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Asymptotic behaviour of matrix coefficients related to reductive symmetric spaces

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0. INTRODUCTION

Let G be a real reductive Lie group (of Harish-Chandra's class), τ an involution of G and H an open subgroup of the group G^τ of fixed points for τ. Then G has a τ-stable maximal compact subgroup K.

In harmonic analysis on the reductive symmetric space G/H a major role is played by K-finite functions annihilated by a cofinite ideal of the centre $\mathfrak{Z}$ of the universal enveloping algebra of G. Such functions naturally arise as matrix coefficients of K-finite vectors and H-fixed distribution vectors of admissable representations (cf. e.g. [15, 16, 2, 7, 14]). In this paper we study the asymptotic behaviour of such functions, using the methods developed in Harish-Chandra [9, 10] and Casselman-Milicic [4] (actually we allow the functions more generally to be H-spherical from the right). As an application an analogue of Harish-Chandra's space of Schwartz functions is introduced. We prove that a $\mathfrak{Z}$- and K-finite function belongs to this space if and only if it belongs to $L^2(G/H)$ (Theorem 7.3). This generalizes a well known result of Harish-Chandra [11]. A second application will be given elsewhere in joint work with H. Schlichtkrull (cf. [3]). Via Flensted-Jensen's duality (cf. [6]) a K-finite eigenfunction f for the algebra $\mathbb{D}(G/H)$ of invariant differential operators on G/H corresponds to a H^d-finite eigenfunction f^d for $\mathbb{D}(G^d/K^d)$ on a dual Riemannian symmetric space G^d/K^d. The estimates for f obtained in the present paper (note that f is $\mathfrak{Z}$-finite since $\mathbb{D}(G/H)$ is a finite $\mathfrak{Z}$-module, cf. [2]) are a first step towards a new proof that the so called boundary value of f^d (originally defined as a hyperfunction, cf. [18]) is a distribution. This result, originally proved using hyperfunction methods (cf. [8]) is of importance for the theory of the discrete series for G/H (cf. [14]).

A reductive group $'G$ of Harish-Chandra's class may be viewed as a reductive symmetric space G/H: here $G='G\times 'G$, $\tau(x, y)=(y, x)$, $H=\{(x, x);\ x\in 'G\}$ and the map $G\to 'G$, $(x, y)\mapsto xy^{-1}$ induces the identification $G/H\simeq 'G$. If $'K$ is a maximal compact subgroup of $'G$, then $K='K\times 'K$ is a τ-stable maximal compact subgroup of G, and $\mathfrak{Z}$-, K-finite functions on G/H correspond bijectively to $'\mathfrak{Z}$- and $'K\times 'K$-finite functions on $'G$. In this situation ("the group case") the general study of these functions and the differential equations satisfied by them was started by Harish-Chandra in two unpublished papers [9, 10]. Later on the material was made more accessible by Casselman and Milicic [4]. They discovered that in suitable coordinates at infinity the equations become a system of complex partial differential equations of the regular singular type. In fact the singularities are of a very special type called simple, and the equations can be treated by a several variable version of the classical Frobenius method (cf. also [5]). A different approach to asymptotics was followed by Wallach [20].

The methods of [9, 10] and [4] apply very well to our more general situation. In all directions to infinity the asymptotic behaviour of $\mathfrak{Z}$- and K-finite functions on G/H can be described by converging series expansions, similar to those in the group case. There occurs a new phenomenon however which we shall briefly describe. The space G/H admits a Cartan decomposition $G=Kcl(A^-)H$, where A^- is a Weyl chamber corresponding to a root system Σ_+ of a vectorial subgroup A in the subgroup $G_+=\{x\in G;\ \tau\theta(x)=x\}$. A $\mathfrak{Z}$- and K-finite function f satisfies so called radial differential equations on A^- (cf. § 3). However it does not admit a converging series expansion on the whole of A^-. Instead A^- is divided into finitely many Weyl chambers determined by a root system Σ of A in G which contains Σ_+ as a subsystem. The function f admits a converging series expansion on every such smaller chamber (cf. § 2 for a detailed explanation of this phenomenon). Global estimates for f can be obtained from information on the leading exponents of f along each of the subchambers of A^- (cf. § 6).

As we indicated above, the main ideas of this paper stem from [9, 10] and [4]. However, the present situation is sufficiently different from the group case to justify a separate treatment. Often we refer to [4] when proofs would have been essentially the same. On the other hand we have kept this paper as self-contained as possible by not referring to [4] for notations or definitions.

Finally we should mention that for K-finite eigenfunctions of $\mathbb{D}(G/H)$ related results have been obtained by Oshima-Matsuki [14]. Via Flensted-Jensen's duality they transfer the problem to the dual Riemannian symmetric space G^d/K^d and then apply hyperfunction methods (cf. [18] for an introduction to these methods).

1. SYMMETRIC SPACES OF CLASS $\mathscr{H}$

If G is a group of class $\mathscr{H}$, τ an involution of G, H a closed subgroup with

$(G^\tau)^0 \subset H \subset G^\tau$, we call the homogeneous space G/H a symmetric space of the Harish-Chandra class (class $\mathscr{H}$). For the basic structure theory of groups of class $\mathscr{H}$, we refer the reader to [19, pp. 192–198].

PROPOSITION 1.1. *Let G be a group of class $\mathscr{H}$, τ, H as above. Then G carries a Cartan involution θ with $\theta\tau = \tau\theta$. Moreover, $[H:H^0] < \infty$ and $\theta(H) = H$.*

PROOF. Let $X(G)$ denote the space of continuous multiplicative homomorphisms $G \to \mathbb{R}^* = \mathbb{R} \setminus \{0\}$, and put

$$^0G = \bigcap_{\chi \in X(G)} \ker |\chi|.$$

Let $\mathfrak{c}$ be the centre of the Lie algebra $\mathfrak{g}$ of G, $^0\mathfrak{g}$ the Lie algebra of ^0G, and set $^0\mathfrak{c} = {}^0\mathfrak{g} \cap \mathfrak{c}$. Because τ is an automorphism, it leaves ^0G invariant. The associated infinitesimal involution, denoted by the same symbol τ, leaves $^0\mathfrak{g}$, $\mathfrak{c}$ and $^0\mathfrak{c}$ invariant. If we let $\mathfrak{h}$ and $\mathfrak{q}$ denote the $+1$ and -1 eigenspace of τ in $\mathfrak{g}$ respectively, we have decompositions $\mathfrak{c} = \mathfrak{c}_h \oplus \mathfrak{c}_q$ and $^0\mathfrak{c} = {}^0\mathfrak{c}_h \oplus {}^0\mathfrak{c}_q$, where $\mathfrak{c}_h = \mathfrak{c} \cap \mathfrak{h}$, $\mathfrak{c}_q = \mathfrak{c} \cap \mathfrak{q}$, etc. Fix linear subspaces $\mathfrak{v}_+$ and $\mathfrak{v}_-$ of $\mathfrak{c}_h$ and $\mathfrak{c}_q$ such that

$$\mathfrak{c}_h = {}^0\mathfrak{c}_h \oplus \mathfrak{v}_+, \quad \mathfrak{c}_q = {}^0\mathfrak{c}_q \oplus \mathfrak{v}_-,$$

and put $\mathfrak{v} = \mathfrak{v}_+ \oplus \mathfrak{v}_-$. Then $\mathfrak{c} = {}^0\mathfrak{c} \oplus \mathfrak{v}$, and so $V = \exp \mathfrak{v}$ is a τ-stable maximal closed vector subgroup of centre (G). On the other hand, since τ leaves the semisimple algebra $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$ invariant, there exists a τ-stable maximal compact subalgebra $\mathfrak{k}_1$ of $\mathfrak{g}_1$ (cf. [1]). Moreover, there exists a unique maximal compact subgroup K of G, whose Lie algebra contains $\mathfrak{k}_1$ (cf. [19, p. 197, Thm 12]). Finally, there exists a unique Cartan involution θ of G such that $G^\theta = K$ and $\theta(x) = x^{-1}$ for $x \in V$. We claim that $\tau\theta = \theta\tau$. In fact, $\tau(K)$ is a maximal compact subgroup of G, whose Lie algebra $\tau(\mathfrak{k})$ contains $\tau(\mathfrak{k}_1) = \mathfrak{k}_1$. Hence, by the uniqueness referred to above, $\tau(K) = K$. The infinitesimal Cartan involution θ leaves $\mathfrak{g}_1$ and $\mathfrak{c}$ invariant, so that $\mathfrak{p} = \mathfrak{g}^{-\theta} = \mathfrak{g}_1^{-\theta} \oplus \mathfrak{c}^{-\theta} = \mathfrak{g}_1^{-\theta} \oplus \mathfrak{v}$. Therefore $\mathfrak{p}$ is τ-stable, hence $\exp \mathfrak{p}$ is, whence the claim.

Finally, since τ and θ commute, G^τ and $(G^\tau)^0$ are θ-invariant, so that $G^\tau = (G^\tau \cap K) \exp (\mathfrak{p} \cap \mathfrak{h})$ and $(G^\tau)^0 = [(G^\tau)^0 \cap K] \exp (\mathfrak{p} \cap \mathfrak{h})$. It follows that $[H:H^0] \leq [G^\tau : (G^\tau)^0] \leq [G^\tau \cap K : (G^\tau)^0 \cap K] < \infty$ (the latter by compactness of K). It also follows that $(G^\tau)^0 \cap K = (G^\tau \cap K)^0$, hence $H^0 \cap K = (H \cap K)^0$, and $H = (H \cap K) \exp (\mathfrak{p} \cap \mathfrak{h})$. In particular $H = \theta(H)$. □

From now on, let G be a group of class $\mathscr{H}$, τ an involution of G, and θ a commuting Cartan involution. In the sequel we shall use the notations of the above proof without further comments. Moreover, we fix a bilinear form B on $\mathfrak{g}$ which is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$, coincides with the Killing form on $\mathfrak{g}_1$ and for which $\mathfrak{g}_1$, $\mathfrak{c}_h$ and $\mathfrak{c}_q$ are orthogonal. Then B is non-degenerate and $\mathrm{Ad}(G)$-invariant.

We conclude this section with recalling some known results on the structure of G/H that are relevant for this paper.

LEMMA 1.2. *The map* $\varphi: K\times(\mathfrak{p}\cap\mathfrak{q})\times(\mathfrak{p}\cap\mathfrak{h})\to G$, $(k, X, Y)\mapsto k \exp X \exp Y$ *is a diffeomorphism.*

PROOF. This follows easily from the corresponding result in the semisimple case (i.e. $\mathfrak{g}$ semisimple, cf. [6, proof of Theorem 4.1]). □

Let $\mathfrak{g}_+$ be the $+1$ eigenspace of $\tau\theta$ in $\mathfrak{g}$. Then $\mathfrak{g}_+$ is a reductive subalgebra with polar decomposition

$$\mathfrak{g}_+=(\mathfrak{k}\cap\mathfrak{h})\oplus(\mathfrak{p}\cap\mathfrak{q}).$$

Select a maximal abelian subspace $\mathfrak{a}_{pq}$ of $\mathfrak{p}\cap\mathfrak{q}$. Then from the corresponding result in the semisimple case one easily sees that the set $\Sigma=\Sigma(\mathfrak{g},\mathfrak{a}_{pq})$ of restricted roots of $\mathfrak{a}_{pq}$ in $\mathfrak{g}$ is a (possibly non-reduced) root system (cf. [17]).

