bar{h}}\mathscr{C}_{a,X,w}$.*

The proof of this lemma will make use of Lemma 8.2 below. In that lemma, $L_{a,X,h} \in \mathrm{End}(\mathfrak{h})$ is defined as in (41). Let $\bar{k}_a := \pi \circ k_a : H \to K/M$, where $k_a : H \to K$ is defined as in Lemma 7.1 and where π denotes the canonical projection $K \to K/M$.

Lemma 8.2. *Let $h \in H_X^\circ$ and $V \in \mathfrak{h}$. Then the following statements are equivalent.*

(a) $V \in \ker L_{a,X,h}$,

(b) $\mathrm{d}(l_{k_a(h)^{-1}} \circ \bar{k}_a \circ l_h)(e)(V) \in \mathfrak{k}_X/\mathfrak{m}$,

(c) $V \in \mathfrak{h}_X + (\mathfrak{h} \cap \mathfrak{n}_P)$.

Proof. First, we prove that (a) $\Longrightarrow$ (b). Assume (a) holds. In view of (41) this is equivalent to

$$\mathrm{Ad}(a^{-1}) \circ \mathrm{Ad}(k_a(h)) \circ \mathrm{ad}(X) \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(\tau(ah))V \in \mathfrak{q}. \tag{43}$$

Observe that $\mathrm{Ad}(k_a(h)) \circ \mathrm{ad}(X) \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(\tau(ah))V \in \mathfrak{p}$. In view of [4, Lemma 5.7] we see that (43) implies that

$$\mathrm{Ad}(k_a(h)) \circ \mathrm{ad}(X) \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(\tau(ah))V \in \mathfrak{a}_{\mathrm{q}}. \tag{44}$$

Since $h \in H_X$ and $G_X = K_X A N_{P,X}$, see (30), it follows that $k_a(h)$ centralizes X. Thus, $\mathrm{Ad}(k_a(h))$ and $\mathrm{ad}(X)$ commute. Now $\mathrm{Ad}(k_a(h)) \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(\tau(ah))V$ is an element in $\mathfrak{k}$, which decomposes as

$$\mathfrak{k} = \mathfrak{k}_X + \bigoplus_{\substack{\alpha \in \Sigma(P) \\ \alpha(X) \neq 0}} (I + \theta)\mathfrak{g}_\alpha.$$

Furthermore, by (44), we know that $\mathrm{ad}(X)$ maps this element to an element of $\mathfrak{a}_{\mathrm{q}}$. This implies that

$$\mathrm{Ad}(k_a(h)) \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(\tau(ah))V \in \mathfrak{k}_X.$$

Since $k_a(h) \in K_X$, we obtain that

$$E_{\mathfrak{k}} \circ \mathrm{Ad}(\tau(ah))V \in \mathfrak{k}_X. \tag{45}$$

By the use of [4, Lemma 5.2], we may rewrite

$$E_{\mathfrak{k}} \circ \mathrm{Ad}(\tau(ah)) = \mathrm{d}l_{k_a(h)}(e)^{-1} \circ \mathrm{d}k_a(h) \circ \mathrm{d}l_h(e) = \mathrm{d}(l_{k_a(h)^{-1}} \circ k_a \circ l_h)(e).$$

Hence, (45) implies

$$\mathrm{d}(l_{k_a(h)^{-1}} \circ k_a \circ l_h)(e)(V) \in \mathfrak{k}_X. \tag{46}$$

Observe that $\mathrm{d}\pi(e) : \mathfrak{k}_X \to \mathfrak{k}_X/\mathfrak{m}$ is given by the canonical projection and that the maps π and $l_{k_a(h)^{-1}}$ commute. Hence, equation (46) is equivalent to

$$\mathrm{d}(l_{k_a(h)^{-1}} \circ \bar{k}_a \circ l_h)(e)(V) \in \mathfrak{k}_X/\mathfrak{m} \tag{47}$$

and (b) follows.

Next, we prove that (b) $\Longrightarrow$ (c). Assume (b) and denote by φ the diffeomorphism $\varphi : K/M \to G/P$ arising from the Iwasawa decomposition $G = KAN_P$. The inclusion $H \hookrightarrow G$ induces the map $\psi : H \to G/P$. It is easy to check that the diagram given below commutes.

$$\begin{array}{ccc} H & \xrightarrow{\quad\psi\quad} & G/P \\ {\scriptstyle \bar{k}_a}\big\downarrow & & \big\downarrow{\scriptstyle l_a} \\ K/M & \xrightarrow{\quad\varphi\quad} & G/P \end{array} \tag{48}$$

The map ψ commutes with the left multiplication by an element $h \in H$, viewed either as the map $l_h : H \to H$ or as the map $l_h : G/P \to G/P$. On the other hand, the diffeomorphism φ introduced above, commutes with the left multiplication $l_k : K/M \to K/M$, where $k \in K$. Hence, the commutative diagram (48) gives rise to the following commutative diagram. We use the notation $k := k_a(h)$.

$$\begin{array}{ccc} H & \xrightarrow{\quad\psi\quad} & G/P \\ {\scriptstyle l_k^{-1}\circ\bar{k}_a\circ l_h}\big\downarrow & & \big\downarrow{\scriptstyle l_{k^{-1}ah}} \\ K/M & \xrightarrow{\quad\varphi\quad} & G/P \end{array} \tag{49}$$

Note that under each of the four maps in diagram (49), the origin of the domain is mapped to the origin of the codomain. Taking derivatives at the origins we obtain the commutative diagram given below.

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{\quad\psi_\star\quad} & \mathfrak{g}/\underline{\mathfrak{p}} \\ {\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{d}(l_{k^{-1}ah})(eP)} \\ \mathfrak{k}/\mathfrak{m} & \xrightarrow{\quad\varphi_\star\quad} & \mathfrak{g}/\underline{\mathfrak{p}} \end{array} \tag{50}$$

Here

$$\underline{\mathfrak{p}} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}_P$$

denotes the Lie algebra of P and T denotes the map $\mathrm{d}(l_k^{-1} \circ \bar{k}_a \circ l_h)(e) : \mathfrak{h} \to \mathfrak{k}/\mathfrak{m}$. Furthermore, $\varphi_* = d\varphi(eM)$ and $\psi_* = d\psi(e)$.

Observe that $k^{-1}ah = \tau := \tau(ah)$. Since h belongs to H_X, it follows that τ and τ^{-1} belong to $AN_{P,X} \subseteq P$. This in turn implies that $\mathrm{Ad}(\tau^{-1})$ is a bijection from $\mathfrak{g}_X$ to $\mathfrak{g}_X$ which normalizes $\underline{\mathfrak{p}}$. Let $\overline{\mathrm{Ad}(\tau)} : \mathfrak{g}/\underline{\mathfrak{p}} \to \mathfrak{g}/\underline{\mathfrak{p}}$ be the map induced by $\mathrm{Ad}(\tau) : \mathfrak{g} \to \mathfrak{g}$. Then

$$\mathrm{d}(l_{k^{-1}ah})(eP) = \overline{\mathrm{Ad}(\tau)}.$$

We use the commutativity of diagram (50) to compute the pre-image of $\mathfrak{k}_X/\mathfrak{m}$ under the map T :

$$\begin{aligned} T^{-1}(\mathfrak{k}_X/\mathfrak{m}) &= \psi_\star^{-1} \circ \overline{\mathrm{Ad}(\tau^{-1})} \circ \varphi_\star(\mathfrak{k}_X/\mathfrak{m}) \\ &= \psi_\star^{-1}(\overline{\mathrm{Ad}(\tau^{-1})}(\mathfrak{k}_X + \underline{\mathfrak{p}})) \\ &= \psi_\star^{-1}(\overline{\mathrm{Ad}(\tau^{-1})}(\mathfrak{g}_X + \underline{\mathfrak{p}})) \end{aligned}$$

$$
\begin{aligned}
&= \psi_\star^{-1}((\mathrm{Ad}(\tau^{-1})\mathfrak{g}_X) + \underline{\mathfrak{p}}) \\
&= \psi_\star^{-1}(\mathfrak{g}_X + \underline{\mathfrak{p}}) \\
&= \{U \in \mathfrak{h} : \ U + \underline{\mathfrak{p}} \in \mathfrak{g}_X + \underline{\mathfrak{p}}\} \\
&= \mathfrak{h}_X + (\mathfrak{h} \cap \underline{\mathfrak{p}}).
\end{aligned}
$$

Since $\mathfrak{h} \cap \underline{\mathfrak{p}} = (\mathfrak{m} \oplus \mathfrak{a}) \cap \mathfrak{h} \oplus (\mathfrak{n}_P \cap \mathfrak{h})$, see Subsection 2.4, and $(\mathfrak{m} \oplus \mathfrak{a}) \cap \mathfrak{h} \subseteq \mathfrak{h}_X$, we obtain that $\mathfrak{h}_X + (\mathfrak{h} \cap \underline{\mathfrak{p}}) = \mathfrak{h}_X + (\mathfrak{h} \cap \mathfrak{n}_P)$. Thus if (b) holds, then $T(V) \in \mathfrak{k}_X/\mathfrak{m}$ and we infer that $V \in \mathfrak{h}_X + (\mathfrak{h} \cap \underline{\mathfrak{p}})$ hence (c).

Finally, the implication (c) $\Longrightarrow$ (a) is easy. $\square$

Proof of Lemma 8.1. Recall that H is essentially connected. By [4, Prop. 2.3], the centralizer H_X is essentially connected as well (relative to G_X).

Assume first that $\bar{h} = h \in H_X^\circ$. Then, by Lemma 8.2 above, we have that

$$\ker L_{a,X,h} = \mathfrak{h}_X + (\mathfrak{n}_P \cap \mathfrak{h}).$$

Since $dl_h(e)$ is a linear isomorphism $\mathfrak{g} \to T_h G$, mapping $T_e[H_X(N_P \cap H)]$ onto $T_h[H_X(N_P \cap H)]$, we obtain that

$$\ker H(F_{a,X})(h) = dl_h(e)(\ker L_{a,X,h}) = T_h[H_X(N_P \cap H)],$$

which establishes the assertion for $\bar{h} = h \in H_X^\circ$.

Let now $\bar{h} = hn$, with $n \in N_P \cap H$. Then the right-multiplication $r_n : H \to H$ is a diffeomorphism and $F_{a,X} \circ r_n = F_{a,X}$, so that

$$\ker H(F_{a,X})(hn) = dr_n(h)[\ker H(F_{a,X})(h)] = dr_n(h) T_h[H_X(N_P \cap H)].$$

As the latter space equals $T_{hn}[H_X(N_P \cap H)]$ this proves the assertion for $\bar{h} \in H_X^\circ(N_P \cap H)$.