Since $\tau\theta=I$ on $\mathfrak{a}_{pq}$, every root space $\mathfrak{g}^\alpha$ $(\alpha\in\Sigma)$ is $\tau\theta$-invariant, so that we have a corresponding decomposition

$$\mathfrak{g}^\alpha=\mathfrak{g}^\alpha_+\oplus\mathfrak{g}^\alpha_-$$

into $+1$ and -1 eigenspaces. Let

$$\Sigma_+=\{\alpha\in\Sigma;\ \mathfrak{g}^\alpha_+\neq 0\}.$$

Then $\Sigma_+=\Sigma(\mathfrak{g}_+,\mathfrak{a}_{pq})$, the restricted root system of $\mathfrak{a}_{pq}$ in $\mathfrak{g}_+$. Of course $\mathfrak{a}_{pq}$ may be central in $\mathfrak{g}_+$, so that $\Sigma_+=\emptyset$. We fix a choice Σ_+^+ of positive roots for Σ_+ (if $\Sigma_+=\emptyset$, then $\Sigma_+^+=\emptyset$), and put

$$\mathfrak{a}_{pq}^-=\{H\in\mathfrak{a}_{pq};\ \alpha(H)<0 \text{ for all } \alpha\in\Sigma_+^+\},$$

$$A=\exp(\mathfrak{a}_{pq}),\quad A^-=\exp(\mathfrak{a}_{pq}^-).$$

Moreover, we write

$$\mathfrak{a}_{pq}'=\{H\in\mathfrak{a}_{pq};\ \alpha(H)\neq 0 \text{ for all } \alpha\in\Sigma_+\},$$

and $A'=\exp(\mathfrak{a}_{pq}')$. If $\Sigma_+=\emptyset$ this is to be interpreted as $\mathfrak{a}_{pq}^-=\mathfrak{a}_{pq}'=\mathfrak{a}_{pq}$.

LEMMA 1.3. *For every* $X\in\mathfrak{p}\cap\mathfrak{q}$ *there exists a unique* $Y\in\mathrm{cl}(\mathfrak{a}_{pq}^-)$ *such that* $X=\mathrm{Ad}(k)Y$ *for some* $k\in K\cap H^0$.

PROOF. Without loss of generality we may assume that $G={}^0G$, and then the same proof as in [6, p. 118] applies. □

COROLLARY 1.4 (*Cartan decomposition*). *For every* $x\in G$ *there exists a unique* $a\in\mathrm{cl}(A^-)$ *such that* $x\in KaH^0$.

PROOF. This follows from a straightforward combination of Lemmas 1.2 and 1.3. □

Before stating the next result we introduce a few more notations. Let $\mathfrak{l}$ be the centralizer of $\mathfrak{a}_{pq}$ in $\mathfrak{g}$. Since $\mathfrak{a}_{pq}$ is invariant under τ and θ, so is $\mathfrak{l}$, and we have the decomposition

(1.1) $\quad \mathfrak{l}=\mathfrak{l}_{kq}\oplus\mathfrak{l}_{kh}\oplus\mathfrak{a}_{pq}\oplus\mathfrak{l}_{ph},$

where $\mathfrak{l}_{kq}=\mathfrak{l}\cap\mathfrak{k}\cap\mathfrak{q}$, etc. For the rest of this section we fix a choice Σ^+ of positive roots for Σ such that $\Sigma^+\cap\Sigma_+=\Sigma_+^+$ and put

$$\mathfrak{n}=\sum_{\alpha\in\Sigma^+}\mathfrak{g}^\alpha$$

and $\bar{\mathfrak{n}}=\theta\mathfrak{n}$. Then obviously

$$(1.2)\qquad \mathfrak{g}=\bar{\mathfrak{n}}\oplus\mathfrak{l}\oplus\mathfrak{n}.$$

By the same proof as in [2, Lemma 3.4], we also have

$$(1.3)\qquad \mathfrak{g}=\bar{\mathfrak{n}}\oplus\mathfrak{l}_{kq}\oplus\mathfrak{a}_{pq}\oplus\mathfrak{h}.$$

Moreover, the maps $\bar{\mathfrak{n}}\times\mathfrak{l}_k\to\mathfrak{k}$, $(X,U)\mapsto X+\theta X+U$ and $\bar{\mathfrak{n}}\times\mathfrak{l}_h\to\mathfrak{h}$, $(X,U)\mapsto$ $\mapsto X+\tau X+U$ are easily seen to be bijective. Using [2, Lemma 3.5], we now obtain the following.

LEMMA 1.5 (*Infinitesimal Cartan decomposition*). *Let $\mathfrak{h}^c$ be the orthocomplement of $\mathfrak{l}_{kh}$ in $\mathfrak{h}$. Then for every $a\in A'$ we have the direct sum decomposition*

$$\mathfrak{g}=Ad(a^{-1})\mathfrak{k}\oplus\mathfrak{a}_{pq}\oplus\mathfrak{h}^c.$$

Let M be the centralizer of $\mathfrak{a}_{pq}$ in $K\cap H$, and put $d(M)=\{(m,m)\in K\times H^0;$ $m\in M\}$.

LEMMA 1.6. *The map $(K\times H^0)/d(M)\times A^-\to G$ given by*

$$(1.4)\qquad ((k,h)d(M),a)\mapsto kah^{-1}$$

is a diffeomorphism onto the open dense subset $G'=KA^-H^0$ of G.

PROOF. From Lemmas 1.2 and 1.3 it easily follows that G' equals the open dense subset $K\exp(Ad(K\cap H^0)\mathfrak{a}_{pq}^-)\exp(\mathfrak{p}\cap\mathfrak{h})$ of G.

To see that (1.4) is injective, suppose that $k\in K$, $a,b\in A^-$, $h\in H^0$. Then it suffices to show that $kah^{-1}=b$ implies $k=h\in M\cap H^0$ and $a=b$. Now this is seen as follows. Write $h^{-1}=h_1h_2$, where $h_1\in K\cap H^0$, $h_2\in\exp(\mathfrak{p}\cap\mathfrak{h})$. Then by Lemma 1.2 we have $kh_1=1$, $h_2=1$, $h_1^{-1}ah_1=b$. Let Ad_+ be the adjoint representation of $G_+^0=(K\cap H^0)\exp(\mathfrak{p}\cap\mathfrak{q})$ in $\mathfrak{g}_+$. Then it follows that $Ad_+(k)$ maps $\log(a)\in\mathfrak{a}_{pq}^-$ into $\mathfrak{a}_{pq}^-$. But $\mathfrak{a}_{pq}^-$ is a Weyl chamber for $\Sigma_+=\Sigma(\mathfrak{g}_+,\mathfrak{a}_{pq})$ and so, by standard semisimple theory applied to $Ad_+(G_+^0)$, it follows that k centralizes $\mathfrak{a}_{pq}$. Hence $k=h\in M\cap H^0$ and $a=b$.

Finally, fix $k\in K$, $h\in H^0$, $a\in A^-$, and consider the map

$$\psi:\mathfrak{k}\times\mathfrak{h}^c\times\mathfrak{a}_{pq}\to G,\ (X,Y,Z)\mapsto k\exp(X)a\exp(Z)\exp(Y)h.$$

Then the differential $d\psi(0)$ of ψ at $(0,0,0)$ is given by

$$d\psi(0)(U,V,W)=d(\lambda_{ka}\varrho_h)(e)(Ad(a^{-1})U+V+W),$$

where λ_{ka} denotes left multiplication by ka, and ϱ_h right multiplication by h on G. By Lemma 1.5 this differential is bijective. Consequently (1.4) has bijective differential everywhere. $\square$

2. SPHERICAL FUNCTIONS AND THE BASIC EQUATIONS

Let $\mu=(\mu_1,\mu_2)$ be a smooth representation of $K\times H$ in a finite dimensional complex linear space E. If $v\in E$, $h\in H$, we shall often write $v\mu_2(h^{-1})$ instead of $\mu_2(h)v$. A C^∞-function $F:G\to E$ such that for all $x\in G$, $k\in K$, $h\in H$ we have

$$F(kxh)=\mu_1(k)F(x)\mu_2(h)$$

is called μ-spherical. The space of all such functions is denoted by $C_\mu^\infty(G)$.

If $\mathfrak{b}$ is a real Lie algebra, then we denote the universal enveloping algebra of its complexification $\mathfrak{b}_c$ by $U(\mathfrak{b})$. Similarly, we denote the symmetric algebra of $\mathfrak{b}_c$ by $S(\mathfrak{b})$. Unless otherwise specified, $U(\mathfrak{g})$ acts on smooth functions on G via the right regular reprfesentation R. The centre of $U(\mathfrak{g})$ is denoted by $\mathfrak{Z}$. A function f on G is called $\mathfrak{Z}$-finite if the vector space $\{Zf;Z\in\mathfrak{Z}\}$ is finite dimensional over $\mathbb{C}$.

The subspace of $\mathfrak{Z}$-finite elements in $C_\mu^\infty(G)$ is denoted by $A_\mu(G)$. As they are annihilated by an elliptic differential operator with real analytic coefficients (see for instance the argument in [19, p. 310]), the elements of $A_\mu(G)$ are in fact real analytic functions.

A function $F\in C_\mu^\infty(G)$ belongs to $A_\mu(G)$ iff it is annihilated by a cofinite ideal I in $\mathfrak{Z}$. We write $A_\mu(G,I)$ for the space of $F\in A_\mu(G)$ satisfying

$$R_Z F=0 \quad (Z\in I).$$

Here we have used the notation $R_u=R(u)$ for the infinitesimal right regular action of an element $u\in U(\mathfrak{g})$.

For the sake of completeness we list the following lemma which is proved along the same lines as [2, Cor. 3.10], involving a finite basis of $\mathfrak{Z}/I$ over $\mathbb{C}$ (cf. [19, p. 308, Thm. 8]). Let $\mathfrak{a}\subset\mathfrak{l}$ be a Cartan subalgebra containing $\mathfrak{a}_{pq}$, $\Phi=\Sigma(\mathfrak{g}_c,\mathfrak{a}_c)$, $\Phi_0=\Sigma(\mathfrak{l}_c,\mathfrak{a}_c)$, and let $W(\Phi)$, $W(\Phi_0)$ be the Weyl groups of Φ and Φ_0 respectively.

LEMMA 2.1. *Let I be a cofinite ideal in $\mathfrak{Z}$. Then*

$$\dim A_\mu(G,I)\leq \dim(\mathfrak{Z}/I)\dim(\mu)[W(\Phi):W(\Phi_0)].$$

Before proceeding we briefly discuss how spherical functions arise in representation theory. Let π be an admissible representation of finite length of G in a Fréchet space V. Let the space V^∞ of C^∞-vectors in V be equipped with the topology induced by the collection of seminorms

$$N_{p,a}: v\mapsto p(\pi(a)v),$$

where p ranges over a complete set of seminorms for V, and $a\in U(\mathfrak{g})$. As a locally convex space V^∞ is isomorphic to the closed subspace

$$T=\{f\in C^\infty(G,V); f(x)=\pi(x)f(e)\}$$

of the Fréchet space $C^\infty(G, V)$; this follows by a straightforward application of the Banach-Steinhaus theorem. Hence V^∞ is Fréchet. The topological dual $V^{-\infty}$ of V^∞ is a G-module in a natural way; we let $(V^{-\infty})^H$ denote the subspace of H-fixed elements in $V^{-\infty}$.

Given $\varphi \in (V^{-\infty})^H$, $u \in V_K$ (the K-finite vectors in V), we may form the (C^∞-) matrix coefficient

$$(2.1) \qquad m(x) = m_{\varphi, u}(x) = \varphi(\pi(x^{-1})u) \quad (x \in G).$$

Now let $\vartheta \subset \hat{K}$ be the set of K-types occurring in u, V_ϑ the (finite dimensional) span of all K-isotopic vectors in V_K with K-type contained in ϑ, and let $P_\vartheta : V \to V_\vartheta$ be the projection along the other K-types. Let E be the linear dual of V_ϑ, μ_1 the contragredient of the representation of K in V_ϑ. Then the function $F = F_{\varphi, \vartheta} : G \to E$ defined by:

$$F(x) = \varphi \circ \pi(x^{-1}) \circ P_\vartheta \quad (x \in G)$$

is $(\mu_1, 1)$-spherical. Viewing u as an element of E, we have

$$(2.2) \qquad m_{\varphi, u}(x) = \langle u, F_{\varphi, \vartheta}(x) \rangle \quad (x \in G).$$

The annihilator J of V^∞ in $\mathfrak{Z}$ is a cofinite ideal because π has finite length. Let $u \mapsto u^\vee$ denote the principal anti-automorphism of $U(\mathfrak{g})$. Then obviously $F \in A_\mu(G, J^\vee)$, where $\mu = (\mu_1, 1)$.

COROLLARY 2.2. *Let π be an admissible representation of finite length of G in a Fréchet space V. Then* $\dim (V^{-\infty})^H < \infty$.

PROOF. Select a finite set S of generators for the $(\mathfrak{g}, K)$-module V_K and let ϑ be the (finite) set of K-types occurring in the elements of S. Then the linear map $(V^{-\infty})^H \to A_\mu(G, J^\vee)$, $\varphi \mapsto F_{\varphi, \vartheta}$ is injective, and the result follows from Lemma 2.1. □

In the above we have seen how matrix coefficients may be expressed in terms of $(\mu_1, 1)$-spherical functions. We now return to the more general situation of a fixed smooth representation $\mu = (\mu_1, \mu_2)$ of $K \times H$ in a finite dimensional complex linear space E.

In view of the Cartan decomposition $G = Kcl(A^-)H$, a function $F \in C_\mu^\infty(G)$ is determined by its restriction Res (F) to A^-. Let M be the centralizer of A in $K \cap H$, and put

$$(2.3) \qquad E^M = \{u \in E;\, \mu_1(m)u = u\mu_2(m) \text{ for all } m \in M\}.$$

Then obviously the restriction map Res maps $C_\mu^\infty(G)$ injectively into $C^\infty(A^-, E^M)$.

Let now $F \in A_\mu(G, I)$, I being a cofinite ideal in $\mathfrak{Z}$. Following a method of Harish-Chandra (cf. [10]), we shall associate to F a system of first order differential equations along A^-. Using substitutions of variables as in [4], we then obtain systems with simple singularities (in the sense of [4, Appendix]), which enable us to obtain series expansions for Res (F). There is a complication however which does not occur in [4]. Though the differential equations have no singularities on A^-, the series expansions for Res (F) do not converge in the whole of A^-; in general they break down along root hyperplanes ker α, $\alpha\in\Sigma\setminus\Sigma_+$. The explanation for this phenomenon is that the regions of convergence are determined by the singularities of the equations in the complexification of the group A. *Each* root contributes to these singularities.

Let $\mathscr{P}$ be the collection of systems P of positive roots for Σ with $P\cap\Sigma_+= =\Sigma_+^+$. If $P\in\mathscr{P}$, we set

$$\mathfrak{a}_{pq}^-(P)=\{H\in\mathfrak{a}_{pq};\ \alpha(H)<0 \text{ for all } \alpha\in P\},$$

and $A^-(P)=\exp\,\mathfrak{a}_{pq}^-(P)$. Then

$$cl(A^-)=\bigcup_{P\in\mathscr{P}} cl(A^-(P)).$$

Each chamber $A^-(P)$, $P\in\mathscr{P}$ will be a region of convergence of a series expansion for Res (F). In the course of this paper we will see that the expansions for $F|A^-(P)$, $P\in\mathscr{P}$ together completely determine the asymptotics of F. This being said let us fix a particular element Σ^+ of $\mathscr{P}$ and concentrate on the behaviour of F along $A^-(\Sigma^+)$.

Let Δ be the set of simple roots in Σ^+. Then Δ is a basis for $(\mathfrak{a}_{pq}\cap\mathfrak{g}_1)^*$ over $\mathbb{R}$. Select a basis $\Lambda_\mathfrak{c}$ for $(\mathfrak{a}_{pq}\cap\mathfrak{c})^*$ over $\mathbb{R}$. Identifying $(\mathfrak{a}_{pq}\cap\mathfrak{g}_1)^*$ and $(\mathfrak{a}_{pq}\cap\mathfrak{c})^*$ with subspaces of $\mathfrak{a}_{pq}^*$ via B, we put:

$$\Lambda=\Delta\cup\Lambda_\mathfrak{c}.$$

Let $\{H_\lambda;\ \lambda\in\Lambda\}$ be the basis of $\mathfrak{a}_{pq}$ which is dual to the basis Λ of $\mathfrak{a}_{pq}^*$. Then $H_\alpha\in\mathfrak{a}_{pq}\cap\mathfrak{g}_1$ for $\alpha\in\Delta$ and $H_\lambda\in\mathfrak{a}_{pq}\cap\mathfrak{c}$ for $\lambda\in\Lambda_\mathfrak{c}$.

As in [2], let $\mathscr{F}^+$ be the algebra of functions $A'\to\mathbb{R}$ generated by

$$f_+^\alpha(a)=(a^\alpha-a^{-\alpha})^{-1},\quad g_+^\alpha(a)=-a^{-\alpha}f_+^\alpha(a),$$

$$f_-^\beta(a)=(a^\beta+a^{-\beta})^{-1},\quad g_-^\beta(a)=-a^{-\beta}f_-^\beta(a)$$

$(\alpha\in\Sigma_+^+;\ \beta\in\Sigma^+,\ \mathfrak{g}_-^\beta\neq 0)$. Here we have used the notation

$$a^\gamma=e^{\gamma\log a},$$

for $\gamma\in\mathfrak{a}_{pq}^*$, $a\in A$. Moreover, let $\mathscr{F}$ be the ring generated by 1 and $\mathscr{F}^+$.

Let $\mathfrak{Z}(\mathfrak{l})$ denote the centre of $U(\mathfrak{l})$ and let $v_1=1,\ v_2,\dots,v_r\in\mathfrak{Z}(\mathfrak{l})$ be as in [2, Lemma 3.7]. Moreover, fix $D_1=1,\ D_2,\dots,D_s\in\mathfrak{Z}$ such that their canonical images generate $\mathfrak{Z}/I$ over $\mathbb{C}$. Then by [2, Lemma 3.8] there exist finitely many elements $f_{\lambda ij}^{kl}\in\mathscr{F}$, $\xi_{\lambda ij}^{kl}\in U(\mathfrak{k})$, $\eta_{\lambda ij}^{kl}\in U(\mathfrak{h})$ $(\lambda\in\Lambda,\ 1\le i,k\le s,\ 1\le j,l\le r)$, such that

$$H_\lambda D_i v_j'\equiv\sum_{k,l} f_{\lambda ij}^{kl}(a)(\xi_{\lambda ij}^{kl})^a D_k v_l'\eta_{\lambda ij}^{kl} \bmod I \tag{2.4}$$

for all $a\in A'$. Here we have used the notation

$$Y^x=Ad(x^{-1})Y\quad(x\in G,\ Y\in U(\mathfrak{g})), \tag{2.5}$$

which is the technically more convenient notation of [4], but inconsistent with the notation in [2].

The centralizer L of $\mathfrak{a}_{pq}$ in G is of class $\mathscr{H}$, hence centralizes $\mathfrak{Z}(\mathfrak{l})$ (cf. [19, p. 286, Theorem 13]). Therefore M centralizes $\mathfrak{Z}$ and $\mathfrak{Z}(\mathfrak{l})$. Consequently, if $F \in A_\mu(G, I)$ then the functions

$$\Phi_{ij} = \text{Res}\ (R(D_i v_j')F) \tag{2.6}$$

$(1 \leq i \leq s,\ 1 \leq j \leq r)$ map A into E^M. By (2.4) it follows that

$$R(H_\lambda)\Phi_{ij}(a) = \sum_{k,l} f_{\lambda ij}^{kl}(a)\mu_1(\xi_{\lambda ij}^{kl})\Phi_{kl}(a)\mu_2(\eta_{\lambda ij}^{kl}), \tag{2.7}$$

for all $a \in A'$. Now let $\Phi : A' \to (E^M)^{sr}$ be the vector valued function with entries Φ_{ij} $(1 \leq i \leq s,\ 1 \leq j \leq r)$. Then by (2.7) there exist elements

$$G_\lambda \in \mathscr{F} \otimes \text{End}_{\mathbb{C}}\ \{(E^M)^{sr}\} \quad (\lambda \in \Lambda)$$

such that the real analytic map $\Phi : A \to (E^M)^{sr}$ satisfies the differential equations

$$R(H_\lambda)\Phi = G_\lambda \cdot \Phi \quad (\lambda \in \Lambda)$$

on A'.

As in [4] we view A as embedded in $\mathbb{C}^\Lambda$ under the map

$$\underline{\lambda}(a) = (a^\lambda;\ \lambda \in \Lambda).$$

Under this map the differential operators $R(H_\lambda)$ $(\lambda \in \Lambda)$ correspond to $z_\lambda \partial/\partial z_\lambda$ in $\mathbb{C}^\Lambda$. If $\gamma \in \mathbb{Z}\Delta$, then the character $e^\gamma : a \mapsto a^\gamma$ corresponds to a rational function on $\mathbb{C}^\Lambda$. Identifying $\gamma \in \mathbb{Z}\Delta$ with the element $(\gamma_\alpha;\ \alpha \in \Delta)$ of $\mathbb{Z}^\Delta \subset \mathbb{Z}^\Lambda$ determined by $\gamma = \sum_{\alpha \in \Delta} \gamma_\alpha \alpha$, and using the multi-index notation

$$z^t = \prod_{\lambda \in \Lambda} (z_\lambda)^{t_\lambda}$$

for $z \in \mathbb{C}^\Lambda$, $t \in \mathbb{Z}^\Lambda$, we have

$$a^\gamma = \underline{\lambda}(a)^\gamma.$$

Consequently the elements of $\mathscr{F}$ can be viewed as rational functions on $\mathbb{C}^\Lambda$. If $\alpha \in \Sigma_+$, we put

$$Y_+^\alpha = \{z \in \mathbb{C}^\Lambda;\ z^{2\alpha} = 1\}, \tag{2.8}$$

and if $\beta \in \Sigma$, $\mathfrak{g}_-^\beta \neq 0$, we put

$$Y_-^\beta = \{z \in \mathbb{C}^\Lambda;\ z^{2\beta} = -1\}.$$

Moreover, let $Y_+ = \cup\{Y_+^\alpha;\ \alpha \in \Sigma_+^+\}$, $Y_- = \cup\{Y_-^\beta;\ \beta \in \Sigma^+, \mathfrak{g}_-^\beta \neq 0\}$ and

$$Y = Y_+ \cup Y_-. \tag{2.9}$$

Then the elements of $\mathscr{F}$ are regular on $\mathbb{C}^\Lambda \setminus Y$. Being real analytic on A, the map Φ extends to a holomorphic $(E^M)^{sr}$-valued map on an open neighbourhood Ω of $\underline{\lambda}(A)$ in $\mathbb{C}^\Lambda$. We conclude that it satisfies the system of differential equations

$$(2.10) \quad z_\lambda \frac{\partial}{\partial z_\lambda} \Phi = G_\lambda \cdot \Phi \quad (\lambda \in \Lambda)$$

on $\Omega \setminus Y$.