Finally, we discuss the general case $\bar{h} \in wH_X(N_P \cap H)$. Since H, respectively H_X, is essentially connected, see (6), we may write $\bar{h} = x_w hn$, where $h \in H_X^\circ$, $n \in N_P \cap H$ and x_w is a representative of w in $N_{K \cap H}(\mathfrak{a}_\mathrm{q})$ chosen accordingly. Since x_w normalizes A_q,

$$F_{a,X} \circ l_{x_w} = F_{w^{-1}a,X}.$$

Furthermore, from $a \in A_\mathrm{q}^\mathrm{reg}$ it follows that $w^{-1}a \in A_\mathrm{q}^\mathrm{reg}$. Since l_{x_w} is a diffeomorphism from H to itself, it follows that $dl_{x_w}(hn)$ is a linear isomorphism from $T_{hn}H$ onto $T_{\bar{h}}H$ and that

$$
\begin{aligned}
\ker H(F_{a,X})(\bar{h}) &= \ker H(F_{a,X})(x_w hn) \\
&= dl_{x_w}(hn)[\ker H(F_{x_w^{-1}a,X})(hn)] \\
&= dl_{x_w}(hn) T_{hn}[H_X(N_P \cap H)] \\
&= T_{\bar{h}}[x_w H_X(N_P \cap H)] \\
&= T_{\bar{h}} \mathscr{C}_{a,X,w}. \qquad (51)
\end{aligned}
$$

$\square$

We will now determine the set of critical points where the Hessian is transversally positive definite. For the description of our next result we define the following subsets of $\Sigma(P)$. If $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}) \cap \mathfrak{a}_{\mathrm{q}}^*$, then the associated root space $\mathfrak{g}_\alpha$ is $\sigma\theta$-invariant. Hence, for such a root α,

$$\mathfrak{g}_\alpha = \mathfrak{g}_{\alpha,+} \oplus \mathfrak{g}_{\alpha,-} \,,$$

where

$$\mathfrak{g}_{\alpha,\pm} = \{U \in \mathfrak{g}_\alpha : \; \sigma\theta U = \pm U\}.$$

Accordingly, we define

$$\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})_\pm := \{\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}}) : \; \mathfrak{g}_{\alpha,\pm} \neq 0\}.$$

In order to formulate the first main result of this section, we need to specify particular subsets of $\Sigma(P)$.

Definition 8.3.

(a) $\Sigma(P)_+ := \{\alpha \in \Sigma(P) : \; \alpha \in \mathfrak{a}_{\mathrm{q}}^* \implies \mathfrak{g}_{\alpha,+} \neq 0\}$.

(b) $\Sigma(P)_- := \{\alpha \in \Sigma(P, \sigma\theta) : \; \alpha \in \mathfrak{a}_{\mathrm{q}}^* \implies \mathfrak{g}_{\alpha,-} \neq 0\}$.

Note that (b) in this definition is consistent with (4).

Proposition 8.4. *Let $w \in W_{K\cap H}$. Then the Hessian $H(F_{a,X})(x_w)$ is positive definite transversally to $\mathscr{C}_{a,X,w}$ if and only if the following two conditions are fulfilled*

(a) $\forall \alpha \in \Sigma(P)_+ : \; \alpha(X)\alpha(w^{-1}(\log a)) \leq 0$*;*

(b) $\forall \alpha \in \Sigma(P)_- : \; \alpha(X) \geq 0$.

Remark 8.5. For the geometric meaning of these conditions we refer to Lemma 8.14, towards the end of this section.

Proof. We will prove the proposition in a number of steps. As a first step, let $l_w := l_{x_w}$ denote left multiplication by x_w on H. Then the tangent space of $\mathscr{C}_{a,X,w}$ at x_w is the image of $\mathfrak{h}_X + (\mathfrak{h} \cap \mathfrak{n}_P)$ under the tangent map $dl_w(e) : \mathfrak{h} \to T_{x_w}H$. We will denote by H_w the pull-back of the Hessian $H(F_{a,X})(x_w)$ under $dl_w(e)$. Then, in view of (51),

$$\ker H_w = \mathfrak{h}_X + (\mathfrak{n}_P \cap \mathfrak{h}) \tag{52}$$

and the following conditions are equivalent:

(a) the Hessian $H(F_{a,w})(x_w)$ is positive definite transversally to $\mathscr{C}_{a,X,w}$;

(b) the bilinear form H_w is positive definite transversally to $\mathfrak{h}_X + (\mathfrak{h} \cap \mathfrak{n}_P)$.

Accordingly, we will concentrate on deriving necessary and sufficient conditions for (b) to be valid.

Lemma 8.6. *The bilinear form* H_w *on* $\mathfrak{h}$ *is given by*

$$\mathrm{H}_w(U,V) = \langle U, L_w V\rangle, \qquad (U,V \in \mathfrak{h}),$$

where $L_w : \mathfrak{h} \to \mathfrak{h}$ *is the linear map given by*

$$L_w = -\pi_{\mathfrak{h}} \circ \mathrm{ad}(X) \circ \mathrm{Ad}(a^w) \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(a^w).$$

Proof. Let $U,V \in \mathfrak{h}$. Then in view of Lemma 7.1 we have

$$\mathrm{H}_w(U,V) = R_U R_V F_{a,X}(x_w) = B(U, L_{a,X,h}V) = -\langle U, \theta L_{a,X,h} V\rangle$$

with $h = x_w$ and $L_{a,X,h}$ defined as in Lemma 7.1. Now $ah = ax_w = x_w a^w$ and we see that $\tau = \tau(ah) = a^w$ and $k_a(h) = x_w$. Hence,

$$\begin{aligned}
-\theta \circ L_{a,X,h}(V) &= \theta \circ \mathrm{Ad}(x_w^{-1}) \circ \pi_{\mathfrak{h}} \circ \mathrm{Ad}(a^{-1}) \circ \mathrm{Ad}(x_w) \circ \mathrm{ad}(X) \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(a^w)V \\
&= \theta \circ \pi_{\mathfrak{h}} \circ \mathrm{Ad}(x_w^{-1}) \circ \mathrm{Ad}(a^{-1}) \circ \mathrm{Ad}(x_w) \circ \mathrm{ad}(X) \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(a^w)V \\
&= \theta \circ \pi_{\mathfrak{h}} \circ \mathrm{Ad}(a^w)^{-1} \circ \mathrm{ad}(X) \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(a^w)V \\
&= -\pi_{\mathfrak{h}} \circ \mathrm{Ad}(a^w) \circ \mathrm{ad}(X) \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(a^w)V.
\end{aligned}$$

The result now follows since $\mathrm{Ad}(a^w)$ and $\mathrm{ad}(X)$ commute. $\square$

In the sequel it will be useful to consider the finite subgroup

$$F = \{1, \sigma, \theta, \sigma\theta\} \subseteq \mathrm{Aut}(\mathfrak{g}).$$

The natural left action of F on $\mathfrak{g}$ leaves $\mathfrak{a}$ invariant, and induces natural left actions on $\mathfrak{a}^*$ and on $\Sigma(\mathfrak{g},\mathfrak{a})$. Accordingly, if $\tau \in F$ and $\alpha \in \Sigma(\mathfrak{g},\mathfrak{a})$, then

$$\tau(\mathfrak{g}_\alpha) = \mathfrak{g}_{\tau\alpha}$$

If $\mathscr{O}$ is an orbit for the F-action on $\Sigma(\mathfrak{g},\mathfrak{a})$, we write, in accordance with (35),

$$\mathfrak{g}_{\mathscr{O}} = \bigoplus_{\alpha\in\mathscr{O}} \mathfrak{g}_\alpha.$$

Then obviously,

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\mathscr{O}\in\Sigma(\mathfrak{g},\mathfrak{a})/F} \mathfrak{g}_{\mathscr{O}}, \tag{53}$$

with mutually orthogonal summands. Each of the summands is F-invariant, hence σ-invariant. In particular, if we write $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$ and $\mathfrak{h}_{\mathscr{O}} = \mathfrak{h} \cap \mathfrak{g}_{\mathscr{O}}$, then

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \bigoplus_{\mathscr{O}\in\Sigma(\mathfrak{g},\mathfrak{a})/F} \mathfrak{h}_{\mathscr{O}}, \tag{54}$$

with F-stable orthogonal summands.

Lemma 8.7.

(a) *The decomposition* (54) *is orthogonal for* $\langle \cdot , \cdot \rangle$.

(b) *The decomposition* (54) *is preserved by* L_w.

(c) *The decomposition* (54) *is orthogonal for* H_w.

Proof. The validity of (a) follows immediately from the fact that relative to the given inner product, the root spaces are mutually orthogonal, as well as orthogonal to $\mathfrak{g}_0$.

For (b) we note that the decomposition (53) is preserved by $\mathrm{Ad}(A)$, $\mathrm{ad}(\mathfrak{a})$, $E_{\mathfrak{k}}$ and $\pi_{\mathfrak{h}}$. Finally, in view of Lemma 8.6, the validity of (c) follows from (a) and (b). $\square$

It follows from the above lemma that the kernel of H_w decomposes in accordance with (54). Let $\mathfrak{v}_{P,X} := (\ker \mathrm{H}_w)^{\perp} \cap \mathfrak{h}$. Then in view of (52) we have

$$\mathfrak{v}_{P,X} = \mathfrak{h}_X^{\perp} \cap (\mathfrak{h} \cap \mathfrak{n}_P)^{\perp} \cap \mathfrak{h} = \bigoplus_{\mathscr{O} \in \Sigma(\mathfrak{g},\mathfrak{a})/F} \mathfrak{v}_{\mathscr{O}}, \tag{55}$$

with

$$\mathfrak{v}_{\mathscr{O}} := \mathfrak{h}_X^{\perp} \cap (\mathfrak{h} \cap \mathfrak{n}_P)^{\perp} \cap \mathfrak{h}_{\mathscr{O}} \tag{56}$$

From these definitions it follows that H_w is non-degenerate on each of the spaces $\mathfrak{v}_{\mathscr{O}}$. Moreover, H_w is positive definite if and only if the restriction of H_w to $\mathfrak{v}_{\mathscr{O}}$ is positive definite for every $\mathscr{O} \in \Sigma(\mathfrak{g}, \mathfrak{a})/F$. This in turn is equivalent to the condition that the symmetric map $L_w : \mathfrak{h} \to \mathfrak{h}$ has a positive definite restriction to each of the spaces $\mathfrak{v}_{\mathscr{O}}$ (if $\mathfrak{v}_{\mathscr{O}}$ is zero, we agree that the latter is automatic). We will now systematically discuss the types of orbits $\mathscr{O}$ for which $\mathfrak{v}_{\mathscr{O}}$ is non-trivial.

First of all, we note that $\alpha \in \mathscr{O} \implies -\alpha = \theta\alpha \in \mathscr{O}$. Therefore, we see that $\mathscr{O} \cap \Sigma(P) \neq \emptyset$ for all $\mathscr{O} \in \Sigma(\mathfrak{g}, \mathfrak{a})/F$. Let $\sim$ denote the equivalence relation on $\Sigma(P)$ defined by

$$\alpha \sim \beta \iff F\alpha = F\beta,$$

then the map $\alpha \mapsto F\alpha$ induces a bijection from $\Sigma(P)/\sim$ onto $\Sigma(\mathfrak{g}, \mathfrak{a})/F$. The following lemma summarizes all possibilities for the spaces $\mathfrak{v}_{\mathscr{O}}$, as $\mathscr{O} \in \Sigma(\mathfrak{g}, \mathfrak{a})/F$.

Lemma 8.8. *Let* $\alpha \in \Sigma(P)$, *and put* $\mathscr{O} = F\alpha$.

(a) *If* $\alpha(X) = 0$ *then* $\mathfrak{v}_{\mathscr{O}} = 0$.

(b) *If* $\alpha(X) \neq 0$ *then we are in one of the following two cases* (b.1) *and* (b.2).

(b.1) $\alpha \in \Sigma(P, \sigma)$; *in this case* $\mathfrak{v}_{\mathscr{O}} = \{V + \sigma(V) : V \in \mathfrak{g}_{-\alpha}\}$.

(b.2) $\alpha \in \Sigma(P, \sigma\theta)$; *in this case* $\mathfrak{v}_{\mathscr{O}} = \mathfrak{h}_{\mathscr{O}}$.

Proof. (a) If $\alpha(X) = 0$ then $\mathfrak{h}_{\mathscr{O}} \subseteq \mathfrak{g}_X$, so that $\mathfrak{v}_{\mathscr{O}} = \{0\}$.

(b) Assume that $\alpha(X) \neq 0$. Then it follows that $\alpha \notin \mathfrak{a}_{\mathrm{h}}^*$, so that $\alpha \neq \sigma\alpha$. By Lemma 2.2 we are in one of the cases (b.1) and (b.2).

We first discuss case (b.1). Then $\sigma\alpha \in \Sigma(P)$ so that $\sigma\alpha \neq -\alpha$ and $\mathscr{O} = F\alpha$ consists of the four distinct elements $\alpha, \theta\alpha = -\alpha, \sigma\alpha$ and $\sigma\theta\alpha = -\sigma\alpha$. We see that $\mathfrak{h}_{\mathscr{O}}$ consists of sums of elements of the form $U + \sigma(U)$ and $V + \sigma(V)$ with $U \in \mathfrak{g}_\alpha$ and $V \in \mathfrak{g}_{-\alpha}$. The elements $U + \sigma(U)$ belong to $\mathfrak{h} \cap \mathfrak{n}_P$, whereas the elements $V + \sigma(V)$ belong to $\mathfrak{h}_X^\perp \cap (\mathfrak{h} \cap \mathfrak{n}_P)^\perp$. In view of (56) this implies the assertion of (b.1).

Next, we discuss case (b.2). Then $\mathscr{O} \cap \Sigma(P) = \{\alpha, -\sigma\alpha\}$ so that $\mathfrak{n}_P \cap \mathfrak{h} = 0$. Since obviously $\mathfrak{h}_{\mathscr{O}} \perp \mathfrak{h}_X$, we infer the assertion of (b.2). □

We will now proceed by explicitly calculating the restrictions $L_w|_{\mathfrak{v}_{\mathscr{O}}}$ for all these cases. The following lemma will be instrumental in our calculations.

Lemma 8.9. *Let $T_w : \mathfrak{g} \to \mathfrak{g}$ be defined by*

$$T_w = \mathrm{ad}(X) \circ \mathrm{Ad}(a^w) \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(a^w).$$

Let $\beta \in \Sigma(\mathfrak{g}, \mathfrak{a})$ and $U_\beta \in \mathfrak{g}_\beta$.

(a) *If $\beta \in \Sigma(P)$ then $T_w(U_\beta) = 0$.*

(b) *If $\beta \in -\Sigma(P)$ then $T_w(U_\beta) = \beta(X)\,(a^{2w\beta}U_\beta - \theta U_\beta)$.*

Proof. Assume $\beta \in \Sigma(P)$. Then $\mathfrak{g}_\beta \subseteq \mathfrak{n}_P \subseteq \ker E_{\mathfrak{k}}$. Since $\mathrm{Ad}(a^w)$ preserves $\mathfrak{g}_\beta$, (a) follows.

For (b), assume that $\beta \in -\Sigma(P)$. Then U_β equals $U_\beta + \theta U_\beta$ modulo $\mathfrak{n}_P$, so that $E_{\mathfrak{k}}(U_\beta) = U_\beta + \theta U_\beta$. Hence,

$$\begin{aligned} T_w(U_\beta) &= \mathrm{ad}(X) \circ \mathrm{Ad}(a^w)[a^{w\beta}(U_\beta + \theta U_\beta)] \\ &= \mathrm{ad}(X)(a^{2w\beta}U_\beta + \theta U_\beta) \\ &= \beta(X)(a^{2w\beta}U_\beta - \theta U_\beta). \end{aligned}$$

□

In our calculations of $L_w|_{\mathfrak{v}_{\mathscr{O}}}$, we will distinguish between the cases described in Lemma 8.8. Case (a) is trivial.

Lemma 8.10 (Case b.1)**.** *Let $\mathscr{O} = F\alpha$ with $\alpha \in \Sigma(P, \sigma)$ and $\alpha(X) \neq 0$. Then*

$$L_w|_{\mathfrak{v}_{\mathscr{O}}} = \frac{\alpha(X)}{2}(a^{-2w\alpha} - a^{2w\alpha})I.$$

In particular, this restriction is positive definite if and only if $\alpha(X)\alpha(w^{-1}\log a) < 0$.

Proof. Let $V \in \mathfrak{g}_{-\alpha}$ and put $Z := V + \sigma(V)$. Since $-\alpha, -\sigma\alpha \in -\Sigma(P)$, it follows from Lemma 8.9 that

$$\begin{aligned} T_w(Z) &= -\alpha(X)(a^{-2w\alpha}V - \theta V) - \alpha(\sigma X)(a^{2w\alpha}\sigma V - \theta\sigma V) \\ &= \alpha(X)[-a^{-2w\alpha}V + a^{2w\alpha}\sigma V + \theta V - \theta\sigma V] \end{aligned}$$

so that

$$L_w(Z) = -\pi_{\mathfrak{h}} \circ T_w(Z) = \frac{\alpha(X)}{2}(a^{-2w\alpha} - a^{2w\alpha})Z.$$

It follows that L_w restricts to multiplication by a scalar on $\mathfrak{v}_{\mathscr{O}}$. The sign of this scalar equals the sign of $-\alpha(X)\alpha(w^{-1}\log a)$. The result follows. $\square$

We now turn to the calculation of $L_w|_{\mathfrak{v}_{\mathscr{O}}}$ in case (b.2), where $\mathscr{O} = F\alpha$, with $\alpha \in \Sigma(P, \sigma\theta)$ and $\alpha(X) \neq 0$. There are two possibilities between which we will distinguish:

(b.2.1) $\alpha \in \Sigma(P, \sigma\theta) \setminus \mathfrak{a}_{\mathrm{q}}^*$,

(b.2.2) $\alpha \in \Sigma(P) \cap \mathfrak{a}_{\mathrm{q}}^*$.

In each of these cases, $\mathfrak{v}_{\mathscr{O}} = \mathfrak{h}_{\mathscr{O}}$ by Lemma 8.10. We will use the notation

$$\mathfrak{v}(U) = \mathfrak{h}(U) = \mathfrak{h} \cap \mathrm{span}\,(F \cdot U),$$

for $U \in \mathfrak{g}_\alpha$. In case (b.2.1), the orbit $\mathscr{O} = F\alpha$ consists of the four distinct roots $\alpha, \sigma\alpha, \theta\alpha$ and $\sigma\theta\alpha$, and

$$\mathfrak{v}(U) = \mathbb{R}(U + \sigma(U)) \oplus \mathbb{R}(\sigma\theta(U) + \theta(U)).$$

In case (b.2.2), $\mathscr{O} = F\alpha = \{\alpha, -\alpha\}$, and we see that

$$\mathfrak{v}(U) = \mathbb{R}(U + \sigma(U)).$$

In all of these cases, we see that if $U_1, \ldots, U_m$ is an orthonormal basis of $\mathfrak{g}_\alpha$, then

$$\mathfrak{v}_{\mathscr{O}} = \bigoplus_{j=1}^{m} \mathfrak{v}(U_j), \qquad (57)$$

with mutually orthogonal summands.

Lemma 8.11 (Case (b.2.1))**.** *Let $\mathscr{O} = F\alpha$, with $\alpha \in \Sigma(P, \sigma\theta) \setminus \mathfrak{a}_{\mathrm{q}}^*$ and $\alpha(X) \neq 0$. Then $L_w|_{\mathfrak{v}_{\mathscr{O}}}$ is positive definite if and only if $\alpha(X) > 0$ and $\alpha(X)\alpha(w^{-1}\log a) < 0$.*

Proof. Fix an element $U \in \mathfrak{g}_\alpha$ and put $Z_1 = U + \sigma(U)$ and $Z_2 = \theta Z_1 = \theta U + \sigma\theta U$. Then $T_w(U) = 0$ by Lemma 8.9, hence

$$\begin{aligned} T_w(Z_1) &= T_w(\sigma U) \\ &= \sigma\alpha(X)(a^{2w\sigma\alpha}\sigma(U) - \theta\sigma(U)) \\ &= \alpha(X)(\theta\sigma(U) - a^{-2w\alpha}\sigma(U)), \end{aligned}$$

from which we see that

$$L_w(Z_1) = -\pi_{\mathfrak{h}} T_w(Z_1) = \frac{\alpha(X)}{2}(a^{-2w\alpha}Z_1 - Z_2).$$

Likewise,

$$L_w(Z_2) = \frac{\alpha(X)}{2}(a^{-2w\alpha}Z_2 - Z_1).$$

It follows that L_w preserves the subspace $\mathfrak{v}(U)$ of $\mathfrak{v}_{\mathscr{O}}$ spanned by the orthogonal vectors Z_1, Z_2 and that the restriction $L_w|_{\mathfrak{v}(U)}$ has the following matrix with respect to this basis:

$$\mathrm{mat}(L_w|_{\mathfrak{v}(U)}) = \frac{\alpha(X)}{2}\begin{pmatrix} a^{-2w\alpha} & -1 \\ -1 & a^{-2w\alpha} \end{pmatrix}$$

This matrix is positive definite if and only if both its trace and determinant are positive. This is equivalent to

$$\alpha(X) > 0 \quad \text{and} \quad \alpha(X)(a^{-4w\alpha} - 1) > 0.$$

It follows that L_w is positive definite on the subspace $\mathfrak{v}(U)$ if and only if the inequalities $\alpha(X) > 0$ and $\alpha(X)\alpha(w^{-1}\log a) < 0$ are valid.

Let $U_1, \dots, U_m$ be an orthonormal basis for $\mathfrak{g}_\alpha$. Then by (57) we see that map L_w is positive definite if and only if all restrictions $L_w|_{\mathfrak{v}(U_j)}$ are positive definite. This is true if and only if $\alpha(X) > 0$ and $\alpha(X)\alpha(w^{-1}\log a) < 0$. $\square$

Lemma 8.12 (Case (b.2.2)). *Let $\mathscr{O} = F\alpha$ with $\alpha \in \Sigma(P) \cap \mathfrak{a}_{\mathrm{q}}^*$ and $\alpha(X) \neq 0$. Then $L_w|_{\mathfrak{v}_{\mathscr{O}}}$ is positive definite if and only if the following two conditions are fulfilled.*

(a) $\alpha \in \Sigma(P)_+ \cap \mathfrak{a}_{\mathrm{q}}^* \implies \alpha(X)\alpha(w^{-1}\log a)) < 0.$

(b) $\alpha \in \Sigma(P)_- \cap \mathfrak{a}_{\mathrm{q}}^* \implies \alpha(X) > 0.$

Proof. We recall that $\mathfrak{v}_{\mathscr{O}} = \mathfrak{h}_{\mathscr{O}}$ in this case and write $\mathfrak{v}_{\mathscr{O},+} = \mathfrak{v}_{\mathscr{O}} \cap \mathfrak{k}$ and $\mathfrak{v}_{\mathscr{O},-} = \mathfrak{v}_{\mathscr{O}} \cap \mathfrak{p}$. Then

$$\mathfrak{v}_{\mathscr{O}} = \mathfrak{v}_{\mathscr{O},+} \oplus \mathfrak{v}_{\mathscr{O},-},$$

with orthogonal summands. We will show that L_w preserves this decomposition, and determine when both restrictions $L_w|_{\mathfrak{v}_{\mathscr{O},\pm}}$ are positive definite.