The system (2.10) has simple singularities (in the sense of [4, Appendix]) along the coordinate hyperplanes $z_\lambda = 0$ $(\lambda \in \Lambda)$, so that we may apply the theory described in [4, Appendix]. Put

$$D = \{z \in \mathbb{C};\ |z| < 1\}.$$

Then obviously $\underline{\lambda}(A^-(\Sigma^+)) \subset D^\Delta \times \mathbb{C}^{\Lambda \setminus \Delta} \subset \mathbb{C}^\Lambda \setminus Y$, so that a result analogous to [4, Lemma 5.1] holds. To formulate it, we need some definitions and notations. If $m \in \mathbb{N}^\Lambda$ $(\mathbb{N} = \{0, 1, \ldots\})$, $s \in \mathbb{C}^\Lambda$, we put

$$\log^m \underline{\lambda}(a) = \prod_{\lambda \in \Lambda} \{\lambda(\log a)\}^{m_\lambda},$$

$$\underline{\lambda}^s(a) = \prod_{\lambda \in \Lambda} \exp\,(s_\lambda \lambda(\log a)),$$

for $a \in A$. Two elements $s, t \in \mathbb{C}^\Lambda$ are called integrally equivalent iff $s - t \in \mathbb{Z}^\Lambda$.

LEMMA 2.3. *Let $F \in \mathcal{A}_\mu(G, I)$. Then there exist*

(i) *a finite set S of mutually integrally inequivalent elements of $\mathbb{C}^\Lambda$, and*

(ii) *for each $s \in S$ a finite collection $F_{s,m}$ $(m \in \mathbb{N}^\Lambda)$ of non-trivial holomorphic E^M-valued functions on $D^\Delta \times \mathbb{C}^{\Lambda \setminus \Delta}$ such that on each of the coordinate hyperplanes $z_\lambda = 0$ $(\lambda \in \Lambda)$ at least one of them is not identically zero, such that*

$$F = \sum_{s,m} (F_{s,m} \circ \underline{\lambda}) \underline{\lambda}^s \log^m \underline{\lambda}$$

on $A^-(\Sigma^+)$.

This S and the $F_{s,m}$ are unique.

Let $\sum_k c_{s+k,m} z^k$ (summation over $\mathbb{N}^\Lambda$) be the power series expansion of $F_{s,m}$. Then the series expansion

$$(2.11) \quad F = \sum_{t,m} c_{t,m} \underline{\lambda}^t \log^m \underline{\lambda}$$

of F converges absolutely on $A^-(\Sigma^+)$. Any series expansion like (2.11) which converges absolutely to F on a non-empty open subset of $A^-(\Sigma^+)$ must be identical to (2.11). If $c_{t,m} \neq 0$ for some $m \in \mathbb{N}^\Lambda$, then t is called a Σ^+-exponent of F. On $\mathbb{C}^\Lambda$ we define the $\leq_\Delta$-order by

$$s \leq_\Delta t \text{ iff } t - s \in \mathbb{N}^\Delta,$$

for $s, t \in \mathbb{C}^\Lambda$. The $\leq_\Delta$-minimal elements in the set of Σ^+-exponents of F are called the Σ^+-leading exponents of F. Given a Σ^+-leading exponent $t \in \mathbb{C}^\Lambda$, the corresponding character $\underline{\lambda}^t : A \to \mathbb{C}^*$ is called a Σ^+-leading character, and

$$F_t = \sum_m c_{t,m} \underline{\lambda}^t \log^m \underline{\lambda}$$

is called the corresponding Σ^+-leading term of F.

In Section 3 we shall develop the theory of radial components associated with the Cartan decomposition (Cor. 1.4) in order to limit the possible Σ^+-leading terms of F. Let σ be the injective algebra homomorphism of $\mathfrak{Z}$ into $\mathfrak{Z}(\mathfrak{l})$, determined by

(2.12) $\quad Z - \sigma(Z) \in \mathfrak{n}U(\mathfrak{g}),$

for $Z \in \mathfrak{Z}$ ($\theta \circ \sigma \circ \theta$ is the map denoted by $\tilde{\mu}$ in [2, Lemma 3.6]). If I is a cofinite ideal in $\mathfrak{Z}$, then $\mathfrak{Z}(\mathfrak{l})\sigma(I)$ is a cofinite ideal in $\mathfrak{Z}(\mathfrak{l})$ (cf. [2, Lemma 3.7]). Under left multiplication the space $U = \mathfrak{Z}(\mathfrak{l})/\mathfrak{Z}(\mathfrak{l})\sigma(I)$ is an $\mathfrak{a}_{pq}$-module; by exponentiation it becomes an A-module. Being finite dimensional, the A-module U splits into a finite direct sum of generalized A-weight spaces. A character $\omega : A \to \mathbb{C}^*$ is said to lie Σ^+-shifted over the cofinite ideal I in $\mathfrak{Z}$ if it is a generalized A-weight for the A-module $\mathfrak{Z}(\mathfrak{l})/\mathfrak{Z}(\mathfrak{l})\sigma(I)$.

REMARK. Here we do not follow the terminology of [4]. The reason is that we wish to make the dependence on the choice $\Sigma^+ \in \mathscr{P}$ explicit. If I is a cofinite ideal in $\mathfrak{Z}$, then to each $P \in \mathscr{P}$ corresponds the set $X(P, I)$ of characters lying P-shifted over I. The sets $X(P, I)$, $P \in \mathscr{P}$, are mutually different, but related by certain "ϱ-shifts". We discuss this in Section 4.

THEOREM 2.4. *Let I be a cofinite ideal in $\mathfrak{Z}$, $F \in A_\mu(G, I)$. Then all Σ^+-leading characters lie Σ^+-shifted over I.*

We postpone the proof of this theorem to the next section.

In particular, the set of Σ^+-leading characters is finite, so that with essentially the same proof we have the following analogue of [4, Theorem 5.6]. Viewing $\mathbb{C}^\Delta$ as a subspace of $\mathbb{C}^\Lambda$, we call two elements $s, t \in \mathbb{C}^\Lambda$ Δ-integrally equivalent if $s - t \in \mathbb{Z}^\Delta$. Moreover, we define the map $\underline{\alpha} : A \to \mathbb{C}^\Delta$ by

$$\underline{\alpha}(a) = (a^\alpha;\ \alpha \in \Delta).$$

THEOREM 2.5. *Let F be a $\mathfrak{Z}$-finite μ-spherical function on G. Then there exist*

(i) *a finite set S_Δ of mutually Δ-integrally inequivalent elements of $\mathbb{C}^\Lambda$, and*

(ii) *for each $s \in S_\Delta$ a finite set of non-trivial holomorphic functions $F^\Delta_{s,m} : D^\Delta \to E^M$ ($m \in \mathbb{N}^\Lambda$) such that on each coordinate hyperplane $z_\alpha = 0$ ($\alpha \in \Delta$) at least one of them is not identically zero, such that*

(2.13) $\quad F = \sum_{s,m} (F^\Delta_{s,m} \circ \underline{\alpha}) \underline{\lambda}^s \log^m \underline{\lambda}$

on $A^-(\Sigma^+)$.

This S_Δ and the $F^\Delta_{s,m}$ are unique.

REMARK 2.6. As in [4] the set S_Δ can be characterized as follows. For each class Ω of Δ-integrally equivalent Σ^+-leading exponents we define the element $s(\Omega) \in \mathbb{C}^\Lambda$ by

$$s(\Omega)_\lambda = \min\ \{t_\lambda;\ t \in \Omega\}.$$

Then S_Δ is the set of all $s(\Omega)$.

REMARK 2.7. Using arguments involving monodromy as in [4, Appendix] one can actually show that the functions $F^{\Delta}_{s,m}\circ \underline{\alpha}$ (and hence equation 2.13) admit real analytic extensions to the bigger Weyl chamber A^-. This is a consequence of the fact that the system (2.10) is regular at points of $\underline{\alpha}(A^-)$.

3. RADIAL COMPONENTS AND LEADING CHARACTERS

In this section we develop the theory of the μ-radial component of a differential operator in order to prove Theorem 2.4. We start with a result related to the infinitesimal Cartan decomposition (see Lemma 1.5). Let $\mathscr{R}$ be the ring of functions $A'\to\mathbb{R}$ generated by 1, a^{α} ($\alpha\in\Delta$) and

$$(1-a^{2\alpha})^{-1}\quad (\alpha\in\Sigma^{+}_{+}),$$

$$(1+a^{2\beta})^{-1}\quad (\beta\in\Sigma^{+},\ \mathfrak{g}^{\beta}_{-}\neq 0).$$

Moreover, let $\mathscr{R}^+$ be the ideal in $\mathscr{R}$ generated by the functions a^{α}, $\alpha\in\Delta$.

LEMMA 3.1. *Let $X_\alpha\in\mathfrak{g}^{\alpha}_{+}$ or $X_\alpha\in\mathfrak{g}^{\alpha}_{-}$ ($\alpha\in\Sigma^+$). Then there exist $f_1, f_2\in\mathscr{R}^+$, such that for all $a\in A'$ we have*

$$(3.1)\qquad X_\alpha=f_1(a)(X_\alpha+\theta X_\alpha)^a+f_2(a)(X_\alpha+\tau X_\alpha).$$

PROOF. First recall that we use the notation (2.5). If $X_\alpha\in\mathfrak{g}^{\alpha}_{+}$, then $\theta X_\alpha=\tau X_\alpha$ so that (3.1) holds with $f_1=a^{\alpha}(1-a^{2\alpha})^{-1}$, $f_2=-a^{2\alpha}(1-a^{2\alpha})^{-1}$. On the other hand, if $X_\alpha\in\mathfrak{g}^{\alpha}_{-}$, then $\theta X_\alpha=-\tau X_\alpha$ and (3.1) holds with $f_1=a^{\alpha}(1+a^{2\alpha})^{-1}$, $f_2=a^{2\alpha}(1+a^{2\alpha})^{-1}$. In both cases it is clear that $f_1,f_2\in\mathscr{R}^+$. □

After this, we are prepared for the radial decomposition of differential operators. As in [4], we define trilinear maps

$$B_a: U(\mathfrak{a}_{pq})\times U(\mathfrak{k})\times U(\mathfrak{h})\to U(\mathfrak{g})$$

($a\in A$) by $B_a(H,X,Y)=X^aHY$. Now let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}_{pq}$ in $\mathfrak{k}\cap\mathfrak{h}$. Then $\mathfrak{m}$ is the Lie algebra of M. If $U\in U(\mathfrak{m})$ then obviously $B_a(H,XU,Y)=$ $=B_a(H,X,UY)$, so that B_a induces the linear map

$$\Gamma_a: U(\mathfrak{a}_{pq})\otimes U(\mathfrak{k})\otimes_{U(\mathfrak{m})}U(\mathfrak{h})\to U(\mathfrak{g})$$

determined by $\Gamma_a(H\otimes X\otimes Y)=X^aHY$ for $a\in A$, $H\in\mathfrak{a}_{pq}$, $X\in U(\mathfrak{k})$, $Y\in U(\mathfrak{h})$. Let $\mathscr{A}$ denote $U(\mathfrak{a}_{pq})\otimes U(\mathfrak{k})\otimes_{U(\mathfrak{m})}U(\mathfrak{h})$, viewed as a linear space.