Let $U_\pm \in \mathfrak{g}_{\alpha,\pm}$ and put $Z_\pm = U_\pm + \sigma(U_\pm)$. Then $Z_\pm \in \mathfrak{v}_{\mathscr{O},\pm}$, and every element of $\mathfrak{v}_{\mathscr{O},\pm}$ can be expressed in this way.

By a straightforward computation, involving Lemma 8.9, we find

$$L_w(Z_\pm) = \frac{1}{2}\alpha(X)(a^{-2w\alpha} \mp 1)Z_\pm.$$

This shows that L_w acts by a real scalar $C_\pm$ on $\mathfrak{v}_{\mathscr{O},\pm}$. The restriction of L_w to $\mathfrak{v}_{\mathscr{O}_\pm}$ is positive definite if and only if the restrictions of L_w to both subspaces $\mathfrak{v}_{\mathscr{O},\pm}$ are positive definite. The latter condition is equivalent to

$$\mathfrak{v}_{\mathscr{O},+} \neq 0 \implies C_+ > 0 \quad \text{and} \quad \mathfrak{v}_{\mathscr{O},-} \neq 0 \implies C_- > 0.$$

The space $\mathfrak{v}_{\mathscr{O},\pm}$ is non-trivial if and only if $\mathfrak{g}_{\alpha,\pm} \neq 0$, which in turn is equivalent to $\alpha \in \Sigma(P)_\pm \cap \mathfrak{a}_{\mathrm{q}}^*$. On the other hand, the sign of C_+ equals that of $-\alpha(X)\alpha(w^{-1}\log a)$ whereas the sign of C_- equals that of $\alpha(X)$. From this the desired result follows. $\square$

Completion of the proof of Proposition 8.4. First assume that H_w is positive definite. Then L_w restricts to a positive definite symmetric map on each of the spaces $\mathfrak{v}_{\mathscr{O}}$ for $\mathscr{O} = F\alpha$, $\alpha \in \Sigma(P)$. First assume that $\alpha \in \Sigma(P)_+$. If $\alpha(X) = 0$, then

$$\alpha(X)\alpha(w^{-1}\log a) \leq 0 \tag{58}$$

holds. If $\alpha(X) \neq 0$, we are in one of the cases (b.1) or (b.2) of Lemma 8.8. In the latter case, we are either in the subcase (b.2.1) or in (b.2.2) with $\alpha \in \Sigma(P)_+ \cap \mathfrak{a}_{\mathrm{q}}^*$. In all of these cases, inequality (58) is valid. We conclude that assertion (a) of the proposition is valid.

For the validity of assertion (b), assume that $\alpha \in \Sigma(P)_-$. If $\alpha(X) = 0$, then

$$\alpha(X) \geq 0. \tag{59}$$

If $\alpha(X) \neq 0$, then we must be in case (b.2) of Lemma 8.8, since $\Sigma(P)_- \cap \Sigma(P,\sigma) = \emptyset$. We are either in subcase (b.2.1) or in subcase (b.2.2) with $\alpha \in \Sigma(P)_+ \cap \mathfrak{a}_{\mathrm{q}}^*$. In both subcases, (59) holds. This establishes condition (b) of the proposition, and the implication in one direction.

For the converse implication, assume that conditions (a) and (b) of the proposition hold. Let $\alpha \in \Sigma(P)$ and put $\mathscr{O} = F\alpha$. Then it suffices to show that H_w is positive definite on $\mathfrak{v}_{\mathscr{O}}$.

If $\alpha(X) = 0$, then $\mathfrak{v}_{\mathscr{O}} = 0$ by Lemma 8.8 and it follows that H_w is positive definite on $\mathfrak{v}_{\mathscr{O}}$. Thus, assume that $\alpha(X) \neq 0$. Then by regularity of $\log a$, the expression $\alpha(X)\alpha(w^{-1}\log a)$ is different from zero. Hence if any of the inequalities (58) or (59) holds, it holds as a strict inequality.

In case (b.1), $\alpha \in \Sigma(P,\sigma) \subseteq \Sigma(P)_+$ so that (58) is valid. Therefore, $\mathrm{H}_w|_{\mathfrak{v}_{\mathscr{O}}}$ is positive definite by Lemma 8.10. In case (b.2.1), $\alpha \in \Sigma(P,\sigma\theta) \setminus \mathfrak{a}_{\mathrm{q}}^* \subseteq \Sigma(P)_+ \cap \Sigma(P)_-$ so that (58) and (59) are both valid. Hence, $\mathrm{H}_w|_{\mathfrak{v}_{\mathscr{O}}}$ is positive definite by Lemma 8.11.

Finally, assume we are in case (b.2.2). Then $\alpha \in \mathfrak{a}_{\mathrm{q}}^*$, hence it follows from hypotheses (a) and (b) of the proposition that conditions (a) and (b) of Lemma 8.12 are fulfilled. Hence, $\mathrm{H}_w|_{\mathfrak{v}_{\mathscr{O}}}$ is positive definite. ☐

Corollary 8.13. *Let $w \in W_{K\cap H}$. Then the function $F_{a,X}$ as well as the signature and rank of its Hessian are constant on the immersed submanifold $wH_X(N_P \cap H)$.*

Proof. As the group H is essentially connected, $H_X = Z_{K\cap H}(\mathfrak{a}_{\mathrm{q}})H_X^\circ$. Let x_w be a representative of w in $N_{K\cap H}$. Since $Z_{K\cap H}(\mathfrak{a}_{\mathrm{q}})$ is normal in $N_{K\cap H}(\mathfrak{a}_{\mathrm{q}})$, it follows that

$$wH_X(N_P \cap H) = x_w Z_{K\cap H}(\mathfrak{a}_{\mathrm{q}})H_X^\circ(N_P \cap H) = Z_{K\cap H}(\mathfrak{a}_{\mathrm{q}})x_w H_X^\circ(N_P \cap H).$$

The function $F_{a,X} : H \to \mathbb{R}$ is left $Z_{K\cap H}(\mathfrak{a}_{\mathrm{q}})$- and right $(N_P \cap H)$-invariant. Hence, it suffices to prove the assertions for the set $x_w H_X^\circ$ of critical points. This set is connected, so that $F_{a,X}$ is constant on it. From Lemma 8.1 it follows that rank and signature of its Hessian remain constant along this set as well. ☐

As in (10) we define

$$\Omega := \mathrm{conv}(W_{K\cap H} \cdot \log a) + \Gamma(P).$$

Lemma 8.14. *Let $a \in A_{\mathrm{q}}^{\mathrm{reg}}$ and $X \in \mathfrak{a}_{\mathrm{q}}$. Assume that the function $F_{a,X}$ has a local minimum at the critical point $h \in \mathscr{C}_{a,X}$. Then for every $U \in \Omega$*

$$\langle X, U \rangle \geq \langle X, \mathfrak{H}_{P,\mathrm{q}}(ah) \rangle.$$

In particular, Ω lies on one side of the hyperplane $\mathfrak{H}_{P,\mathrm{q}}(ah) + X^{\perp}$.

Proof. The critical point h belongs to a connected immersed submanifold of the form $x_w H_X^{\circ}(H \cap N_P)$. All points of this submanifold are critical for $F_{a,X}$, so that $F_{a,X}$ is constant along it. We see that

$$F_{a,X}(h) = F_{a,X}(x_w) = \langle X, \mathfrak{H}_{P,\mathrm{q}}(x_w^{-1} a x_w) \rangle = \langle X, w^{-1} \log a \rangle.$$

The Hessian of $F_{a,X}$ at the critical point h must be positive semidefinite. It now follows from Proposition 8.4 that

(a) $\forall \alpha \in \Sigma(P)_+ : \ \alpha(X)\alpha(w^{-1}(\log a)) \leq 0;$

(b) $\forall \alpha \in \Sigma(P)_- : \ \alpha(X) \geq 0.$

By (a) and Lemma 8.17 below (applied to $-X$), it follows that

$$\langle X, U_1 \rangle \geq \langle X, w^{-1} \log a \rangle = F_{a,X}(h),$$

for all $U_1 \in \mathrm{conv}(W_{K\cap H} \cdot w^{-1} \log a)$. From (b) it follows that $\langle X, H_\alpha \rangle = \langle H_\alpha, H_\alpha \rangle \alpha(X)/2 \geq 0$ for all $\alpha \in \Sigma(P)_-$, so that

$$\langle X, U_2 \rangle \geq 0 \qquad (\forall\, U_2 \in \Gamma(P)).$$

Since every element $U \in \Omega$ may be decomposed as $U = U_1 + U_2$ with U_1 and U_2 as above, the assertion follows. ☐

Remark 8.15. It can be readily shown that the converse implication also holds. Indeed if for every $U \in \Omega$

$$\langle X, U \rangle \geq \langle X, w^{-1}(\log a)) \rangle,$$

then the two conditions of Proposition 8.4 hold.

Lemma 8.16. *The set $\Sigma(P)_+$ consists of all roots $\alpha \in \Sigma(P)$ with $\alpha \in \mathfrak{a}_{\mathrm{h}}^*$ or $\alpha|_{\mathfrak{a}_{\mathrm{q}}} \in \Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})_+$.*

Proof. In view of Definition 8.3 it suffices to show that for $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}) \setminus (\mathfrak{a}_{\mathrm{h}}^* \cup \mathfrak{a}_{\mathrm{q}}^*)$ we have $\alpha|_{\mathfrak{a}_{\mathrm{q}}} \in \Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})_+$. Assume $\alpha \notin \mathfrak{a}_{\mathrm{h}}^* \cup \mathfrak{a}_{\mathrm{q}}^*$. Then α and $\sigma\theta\alpha$ are distinct roots that restrict to the same root $\bar{\alpha}$ of $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$. Thus, the sum $\mathfrak{g}_\alpha + \sigma\theta\mathfrak{g}_\alpha$ is direct and contained in $\mathfrak{g}_{\bar{\alpha}}$ and we see that $\mathfrak{g}_{\bar{\alpha},+} \neq 0$. ☐

Lemma 8.17. *Let $P \in \mathscr{P}(A)$. Let $X, Y \in \mathfrak{a}_{\mathrm{q}}$ and assume that $\alpha(X)\alpha(Y) \geq 0$ for all $\alpha \in \Sigma(P)_+$. Then*

$$\langle X, U \rangle \leq \langle X, Y \rangle, \qquad \text{for all} \quad U \in \mathrm{conv}(W_{K\cap H} \cdot Y).$$

Proof. In view of Lemma 8.16, the hypothesis is equivalent to

$$\alpha(X)\,\alpha(Y) \geq 0$$

for all roots $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})_+$. We may now fix a Weyl chamber $\mathfrak{a}_{\mathrm{q}}^+$ for the root system $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})_+$ such that X and Y belong to the closure of $\mathfrak{a}_{\mathrm{q}}^+$. Then it is well known that $\langle X, wY\rangle \leq \langle X, Y\rangle$ for all w in the reflection group $W(\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})_+)$ generated by $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})_+$. Since this reflection group is equal to $W_{K\cap H}$, by Proposition 2.2 in [4], the result follows. □

9 Reduction by a limit argument

Before turning to the proof of our main theorem, Theorem 10.1, we will first prove a lemma that reduces the validity of the theorem to its validity under the additional assumption that the element a be regular in A_{q}. We assume that $P \in \mathscr{P}(A)$ and recall the definition of the closed convex polyhedral cone $\Gamma(P)$ given in Definition 1.4.

Lemma 9.1. *Assume that the assertion*

$$\mathrm{pr}_{\mathrm{q}} \circ \mathfrak{H}_P(aH) = \mathrm{conv}(W_{K\cap H} \cdot \log a) + \Gamma(P) \tag{60}$$

is valid for all $a \in A_{\mathrm{q}}^{\mathrm{reg}}$. Then assertion (60) holds for all $a \in A_{\mathrm{q}}$.

Proof. Assume the assertion is valid for all $a \in A_{\mathrm{q}}^{\mathrm{reg}}$, and let $a \in A_{\mathrm{q}}$ be an arbitrary fixed element. Fix a sequence $(a_j)_{j\geq 1}$ in $A_{\mathrm{q}}^{\mathrm{reg}}$ with limit a. We will establish the equality (60) for a.

First we will show that the set on the left-hand side of the equality is contained in the set on the right-hand side. For this, assume that $h \in H$. By the validity of (60) for a_j in place of a, there exist, for each $j \geq 1$, elements $\lambda_{w,j} \in [0,1]$ with $\sum_{w\in W_{K\cap H}} \lambda_{w,j} = 1$ and elements $\gamma_j \in \Gamma(P)$ such that

$$\mathfrak{H}_{P,\mathrm{q}}(a_j h) = \sum_{w\in W_{K\cap H}} \lambda_{w,j} w(\log a_j) + \gamma_j.$$

By passing to a subsequence of indices we may arrange that the sequence $(\lambda_{w,j})_j$ converges with limit $\lambda_w \in [0,1]$ for each $w \in W_{K\cap H}$. It follows that the sequence (γ_j) must have a limit $\gamma \in \mathfrak{a}_{\mathrm{q}}$ such that

$$\mathfrak{H}_{P,\mathrm{q}}(ah) = \lim_{j\to\infty} \mathfrak{H}_{P,\mathrm{q}}(a_j h) = \sum_{w\in W_{K\cap H}} \lambda_w w(\log a) + \gamma.$$

By taking the limit we see that $\sum_w \lambda_w = 1$ and since $\Gamma(P)$ is closed, $\gamma \in \Gamma(P)$. Hence, $\mathfrak{H}_{P,\mathrm{q}}(ah) \in \mathrm{conv}(W_{K\cap H} \cdot \log a) + \Gamma(P)$, and we obtain the desired first inclusion.

For the converse inclusion, assume that $Y \in \mathrm{conv}(W_{K\cap H} \cdot \log a) + \Gamma(P)$. Then there exist $\gamma \in \Gamma(P)$ and $\lambda_w \in [0,1]$ with $\sum_{w\in W_{K\cap H}} \lambda_w = 1$ such that

$$Y = \sum_{w\in W_{K\cap H}} \lambda_w w(\log a) + \gamma.$$

Put

$$Y_j = \sum_{w\in W_{K\cap H}} \lambda_w w(\log a_j) + \gamma.$$

Then for every j there exists $h_j \in H$ such that $\mathfrak{H}_{P,\mathfrak{q}}(a_j h_j) = Y_j$. The sequence (Y_j) is convergent, hence contained in a compact set of $\mathfrak{a}_\mathfrak{q}$. Likewise, the sequence (a_j) is contained in a compact subset $\mathscr{A} \subseteq A_\mathfrak{q}$. By Corollary 4.12 (b) there exists a compact subset $\mathscr{K}$ of $H/H\cap P$ such that $h_j(H\cap P) \in \mathscr{K}$ for all j. By passing to a subsequence we may arrange that $h_j(H\cap P)$ converges in $H/H\cap P$. By continuity of the induced map $\overline{\mathfrak{H}}_{P,\mathfrak{q}} : H/H\cap P \to \mathfrak{a}_\mathfrak{q}$, see (21), it now follows that

$$Y = \lim_{j\to\infty} Y_j = \lim_{j\to\infty} \mathfrak{H}_{P,\mathfrak{q}}(a_j h_j) = \mathfrak{H}_{P,\mathfrak{q}}(ah) \in \mathfrak{H}_{P,\mathfrak{q}}(aH).$$

□

10 Proof of the main theorem

In this section we will prove our main result. For $P \in \mathscr{P}(A)$ we recall the definition of the closed convex polyhedral cone $\Gamma(P)$ given in Definition 1.4.

Theorem 10.1. *Let P be a minimal parabolic subgroup of G containing A and let $a \in A_\mathfrak{q}$. Then*

$$\mathrm{pr}_\mathfrak{q} \circ \mathfrak{H}_P(aH) = \mathfrak{H}_{P,\mathfrak{q}}(aH) = \mathrm{conv}(W_{K\cap H} \cdot \log a) + \Gamma(P). \tag{61}$$

The proof of our main theorem proceeds by induction, for whose induction step the following lemma is a key ingredient.

If $X \in \mathfrak{a}_\mathfrak{q}$, we denote by G_X the centralizer of X in G. This group belongs to the Harish-Chandra class and is σ-stable. Moreover, by [4, Prop. 2.3], the centralizer $H_X := H \cap G_X$ is an essentially connected open subgroup of $(G_X)^\sigma$. From

$$P \cap G_X = (Z_K(\mathfrak{a})AN_P) \cap (K_X A N_{P,X}) = Z_K(\mathfrak{a})AN_{P,X},$$

see (30) for notation, we see that $P_X := P \cap G_X$ is a minimal parabolic subgroup of G_X.

We agree to write $\Gamma(P_X)$ for the cone in $\mathfrak{a}_\mathfrak{q}$ spanned by $\mathrm{pr}_\mathfrak{q} H_\alpha$, for $\alpha \in \Sigma(P)_-$ with $\alpha(X) = 0$. Furthermore, for a given $a \in A_\mathfrak{q}$, we define $\Omega_{a,X} = \Omega_X$ by

$$\Omega_X := \bigcup_{w \in W_{K\cap H}} \Omega_{X,w}, \qquad \text{where} \tag{62}$$

$$\Omega_{X,w} := \mathrm{conv}(W_{K\cap H_X} \cdot w^{-1} \log a) + \Gamma(P_X). \tag{63}$$

We write $\Omega := \Omega_0$ and note that this set equals $\mathrm{conv}(W_{K\cap H} \cdot \log a) + \Gamma(P)$ hence contains Ω_X for every $X \in \mathfrak{a}_\mathfrak{q}$.

Remark 10.2. It is clear from the definition that the set $\Omega_{X,w}$, for $w \in W_{K\cap H}$, is a closed convex polyhedral set, contained in the affine subset $w^{-1}\log a + \mathrm{span}\{H_\alpha : \alpha \in \Sigma(\mathfrak{g}_X, \mathfrak{a}_\mathfrak{q})\}$ of $\mathfrak{a}_\mathfrak{q}$. In particular,

$$\Omega_{X,w} \subseteq w^{-1} \log a + X^\perp.$$

Lemma 10.3. *Let $X \in S$, $a \in A_{\mathrm{q}}^{\mathrm{reg}}$ and let $\mathscr{C}_{a,X} \subseteq H$ be the set of critical points of the function $F_{a,X} : H \to \mathbb{R}$; cf. Lemma 5.5 and* (38). *If the analogue of the assertion of Theorem 10.1 holds for the data G_X, H_X, K_X and P_X in place of G, H, K and P then*

$$\mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_{a,X}) = \Omega_X. \tag{64}$$

Proof. Using the characterization of $\mathscr{C}_{a,X}$ given in Lemma 5.5, we obtain

$$\begin{aligned} \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_{a,X}) &= \bigcup_{w\in W_{K\cap H}} \mathfrak{H}_{P,\mathrm{q}}(awH_X(N_P\cap H)) \\ &= \bigcup_{w\in W_{K\cap H}} \mathfrak{H}_{P,\mathrm{q}}(a^w H_X), \end{aligned} \tag{65}$$

where $a^w = w^{-1}aw$ is regular in A_{q}, for each $w \in W_{K\cap H}$.

By the compatibility of the Iwasawa decompositions for the two groups G and G_X we see that the restriction of $\mathfrak{H}_{P,\mathrm{q}} : G \to \mathfrak{a}_{\mathrm{q}}$ to G_X equals the similar projection $G_X \to \mathfrak{a}_{\mathrm{q}}$ associated with P_X; we denote the latter by $\mathfrak{H}_{P_X,\mathrm{q}}$. Hence,

$$\mathfrak{H}_{P,\mathrm{q}}(a^w H_X) = \mathfrak{H}_{P_X,\mathrm{q}}(a^w H_X).$$

In view of the hypothesis that the convexity theorem holds for the data G_X, H_X, P_X, we infer that

$$\mathfrak{H}_{P,\mathrm{q}}(a^w H_X) = \mathrm{conv}(W_{K\cap H_X}\cdot \log a^w) + \Gamma(P_X) = \Omega_{X,w}.$$

In view of (65) and (62) we now obtain (64). ☐

Proof of Theorem 10.1. The proof relies on an inductive procedure, with induction over the rank of the root system $\Sigma(\mathfrak{g},\mathfrak{a}_{\mathrm{q}})$. The legitimacy of this procedure has been discussed at length in [4, Sect. 2].

We start the induction with $\mathrm{rk}\,\Sigma(\mathfrak{g},\mathfrak{a}_{\mathrm{q}}) = 0$. In this case,

$$\forall \alpha \in \Sigma(\mathfrak{g},\mathfrak{a}): \quad \alpha|_{\mathfrak{a}_{\mathrm{q}}} = 0. \tag{66}$$

This implies that $\mathfrak{a}_{\mathrm{q}}$ is central in $\mathfrak{g}$. As G is of the Harish-Chandra class, $\mathrm{Ad}(G) \subset \mathrm{Int}(\mathfrak{g}_{\mathbb{C}})$ so that G centralizes $\mathfrak{a}_{\mathrm{q}}$. Hence A_{q} is central in G. Furthermore, (66) also implies that every root $\alpha \in \Sigma(\mathfrak{g},\mathfrak{a})$ is fixed by σ, so that $\mathfrak{g}_\alpha$ is σ-invariant. This implies that the Iwasawa decomposition $G = KAN_P$ is σ-stable, so that $H = (H\cap K)(H\cap A)(H\cap N_P)$. We conclude that

$$\mathfrak{H}_{P,\mathrm{q}}(aH) = \mathfrak{H}_{P,\mathrm{q}}(Ha) = \mathfrak{H}_{P,\mathrm{q}}(H\cap A) + \log a = \log a. \tag{67}$$

On the other hand, it follows from (66) that $\Sigma(P)_- = \emptyset$, so that $\Gamma(P) = \{0\}$. Furthermore, since G centralizes $\mathfrak{a}_{\mathrm{q}}$, we see that $W_{K\cap H} = \{e\}$, so that

$$\mathrm{conv}(W_{K\cap H}\cdot \log a) + \Gamma(P) = \log a. \tag{68}$$

From (67) and (68) we see that the equality (61) holds in case $\mathrm{rk}\,\Sigma(\mathfrak{g},\mathfrak{a}_{\mathrm{q}}) = 0$.

Now assume that m is a positive integer, that $\mathrm{rk}\,\Sigma(\mathfrak{g},\mathfrak{a}_{\mathrm{q}}) = m$ and that the assertion of the theorem has already been established for the case that $\mathrm{rk}\,\Sigma(\mathfrak{g},\mathfrak{a}_{\mathrm{q}}) < m$.