LEMMA 3.2. *If $a\in A'$ then the map $\Gamma_a:\mathscr{A}\to U(\mathfrak{g})$ is a linear isomorphism. For each $D\in U(\mathfrak{g})$ there exists a unique $\Pi(D)\in\mathscr{R}\otimes\mathscr{A}$ such that, for all $a\in A'$:*

$$(3.2)\qquad \Gamma_a(\Pi(D))=D.$$

PROOF. Since $\mathfrak{m}=\mathfrak{l}_{kh}$, we have $\mathfrak{h}=\mathfrak{h}^c\oplus\mathfrak{m}$, and the first assertion follows from the infinitesimal Cartan decomposition (see Lemma 1.5) and the Poincaré-Birkhoff-Witt theorem.

The uniqueness part of the last assertion will follow from (3.2) and the first assertion. Therefore it suffices to prove the existence part. We proceed by induction on the degree deg (D) of D. If deg $(D)=0$ the assertion is trivial, so let $m>0$ and assume that the assertion has been proved already for deg $(D)<m$. Let $D\in U(\mathfrak{g})_m$ (the subalgebra of elements of degree $\leq m$). By the direct sum decomposition

(3.3) $\quad \mathfrak{g}=\mathfrak{n}\oplus\mathfrak{l}_{kq}\oplus\mathfrak{a}_{pq}\oplus\mathfrak{h}$

(cf. also (1.3)) and the Poincaré-Birkhoff-Witt theorem, there exists a $D_0\in U(\mathfrak{l}_k)U(\mathfrak{a}_{pq})U(\mathfrak{h})$ such that

$$D-D_0\in\mathfrak{n}U(\mathfrak{g})_{m-1}.$$

Since A centralizes $U(\mathfrak{l}_k)$, the assertion is true for D_0, so that we may restrict ourselves to the case $D_0=0$. Without loss of generality we may even assume that $D=X_\alpha V$, where $X_\alpha\in\mathfrak{g}_+^\alpha$ or $X_\alpha\in\mathfrak{g}_-^\alpha$ $(\alpha\in\Sigma^+)$, and $V\in U(\mathfrak{g})_{m-1}$. By Lemma 3.1 there exist $f_1,f_2\in\mathscr{R}^+$ such that

$$X_\alpha=f_1(a)(X_\alpha+\theta X_\alpha)^a+f_2(a)(X_\alpha+\tau X_\alpha),$$

for all $a\in A'$. Hence

$$D=f_1(a)(X_\alpha+\theta X_\alpha)^aV+f_2(a)\{V(X_\alpha+\tau X_\alpha)+\tilde{V}\},$$

where $\tilde{V}=[X_\alpha+\tau X_\alpha,V]\in U(\mathfrak{g})_{m-1}$, so that the assertion follows if we apply the induction hypothesis to V and $\tilde{V}$. $\square$

In a natural way $\mathscr{R}\otimes\mathscr{A}$ may be viewed as a M-module, the multiplication being given by

$$m(f\otimes H\otimes X\otimes_{U(\mathfrak{m})}Y)=f\otimes H\otimes Ad(m)X\otimes_{U(\mathfrak{m})}Ad(m)Y,$$

if $m\in M$, $f\in\mathscr{R}$, $H\in U(\mathfrak{a}_{pq})$, $X\in U(\mathfrak{k})$, $Y\in U(\mathfrak{h})$. Viewing $U(\mathfrak{g})$ as a M-module for the adjoint action, we now have the analogue of [4, Proposition 2.5]. We omit the proof, which is essentially the same.

PROPOSITION 3.3. *The linear map $\Pi: U(\mathfrak{g})\to\mathscr{R}\otimes\mathscr{A}$ is a M-module homomorphism.*

The filtration by degree on $U(\mathfrak{a}_{pq})$ naturally induces a filtration on $\mathscr{R}\otimes\mathscr{A}$, which we call the $\mathfrak{a}_{pq}$-filtration. The corresponding degree is called the $\mathfrak{a}_{pq}$-degree.

LEMMA 3.4. *If $X\in\mathfrak{n}U(\mathfrak{g})_m$ $(m\in\mathbb{N})$, then $\Pi(X)\in\mathscr{R}^+\otimes\mathscr{A}$ and $\Pi(X)$ has $\mathfrak{a}_{pq}$-degree $\leq m$.*

PROOF. This is easily verified in the course of the proof of Lemma 3.2. $\square$

By the definition (2.3) of E^M we have

$$\mu_1(XZ)\mu_2(Y^\vee)u = \mu_1(X)\mu_2((ZY)^\vee)u,$$

for $X \in U(\mathfrak{k})$, $Y \in U(\mathfrak{h})$, $Z \in U(\mathfrak{m})$ and $u \in E^M$. Therefore the bilinear map $U(\mathfrak{k}) \times U(\mathfrak{h}) \to \mathrm{Hom}_{\mathbb{C}}(E^M, E)$ given by $(X, Y) \mapsto \mu_1(X)\mu_2(Y^\vee)$ naturally induces a linear map $\xi_\mu : U(\mathfrak{k}) \otimes_{U(\mathfrak{m})} U(\mathfrak{h}) \to \mathrm{Hom}_{\mathbb{C}}(E^M, E)$, determined by

$$\xi_\mu(X \otimes Y) = \mu_1(X)\mu_2(Y^\vee),$$

for $X \in U(\mathfrak{k})$, $Y \in U(\mathfrak{h})$. We now define the linear map

$$\Pi_\mu : U(\mathfrak{g}) \to \mathscr{R} \otimes U(\mathfrak{a}_{pq}) \otimes \mathrm{Hom}_{\mathbb{C}}(E^M, E)$$

by

$$\Pi_\mu = (1 \otimes 1 \otimes \xi_\mu) \circ \Pi.$$

The elements of $\mathscr{R} \otimes U(\mathfrak{a}_{pq}) \otimes \mathrm{Hom}_{\mathbb{C}}(E^M, E)$ may be viewed as differential operators on A', mapping $C^\infty(A', E^M)$ into $C^\infty(A', E)$, in the following way. If $f \in \mathscr{R}$, $H \in U(\mathfrak{a}_{pq})$, $T \in \mathrm{Hom}_{\mathbb{C}}(E^M, E)$, then for $F \in C^\infty(A', E^M)$ we have

$$(f \otimes H \otimes T)F = fR_H(T \circ F).$$

Thus, if $X \in U(\mathfrak{g})$, then $\Pi_\mu(X)$ may be viewed as a differential operator on A', called the μ-radial component of X. We now have the following analogue of [4, Theorem 3.1], the proof being essentially the same.

PROPOSITION 3.5. *If $F \in C_\mu^\infty(G)$ and $X \in U(\mathfrak{g})$, then*

$$\mathrm{Res}\,(R_X F) = \Pi_\mu(X)\,\mathrm{Res}\,(F).$$

We also have the following analogue of [4, Proposition 3.2], which is an immediate consequence of Proposition 3.3 and the definition of E^M.

LEMMA 3.6. *The map Π_μ maps $U(\mathfrak{g})^M$ into $\mathscr{R} \otimes U(\mathfrak{a}_{pq}) \otimes \mathrm{End}_{\mathbb{C}}(E^M)$.*

Let $\mathscr{D}$ denote $\mathscr{R} \otimes U(\mathfrak{a}_{pq}) \otimes \mathrm{End}_{\mathbb{C}}(E^M)$, viewed as a subalgebra of the algebra of differential operators mapping $C^\infty(A', E^M)$ into itself.

PROPOSITION 3.7. *The map $\Pi_\mu : U(\mathfrak{g})^H \to \mathscr{D}$ is an algebra homomorphism.*

PROOF. If $X, Y \in U(\mathfrak{g})^H$, $F \in C_\mu^\infty(G)$, then $R_Y F \in C_\mu^\infty(G)$, so that by Proposition 3.5 we have

$$\mathrm{Res}\,(R_X R_Y F) = \Pi_\mu(X)\,\mathrm{Res}\,(R_Y F) = \Pi_\mu(X)\Pi_\mu(Y)\,\mathrm{Res}\,(F).$$

Hence $\Pi_\mu(XY) = \Pi_\mu(X)\Pi_\mu(Y)$ on $\mathrm{Res}\,(C_\mu^\infty(G))$. Using Lemma 1.6, we may now complete the proof just as in [4, Theorem 3.3]. □

Since L is of class $\mathscr{H}$ (cf. [19, p. 286, Thm. 13]), its subgroup M centralizes $\mathfrak{Z}(\mathfrak{l})$, so that $\mathfrak{Z}(\mathfrak{l}) \subset U(\mathfrak{g})^M$. By Lemma 3.6 it follows that Π_μ maps $\mathfrak{Z}(\mathfrak{l})$ into $\mathscr{D}$. Moreover, we have the following.