By Lemma 9.1 it suffices to prove the validity of (61) under the assumption that $a \in A_{\mathrm{q}}^{\mathrm{reg}}$. We will first do so under the additional assumption that $\Sigma(\mathfrak{g},\mathfrak{a}_{\mathrm{q}})$ spans $\mathfrak{a}_{\mathrm{q}}^*$. In the end, the general case will be reduced to this.

Our assumption that $\Sigma(\mathfrak{g},\mathfrak{a}_{\mathrm{q}})$ is spanning guarantees that for each non-zero $X \in \mathfrak{a}_{\mathrm{q}}$ not all roots of $\Sigma(\mathfrak{g},\mathfrak{a}_{\mathrm{q}})$ vanish on X. Therefore, the rank of $\Sigma(\mathfrak{g}_X,\mathfrak{a}_{\mathrm{q}})$ is strictly smaller than $m = \mathrm{rk}\Sigma(\mathfrak{g},\mathfrak{a}_{\mathrm{q}})$. By the induction hypothesis, the convexity theorem holds for (G_X,H_X,K_X,P_X). Hence, by Lemma 10.3 we have that

$$\mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_{a,X}) = \Omega_X. \tag{69}$$

By Remark 10.2 the complement $\mathfrak{a}_{\mathrm{q}} \setminus \Omega_X$ is open and dense in $\mathfrak{a}_{\mathrm{q}}$.

Let $S_0 \subseteq S$ be a finite subset as in Lemma 6.6. Then it follows by application of Lemma 10.3 that

$$\mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a) = \cup_{X\in S_0}\Omega_X. \tag{70}$$

In particular, the complement of this set in $\mathfrak{a}_{\mathrm{q}}$ is dense. Moreover, it follows from (70) and the text below (63) that

$$\mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a) \subseteq \Omega = \mathrm{conv}(W_{K\cap H}\cdot \log a) + \Gamma(P). \tag{71}$$

From Lemma 6.8 we see that $\mathfrak{H}_{P,\mathrm{q}}(aH)$ and $\mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a)$ are closed subsets of $\mathfrak{a}_{\mathrm{q}}$ and that $\mathfrak{H}_{P,\mathrm{q}}(aH) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a)$ is an open and closed subset of the (open and dense) subset $\mathfrak{a}_{\mathrm{q}} \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a)$, hence a union of connected components of the latter set. Lemma 6.9 ensures that at least one connected component of $\mathfrak{a}_{\mathrm{q}} \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a)$ must belong to $\mathfrak{H}_{P,\mathrm{q}}(aH) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a)$.

From (71) it follows that

$$\mathfrak{a}_{\mathrm{q}} \setminus \Omega \subseteq \mathfrak{a}_{\mathrm{q}} \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a).$$

Now $\mathfrak{a}_{\mathrm{q}} \setminus \Omega$ is connected hence must be contained in a connected component Λ of $\mathfrak{a}_{\mathrm{q}} \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a)$.

There are two possibilities:

(a) $\Lambda \subseteq \mathfrak{H}_{P,\mathrm{q}}(aH) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a)$;

(b) $\Lambda \cap (\mathfrak{H}_{P,\mathrm{q}}(aH) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a)) = \emptyset$.

From its definition, one sees that Ω is strictly contained in a half-space, which implies that $\mathfrak{a}_{\mathrm{q}} \setminus \Omega$, and therefore Λ, must contain a line of $\mathfrak{a}_{\mathrm{q}}$. From Corollary 4.15 we know that $\mathfrak{H}_{P,\mathrm{q}}(aH)$ does not contain such a line, so that we may exclude case (a) above. From (b) it follows that

$$(\mathfrak{a}_{\mathrm{q}} \setminus \Omega) \cap \mathfrak{H}_{P,\mathrm{q}}(aH) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a) = \emptyset,$$

which implies that $\mathfrak{H}_{P,\mathrm{q}}(aH) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a) \subseteq \Omega$. Combining this with (71) we conclude that

$$\mathfrak{H}_{P,\mathrm{q}}(aH) \subseteq \Omega. \tag{72}$$

We now turn to the proof of the converse inclusion.

In the above we concluded that the set $\mathfrak{H}_{P,\mathrm{q}}(aH) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a)$ is open and closed as a subset of $\mathfrak{a}_{\mathrm{q}} \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a)$. In view of (72) the set is also open and closed as a subset of $\Omega \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a)$. Thus,

$\mathfrak{H}_{P,\mathrm{q}}(aH) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a)$ is a union of connected components of $\Omega \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a)$. We will establish the converse of (72) by showing that all connected components of $\Omega \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a)$ are contained in $\mathfrak{H}_{P,\mathrm{q}}(aH)$.

Again by the use of Lemma 6.9 we infer that at least one connected component Λ_1 of $\Omega \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a)$ is contained in $\mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a)$. Arguing by contradiction, assume this were not the case for all components. Then there exists a second connected component Λ_2 of $\Omega \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a) = \Omega \setminus \cup_{X\in S_0}\Omega_X$ such that

$$\Lambda_2 \cap \mathfrak{H}_{P,\mathrm{q}}(aH) = \emptyset. \tag{73}$$

In view of Remark 10.2, we may apply Lemma 10.4 below to the set Ω and the finite collection of subsets $\Omega_{X,w}$, where $X \in S_0$ and $w \in W_{K\cap H}$, and obtain a line segment with the properties of Lemma 10.4, connecting Λ_1 and Λ_2. By following intersections along this line segment, we see that we may assume that the connected components Λ_1 and Λ_2 exist with the additional property that they are adjacent, i.e., there exists a codimension 1 subset $\Omega_{X,w} \subseteq \Omega$ together with a point $Y \in \Omega_{X,w}$ and a positive number $\varepsilon > 0$ such that $B(Y;\varepsilon) \setminus \Omega_{X,w}$ consists of two connected components Λ_1' and Λ_2' such that $\Lambda_j' \subseteq \Lambda_j$ for $j = 1,2$. In particular, this implies that Λ_1' and Λ_2' are on different sides of the hyperplane $\mathrm{aff}(\Omega_{X,w}) = Y + X^\perp$. We may replace X by $-X$ if necessary, to arrange that $Y + tX \in \Lambda_1$ for $t \downarrow 0$. Then

$$\langle X, \cdot\rangle \geq \langle X, Y\rangle \quad \text{on} \quad \mathrm{cl}\,(\Lambda_1'). \tag{74}$$

By (69) there exists a point $h \in \mathscr{C}_{a,X}$ such that $\mathfrak{H}_{P,\mathrm{q}}(ah) = Y$. For a sufficiently small neighborhood U of h in H we have $\mathfrak{H}_{P,\mathrm{q}}(aU) \subseteq B(Y;\varepsilon)$. Combined with (73) this implies $\mathfrak{H}_{P,\mathrm{q}}(aU) \subseteq \mathrm{cl}(\Lambda_1')$. In view of (74) we now infer that $F_{a,X} \geq \langle X, Y\rangle = F_{a,X}(h)$ on U. Hence, $F_{a,X}$ has a local minimum at h. By what we established in Lemma 8.14 this implies that Ω should be on one side of the hyperplane $Y + X^\perp$, contradicting the observation that Λ_1' and Λ_2' are non-empty open subsets on different sides of this hyperplane, but both contained in Ω.

In view of this contradiction we conclude that all components of $\Omega \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathscr{C}_a)$ are contained in $\mathfrak{H}_{P,\mathrm{q}}(aH)$.

This finishes the proof in case $\Sigma(\mathfrak{g},\mathfrak{a}_\mathrm{q})$ has rank m and spans $\mathfrak{a}_\mathrm{q}^*$. We finally consider the case with $\mathrm{rk}\,\Sigma(\mathfrak{g},\mathfrak{a}_\mathrm{q}) = m$ in general.

Let $\mathfrak{c}$ be the intersection of the root hyperplanes $\ker\alpha \subseteq \mathfrak{a}_\mathrm{q}$ for $\alpha \in \Sigma(\mathfrak{g},\mathfrak{a}_\mathrm{q})$. Then $\mathfrak{c}$ is contained in $\mathfrak{a}_\mathrm{q}$ and central in $\mathfrak{g}$. Since G is of the Harish-Chandra class, $\mathrm{Ad}(G)$ is contained in $\mathrm{Int}\,(\mathfrak{g}_\mathbb{C})$, hence centralizes $\mathfrak{c}$. Therefore, the subgroup $C := \exp(\mathfrak{c})$ is central in G.

Let ${}^\backprime\mathfrak{p}$ be the orthocomplement of $\mathfrak{c}$ in $\mathfrak{p}$. Then ${}^\backprime\mathfrak{g} = \mathfrak{k} \oplus {}^\backprime\mathfrak{p}$ is an ideal of $\mathfrak{g}$ which is complementary to $\mathfrak{c}$.

By the Cartan decomposition and the fact that $\mathfrak{c}$ is central, it follows that the map $K \times {}^\backprime\mathfrak{p} \times \mathfrak{c} \to G$, $(k,X,Z) \mapsto k\exp X \exp Z$ is a diffeomorphism onto. It readily follows that ${}^\backprime G = K\exp{}^\backprime\mathfrak{p}$ is a group of the Harish-Chandra class, with the indicated Cartan decomposition for the Cartan involution ${}^\backprime\theta = \theta|_{{}^\backprime G}$. The restricted map ${}^\backprime\sigma := \sigma|_{{}^\backprime G}$ is an involution of ${}^\backprime G$ which commutes with ${}^\backprime\theta$. The group ${}^\backprime H := H$ is an open subgroup of $({}^\backprime G)^{{}^\backprime\sigma}$, which is essentially connected. Furthermore, ${}^\backprime\mathfrak{a}_\mathrm{q} := {}^\backprime\mathfrak{p} \cap \mathfrak{a}_\mathrm{q}$ is maximal abelian in ${}^\backprime\mathfrak{p} \cap \mathfrak{q}$ and ${}^\backprime\mathfrak{a} = {}^\backprime\mathfrak{p} \cap \mathfrak{a}$ is maximal abelian in ${}^\backprime\mathfrak{p}$. The root system $\Sigma({}^\backprime\mathfrak{g},{}^\backprime\mathfrak{a}_\mathrm{q})$ consists of the restrictions of the roots from $\Sigma(\mathfrak{g},\mathfrak{a}_\mathrm{q})$, hence spans the dual of ${}^\backprime\mathfrak{a}_\mathrm{q}$.

The group ${}^{\backprime}P = {}^{\backprime}G \cap P$ is a minimal parabolic subgroup of ${}^{\backprime}G$ containing ${}^{\backprime}A$. We note that ${}^{\backprime}P = M{}^{\backprime}AN_P$.

We note that $A_{\mathrm{q}}^{\mathrm{reg}} \simeq {}^{\backprime}A_{\mathrm{q}}^{\mathrm{reg}} \times C$. Let $a \in A_{\mathrm{q}}^{\mathrm{reg}}$. Then we may write $a = {}^{\backprime}a \cdot c$, with ${}^{\backprime}a \in {}^{\backprime}A_{\mathrm{q}}^{\mathrm{reg}}$ and $c \in C$. By the convexity theorem for ${}^{\backprime}G$ and since c is central in G, it now follows that

$$\begin{aligned}
\mathfrak{H}_{P,\mathrm{q}}(aH) &= \mathfrak{H}_{P,\mathrm{q}}({}^{\backprime}aHc) \\
&= \mathfrak{H}_{{}^{\backprime}P,\mathrm{q}}({}^{\backprime}aH) + \log c \\
&= \mathrm{conv}(W_{K\cap H} \cdot \log{}^{\backprime}a) + \Gamma({}^{\backprime}P) + \log c \\
&= \mathrm{conv}(W_{K\cap H} \cdot \log a) + \Gamma(P).
\end{aligned}$$

□

We recall that the relative interior of a convex subset S of a finite dimensional real linear space is defined to be the interior of S in its affine span $\mathrm{aff}(S)$.