PROPOSITION 3.8. *The map $\Pi_\mu : \mathfrak{Z}(\mathfrak{l}) \to \mathcal{D}$ is an algebra homomorphism.*

PROOF. Let $X, Y \in \mathfrak{Z}(\mathfrak{l})$. Then by (1.1) and the Poincaré-Birkhoff-Witt theorem X can be written as a sum $\sum_i U_i H_i V_i$, and Y as a sum $\sum_j \tilde{U}_j \tilde{H}_j \tilde{V}_j$, where $U_i, \tilde{U}_j \in U(\mathfrak{l}_k)$, $H_i, \tilde{H}_j \in U(\mathfrak{a}_{pq})$ and $V_i, \tilde{V}_j \in U(\mathfrak{l}_h)$. Hence

$$(3.4)\quad \begin{cases} \Pi_\mu(X)\Pi_\mu(Y) = \sum\limits_{i,j} (1 \otimes H_i \otimes \mu_1(U_i)\mu_2(V_i^\vee))(1 \otimes \tilde{H}_j \otimes \mu_1(\tilde{U}_j)\mu_2(\tilde{V}_j^\vee)) \\ \qquad = \sum\limits_{i,j} 1 \otimes H_i \tilde{H}_j \otimes \mu_1(U_i \tilde{U}_j)\mu_2(V_i^\vee \tilde{V}_j^\vee) \\ \qquad = \sum\limits_{i,j} 1 \otimes H_i \tilde{H}_j \otimes \mu_1(U_i \tilde{U}_j)\mu_2([\tilde{V}_j V_i]^\vee). \end{cases}$$

On the other hand, since $Y \in \mathfrak{Z}(\mathfrak{l})$, we have $XY = \sum_i U_i H_i V_i Y = \sum_i U_i H_i Y V_i = \sum_{i,j} U_i H_i \tilde{U}_j \tilde{H}_j \tilde{V}_j V_i = \sum_{i,j} (U_i \tilde{U}_j)(H_i \tilde{H}_j)(\tilde{V}_j V_i)$, from which we infer that $\Pi_\mu(XY)$ equals (3.4). Hence the proposition. □

LEMMA 3.9. *If $X \in \mathfrak{n} U(\mathfrak{g})_m$ ($m \in \mathbb{N}$), then $\Pi_\mu(X)$ lies in $\mathcal{R}^+ \otimes U(\mathfrak{a}_{pq}) \otimes \otimes \mathrm{Hom}_{\mathbb{C}}(E^M, E)$ and its degree as a differential operator is $\leq m$.*

PROOF. The degree of the differential operator $\Pi_\mu(X)$ is less than or equal to the $\mathfrak{a}_{pq}$-degree of $\Pi(X)$. Hence the assertion is an immediate consequence of Lemma 3.4. □

COROLLARY 3.10. *If $Z \in \mathfrak{Z}$, then $\Pi_\mu(Z) - \Pi_\mu(\sigma(Z))$ lies in $\mathcal{R}^+ \otimes U(\mathfrak{a}_{pq}) \otimes \otimes \mathrm{End}_{\mathbb{C}}(E^M)$.*

PROOF. This follows immediately from Propositions 3.7,8, Lemma 3.9 and definition (2.12) of σ. □

THEOREM 3.11. *Let $F \in A_\mu(G, I)$, $t \in \mathbb{C}^A$ a Σ^+-leading exponent of F, and F_t the corresponding leading term. Then for $Z \in I$, we have:*

$$\Pi_\mu(\sigma(Z))F_t = 0.$$

For a proof the reader is referred to the proof of the analogous [4, Theorem 5.2].

PROOF OF THEOREM 2.4. The proof is essentially identical to the proof of [4, Proposition 5.4]. Proposition 3.8 and Theorem 3.11 have to be used instead of [4, Proposition 3.6 and Theorem 5.2]. □

4. RELATIONS BETWEEN THE P-SHIFTED CHARACTERS

Let I be a cofinite ideal in $\mathfrak{Z}$. If $P \in \mathcal{P}$, we let $X(P, I)$ denote the set of A-characters lying P-shifted over I (for the definition see the remark preceding Theorem 2.4). In this section we discuss the relations between the sets $X(P, I)$, for different $P \in \mathcal{P}$.

Let σ_P be the homomorphism $\mathfrak{Z}\to\mathfrak{Z}(\mathfrak{l})$ defined as in (2.12), with Σ^+ replaced by P. Thus, writing

$$\mathfrak{n}(P)=\sum_{\alpha\in P}\mathfrak{g}^{\alpha}$$

we have

$$Z-\sigma_P(Z)\in\mathfrak{n}(P)U(\mathfrak{g})$$

for $Z\in\mathfrak{Z}$. Let T_P be the automorphism of $U(\mathfrak{l})$ determined by

$$T_P(X)=X+\tfrac{1}{2}tr(ad(X)|\mathfrak{n}(P)),\quad X\in\mathfrak{l}.$$

Being an automorphism, T_P leaves $\mathfrak{Z}(\mathfrak{l})$ invariant and maps the ideal $\bar{I}_P=\mathfrak{Z}(\mathfrak{l})\sigma_P(I)$ of $\mathfrak{Z}(\mathfrak{l})$ onto the ideal $\bar{I}=\mathfrak{Z}(\mathfrak{l})T_P\sigma_P(I)$. Now the map

$$\mu=T_P\circ\sigma_P$$

is Harish-Chandra's isomorphism of $\mathfrak{Z}$ into $\mathfrak{Z}(\mathfrak{l})$, hence independent of P (cf. [19, p. 228]).

Therefore the ideal $\bar{I}$ is independent of the choice of $P\in\mathscr{P}$. We denote the set of generalized A-weights of $\mathfrak{Z}(\mathfrak{l})/\bar{I}$ by $X(I)$.

Define the element ϱ_P of $\mathfrak{a}_{pq}^*$ by

$$\varrho_P(X)=\tfrac{1}{2}tr(ad(X)|\mathfrak{n}(P)),$$

and let e^{ϱ_P} denote the positive character of A given by

$$a\mapsto a^{\varrho_P}=\exp(\varrho_P\log a).$$

PROPOSITION 4.1. *Let I be a cofinite ideal in $\mathfrak{Z}$, $P\in\mathscr{P}$. Then the set $X(P,I)$ of characters lying P-shifted over I is given by*

$$X(P,I)=e^{\varrho_P}\cdot X(I).$$

PROOF. If $H\in\mathfrak{a}_{pq}$, then it easily follows from the definition of T_P that for $Z\in U(\mathfrak{l})$ we have

$$T_P(HZ)=(H+\varrho_P(H))T_P(Z).$$

Hence ν is a generalized $\mathfrak{a}_{pq}$-weight of $\mathfrak{Z}(\mathfrak{l})/\bar{I}_P$ iff $\nu-\varrho_P$ is a generalized $\mathfrak{a}_{pq}$-weight of $\mathfrak{Z}(\mathfrak{l})/\bar{I}$. The assertion now follows by exponentiation. □

5. ASYMPTOTIC BEHAVIOUR ALONG THE WALLS

In this section we study the asymptotic behaviour of a $\mathfrak{Z}$-finite μ-spherical function $F:G\to E$ along the walls of $A^-(\Sigma^+)$, following the methods of [4].

Recall that Δ is the set of simple roots for the fixed choice Σ^+ from $\mathscr{P}$. To a subset Θ of Δ we associate the wall

$$A_\Theta^-(\Sigma^+)=\{a\in A;\ a^\alpha=1(\alpha\in\Theta),\ a^\alpha<1(\alpha\in\Delta\setminus\Theta)\}.$$

Thus $A_\emptyset^-(\Sigma^+)=A^-(\Sigma^+)$ and we have the disjoint union

$$cl(A^-(\Sigma^+))=\bigcup_{\Theta\subset\Delta}A_\Theta^-(\Sigma^+).$$

Moreover, we write

$$A^{-}(\Theta,\Sigma^{+})=\{a\in A;\ a^{\alpha}\leq 1(\alpha\in\Theta),\ a^{\alpha}<1(\alpha\in\Delta\setminus\Theta)\}.$$

So $A^{-}(\Delta,\Sigma^{+})=cl(A^{-}(\Sigma^{+}))$, and we have the disjoint union

$$A^{-}(\Theta,\Sigma^{+})=\bigcup_{\Psi\subset\Theta}A_{\Psi}^{-}(\Sigma^{+}).$$

We now fix a subset Θ of Δ and describe the grouping of terms procedure of [4], which will provide us with the expansion along $A_{\Theta}^{-}(\Sigma^{+})$.

Using the notations of Section 2, we have

$$A^{-}(\Theta,\Sigma^{+})=\underline{\lambda}((0,1]^{\Theta}\times(0,1)^{\Delta\setminus\Theta}\times(0,\infty)^{\Lambda\setminus\Delta}).$$

Following [4], we view $\mathbb{C}^{\Lambda\setminus\Theta}$ as embedded in $\mathbb{C}^{\Lambda}$, and let

$$pr_{\Lambda\setminus\Theta}:\mathbb{C}^{\Lambda}\to\mathbb{C}^{\Lambda\setminus\Theta}$$

denote the projection map. A notion of $(\Delta\setminus\Theta)$-integral equivalence in $\mathbb{C}^{\Lambda\setminus\Theta}$ is defined by

$$s\sim_{\Delta\setminus\Theta}t\ \text{iff}\ t-s\in\mathbb{Z}^{\Delta\setminus\Theta}$$

and the $(\Delta\setminus\Theta)$-order on $\mathbb{C}^{\Lambda\setminus\Theta}$ is defined by

$$s\leq_{\Delta\setminus\Theta}t\ \text{iff}\ t-s\in\mathbb{N}^{\Delta\setminus\Theta}.$$

The set $pr_{\Lambda\setminus\Theta}(S_{\Delta})$ splits into a finite number of $\sim_{\Delta\setminus\Theta}$-equivalence classes. To each such a class Ω we associate the element $\sigma(\Omega)$ of $\mathbb{C}^{\Lambda\setminus\Theta}$ defined by

$$\sigma(\Omega)_{\gamma}=\min\{t_{\gamma};\ t\in\Omega\}\quad(\gamma\in\Lambda\setminus\Theta).$$

Obviously $\sigma(\Omega)\leq_{\Delta\setminus\Theta}t$ for all $t\in\Omega$. Let $S_{\Delta\setminus\Theta}$ be the set of all $\sigma(\Omega)$, Ω as above. Then the elements of $S_{\Delta\setminus\Theta}$ are mutually $(\Delta\setminus\Theta)$-integrally inequivalent.

If $\lambda\in\Lambda$, we view $\log z_{\lambda}$ as a multivalued holomorphic function on $(\mathbb{C}^{*})^{\Lambda}$. Moreover, for $m\in\mathbb{N}^{\Lambda}$, $s\in\mathbb{C}^{\Lambda}$ we define

$$\log^{m}z=\prod_{\lambda\in\Lambda}(\log z_{\lambda})^{m_{\lambda}},$$

$$z^{s}=\prod_{\lambda\in\Lambda}\exp(s_{\lambda}\log z_{\lambda}).$$

For $s\in S_{\Delta\setminus\Theta}$, $m\in\mathbb{N}^{\Lambda\setminus\Theta}$, we define

$$F_{s,m}^{\Delta\setminus\Theta}=\sum_{t,n}F_{t,m+n}^{\Delta}z^{t-s}\log^{n}z,$$

the sum being taken over $n\in\mathbb{N}^{\Theta}$ and over all $t\in S_{\Delta}$ with $pr_{\Lambda\setminus\Theta}(t)$ $(\Delta\setminus\Theta)$-integrally equivalent to s. Obviously $t-s\in\mathbb{C}^{\Theta}\times\mathbb{N}^{\Delta\setminus\Theta}$, so that $F_{s,m}^{\Delta\setminus\Theta}$ is well defined on $(0,1)^{\Theta}\times D^{\Delta\setminus\Theta}$ and extends holomorphically to any simply connected open subset of $(D^{*})^{\Theta}\times D^{\Delta\setminus\Theta}$ containing $(0,1)^{\Theta}\times D^{\Delta\setminus\Theta}$. By the above and Theorem 2.5 (ii) it is now straightforward to check the following.

LEMMA 5.1. *There exist a finite set $S_{\Delta\setminus\Theta}$ of mutually $(\Delta\setminus\Theta)$-integrally inequivalent elements of $\mathbb{C}^{\Lambda\setminus\Theta}$ and for each $s\in S_{\Delta\setminus\Theta}$ a finite set $F_{s,m}^{\Delta\setminus\Theta}$*

($m \in \mathbb{N}^{\Delta \setminus \Theta}$) *of non-trivial holomorphic funlctions defined on a neighbourhood of* $(0,1)^{\Theta} \times D^{\Delta \setminus \Theta}$ *in* $(D^*)^{\Theta} \times D^{\Delta \setminus \Theta}$, *such that the following conditions are fulfilled.*

(i) *If* $s \in S_{\Delta \setminus \Theta}$, $\gamma \in \Delta \setminus \Theta$, *then there exists a* $m \in \mathbb{N}^{\Delta \setminus \Theta}$ *such that* $F_{s,m}^{\Delta \setminus \Theta}$ *does not vanish identically on the coordinate hyperplane* $z_{\gamma} = 0$.

(ii) *On* $A^-(\Sigma^+)$ *we have:*

$$F = \sum_{s,m} (F_{s,m}^{\Delta \setminus \Theta} \circ \underline{\alpha}) \underline{\lambda}^s \log^m \underline{\lambda}.$$

We also have the following analogues of [4, Lemma 6.1, Theorem 6.2]. We omit the proofs, since they are essentially the same.

LEMMA 5.2. *There exists an open subset* $C(\Theta)$ *of* $(\mathbb{C}^*)^{\Theta} \times D^{\Delta \setminus \Theta}$ *containing* $(0,1]^{\Theta} \times D^{\Delta \setminus \Theta}$, *such that the functions* $F_{s,m}^{\Delta \setminus \Theta}$ *extend to holomorphic functions* $C(\Theta) \to E^M$.

THEOREM 5.3. *Let* $F: G \to E$ *be a* $\mathfrak{Z}$*-finite* μ*-spherical function. Then for any set* $\Theta \subset \Delta$, *we have*

$$F = \sum_{s,m} (F_{s,m}^{\Delta \setminus \Theta} \circ \underline{\alpha}) \underline{\lambda}^s \log^m \underline{\lambda}$$

on $A^-(\Theta, \Sigma^+)$. *Here the summation extends over* $s \in S_{\Delta \setminus \Theta}$ *and finitely many* $m \in \mathbb{N}^{\Delta \setminus \Theta}$.

6. LEADING CHARACTERS AND GLOBAL ESTIMATES

Using the results of the preceding sections we are now able to describe the connections between leading characters and the global growth of $\mathfrak{Z}$-finite μ-spherical functions on G/H. Our results will be analogous to those of [4]. In fact they can be considered as more general, since every group of class $\mathscr{H}$ can be viewed as a symmetric space of class $\mathscr{H}$ (see also the introduction).

From now on, we will restrict ourselves to right H-invariant μ-spherical functions. Here μ is a smooth representation of K in a finite dimensional complex linear space E. We equip E with an inner product such that μ is unitary, and let $\|\cdot\|$ denote the corresponding norm. If $F \in A_{\mu}(G/H)$, then

$$\|F(kah)\| = \|F(a)\|,$$

for $h \in H$, $k \in K$, $a \in A$. Thus by the Cartan decomposition (Corollary 1.4), we see that $\|F\|$ can be estimated once its behaviour on $cl(A^-)$ is known. As we saw in the preceding sections, we cannot associate leading characters to F on the whole of A^-. However, for each $P \in \mathscr{P}$ we defined a finite set of P-leading characters, connected with the asymptotic behaviour of F on $A^-(P)$. As we will see, these govern the behaviour of F on the closed Weyl chamber $cl(A^-(P))$.

In view of the union

$$cl(A^-) = \bigcup_{P \in \mathscr{P}} cl(A^-(P)),$$

this enables us to connect global estimates for F with estimates of the P-leading characters for every $P \in \mathscr{P}$.

We start with some notations. If $P \in \mathscr{P}$, we define the ordering $\leq_P$ on positive characters of A by

$$\chi_1 \leq_P \chi_2 \text{ iff } \chi_1(a) \leq \chi_2(a) \text{ for all } a \in A^-(P).$$

Put:

$$A_\Delta = \{a \in A;\ a^\alpha = 1 \text{ for all } \alpha \in \Delta\}.$$

With the notations of Section 1, we have that

$$(6.1) \qquad G/H \simeq A_\Delta \times {}^0G/({}^0G \cap H).$$

Also, $A_\Delta \subset cl(A^-(P))$, so that $\chi_1 \leq_P \chi_2$ implies that $\chi_1 = \chi_2$ on A_Δ. We put

$$\chi_1 <_P \chi_2 \text{ iff } \chi_1(a) < \chi_2(a) \text{ for all } a \in cl(A^-(P)) \setminus A_\Delta.$$

We now have the analogue of [4, Theorem 7.1]. We omit the proof, which is essentially the same.

THEOREM 6.1. *Let F be a $\mathfrak{Z}$-finite μ-spherical function on G/H, let $P \in \mathscr{P}$, and let ω be a positive character of A. Then the following conditions are equivalent.*

(i) *for every P-leading character ν of F, we have*

$$|\nu| \leq_P \omega;$$

(ii) *there exist $M \geq 0$ and $m \geq 0$ such that*

$$\|F(a)\| \leq M\omega(a)(1 + \|\log a\|)^m$$

for all $a \in cl(A^-(P))$.

A character ζ of A_Δ is called the A_Δ-character of the μ-spherical function $F: G/H \to E$ if

$$F(ax) = \zeta(a)F(x) \quad (x \in G, a \in A_\Delta).$$

From the uniqueness statement in Theorem 2.5 we immediately obtain:

LEMMA 6.2. *Let F be a $\mathfrak{Z}$-finite μ-spherical function on G/H with the A_Δ-character ζ. Then the expansion of F in $A^-(\Sigma^+)$ has the form*

$$F = \sum (F^\Delta_{s,m} \circ \underline{\alpha}) \underline{\lambda}^s \log^m \underline{\lambda},$$

where the restrictions of $\underline{\lambda}^s$, $s \in S_\Delta$, to A_Δ are equal to ζ, and where $m \in \mathbb{N}^\Lambda$.

We now come to results concerning L^p-integrability. We could set up the theory for μ-spherical functions with a unitary A_Δ-character (see also [4]). But because of the decomposition (6.1) and the above lemma, we may as well assume that

$$G = {}^0G.$$

So let this be assumed from now on.

Given $P \in \mathscr{P}$, we define the positive character δ_P of A by

$$\delta_P(a) = \det\,(Ad(a)|\mathfrak{n}(P)) \quad (a \in A).$$

Thus, writing $m(\alpha) = \dim\, \mathfrak{g}^{\alpha}$ for $\alpha \in \Sigma$, we have

$$\delta_P(a) = \prod_{\alpha \in P} (a^{\alpha})^{m(\alpha)} \quad (a \in A).$$

A function f on A with values in a normed linear space is said to vanish at infinity in $A^{-}(P)$ if for every $\eta > 0$ there exists a $0 < \varepsilon < 1$ such that $\|f(a)\| < \eta$ for all $a \in A^{-}(P)$ with $\delta_P(a) < \varepsilon$.

THEOREM 6.3. *Let F be a $\mathfrak{Z}$-finite μ-spherical function on G/H, let $P \in \mathscr{P}$ and let ω be a positive character of A. Then the following conditions are equivalent:*

(i) *for every P-leading character ν of F we have*

$$|\nu| <_P \omega;$$

(ii) *the function $\omega^{-1}F$ vanishes at infinity in $A^{-}(P)$.*

PROOF. Without loss of generality, we may assume that $P = \Sigma^{+}$, and use the notations and results of Sections 2, 3, 5. It is then easy to see how to transfer the proof of [4, Theorem 7.4] to the present case, using $\delta_{\Sigma^{+}}$ instead of the function δ defined there.

Recalling Lemma 1.2, we define the function $\sigma = \sigma_{G/H}$ from G into $[0, \infty)$ by

$$\sigma(k \exp X \exp Y) = \|X\| = [-B(X, \theta X)]^{\frac{1}{2}}$$

for $k \in K$, $X \in \mathfrak{p} \cap \mathfrak{q}$, $Y \in \mathfrak{p} \cap \mathfrak{h}$. Then σ is left K- and right H-invariant, and

$$\sigma(kah) = \|\log a\|,$$

for $k \in K$, $a \in A$, $h \in H$ (see also [2]).

THEOREM 6.4. *Let F be a $\mathfrak{Z}$-finite μ-spherical function on G/H and let $1 \leq p < \infty$. Then the following conditions are equivalent:*

(i) *for each $P \in \mathscr{P}$ and every P-leading character ν of F, we have*

$$(6.2) \qquad |\nu| <_P \delta_P^{1/p};$$

(ii) *for every $l \geq 0$ the function $(1+\sigma)^{l}F$ is L^{p}-integrable;*

(iii) *F is L^{p}-integrable.*

PROOF. If $\alpha \in \Sigma$, we let

$$m_{+}(\alpha) = \dim\,(\mathfrak{g}^{\alpha}_{+}), \quad m_{-}(\alpha) = \dim\,(\mathfrak{g}^{\alpha}_{-}).$$

Thus $m(\alpha) = m_{+}(\alpha) + m_{-}(\alpha)$. Now let

$$(6.3) \qquad D(a) = \prod_{\alpha \in \Sigma^{+}} |a^{-\alpha} - a^{\alpha}|^{m_{+}(\alpha)} |a^{-\alpha} + a^{\alpha}|^{m_{-}(\alpha)}.$$

Then by [7, Theorem 2.6] we can fix normalizations of Haar measures dx on G/H and da on A, such that for $f \in L^1(G/H)$ we have

$$\int_{G/H} f(x)dx = \int_{K \times cl(A^-)} f(kaH)\, D(a)\, dk\, da.$$

Therefore $(1+\sigma)^l F$ is L^p-integrable on G iff for each $P \in \mathscr{P}$ we have

$$\int_{cl(A^-(P))} (1+\|\log a\|)^{lp} \|F(a)\|^p D(a)\, da < \infty. \tag{6.4}$$

Consequently it suffices to prove for a fixed $P \in \mathscr{P}$ the equivalence of the following statements:

(i)′ every P-leading character ν of F satisfies (6.2),

(ii)′ the estimate (6.4) holds for all $l \geq 0$,

(iii)′ the estimate (6.4) holds for $l = 0$.

Moreover, it is immediate that (6.3) remains valid if we replace Σ^+ by P, so that we may restrict ourselves to proving the equivalence of (i)′–(iii)′ for $P = \Sigma^+$.

Put $\delta = \delta_{\Sigma^+}$, suppose (i)′ and fix $l \geq 0$. In the notations of Section 3 we have $\Lambda = \Delta$. As in [4, proof of Theorem 7.5] we can find a positive character ω on A such that for every Σ^+-leading character ν of F we have

$$|\nu| \leq_{\Sigma^+} \omega \text{ and } \omega <_{\Sigma^+} \delta^{1/p}.$$

Moreover, one easily checks that there exist constants $M_1, M_2 > 0$ such that for all $a \in cl(A^-(\Sigma^+))$ we have

$$1 + \|\log a\| \leq M_1(1 + |\log \delta(a)|)$$

and

$$D(a) \leq M_2 \delta(a)^{-1}.$$

Using Theorem 6.1 we infer that the integral at the left of (6.4) may be estimated upon a positive constant times

$$\int_{cl(A^-(\Sigma^+))} \omega(a)^p (1 + |\log \delta(a)|)^{(l+m)p} \delta(a)^{-1} da,$$

which is finite because $\omega^p \delta^{-1} <_{\Sigma^+} 1$.