Lemma 10.4. *Let V be a finite dimensional real linear space and $C \subseteq V$ a closed convex polyhedral subset with non-empty interior. Let C_i ($i \in \{1, \dots, n\}$) be closed convex polyhedral subsets of C, of positive codimension. Then the following statements are true.*

(a) *The complement $C' := C \setminus \cup_{i=1}^{n} C_i$ is dense in C.*

(b) *Let A and B be open subsets of V contained in C'. Then for each $a \in A$ there exists $b \in B$ such that for each i with $C_i \cap [a,b] \neq \emptyset$ the following assertions are valid,*

  (1) $\mathrm{codim}(C_i) = 1$;

  (2) *$[a,b] \cap C_i$ consists of a single point p which belongs to the relative interior of C_i. Furthermore, if $p \in C_j$ for some $1 \leq j \leq n$, then* $\mathrm{aff}(C_j) = \mathrm{aff}(C_i)$.

Proof. Standard, and left to the reader. □

A Proof of Lemma 2.11

Finally, we prove Lemma 2.11.

We begin by showing that the result holds for G a complex semi-simple Lie group, connected with trivial center. That proof will be based on the following general lemma, inspired by [23, Prop. 1].

Let $\mathfrak{h}$ be a complex abelian Lie algebra and let $\mathscr{N}$ be the class of complex finite dimensional nilpotent Lie algebras $\mathfrak{n}$, equipped with a representation of $\mathfrak{h}$ by derivations, such that the following conditions are fulfilled

(a) the representation of $\mathfrak{h}$ in $\mathfrak{n}$ is semi-simple;

(b) all weight spaces of $\mathfrak{h}$ in $\mathfrak{n}$ have complex dimension one.

If $\mathfrak{n}$ belongs to the class $\mathscr{N}$, we write $\Lambda(\mathfrak{n})$ for the set of $\mathfrak{h}$-weights in $\mathfrak{n}$. If $\lambda \in \Lambda(\mathfrak{n})$, then the associated weight space is denoted by $\mathfrak{n}_\lambda$.

Lemma A.1. *Let $\mathfrak{n} \in \mathscr{N}$ and let N be the connected, simply-connected Lie group with Lie algebra $\mathfrak{n}$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct weights of $\mathfrak{h}$ in $\mathfrak{n}$. Then the map*

$$\psi : (X_1, \ldots, X_m) \mapsto \exp X_1 \cdot \cdots \cdot \exp X_m$$

defines a diffeomorphism

$$\mathfrak{n}_{\lambda_1} \times \ldots \times \mathfrak{n}_{\lambda_m} \xrightarrow{\simeq} N.$$

Proof. We will use induction on $\dim_{\mathbb{C}}(\mathfrak{n})$. If $\dim_{\mathbb{C}} \mathfrak{n} = 1$ then $\mathfrak{n}$ is abelian and the result holds trivially.

Next, assume that $m > 1$ and assume that the result has been established for $\mathfrak{n}$ with $\dim_{\mathbb{C}} \mathfrak{n} < m$. Assume that $\mathfrak{n} \in \mathscr{N}$ has dimension m.

Denote by $\mathfrak{n}_1$ the center of $\mathfrak{n}$, which is non-trivial. If $\mathfrak{n}_1 = \mathfrak{n}$ then $\mathfrak{n}$ is abelian and the result is trivially true. Thus, we may as well assume that $0 \subsetneq \mathfrak{n}_1 \subsetneq \mathfrak{n}$. In particular, this implies that both $\mathfrak{n}_1$ and $\mathfrak{n}/\mathfrak{n}_1$ have dimensions at most $m-1$. Put $l := \dim \mathfrak{n}_1$.

The ideal $\mathfrak{n}_1$ is stable under the action of $\mathfrak{h}$ and it is readily verified that $\mathfrak{n}_1$ and $\mathfrak{n}/\mathfrak{n}_2$ with the natural $\mathfrak{h}$-representations belong to $\mathscr{N}$. Furthermore, since all weight spaces are 1-dimensional, we see that

$$\Lambda(\mathfrak{n}) = \Lambda(\mathfrak{n}_1) \sqcup \Lambda(\mathfrak{n}/\mathfrak{n}_1).$$

We will first prove that ψ is a diffeomorphism under the assumption that the $\mathfrak{h}$-weights in $\mathfrak{n}$ are numbered in such a way that

$$\Lambda(\mathfrak{n}_1) = \{\lambda_1, \ldots, \lambda_l\} \quad \text{and } \Lambda(\mathfrak{n}/\mathfrak{n}_1) = \{\lambda_{l+1}, \ldots, \lambda_m\}.$$

Since N is simply-connected, the map $\exp : \mathfrak{n} \to N$ is a diffeomorphism; hence, $N_1 := \exp(\mathfrak{n}_1)$ is the connected subgroup of N with Lie algebra $\mathfrak{n}_1$. In particular, N_1 is simply connected as well. Since $\mathfrak{n}_1$ is an ideal, N/N_1 has a unique structure of Lie group for which the natural map $N \to N/N_1$ is a Lie group homomorphism. We now observe that $N \to N/N_1$ is a principal fiber bundle with fiber N_1. By standard homotopy theory we have a natural exact sequence

$$\pi_1(N) \to \pi_1(N/N_1) \to \pi_0(N_1).$$

Since N is simply-connected, and N_1 connected, we conclude that N/N_1 is the simply connected group with Lie algebra $\mathfrak{n}/\mathfrak{n}_1$.

By the induction hypothesis, the maps

$$\begin{aligned} \psi_{\mathfrak{n}_1} \quad &: \mathfrak{n}_{\lambda_1} \times \ldots \times \mathfrak{n}_{\lambda_l} \to N_1 \\ \psi_{\mathfrak{n}/\mathfrak{n}_1} \quad &: (\mathfrak{n}/\mathfrak{n}_1)_{\lambda_{l+1}} \times \ldots \times (\mathfrak{n}/\mathfrak{n}_1)_{\lambda_m} \to N/N_1 \end{aligned}$$

are diffeomorphisms. For every $j \in \{l+1, \ldots, m\}$ the canonical projection $\mathfrak{n} \to \mathfrak{n}/\mathfrak{n}_1$ induces the isomorphisms of weight spaces $\mathfrak{n}_{\lambda_j} \to (\mathfrak{n}/\mathfrak{n}_1)_{\lambda_j}$. Let $\bar{\psi} : \mathfrak{n}_{\lambda_{l+1}} \times \ldots \times \mathfrak{n}_{\lambda_m} \to N/N_1$ be defined by $\bar{\psi}(X_{l+1}, \ldots, X_m) = \exp X_{l+1} \cdot \ldots \cdot \exp X_m \cdot N_1$. Then the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{n}_{\lambda_{l+1}} \times \ldots \times \mathfrak{n}_{\lambda_m} & \xrightarrow{\quad \bar{\psi} \quad} & N/N_1 \\ \simeq \downarrow & & \| \\ (\mathfrak{n}/\mathfrak{n}_1)_{\lambda_{l+1}} \times \ldots \times (\mathfrak{n}/\mathfrak{n}_1)_{\lambda_m} & \xrightarrow{\quad \psi_{\mathfrak{n}/\mathfrak{n}_1} \quad} & N/N_1 \end{array}$$

From this we infer that $\bar{\psi}$ is a diffeomorphism. We now obtain that the map $\tilde{\psi} : \mathfrak{n}_{\lambda_{l+1}} \times \ldots \times \mathfrak{n}_{\lambda_m} \times N_1 \to N$,

$$(X_{l+1}, \ldots, X_m, n_1) \mapsto (\exp X_{l+1} \cdot \ldots \cdot \exp X_m) n_1,$$

is a diffeomorphism onto N. Since

$$\psi(X_1, \ldots, X_l, X_{l+1}, \ldots, X_m) = \tilde{\psi}(X_{l+1}, \ldots, X_m, \psi_{\mathfrak{n}_1}(X_1, \ldots, X_l))$$

it follows that ψ is a diffeomorphism as well. Clearly, the above proof works for every enumeration of the weights in $\Lambda(\mathfrak{n}/\mathfrak{n}_1)$. Since the weight spaces $(\mathfrak{n}_1)_\lambda$ for $\lambda \in \Lambda(\mathfrak{n}_1)$ are all central in $\mathfrak{n}$, we conclude that the result holds for any enumeration of the weights in $\Lambda(\mathfrak{n})$. □

Corollary A.2. *Let G be a connected complex semi-simple Lie group and $\mathfrak{n}_B$ the nilpotent radical of a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra contained in $\mathfrak{b}$. Let $\mathfrak{n}_1, \ldots, \mathfrak{n}_k$ be linearly independent subalgebras of $\mathfrak{n}_B$, each of which is a direct sum of $\mathfrak{h}$-root spaces, and assume that their direct sum $\mathfrak{n} := \mathfrak{n}_1 \oplus \ldots \oplus \mathfrak{n}_k$ is again a subalgebra. Put $N := \exp \mathfrak{n}$ and $N_j := \exp(\mathfrak{n}_j)$, for $1 \leq j \leq k$.*

Then the multiplication map

$$\mu : N_1 \times \ldots \times N_k \to N$$

is a diffeomorphism.

Proof. This is an immediate consequence of Lemma A.1. □

Proof of Lemma 2.11. We assume that G is a real reductive Lie group of the Harish-Chandra class. Define

$$\mathfrak{g}_1 := [\mathfrak{g}, \mathfrak{g}],$$

the semi-simple part of the Lie algebra of G. Let G_1 be the analytic subgroup of G with Lie algebra $\mathfrak{g}_1$. Since the nilpotent radical N_P of P is completely contained in G_1, we may assume from the start that $G = G_1$, i.e. G is connected semi-simple with finite center.