The implication (ii)′ ⇒ (iii)′ is obvious. For the remaining implication suppose that (iii)′ holds. Fix $0 < \varepsilon < 1$ and put

$$A_\varepsilon^-(\emptyset, \Sigma^+) = \{a \in A;\, a^\alpha < \varepsilon \quad (\alpha \in \Delta)\}.$$

There exists a constant $0 < C < 1$ such that

$$D(a) \geq C\delta(a)^{-1}, \quad a \in A_\varepsilon^-(\emptyset, \Sigma^+).$$

Combined with the estimate in (iii)′, this implies that

$$\int_{A_\varepsilon^-(\emptyset, \Sigma^+)} \|F(a)\|^p \delta(a)^{-1} da < \infty.$$

Now the proof of the implication (ii) ⇒ (i) in [4, Theorem 7.5] applies here too and gives us (i)′. □

7. SCHWARTZ FUNCTIONS ON G/H

In this section we assume that $G={}^0G$, so that $A_\Delta=\{1\}$. Given $1\leq p<\infty$ we define the space $\mathscr{C}^p(G/H)$ as the space of functions $f\in C^\infty(G/H)$ for which all the seminorms

$$N_{r,u}(f)=(1+\sigma)^r\|L_u f\|_{L^p(G/H)}$$

($r\geq 0$, $u\in U(\mathfrak{g})$) are bounded. By the classical Sobolev inequalities the space $\mathscr{C}^p(G/H)$, equipped with the topology induced by the above seminorms, is a Fréchet space. We call $\mathscr{C}(G/H)=\mathscr{C}^2(G/H)$ the space of rapidly decreasing, or Schwartz functions on G/H. In the group case our definition coincides with Harish-Chandra's definition of Schwartz space (cf. [19, p. 348]). We leave it to the reader to check that by slightly modified proofs, we have the following analogues of [2, Lemmas 1.1, 1.2].

LEMMA 7.1. *Let $1\leq p<\infty$. Then $C_c^\infty(G/H)$ is dense in $\mathscr{C}^p(G/H)$.*

LEMMA 7.2. *Let $1\leq p<\infty$. Then the algebra $\mathbb{D}(G/H)$ maps $\mathscr{C}^p(G/H)$ continuously into itself.*

The main result of this section is the following generalization of a well known result of Harish-Chandra (cf. [11, Lemma 43]).

THEOREM 7.3. *Let G be a group of class $\mathscr{H}$ with $G={}^0G$, and let f be a $\mathfrak{Z}$-finite and K-finite function on G/H. Fix $1\leq p<\infty$. Then f belongs to $L^p(G/H)$ if and only if it belongs to $\mathscr{C}^p(G/H)$.*

For the proof of this theorem we need a few lemmas. Let $\mathscr{R}$ and $\mathscr{R}^+$ be as in Lemma 3.1. One easily verifies that the following result can be proved in the same fashion as Lemma 3.1. Recall that we use the notation (2.5).

LEMMA 7.4. *Let $X_\alpha\in\mathfrak{g}_+^\alpha$ or $X_\alpha\in\mathfrak{g}_-^\alpha$ $(\alpha\in\Sigma^+)$. Then there exist $f_1,f_2\in\mathscr{R}^+$ such that*

$$\theta X_\alpha=f_1(a)(X_\alpha+\theta X_\alpha)+f_2(a)(X_\alpha+\tau X_\alpha)^{a^{-1}},\quad a\in A^-.$$

LEMMA 7.5. *Let $D\in U(\mathfrak{g})$. Then there exist finitely many $f_i\in\mathscr{R}$, $X_i\in U(\mathfrak{k})$, $H_i\in U(\mathfrak{a}_{pq})$, $Y_i\in U(\mathfrak{h})$ $(1\leq i\leq I)$, such that for all $a\in A^-$ we have*

$$D=\sum_{1\leq i\leq I} f_i(a)Y_i^{a^{-1}}H_iX_i.$$

PROOF. The proof goes by induction on deg (D), in the same fashion as the proof of Lemma 3.2. Here one has to use the decomposition

$$=\bar{\mathfrak{n}}\oplus\mathfrak{a}_{pq}\oplus\mathfrak{l}_{ph}\oplus\mathfrak{k}$$

instead of (3.3), and Lemma 7.4 instead of Lemma 3.1. □

LEMMA 7.6. *Let F be a $\mathfrak{Z}$-finite μ-spherical function $G/H \to E$, and let $u \in U(\mathfrak{g})$. Then there exists a $\mathfrak{Z}$-finite spherical function $\tilde{F}$ on G/H with values in a finite dimensional vector space $\tilde{E}$ such that the following conditions are fulfilled.*

(i) *There exists a $\xi \in \mathrm{Hom}_{\mathbb{C}}(\tilde{E}, E)$ such that $L_u F = \xi \circ \tilde{F}$.*

(ii) *For each $P \in \mathscr{P}$ and every P-leading exponent $\tilde{t}$ of $\tilde{F}$ there exists a P-leading exponent t of F with $\tilde{t} \in t + \mathbb{N}P$.*

REMARK. Observe that $\tilde{t} \in t + \mathbb{N}P$ implies $\underline{\lambda}^{\tilde{t}} \leq_P \underline{\lambda}^t$.

PROOF. Let U be the finite dimensional linear subspace of $U(\mathfrak{g})$ spanned by the elements $Ad(k)u$, $k \in K$. Let τ denote the adjoint representation of K restricted to U, and let τ^* be the contragredient representation of K in U^*. Fix a basis $\{u_j;\ 1 \leq j \leq J\}$ of U and let $\{u_j^*\}$ be the dual basis of U^*. Finally, put $\tilde{E} = U^* \otimes E$ and define $\tilde{F}: G/H \to \tilde{E}$ by:

$$\tilde{F}(x) = \sum_{1 \leq j \leq J} u_j^* \otimes L_{u_j} F(x). \tag{7.1}$$

Then the annihilator of F in $\mathfrak{Z}$ annihilates $\tilde{F}$ too, so that $\tilde{F}$ is $\mathfrak{Z}$-finite. Moreover, one easily checks that $\tilde{F}$ is $\tau^* \otimes \mu$-spherical. Since (i) is evident, it remains to prove (ii).

Without loss of generality we may assume that $P = \Sigma^+$. By the results of Section 2, F has a unique series expansion

$$F = \sum_{s,m} c_{s,m} \underline{\lambda}^s \log^m \underline{\lambda} \tag{7.2}$$

which converges absolutely on $A^-(\Sigma^+)$. Here $c_{s,m} \in E$, $s \in \mathbb{C}^\Delta$, $m \in \mathbb{N}^\Delta$ (recall that $\Lambda = \Delta$). We call $s \in \mathbb{C}^\Delta$ an exponent of F if $c_{s,m} \neq 0$ for some $m \in \mathbb{N}^\Delta$, and denote the set of exponents by $\mathscr{E}(F)$. Being $\mathfrak{Z}$-finite and spherical, $\tilde{F}$ also has a unique absolutely converging series expansion

$$\tilde{F} = \sum_{j,s,m} (u_j^* \otimes d_{s,m}^j) \underline{\lambda}^s \log^m \underline{\lambda} \tag{7.3}$$

on $A^-(\Sigma^+)$. Here $d_{s,m}^j \in E$, $s \in \mathbb{C}^\Delta$, $m \in \mathbb{N}^\Delta$, $1 \leq j \leq J$. Clearly

$$\mathscr{E}(\tilde{F}) = \bigcup_{1 \leq j \leq J} \mathscr{E}_j(\tilde{F}),$$

where $\mathscr{E}_j(\tilde{F})$ is the set of $s \in \mathbb{C}^\Delta$ such that $d_{s,m}^j \neq 0$ for some $m \in \mathbb{N}^\Delta$. From (7.1) and (7.3) it is immediate that

$$L_{u_j} F = \sum_{s,m} d_{s,m}^j \underline{\lambda}^s \log^m \underline{\lambda} \tag{7.4}$$

on $A^-(\Sigma^+)$, for each $1 \leq j \leq J$. Now fix j. Then by Lemma 7.5 there exist $f_i \in \mathscr{R}$, $X_i \in U(\mathfrak{k})$, $H_i \in U(\mathfrak{a}_{pq})$ $(1 \leq i \leq I)$, such that

$$u_j - \sum_{1 \leq i \leq I} f_i(a) H_i X_i \in \mathfrak{h}^{a^{-1}} U(\mathfrak{g})$$

for all $a \in A^-(\Sigma^+)$. Since F is right H-invariant it now follows that

$$L_{u_j}F(a) = \sum_i f_i(a)\mu(X_i^\vee)L(H_i)F(a)$$

for all $a \in A^-(\Sigma^+)$. From the definition of $\mathscr{R}$ it easily follows that there exist holomorphic functions $\varphi_i : D^\Delta \to \mathbb{C}$ such that $f = \varphi_i \circ \underline{\lambda}$ on $A^-(\Sigma^+)$. Moreover, via $\underline{\lambda}$, the differential operators H_i correspond to polynomials in the differential operators $z_\alpha \partial/\partial z_\alpha$ $(\alpha \in \Delta)$ on $\mathbb{C}^\Delta$. Since the expansion (7.2) arises from power series expansions in (z_α), it now follows that we may find an absolutely converging series expansion for $L(u_j)F$ on $A^-(\Sigma^+)$ by formally applying the expansion for the differential operator $\Sigma_i \varphi_i \circ \underline{\lambda}\mu(X_i^\vee)L(H_i)$ to the expansion (7.2) for F. By uniqueness this must give the expansion (7.4). Now, if $\alpha \in \Delta$, let $e_\alpha \in \mathbb{C}^\Delta$ be defined by $(e_\alpha)_\beta = 0$ if $\beta \neq \alpha$, $= 1$ if $\beta = \alpha$. Then obviously

$$z_\alpha \frac{\partial}{\partial z_\alpha} (z^s \log^m z) = s_\alpha z^s \log^m z + m_\alpha z^s \log^{m - e_\alpha} z,$$

for $\alpha \in \Delta$. We infer that $\mathscr{E}_j(\tilde{F}) \subset \mathscr{E}(F) + \mathbb{N}\Delta$. Hence $\mathscr{E}(\tilde{F}) \subset \mathscr{E}(F) + \mathbb{N}\Sigma^+$ and (ii) follows from the definition of leading exponent. □

PROOF OF THEOREM 7.3. Fix $r \in \mathbb{N}, u \in U(\mathfrak{g})$. We must show that

$$(1+\sigma)^r L_u f \in L^p(G/H).$$

Now let V be the finite dimensional span of the functions $L_k f$, $k \in K$. Via the left regular representation K acts on V. Let μ be the contragredient representation of K on the linear dual E of V, and define the function $F : G \to E$ by $F(x)v = v(x)$, for $x \in G$, $v \in V$. Then $F \in A_\mu(G/H)$. Moreover, let $\eta \in E^*$ be the element which canonically corresponds to $f \in V$. Then $f = \eta \circ F$. Hence $L_u f = \eta \circ L_u F$ and it suffices to show that

$$(1+\sigma)^r L_u F \in L^p(G/H, E).$$

Now select $\tilde{F} : G \to U^* \otimes E$ as in Lemma 7.6. Then for some $M > 0$ we have $\|L_u F(x)\| \leq M \|\tilde{F}(x)\|$ $(x \in G/H)$. Therefore it suffices to show that

$$(1+\sigma)^r \tilde{F} \in L^p(G/H, E).$$

Now this follows immediately from Theorem 6.4 and Lemma 7.6. □

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