Since Ad is a finite covering homomorphism from G onto $\mathrm{Aut}(\mathfrak{g})^\circ$, mapping N diffeomorphically onto $\mathrm{Ad}(N)$, whereas $\mathrm{Aut}(\mathfrak{g})^\circ$ is a connected real form of $\mathrm{Int}(\mathfrak{g}_\mathbb{C})$, we may assume that G is a connected real form of a connected complex semi-simple Lie group $G_\mathbb{C}$ with trivial center. Let τ be the conjugation on $G_\mathbb{C}$, such that

$$G = (G_\mathbb{C}^\tau)^\circ.$$

Let $\mathfrak{g}_\mathbb{C}$ denote the Lie algebra of $G_\mathbb{C}$, then $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}$. Note that the complexification $\mathfrak{n}_{P\mathbb{C}}$ of $\mathfrak{n}_P$ equals $\mathfrak{n}_P \oplus i\mathfrak{n}_P$ and that

$$N_P = (N_{P\mathbb{C}})^\tau.$$

Take a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$, containing $\mathfrak{a}_\mathbb{C} = \mathfrak{a} \oplus i\mathfrak{a}$. It is of the form

$$\mathfrak{h}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \mathfrak{a}_\mathbb{C},$$

where $\mathfrak{t}$ is a maximal abelian subspace of $\mathfrak{m} := Z_{\mathfrak{k}}(\mathfrak{a})$. Since $\mathfrak{t}$ centralizes $\mathfrak{a}$, all $\mathfrak{a}$-root spaces are invariant under $\mathrm{ad}(\mathfrak{t})$. This implies that the subalgebras $\mathfrak{n}_{j\mathbb{C}} := \mathfrak{n}_j \oplus i\mathfrak{n}_j$ ($j \in \{1,\dots,k\}$) of $\mathfrak{n}_{P\mathbb{C}}$ are direct sums of $\mathfrak{h}_{\mathbb{C}}$-root spaces. Furthermore, their direct sum equals $\mathfrak{n}_{\mathbb{C}} = \mathfrak{n} \oplus i\mathfrak{n}$, hence is a subalgebra. Finally, there exists a Borel subalgebra containing $\mathfrak{h}_{\mathbb{C}} + \mathfrak{n}_{\mathbb{C}}$. By Corollary A.2, the multiplication map

$$\mu_{\mathbb{C}} : N_{1\mathbb{C}} \times \ldots \times N_{k\mathbb{C}} \to N_{\mathbb{C}}$$

is a diffeomorphism. It readily follows that $\mu_{\mathbb{C}}$ restricts to a bijection from $(N_{1\mathbb{C}})^{\tau} \times \cdots \times (N_{k\mathbb{C}})^{\tau}$ onto $(N_{\mathbb{C}})^{\tau}$. Since

$$(N_{\mathbb{C}})^{\tau} = N \quad \text{and} \quad (N_{j\mathbb{C}})^{\tau} = N_j \quad \text{for all } 1 \leq j \leq k,$$

it follows that μ is a bijective embedding from $N_1 \times \cdots \times N_k$ onto N, hence a diffeomorphism.

B The case of the group

Every semisimple Lie group G can be viewed as a semi-simple symmetric space for the group $G \times G$. In this section we investigate what our convexity theorem means for this particular example. An independent proof for this case is presented in [3, Section 3.2.2].

More generally, let G be real reductive group of the Harish-Chandra class, θ a Cartan involution, $K := G^{\theta}$ the associated maximal compact subgroup and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the associated Cartan decomposition as in Section 1. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, $A = \exp \mathfrak{a}$ and $\Sigma(\mathfrak{g}, \mathfrak{a})$ the associated root system.

Let

$$G' := G \times G.$$

Then $\theta' := \theta \times \theta$ is a Cartan decomposition of G' with associated maximal compact subgroup $K' := K \times K$. The involution

$$\sigma' : G' \to G', \;\; (x,y) \mapsto (y,x),$$

commutes with θ'. Its fixed point group H' equals the diagonal in $G \times G$ and is essentially connected in G', see [3, Example 2.3.7].

The associated space $\mathfrak{p}' \cap \mathfrak{q}'$ equals $\{(X,-X) : \;\; X \in \mathfrak{p}\}$ and has

$$\mathfrak{a}'_{\mathrm{q}} := \{(X,-X) : \;\; X \in \mathfrak{a}\}$$

as a maximal abelian subspace. Its root system is given by

$$\Sigma(\mathfrak{g}', \mathfrak{a}'_{\mathrm{q}}) = \Sigma(\mathfrak{g}, \mathfrak{a}) \times \{0\} \cup \{0\} \times \Sigma(\mathfrak{g}, \mathfrak{a}).$$

Finally, $\mathfrak{a}'_{\mathrm{q}}$ is contained in the maximal abelian subspace $\mathfrak{a}' := \mathfrak{a} \times \mathfrak{a}$ of $\mathfrak{p}'$. We put $A' := \exp(\mathfrak{a}') = A \times A$. Note that the projection map $\mathrm{pr}_q : \mathfrak{a}' \to \mathfrak{a}'_{\mathrm{q}}$ is given by

$$\mathrm{pr}_{\mathrm{q}}(U,V) = (\tfrac{1}{2}(U-V), \tfrac{1}{2}(V-U)). \tag{75}$$

Let P and Q be minimal parabolic subgroups of G containing A, i.e. $P,Q \in \mathscr{P}(A)$. Then $P \times Q$ is a minimal parabolic subgroup of G' containing A'. Moreover, any minimal parabolic subgroup of G' containing A' is of this form. The positive system of $\mathfrak{a}'$-roots associated with $P \times Q$ is given by

$$\Sigma(P \times Q) := \Sigma(P) \times \{0\} \cup \{0\} \times \Sigma(Q),$$

where $\Sigma(P)$ and $\Sigma(Q)$ are positive systems for $\Sigma(\mathfrak{g},\mathfrak{a})$ corresponding to the minimal parabolic subgroups P and Q.

In the present setting, our main result, Theorem 10.1, tells us that for $a \in A'_q$ we have

$$\mathrm{pr}_{\mathrm{q}} \circ \mathfrak{H}_{P\times Q}(aH') = \mathrm{conv}(W_{K'\cap H'} \cdot \log a) + \Gamma(P \times Q).$$

In order to determine the cone $\Gamma(P \times Q)$, we need to determine the set $\Sigma(P \times Q, \sigma'\theta')$ of roots $\gamma \in \Sigma(P \times Q)$ for which $\sigma'\theta'\gamma \in \Sigma(P \times Q)$. Let $\gamma = (\alpha, 0)$ be such a root. Then $\alpha \in \Sigma(P)$ and $\sigma'\theta'\gamma = (0,-\alpha)$ must be an element of $\{0\} \times \Sigma(Q)$ so that $\alpha \in \Sigma(P) \cap \Sigma(\bar{Q})$. Likewise, if $(0,\beta)$ belongs to this set then $\beta \in \Sigma(Q) \cap \Sigma(\bar{P})$. We thus see that

$$\Sigma(P \times Q, \sigma\theta') = (\Sigma(P) \cap \Sigma(\bar{Q})) \times \{0\} \cup \{0\} \times (\Sigma(\bar{P}) \cap \Sigma(Q)).$$

Notice that there are no roots $\gamma \in \Sigma(P \times Q)$ for which $\sigma'\theta'\gamma = \gamma$. Thus, $\Sigma(P \times Q)_- = \Sigma(P \times Q, \sigma'\theta')$ and we conclude that

$$\Gamma(P \times Q) = \Gamma_{\mathfrak{a}'_{\mathrm{q}}}(\Sigma(P \times Q, \sigma'\theta')) = \sum_{\gamma \in \Sigma(P\times Q,\sigma'\theta')} \mathbb{R}_{\geq 0}\, \mathrm{pr}_{\mathrm{q}} H'_\gamma.$$

If γ is of the form $(\alpha,0)$ then $H'_\gamma = (H_\alpha, 0)$ and if $\gamma = (0,\alpha)$, then $H'_\gamma = (0, H_\alpha)$. In view of (75) we now obtain

$$\begin{aligned}
\Gamma(P \times Q) &= \\
&= \sum_{\alpha \in \Sigma(P)\cap\Sigma(\bar{Q})} \mathbb{R}_{\geq 0}(\tfrac{1}{2}H_\alpha, -\tfrac{1}{2}H_\alpha) + \sum_{\alpha \in \Sigma(\bar{P})\cap\Sigma(Q)} \mathbb{R}_{\geq 0}(-\tfrac{1}{2}H_\alpha, \tfrac{1}{2}H_\alpha) \\
&= \sum_{\alpha \in \Sigma(P)\cap\Sigma(\bar{Q})} \mathbb{R}_{\geq 0}(H_\alpha, -H_\alpha) + \sum_{\alpha \in \Sigma(\bar{P})\cap\Sigma(Q)} \mathbb{R}_{\geq 0}(H_{-\alpha}, -H_{-\alpha}) \\
&= \sum_{\alpha \in \Sigma(P)\cap\Sigma(\bar{Q})} \mathbb{R}_{\geq 0}(H_\alpha, -H_\alpha) + \sum_{\alpha \in \Sigma(P)\cap\Sigma(\bar{Q})} \mathbb{R}_{\geq 0}(H_\alpha, -H_\alpha) \\
&= \sum_{\alpha \in \Sigma(P)\cap\Sigma(\bar{Q})} \mathbb{R}_{\geq 0}(H_\alpha, -H_\alpha).
\end{aligned}$$

We will identify $\mathfrak{a}'_{\mathrm{q}}$ with $\mathfrak{a}$ via the map $(X,-X) \mapsto X$. Thus,

$$\Gamma_{\mathfrak{a}'_{\mathrm{q}}}(\Sigma(P \times Q)) = \Gamma_{\mathfrak{a}}(\Sigma(P) \cap \Sigma(\bar{Q})).$$

For $Q = P$, the resulting cone is the zero one, and we retrieve the non-linear convexity theorem of Kostant [22] for the group G. At the other extreme, for $Q = \bar{P}$, the resulting cone is maximal, and we retrieve the convexity theorem of [4] for the pair (G', H').

Taking $a=e$ we obtain, with the same identification $\mathfrak{a}'_{\mathrm{q}} \simeq \mathfrak{a}$,

$$\mathrm{pr}_{\mathrm{q}} \circ \mathfrak{H}_{P\times Q}(H') = \Gamma_{\mathfrak{a}}(\Sigma(P)\cap\Sigma(\bar{Q})) \tag{76}$$

On the other hand,

$$\begin{aligned}
\mathrm{pr}_{\mathrm{q}} \circ \mathfrak{H}_{P\times Q}(H') &= \mathrm{pr}_{\mathrm{q}} \circ \mathfrak{H}_{P\times Q}(\mathrm{diag}(G\times G)) \\
&= \mathrm{pr}_{\mathrm{q}}(\{(\mathfrak{H}_P(g),\mathfrak{H}_Q(g)) :\ g\in G\}) \\
&= \mathrm{pr}_{\mathrm{q}}(\{(\mathfrak{H}_P(kan_p),\mathfrak{H}_Q(kan_p)) :\ k\in K, a\in A, n_p\in N_P\}) \\
&= \mathrm{pr}_{\mathrm{q}}(\{(\log a,\mathfrak{H}_Q(an_pa^{-1})+\log a) :\ a\in A, n_p\in N_P\}) \\
&= \mathrm{pr}_{\mathrm{q}}(\{(\log a,\mathfrak{H}_Q(n_p)+\log a) :\ a\in A, n_p\in N_P\}) \\
&= \{(-\tfrac{1}{2}\mathfrak{H}_Q(n_p),\tfrac{1}{2}\mathfrak{H}_Q(n_p)) :\ n_p\in N_P\}.
\end{aligned}$$

Using the same identification $\mathfrak{a}'_{\mathrm{q}} \simeq \mathfrak{a}$ as above, we conclude that

$$\begin{aligned}
\mathrm{pr}_{\mathrm{q}} \circ \mathfrak{H}_{P\times Q}(H') &= -\frac{1}{2}\mathfrak{H}_Q(N_P) \\
&= -\tfrac{1}{2}\mathfrak{H}_Q((N_P\cap\bar{N}_Q)(N_P\cap N_Q)) \\
&= -\tfrac{1}{2}\mathfrak{H}_Q(N_P\cap\bar{N}_Q).
\end{aligned}$$

Thus, by equation (76), we obtain that

$$-\frac{1}{2}\mathfrak{H}_Q(N_P\cap\bar{N}_Q) = \Gamma_{\mathfrak{a}}(\Sigma(P)\cap\Sigma(\bar{Q})),$$

which is equivalent to

$$\mathfrak{H}_Q(N_P\cap\bar{N}_Q) = \Gamma_{\mathfrak{a}}(\Sigma(\bar{P})\cap\Sigma(Q)).$$

Thus we retrieve the identity of Lemma 4.9, which, of course, was used in the proof of our main theorem.

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